Preferences, Homophily, and Social Learning*

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Abstract

We study a sequential model of Bayesian social learning in networks in which agents have heterogeneous preferences, and neighbors tend to have similar preferences—a phenomenon known as homophily. We find that the density of network connections determines the impact of preference diversity and homophily on learning. When connections are sparse, diverse preferences are harmful to learning, and homophily may lead to substantial improvements. In contrast, in a dense network, preference diversity is beneficial. Intuitively, diverse ties introduce more independence between observations while providing less information individually. Homophilous connections individually carry more useful information, but multiple observations become redundant.

1 Introduction

We rely on the actions of our friends, relatives, and neighbors for guidance in many decisions. This is true of both minor decisions, like where to go for dinner or what movie to watch tonight, and major, life-altering ones, such as whether to go to college or get a job after high school. This reliance may be justified because the decisions made by others are informative: our friends’ choices reveal some of their knowledge, which potentially enables us to make better decisions for ourselves. Nevertheless, in the vast majority of real-life situations, we view the implicit advice of our friends with at least some skepticism. In essentially any setting in which we would expect to find social learning, choices are influenced by preferences as much as by information. For instance, upon observing a friend’s decision to patronize a particular restaurant, we learn something about the restaurant’s quality, but

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her decision is also influenced by her preference over cuisines. Similarly, a stock purchase signals both company quality and risk preferences. In each case, two individuals with the same information may reasonably choose opposite actions.

The structure of the social network in which we are embedded determines what information is available through social ties. This structure comprises not only the pattern of links between individuals, but also patterns of preferences among neighbors. Since differences in preferences impact our ability to infer information from our friends’ choices, these preference patterns are potentially relevant to the flow of information in a network. In real-world networks, homophily is a widespread structural regularity: individuals interact much more frequently with others who share similar characteristics or preferences. Understanding the effects of network structure on information transmission requires that we consider how varying preference distributions and the extent of homophily influence social learning.

In this paper, we study a sequential model of social learning to elucidate how link structure and preference structure interact to determine learning outcomes. We adopt a particularly simple structure for individual decisions, assuming binary states and binary actions, to render a clear intuition on the role of network structure. Our work builds directly on that of Acemoglu et al. [2011] and Lobel and Sadler [2014], introducing two key innovations. First, we significantly relax the assumption of homogeneous preferences, considering a large class of preference distributions among the agents. We assume that any agent would choose an action matching the underlying state if given enough information in favor of that state, but individuals can differ arbitrarily in how they weigh the risk of error in each state. Second, we allow correlations between the link structure of the network and individual preferences, enabling our study of homophily.

Our principal finding is that the long-run impact of preference heterogeneity and homophily on learning depends crucially on network link density. In a relatively sparse network, diverse preferences present a clear barrier to information transmission. Heterogeneity between neighbors introduces an additional source of signal noise, but sufficiently strong homophily can ameliorate this noise, leading to learning results that are comparable to those in networks with homogeneous preferences. In a dense network, the situation is quite different. If preferences are sufficiently diverse, dense connectivity facilitates learning that is highly robust, and too much homophily may allow herd behavior to emerge. Our analysis shows that homophilous connections offer more useful information, but the information provided through additional ties quickly becomes redundant. Conversely, diverse ties provide less information individually, but they introduce more independence between observations. Hence, the value of a homophilous versus a diverse connection is highly dependent on the other social information available to an individual.

After describing the model, we first present a specialized example to highlight key mechanisms underlying our results for sparse networks. There are two distinct reasons why preference heterogeneity reduces the informational value of an observation. One source of inefficiency is uncertainty about how our neighbors make tradeoffs. Perhaps surprisingly, there is additional inefficiency simply by virtue of the opposing tradeoffs agents with differ-

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1See Marsden [1988], McPherson et al. [2001], and Curcurini et al. [2009, 2010].
ent preferences make. Consider a diner who prefers Japanese food to Italian. When choosing between restaurants, this diner might default to a Japanese restaurant unless there is strong evidence that a nearby Italian option is of higher quality. Observing this individual enter a Japanese restaurant is a much weaker signal of quality than observing the same individual enter an Italian restaurant. Consequently, this observation is unlikely to influence a person who strongly prefers Italian food. When connections are scarce, long-run learning depends upon individual observations carrying a greater amount of information, and homophily reduces inefficiency from both sources.

We next present general results for sparse networks; we say a network is sparse if agents have neighborhoods that are uniformly bounded in size. Our first two theorems consider learning outcomes in networks with no homophily, assuming that the preferences of each agent are independent and identically distributed. Theorem 1 is a negative result, showing that too much preference diversity renders asymptotic learning impossible in a sparse network, even if private signals are unbounded in strength. This contrasts sharply with what happens with homogeneous preferences. Acemoglu et al. [2011] demonstrate that unbounded private signals and a simple connectivity condition are sufficient for asymptotic learning in the latter case, but we find that there is always a preference distribution such that asymptotic learning fails no matter how connections are structured.

Theorem 2 refines our analysis for the special case in which agents observe at most one neighbor. Here, successful learning depends on an improvement principle, whereby agents can use their signals to obtain strictly higher ex-ante utility than a neighbor (Banerjee and Fudenberg [2004]). We find the improvement principle breaks down with even a little preference diversity; learning requires that strong preferences occur much less frequently than strong signals. Early work on learning suggests that diverse preferences can slow learning (Vives [1993, 1995]), even if information still aggregates in the long run. Perhaps surprisingly, Proposition 1 shows in a line network that as long as an improvement principle still holds, asymptotic learning rates are identical to the homogeneous preferences case.

We introduce the concept of a strongly homophilous network to state our positive findings. In a strongly homophilous network, each agent can find a neighbor with preferences arbitrarily close to her own in the limit as the network grows. Theorem 3 shows this rescues the improvement principle, and we obtain a learning result comparable to the case with homogeneous preferences. Further analysis provides a partial converse and shows that adding homophily to a network can only help learning in this model. In a family of networks we call simple $\kappa$-homophilous networks, we find that strong homophily is both necessary and sufficient for learning. The parameter $\kappa$ measures the extent of homophily in these networks, with larger values of $\kappa$ corresponding to greater homophily. Proposition 2 demonstrates that asymptotic learning succeeds in these networks if and only if $\kappa$ is strictly larger than 1. Proposition 3 then shows that adding homophily to a network that already satisfies the improvement principle will never disrupt learning. These results provide additional support for the contention that homophily is beneficial for long-run learning in sparse networks.

In contrast, when networks are dense, preference heterogeneity plays a positive role in the learning process. When agents have many sources of information, the law of large numbers
enables them to learn as long as there is some independence between the actions of their neighbors. If the support of the preference distribution is sufficiently broad, there are always some types that act on their private information, creating the needed independence. Goeree et al. [2006] find this leads to asymptotic learning in a complete network even with bounded private beliefs. Theorem 4 generalizes this insight to a broad class of network structures, showing that learning robustly succeeds within a dense cluster of agents as long as two conditions are satisfied: there is sufficient diversity within the cluster, and agents in the cluster are aware of its existence. This last condition, which we call identification, is a technical contribution of our paper. In a complete network the condition is unnecessary, but in a complex network in which only some agents constitute the cluster, learning may fail without it.

Since homophily reduces the diversity of preferences among an agent’s neighbors, it has the potential to interfere with learning in a dense network. Example 4 shows that if homophily is especially extreme, with agents sorting themselves into two isolated clusters based on preferences, each cluster has the potential to herd, and learning is incomplete. We might imagine a network of individuals with strongly polarized political beliefs, in which agents on each side never interact with agents on the other. Interestingly, Proposition 4 shows that a small amount of bidirectional communication between the two clusters is sufficient to overcome this failure. The seminal paper of Bikhchandani et al. [1992] emphasized the fragility of herds, showing that introducing a little outside information can quickly reverse herding. Our result is similar in spirit, showing that in the long run, the negative impact of homophily on learning in dense networks is also fragile.

Our work makes several contributions to the study of social learning and the broader literature on social influence. First, we show clearly how preference heterogeneity has distinct effects on two key learning mechanisms, leading to different long-run outcomes as a function of network structure. Most of the social learning literature has assumed that all individuals in society have identical preferences, differing only in their knowledge of the world.² In one exception, Smith and Sorensen [2000] show in a complete network that diverse preferences generically lead to an equilibrium they call confounded learning, in which observed choices become uninformative. However, a key assumption behind this result is that preferences are non-monotonic in the state: different agents may change their actions in opposite directions in response to the same new information. This excludes instances in which utilities contain a common value component. Goeree et al. [2006] consider a model with private and common values in a complete network, obtaining a result that we generalize to complex dense networks. Work that situates learning in the context of investment decisions has also considered preference heterogeneity, finding that it hinders learning and may increase the incidence of cascades, leading to pathological spillover effects (Cipriani and Guarino [2008]). Our innovation is the incorporation preference heterogeneity and complex network structure into the same model. We obtain results for general classes of networks, providing new insights on the

²This includes the foundational papers by Banerjee [1992] and Bikhchandani et al. [1992] as well as more recent work by Bala and Goyal [1998], Gale and Kariv [2003], Celen and Kariv [2004], Guarino and Ianni [2010], Acemoglu et al. [2011, 2013], Mossel et al. [2012], Mueller-Frank [2013], and Lobel and Sadler [2014].
underlying learning mechanisms.

In tandem with our study of preference diversity, we shed light on the potential impact of homophily on social learning. Despite its prevalence in social networks, few papers on learning have considered homophily. Golub and Jackson [2012] study group-based homophily in a non-Bayesian framework, finding that homophily slows down the convergence of beliefs in a network. To the best of our knowledge, this is the first paper on learning in a context with preference-based homophily. We offer a nuanced understanding of its effects, showing two ways it can impact information transmission: homophily increases the information content of observations while decreasing independence between observations. The first effect suggests one reason why empirical studies point to homophilous ties having greater influence over the learning process. For instance, Centola [2011] provides experimental evidence that positive health behaviors spread slower in networks with diverse types, while propagating faster and more broadly in networks with greater homophily. In an empirical study of learning about health plan choices, Sorensen [2006] finds that employees learn more from peers in a similar demographic. Similarly, Conley and Udry [2010] find that farmers in their study learn more about agricultural techniques from other farmers with similar wealth levels.

We can also fruitfully compare our results to findings in the empirical literature on tie strength and social influence. There has been much debate on the relative importance of weak versus strong ties in propagating behaviors in a social network. Notable work has found that an abundance of weak ties bridging structural holes often accounts for the majority of diffusion or information flows in a network (see, for instance, Granovetter [1973], Burt [2004], and Bakshy et al. [2012]), but others have emphasized the importance of strong ties that carry greater bandwidth, especially when the information being transmitted is complex (see Hansen [1999], Centola and Macy [2007], and Aral and Walker [2013]). Aral and Van Alstyne [2011] analyze the diversity-bandwidth tradeoff in behavior diffusion, arguing that weak, diverse ties are more likely to provide access to novel information, but strong, homophilous ties transmit a greater volume of information, potentially resulting in more novel information flowing through strong ties. Our work offers an alternative theoretical basis for this tradeoff, showing how rationality implies that decisions made by a homophilous contact convey less noisy information and reflect similar priorities to our own. Furthermore, our analysis of the impact of network structure provides guidance on which contexts should lead strong or weak ties to be most influential.

The next section details our model. Section 3 analyzes preferences and homophily in sparse networks, emphasizing the difficulties created by preference diversity and the benefits conferred by homophily. Section 4 then provides general results on dense networks, and section 5 concludes. Proofs of the more technical results are given in the appendix.

2 Model

Each agent in a countably infinite set, indexed by \( n \in \mathbb{N} \), sequentially chooses between two mutually exclusive actions, 0 and 1. That is, agents make choices in indexed order, with agents \( m < n \) choosing an action prior to agent \( n \)'s decision. The decision of agent \( n \)
is denoted by \( x_n \). Each agent has a random, privately observed type \( t_n \in (0, 1) \), and the payoff to agent \( n \) is a function of \( n \)'s decision, \( n \)'s type, and the underlying state of the world \( \theta \in \{0, 1\} \). For simplicity, we assume that a priori the two possible states are equally likely.

Agent \( n \)'s payoff is

\[
u(t_n, x_n, \theta) = \begin{cases} 
(1 - \theta) + t_n, & \text{if } x_n = 0, \\
\theta + (1 - t_n), & \text{if } x_n = 1.
\end{cases}
\]

Observe that this utility function has two components: the first depends on whether \( x_n = \theta \) and is common to all types, while the second is independent of the state, depending only on the private type \( t_n \) and the action chosen \( x_n \). Agents balance between two objectives: each agent \( n \) wishes to choose \( x_n = \theta \), but the agent also has a type-dependent preference for a particular action. The type \( t_n \) determines how agent \( n \) makes this tradeoff. If \( t_n = \frac{1}{2} \), agent \( n \) is neutral between the two actions and chooses solely based on which action is more likely to realize \( x_n = \theta \). Higher values of \( t_n \) correspond to stronger preferences towards action 0, and lower values of \( t_n \) correspond to stronger preferences towards action 1. Restricting types to the interval \((0, 1)\) implies that no agent is pre-committed to either action.

In line with earlier work in sequential learning models, we study asymptotic outcomes of the learning process. We interpret these outcomes as measures of long-run efficiency and information aggregation, while acknowledging that our understanding of short and medium-run behavior is necessarily limited. We say that asymptotic learning occurs if, in the limit as \( n \) grows, agents act as though they have perfect knowledge of the state. Equivalently, asymptotic learning occurs if actions converge in probability on the true state:

\[
\lim_{n \to \infty} \mathbb{P}_\sigma(x_n = \theta) = 1.
\]

This is the strongest limiting result achievable in our framework. Asymptotic learning implies almost sure convergence of individual beliefs, but almost sure convergence of actions is impossible because agents must continue to act based on their signals for full information aggregation to occur. Whether asymptotic learning obtains is the main focus of our analysis, but we also comment on learning rates to shed some light on shorter term outcomes.

Agent \( n \) is endowed with a private signal \( s_n \), a random variable taking values in an arbitrary metric space \( S \). Conditional on the state \( \theta \), each agent’s signal is independently drawn from a distribution \( \mathbb{F}_\theta \). The pair of measures \( (\mathbb{F}_0, \mathbb{F}_1) \) constitutes the signal structure of the model and is common knowledge. We assume these measures are not almost everywhere equal, meaning agents have a positive probability of receiving an informative signal. The private belief \( p_n \equiv \mathbb{P}(\theta = 1 | s_n) \) of agent \( n \) is a sufficient statistic for the information contained in \( s_n \); we write \( \mathcal{G}_i(r) \equiv \mathbb{P}(p_n \leq r | \theta = i) \) for the state-conditional private belief distribution. We assume the private beliefs have full support over an interval \((\beta, \overline{\beta})\), where \( 0 \leq \beta < \frac{1}{2} < \overline{\beta} \leq 1 \). As in previous papers,\(^3\) we say private beliefs are unbounded if \( \beta = 0 \) and \( \overline{\beta} = 1 \) and bounded if \( \beta > 0 \) and \( \overline{\beta} < 1 \).

In addition to the signal \( s_n \), agent \( n \) also has access to social information. Agent \( n \) observes the actions of the agents in her neighborhood \( B(n) \subseteq \{1, 2, \ldots, n-1\} \). That is, agent \( n \)

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\(^3\)See Smith and Sorensen [2000], Acemoglu et al. [2011], and Lobel and Sadler [2014].
observes the value of $x_m$ for all $m \in B(n)$. The neighborhood $B(n)$ is a random variable, and the sequence of neighborhood realizations describe a social network of connections between the agents. We call the probability $q_n = \mathbb{P}(\theta = 1 \mid B(n), x_m$ for $m \in B(n)$) agent n’s social belief.

The structure of the network is represented as a joint distribution $Q$ over all possible sequences of neighborhoods and types; we call $Q$ the network topology and assume $Q$ is common knowledge. We allow correlations across neighborhoods and types since this is necessary to model homophily, but we assume $Q$ is independent of the state $\theta$ and the private signals. The types $t_n$ share a common marginal distribution $H$, and the distribution $H$ has full support in some range $(\gamma, \overline{\gamma})$, where $0 \leq \gamma \leq \frac{1}{2} \leq \overline{\gamma} \leq 1$. The range $(\gamma, \overline{\gamma})$ provides one measure of the diversity of preferences. If both $\gamma$ and $\overline{\gamma}$ are close to $\frac{1}{2}$, then all agents are nearly indifferent between the two actions prior to receiving information about the state $\theta$. On the other hand, if $\gamma$ is close to 0 and $\overline{\gamma}$ is close to 1, then there are agents with strong biases towards particular actions. Analogous to our definition for private beliefs, we say preferences are unbounded if $\gamma = 0$ and $\overline{\gamma} = 1$ and bounded if $\gamma > 0$ and $\overline{\gamma} < 1$.

For algebraic simplicity, we often impose the following symmetry and density assumptions, but where they save little effort, we refrain from invoking them.

**Assumption 1.** The private belief distributions are anti-symmetric: $G_0(r) = 1 - G_1(1 - r)$ for all $r \in [0,1]$; the marginal type distribution $H$ is symmetric around $1/2$; the distributions $G_0$, $G_1$, and $H$ have densities.

Agent n’s information set, denoted by $I_n \in \mathcal{I}_n$, consists of the type $t_n$, the signal $s_n$, the neighborhood realization $B(n)$, and the decisions $x_m$ for every agent $m \in B(n)$. Agent n’s strategy $\sigma_n$ is a function mapping realizations of $I_n$ to decisions in $\{0,1\}$. A strategy profile $\sigma$ is a sequence of strategies for each agent. We use $\sigma_{-n}$ to denote the set of all strategies other than agent n’s, $\sigma_{-n} = \{\sigma_1, \ldots, \sigma_{n-1}, \sigma_{n+1}, \ldots\}$, and we can represent the strategy profile as $\sigma = (\sigma_n, \sigma_{-n})$. Given a strategy profile $\sigma$, the sequence of actions $\{x_n\}_{n \in \mathbb{N}}$ is a stochastic process with measure $\mathbb{P}_\sigma$. We analyze the perfect Bayesian equilibria of the social learning game, denoting the set of equilibria by $\Sigma$.

We conclude this section with two basic lemmas required in our subsequent analysis. These lemmas provide provide useful properties of belief distributions and characterize best response behavior.

**Lemma 1.** The private belief distributions $G_0$ and $G_1$ satisfy the following properties:

(a) For all $r \in (0,1)$, we have $\frac{dG_0}{dr}(r) = \frac{1-r}{r}$.

(b) For all $0 < z < r < 1$, we have $G_0(r) \geq \frac{1-r}{r}G_1(r) + \frac{r-z}{2}G_1(z)$.

(c) For all $0 < r < w < 1$, we have $1 - G_1(r) \geq \frac{r}{1-r} (1 - G_0(r)) + \frac{w-r}{2} (1 - G_0(w))$.

(d) The ratio $\frac{G_0(r)}{G_1(r)}$ is non-increasing in $r$ and is strictly larger than 1 for all $r \in (\beta, \overline{\beta})$.

(e) Under Assumption 1, we have $dG_0(r) = \frac{1-r}{r}dG_0(1-r)$, and $dG_1(r) = \frac{r}{1-r}dG_1(1-r)$. 

Proof. Parts (a) through (d) comprise Lemma 1 in Acemoglu et al. [2011]. Part (e) follows immediately from part (a) together with Assumption 1.

Lemma 2. Let $\sigma \in \Sigma$ be an equilibrium, and let $I_n \in \mathcal{I}_n$ be a realization of agent $n$’s information set. The decision of agent $n$ satisfies

$$x_n = \begin{cases} 0, & \text{if } \mathbb{P}_{\sigma}(\theta = 1 | I_n) < t_n, \\ 1, & \text{if } \mathbb{P}_{\sigma}(\theta = 1 | I_n) > t_n, \end{cases}$$

and $x_n \in \{0, 1\}$ otherwise. Equivalently, the decision of agent $n$ satisfies

$$x_n = \begin{cases} 0, & \text{if } p_n < \frac{t_n(1-q_n)}{t_n(1-q_n) + q_n(1-t_n)}, \\ 1, & \text{if } p_n > \frac{t_n(1-q_n)}{t_n(1-q_n) + q_n(1-t_n)}, \end{cases}$$

and $x_n \in \{0, 1\}$ otherwise.

Proof. Agent $n$ maximizes her expected utility given her information set $I_n$ and the equilibrium $\sigma \in \Sigma$. Therefore, she selects action 1 if $\mathbb{E}_\sigma[\theta + (1 - t_n) | I_n] > \mathbb{E}_\sigma[(1 - \theta) + t_n | I_n]$, where $\mathbb{E}_\sigma$ represents the expected value in a given equilibrium $\sigma \in \Sigma$. The agent knows her type $t_n$, so this condition is equivalent to $\mathbb{E}_\sigma[\theta | I_n] > t_n$. Since $\theta$ is an indicator function and $t_n$ is independent of $\theta$, we have $\mathbb{E}_\sigma[\theta | I_n] = \mathbb{P}_\sigma(\theta = 1 | s_n, B(n), x_m $ for all $m \in B(n))$, proving the clause for $x_n = 1$. The proof for $x_n = 0$ is identical with the inequalities reversed. The second characterization follows immediately from the first and an application of Bayes’ rule.

An agent chooses action 1 whenever the posterior probability that the state is 1 is higher than her type, and she chooses action 0 whenever the probability is lower than her type. Therefore, agent $n$’s type $t_n$ can be interpreted as the minimum belief agent $n$ must have that the true state is $\theta = 1$ before she will choose action $x_n = 1$.

3 Sparsely Connected Networks

We call a network sparse if there exists a uniform bound on the size of all neighborhoods; we say the network is $M$-sparse if, with probability one, no agent has more than $M$ neighbors. We begin our exploration of learning in sparse networks with an example.

Example 1. Suppose the signal structure is such that $G_0(r) = 2r - r^2$ and $G_1(r) = r^2$. Consider the network topology $Q$ in which each agent observes her immediate predecessor with probability one, agent 1 has type $t_1 = \frac{1}{5}$ with probability 0.5 and type $t_1 = \frac{4}{5}$ with probability 0.5. Any other agent $n$ has type $t_n = 1 - t_{n-1}$ with probability 1.

In this network, agents fail to asymptotically learn the true state, despite having unbounded beliefs and satisfying the connectivity condition from Acemoglu et al. [2011].
Without loss of generality, suppose $t_1 = \frac{1}{5}$. We show inductively that all agents with odd indices err in state 0 with probability at least $\frac{1}{4}$, and likewise agents with even indices err in state 1 with probability at least $\frac{1}{4}$. For the first agent, observe that $G_0 \left( \frac{1}{5} \right) = \frac{9}{25} < \frac{3}{4}$, so the base case holds. Now suppose the claim holds for all agents of index less than $n$, and $n$ is odd. The social belief $q_n$ is minimized if $x_{n-1} = 0$, taking the value

$$
\frac{\mathbb{P}_\sigma(x_{n-1} = 0 | \theta = 1)}{\mathbb{P}_\sigma(x_{n-1} = 0 | \theta = 1) + \mathbb{P}_\sigma(x_{n-1} = 0 | \theta = 0)} \geq \frac{1/4}{1/4 + 1} = \frac{1}{5}.
$$

It follows from Lemma 2 that agent $n$ will choose action 1 whenever $p_n > \frac{1}{2}$. We obtain the bound

$$
\mathbb{P}_\sigma(x_n = 1 | \theta = 0) \geq 1 - G_0 \left( \frac{1}{2} \right) = \frac{1}{4}.
$$

An analogous calculation proves the inductive step for agents with even indices. Hence, all agents err with probability bounded away from zero, and asymptotic learning fails. Note this failure is quite different from the classic results on herding and information cascades. Agents continue acting on their private signals in perpetuity, and both actions are chosen infinitely often. Here, the difference in preferences between neighbors confounds the learning process, bounding the informational content of any observation even though there are signals of unbounded strength.

This example sheds light on the mechanisms that drive our results in this section. Given a simple network structure in which each agent observes one neighbor, long-run learning hinges on whether an individual can improve upon a neighbor’s decision. We can decompose an agent’s utility into two components: utility obtained through copying her neighbor and the improvement her signal allows over this. The improvement component is always positive when private beliefs are unbounded, creating the possibility for improvements to accumulate towards complete learning over time. With homogeneous preferences, this is exactly what happens because copying a neighbor implies earning the same utility as that neighbor, so utility is strictly increasing along a chain of agents. In our example, differing preferences mean that an agent earns less utility than her neighbor if she copies. The improvement component is unable to make up this loss, and asymptotic learning fails.

Although we shall model far more general network structures, in which agents have multiple neighbors and are uncertain about their neighbors’ types, this insight is important throughout this section. Learning fails if there is a chance that an agent’s neighbors have sufficiently different preferences because opposing risk tradeoffs obscure the information that is important to the agent. Homophily in the network allows individuals to identify neighbors who are similar to themselves, and an improvement principle can operate along homophilous connections.

### 3.1 Failure Caused by Diverse Preferences

This subsection analyzes the effects of diverse preferences in the absence of homophily. We assume that all types and neighborhoods are independently distributed.
Assumption 2. The neighborhoods \( \{ B(n) \}_{n \in \mathbb{N}} \) and types \( \{ t_n \}_{n \in \mathbb{N}} \) are mutually independent.

Definition 1. Define the ratio \( R_t \equiv \frac{t(1-\epsilon)}{t(1-\epsilon)+(1-t)\epsilon} \). Given private belief distributions \( \{ G_\theta \} \), we say that the preference distribution \( H \) is \( M \)-diverse with respect to beliefs if there exists \( \epsilon \) with \( 0 < \epsilon < \frac{1}{2} \) such that

\[
\int_0^1 G_0 (R_t) - \left( \frac{1-\epsilon}{\epsilon} \right)^{\frac{1}{M}} G_1 (R_t^{1-\epsilon}) \, dH(t) \leq 0.
\]

A type distribution with \( M \)-diverse preferences has at least some minimal amount of probability mass located near the endpoints of the unit interval. We could also interpret this as a polarization condition, ensuring there are many people in society with strongly opposed preferences. As \( M \) increases, the condition requires that more mass is concentrated on extreme types. For any \( N < M \), an \( M \)-diverse distribution is also \( N \)-diverse. Our first theorem shows that without any correlation between types and neighborhoods, \( M \)-diversity precludes learning in an \( M \)-sparse network.

Theorem 1. Suppose the network is \( M \)-sparse, and let Assumptions 1 and 2 hold. If preferences are \( M \)-diverse with respect to beliefs, then asymptotic learning fails. Moreover, for a given signal structure, the probability that an agent’s action matches the state is uniformly bounded away from \( 1 \) across all \( M \)-sparse networks.

Proof. Given the \( \epsilon \) in definition 1, we show the social belief \( q_n \) is contained in \( [\epsilon, 1-\epsilon] \) with probability 1 for all \( n \), which immediately implies all of the stated results. Proceed inductively; the case \( n = 1 \) is clear with \( q_1 = \frac{1}{2} \). Suppose the result holds for all \( n \leq k \). Let \( B(k+1) = B \) with \( |B| \leq M \), and let \( x_B \) denote the random vector of observed actions. If \( x \in \{0,1\}^{|B|} \) is the realized vector of decisions that agent \( k+1 \) observes, we can write the social belief \( q_{k+1} \) as \( \frac{1}{1+l} \), where \( l \) is a likelihood ratio:

\[
l = \frac{\mathbb{P}_\sigma (x_B = x | \theta = 0)}{\mathbb{P}_\sigma (x_B = x | \theta = 1)} = \prod_{m \in B} \frac{\mathbb{P}_\sigma (x_m = x_m | \theta = 0, x_i = x_i, i < m)}{\mathbb{P}_\sigma (x_m = x_m | \theta = 1, x_i = x_i, i < m)}.
\]

Conditional on a fixed realization of the social belief \( q_m \), the decision of each agent \( m \) in the product terms is independent of the actions of the other agents. Since \( q_m \in [\epsilon, 1-\epsilon] \) with probability 1, we can fix the social belief of agent \( m \) at the endpoints of this interval to obtain bounds. Each term of the product is bounded above by

\[
\int_0^1 G_0 (R_t) \, dH(t) \leq \left( \frac{1-\epsilon}{\epsilon} \right)^{\frac{1}{M}}.
\]

This in turn implies that \( q_{k+1} \geq \epsilon \). A similar calculation using a lower bound on the likelihood ratio shows that \( q_{k+1} \leq 1 - \epsilon \). \( \square \)
Regardless of the signal or network structure, we can always find a preference distribution that will disrupt asymptotic learning in an \(M\)-sparse network. Moreover, due to the uniform bound, this is a more severe failure than what occurs with bounded beliefs and homogeneous preferences. Acemoglu et al. [2011] show that asymptotic learning often fails in an \(M\)-sparse network if private beliefs are bounded, but the point at which learning stops depends on the network structure and may still be arbitrarily close to complete learning. In this sense, the challenges introduced by preference heterogeneity are more substantial than those introduced by weak signals.

An analysis of 1-sparse networks allows a more precise characterization of when diverse preferences cause the improvement principle to fail. In the proof of our result, we employ the following convenient representation of the improvement function under Assumptions 1 and 2. Recall the notation \( R_t^y \equiv \frac{t(1-y)}{(1-y) + (1-t)y} \).

**Lemma 3.** Suppose Assumptions 1 and 2 hold. Define \( \mathcal{Z} : [\frac{1}{2}, 1] \to [\frac{1}{2}, 1] \) by

\[
\mathcal{Z}(y) = y + \int_0^1 (1 - y) G_0(R_t^y) - y G_1(R_t^y) \ d\mathbb{H}(t). \tag{1}
\]

In any equilibrium \( \sigma \), we have \( \mathbb{P}_\sigma(x_n = \theta | B(n) = \{m\}) = \mathcal{Z}(\mathbb{P}_\sigma(x_m = \theta)) \).

**Proof.** Observe under Assumption 1 we have \( \mathbb{P}_\sigma(x_n = \theta) = \mathbb{P}_\sigma(x_n = \theta | \theta = i) \) for each \( i \in \{0, 1\} \), and \( \int_0^1 f(t) d\mathbb{H}(t) = \int_0^1 f(1 - t) d\mathbb{H}(t) \) for any \( f \). Taking \( y = \mathbb{P}_\sigma(x_m = \theta) \), we have

\[
\int_0^1 \sum_{i=0}^1 \mathbb{P}_\sigma(x_n = \theta | B(n) = \{m\}, x_m = i, \theta = 0, t_n = t) \mathbb{P}_\sigma(x_m = i | \theta = 0) d\mathbb{H}(t)
\]

\[= \int_0^1 \sum_{i=0}^1 \mathbb{P}(p_n \leq R_t^y) \mathbb{P}_\sigma(x_m = i | \theta = 0) d\mathbb{H}(t) \]

\[= \int_0^1 y G_0(R_t^1 - y) + (1 - y) G_0(R_t^y) d\mathbb{H}(t) \]

\[= y + \int_0^1 (1 - y) G_0(R_t^y) - y (1 - G_0(R_t^1 - y)) d\mathbb{H}(t) \]

\[= y + \int_0^1 (1 - y) G_0(R_t^y) - y G_1(R_t^1 - t) d\mathbb{H}(t) \]

\[= y + \int_0^1 (1 - y) G_0(R_t^y) - y G_1(R_t^y) d\mathbb{H}(t), \]

as desired. \( \square \)

**Theorem 2.** Suppose the network is 1-sparse, and let Assumptions 1 and 2 hold. If either of the following conditions is met, then asymptotic learning fails.

\[
\text{Proof.}
\]
(a) We have
\[
\liminf_{t \to 0} \frac{\mathbb{H}(t)}{t} > 0.
\]

(b) For some \( K > 1 \) we have
\[
\lim_{r \to 0} \frac{G_0(r)}{r^{K-1}} = c > 0, \quad \text{and} \quad \int_0^1 \frac{t^{K-1}(K - (2K - 1)t)}{(1 - t)^K} d\mathbb{H}(t) < 0.
\]

Proof. Each part follows from the existence of an \( \epsilon > 0 \) such that \( Z(y) \) given in Lemma 3 satisfies \( Z(y) < y \) for \( y \in [1 - \epsilon, 1) \). This immediately implies that learning is incomplete.

Assume the condition of part (a) is met. Divide by \( 1 - y \) to normalize; for any \( \epsilon > 0 \) we obtain the bound
\[
\int_0^1 G_0(R_t^y) - \frac{y}{1 - y} G_1(R_t^y) d\mathbb{H}(t) \leq \int_0^{1-\epsilon} G_0(R_t^y) d\mathbb{H}(t) + \int_{1-\epsilon}^1 G_0(R_t^y) d\mathbb{H}(t)
\]
\[
- \frac{y}{1 - y} \int_y^1 G_1(R_t^y) d\mathbb{H}(t)
\]
\[
\leq G_0(R_{1-\epsilon}^y) + \mathbb{H}(\epsilon) - G_1 \left( \frac{1}{2} \right) \frac{y\mathbb{H}(1 - y)}{1 - y}.
\]

Choosing \( \epsilon \) sufficiently small, the second term is negligible, while for large enough \( y \), the first term approaches zero, and the third is bounded above by a constant less than zero. Hence the improvement term in Eq. (1) is negative in \([1 - \epsilon, 1)\) for sufficiently small \( \epsilon \).

The proof of part (b) is presented in the appendix.

These conditions are significantly weaker than 1-diversity. We establish a strong connection between the tail thicknesses of the type and signal distributions. The improvement principle fails to hold unless the preference distribution has sufficiently thin tails; for instance, part (a) implies learning fails under any signal structure if types are uniformly distributed on \((0, 1)\). As the tails of the belief distributions become thinner, the type distribution must increasingly concentrate around \( \frac{1}{2} \) in order for learning to remain possible. In many cases, little preference diversity is required to disrupt the improvement principle.

Even if preferences are not so diverse that learning fails, we might expect heterogeneity to significantly slow the learning process. Interestingly, once the distribution of preferences is concentrated enough to allow an improvement principle, learning rates are essentially the same as with homogeneous preferences. Lobel et al. [2009] show in a line network with homogeneous preferences that the probability of error decreases as \( n^{-\frac{K}{1+K}} \), where \( K \) is a tail thickness parameter; we adapt their techniques to obtain the following result.

**Proposition 1.** Suppose \( B(n) = \{n-1\} \) with probability one for all \( n \geq 2 \), and let Assumptions 1 and 2 hold. Suppose an improvement principle holds, meaning that the function \( Z \)
from Lemma 3 satisfies \( Z(y) > y \) for all \( y \in [\frac{1}{2}, 1) \). If for some \( K > 1 \) we have

\[
\lim_{r \to 0} \frac{G_0(r)}{r^{K-1}} = c > 0, \quad \text{and} \quad \int_0^1 \frac{t^{K-1}(K - (2K - 1)t)}{(1-t)^K} d\mathbb{H}(t) > 0,
\]

then the probability of error decreases as \( n^{-\frac{1}{K+1}} \):

\[
P_\sigma(x_n \neq \theta) = O \left( n^{-\frac{1}{K+1}} \right).
\]

**Proof.** See Appendix. \(\Box\)

Once an improvement principle holds, the size of the improvements, and hence the rate of learning, depends on the tails of the signal structure. This result suggests that the learning speed we find if preferences are not too diverse, or if we have homophily as in the next section, is comparable to that in the homogeneous preferences case.

### 3.2 The Benefits of Homophily

In this subsection, we extend our analysis to sparse networks with homophily: we consider networks in which agents are more likely to connect to neighbors with similar preferences to their own. We find that homophily resolves some of the learning challenges preference heterogeneity introduces. Analyzing homophily in our model presents technical challenges because we cannot retain the independence assumptions from the last subsection; we need to allow correlations to represent homophily. In a model with homogeneous preferences, Lobel and Sadler [2014] highlight unique issues that arise when neighborhoods are correlated. This creates information asymmetries leading different agents to have different beliefs about the overall structure of connections in the network. To avoid the complications this creates, and focus instead on issues related to preferences and homophily, we assume the following.

**Assumption 3.** For every agent \( n \), the neighborhood \( B(n) \) is independent of the past neighborhoods and types \( \{(t_m, B(m))\}_{m < n} \).

This assumption allows significant correlations in the sequence of types and neighborhoods \( \{(t_n, B(n))\} \), but the neighborhoods \( \{B(n)\} \) by themselves form a sequence of independent random variables. This representation of homophily is a technical contribution of our paper. Instead of first realizing independent types and then realizing links as a function of these types, we reverse the order of events to obtain a more tractable problem. While the assumption is not costless, we retain the ability to explore the role of preferences and homophily in a rich class of network topologies. In many cases, we can translate a network of interest into an identical (or at least similar) one that satisfies Assumption 3. For instance, Examples 2 and 3 in Section 4 show two different, but equivalent ways to model a network with two complete subnetworks, one with low types and one with high types. The first example is perhaps more intuitive, but the second, which satisfies Assumption 3, is entirely equivalent.
If agent \(m\) is a neighbor of agent \(n\), then under our previous assumption, the distribution of \(t_n\) conditioned on the value of \(t_m\) is \(H\), regardless of agent \(m\)’s type. With homophily, we would expect this conditional distribution to concentrate around the realized value of \(t_m\). We define a notion of strong homophily to capture the idea that agents are able to find neighbors with similar types to their own in the limit as the network grows large.

To formalize this, we recall terminology introduced by Lobel and Sadler [2014]. We use \(\hat{B}(n)\) to denote an extension of agent \(n\)’s neighborhood, comprising agent \(n\)’s neighbors, her neighbors’ neighbors, and so on.

**Definition 2.** A network topology \(Q\) features expanding subnetworks if, for all positive integers \(K\), \(\limsup_{n \to \infty} Q(\mid \hat{B}(n) \mid < K) = 0\).

Let \(N_n = \{1, 2, ..., n - 1\}\). A function \(\gamma_n : 2^{N_n} \to N_n \cup \{\emptyset\}\) is a neighbor choice function for agent \(n\) if for all sets \(B_n \in 2^{N_n}\) we have either \(\gamma_n(B_n) \in B_n\) or \(\gamma_n(B_n) = \emptyset\).

A chosen neighbor topology, denoted by \(Q_{\gamma}\), is derived from a network topology \(Q\) and a sequence of neighbor choice functions \(\{\gamma_n\}_{n \in N}\). It consists only of the links in \(Q\) selected by the neighbor choice functions \(\{\gamma_n\}_{n \in N}\).

Without the expanding subnetworks condition, there is some subsequence of agents acting based on a bounded number of signals, which clearly precludes asymptotic learning. We interpret this as a minimal connectivity requirement. Neighbor choice functions allow us to make precise the notion of identifying a neighbor with certain attributes. We call a network topology strongly homophilous if we can form a minimally connected chosen neighbor topology in which neighbors’ types become arbitrarily close. That is, individuals can identify a neighbor with similar preferences, and the subnetwork formed through these homophilous links is itself minimally connected.

**Definition 3.** The network topology \(Q\) is strongly homophilous if there exists a sequence of neighbor choice functions \(\{\gamma_n\}_{n \in N}\) such that \(Q_{\gamma}\) features expanding subnetworks, and for any \(\epsilon > 0\) we have

\[
\lim_{n \to \infty} Q(\mid t_n - t_{\gamma_n(B_n)} \mid > \epsilon) = 0.
\]

**Theorem 3.** Suppose private beliefs are unbounded and Assumption 3 holds. If \(Q\) is strongly homophilous, asymptotic learning obtains.

**Proof.** See Appendix.

The argument utilized to prove this theorem is to repeatedly apply an improvement principle along the links of the chosen neighbor topology. If, in the limit, agents are nearly certain to have neighbors with types in a small neighborhood of their own, they will asymptotically learn the true state. The reasoning behind this result mirrors that of our previous negative result. With enough homophily, we ensure the neighbor shares the agent’s priorities with regard to tradeoffs, so copying this neighbor’s action entails no loss in ex ante expected utility. Unbounded private beliefs are then sufficient for improvements to accumulate over time.
We note that the condition in Theorem 3 does not require the sequence of type realizations to converge. As we now illustrate, sufficient homophily can exist in a network in which type realizations are mutually independent and distributed according to an arbitrary $H$ with full support on $(0,1)$.

To better understand our positive result, consider its application to the following simple class of networks. We shall call a network topology $Q_\kappa$ a simple $\kappa$-homophilous network if it has the following structure. Let $\kappa$ be a non-negative real-valued parameter, and let $H$ be a type distribution with a density. The types are mutually independent and are generated according to the distribution $H$. Given a realization of $\{t_m\}_{m \leq n}$, define $i^*$ such that

$$t_n \in \left[ H^{-1} \left( \frac{i^* - 1}{n - 1} \right), H^{-1} \left( \frac{i^*}{n - 1} \right) \right],$$

and let $m_i$ denote the index of the agent with $i$th smallest type among agents $1, 2, ..., n - 1$. Define the weights $\{w_i\}_{i < n}$ by $w_{i^*} = n^\kappa$, $w_i = 1$ if $i \neq i^*$, and let

$$Q_\kappa (B(n) = \{m_i\}) = \frac{w_i}{\sum_{j<n} w_j}.$$  

For example, let $H$ be a uniform distribution over $(0,1)$, and suppose agent $n = 101$ has randomly drawn type $t_{101} = .552$. By Eq. (2), we have $i^* = 55$ since $.552 \in [.55, .56)$. Suppose agent $n = 80$ has the 55th lowest type among the first 100 agents. Then, agent 101 will observe the decision of agent 80 with probability $\frac{101^\kappa}{101^\kappa + 99}$ and observe any other agent with probability $\frac{99}{101^\kappa + 99}$. Since agent 80, with the 55th lowest type among the first hundred agents, is likely to have a type near $t_{101} = .552$, this network exhibits homophily.

The parameter $\kappa$ neatly captures our concept of homophily. A value of $\kappa = 0$ corresponds to a network with no homophily, in which each agent’s neighborhood is a uniform random draw from the past, independent of realized types. As $\kappa$ increases, agent $n$’s neighborhood places increasing weight on the past agent whose type rank most closely matches agent $n$’s percentile in the type distribution. Moreover, it is easy to check that $B(n)$ contains a past agent drawn uniformly at random, independent of the history, so Assumption 3 is satisfied. We obtain a sharp characterization of learning in these networks as a function of the parameter $\kappa$.

**Proposition 2.** Suppose private beliefs are unbounded, and Assumption 1 holds. Suppose further that $\liminf_{t \to 0} \frac{H(t)}{t} > 0$. Asymptotic learning obtains in a simple $\kappa$-homophilous network if and only if $\kappa > 1$.

**Proof.** For the forward implication, two observations establish that Theorem 3 applies. First,

$$\lim_{n \to \infty} Q_\kappa (B(n) = \{i^*\}) = \frac{n^\kappa}{n - 2 + n^\kappa} = 1$$

whenever $\kappa > 1$. Second, it follows from the Glivenko-Cantelli Theorem that for any $\epsilon > 0$, we have

$$\lim_{n \to \infty} Q_\kappa (|t_n - t_{i^*}| > \epsilon) = 0.$$
Given any $\epsilon > 0$, we can find $N(\epsilon)$ with $\mathbb{Q}_n(B(n) \neq \{i^*\}) \leq \frac{\epsilon}{2}$ and $\mathbb{Q}_n(\{t_n - t_i\} > \epsilon) \leq \frac{\epsilon}{2}$ for all $n \geq N(\epsilon)$. Defining the functions $\{\gamma_n\}_{n \in \mathbb{N}}$ in the natural way, we have

$$\mathbb{Q}_n(\{t_n - t_{\gamma_n(B(n))}\} > \epsilon) \leq \epsilon$$

for all $n \geq N(\epsilon)$. It is a simple exercise to show that $\mathbb{Q}_n$ features expanding subnetworks.

For the converse result, fix $n > m$. Define $y = \mathbb{P}_\sigma(x_m = \theta)$,

$$P_i(t) = \mathbb{P}_\sigma(x_m = \theta \mid \theta = i, B(n) = \{m\}, t_n = t),$$

and

$$q_i(t) = \frac{P_i(t)}{P_i(t) + 1 - P_{1-i}(t)}.$$

Using Assumption 1 and following calculations similar to the proof of Lemma 3, we obtain

$$\mathbb{P}_\sigma(x_n = \theta \mid B(n) = \{m\}) = y + \int_0^1 (1 - P_1(t)) \mathbb{G}_0(R^{\theta_1(t)}) - P_1(t) \mathbb{G}_1(R^{\theta_1(t)}) d\mathbb{H}(t).$$

We can uniformly bound $P_i(t)$, and hence $q_i(t)$, using $y$ as follows. Define $p^* = \frac{n-1}{n-1+\epsilon n}$, and note that the distribution of agent $n$’s neighbor can be expressed as a compound lottery choosing uniformly from the past with probability $p^*$ and choosing agent $k^*$ otherwise. Conditioned on the event that a uniform draw from the past was taken, the probability that $x_m = \theta$ is equal to $y$ when conditioned on either possible value of $\theta$. Thus we have

$$p^* y \leq P_i(t) \leq 1 - p^*(1 - y)$$

for each $i$ and for all $t \in (0,1)$. As long as $p^*$ is uniformly bounded away from zero, we can make a slight modification to the argument of Theorem 2 part (a) to show that learning fails. This is precisely the case if and only if $\alpha \leq 1$. 

With $\kappa > 1$, the likelihood of observing the action of a neighbor with a similar type grows fast as $n$ increases. While types across the entire society could be very diverse—the uniform distribution on $(0,1)$, for instance—agents almost always connect to those who are similar to themselves. If the density of the type distribution is bounded away from zero near the endpoints 0 and 1, the threshold $\kappa = 1$ is sharp. With any less homophily, there is a non-trivial chance of connecting to an agent with substantially different preferences, and asymptotic learning fails.

Our last proposition in this section complements our results on the benefits from homophily. We show that, in a certain sense, making a network more homophilous never disrupts social learning. Under Assumption 3, we can describe a network topology via a sequence of neighborhood distributions together with a sequence of conditional type distributions $\mathbb{H}_{t_n(t_1, \ldots, t_{n-1}, B(1), \ldots, B(n))}(t)$. These conditional distributions may depend on the history of types $t_1, \ldots, t_{n-1}$ and the history of neighborhoods $B(1), \ldots, B(n)$. To simplify notation, we represent the conditional type distribution of agent $n$ by $\mathbb{H}_n$. 

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Definition 4. Let $Q$ and $Q'$ be two network topologies that satisfy Assumption 3. We say that $Q$ and $Q'$ are **modifications** of one another if they share the same neighborhood distributions and the same marginal type distribution $H$.

If the network topologies $Q$ and $Q'$ are modifications of each other, they represent very similar networks. They will have identical neighborhood distributions and equally heterogeneous preferences. However, the conditional distributions $H_n$ and $H'_n$ can differ, so the networks may vary significantly with respect to homophily. This will allow us to make statements about the impact of increasing homophily while keeping other properties of the network topology constant. Given two network topologies that are modifications of each other, we can take convex combinations to create other modifications of the original two.

Definition 5. Let $Q$ and $Q'$ be modifications of one another with conditional type distributions $H_n$ and $H'_n$ respectively. Given a sequence $\lambda = \{\lambda_n\}_{n \in \mathbb{N}} \in [0, 1]^\mathbb{N}$, we define the $\lambda$-**mixture** of $Q$ and $Q'$ as the modification $Q^{(\lambda)}$ of $Q$ with conditional type distributions $H^{(\lambda)}_n = \lambda_n H_n + (1 - \lambda_n) H'_n$.

We consider networks in which asymptotic learning obtains by virtue of an improvement principle, and we find that mixing such networks with strongly homophilous modifications preserves learning.

Definition 6. We say that a network topology $Q$ **satisfies the improvement principle** if any agent can use her signal to achieve strictly higher utility than any neighbor. Formally, this means there is a continuous, increasing function $Z : [1, 3/2] \to [1, 3/2]$ such that $Z(y) > y$ for all $y \in [1, 3/2)$, and $\mathbb{E}_\sigma [u(t_n, x_n, \theta) | B(n) = \{m\}] \leq Z(\mathbb{E}_\sigma [u(t_m, x_m, \theta)])$ for all $m < n$.

This definition expresses the idea that an agent in the network can combine her private signal with information provided by one of her neighbors to arrive at a better decision than that neighbor.

Proposition 3. Suppose private beliefs are unbounded, and Assumption 3 holds. Assume the network topology $Q$ features expanding subnetworks and satisfies the improvement principle. Suppose $Q'$ is any strongly homophilous modification of $Q$. Asymptotic learning obtains in any $\lambda$-mixture of $Q$ and $Q'$.

Proof. See Appendix.

Proposition 3 shows if we take any network topology satisfying the natural sufficient conditions for learning in a sparse network—expanding subnetworks and the applicability of the improvement principle—and consider a convex combination between this network and one with strong homophily, agents will still learn. One interpretation of this statement is that the addition of homophily to a network never harms learning that is already successful without homophily.

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4 The range $[1, 3/2]$ of the function $Z$ in this definition corresponds to the possible range of an agent’s ex ante expected utility, not the possible range of an agent’s probability of matching the state of the world.
Our findings in this section establish a positive role for homophily in social learning. Without homophily, a sparse network with diverse preferences struggles to aggregate information because the meaning of a decision is difficult to interpret, and different agents make different tradeoffs. Homophily counters both sources of inefficiency, leading to better outcomes as the degree of homophily increases. In the next section, we see the other side of homophily’s impact on learning. Networks with dense connections offer an opportunity to overcome the difficulty of learning with bounded private beliefs. The ability to observe a diverse neighborhood is a major driver of this positive result, and an excess of homophily can stifle the learning process.

4 Densely Connected Networks

A classic result by Banerjee [1992] and Bikhchandani et al. [1992] shows that herding occurs in a complete network when private beliefs are bounded and preferences are homogeneous. More recently, Goeree et al. [2006] find that unbounded preferences can remedy the situation, proving a strong positive result in a comparable setting. Intuitively, if the distribution of preferences has full support on (0, 1) and types are independent, then agents always emerge whose preferences roughly balance against their social information. These agents must rely on their private signals to choose an action, so new information is revealed to the rest of the network. Even if an individual decision provides little information, dense connections allow learning to operate through a law-of-large-numbers mechanism in addition to the improvement principle. The presence of this second learning mechanism leads to robust learning.

In contrast to our results for sparse networks, preference diversity appears decisively beneficial with full observation. In this section, we show this insight is generic in a broad class of dense network components that we call clusters.

Definition 7. A cluster \( \mathcal{C} \) is a sequence of stopping times \( \{\alpha_i\}_{i \in \mathbb{N}} \) with respect to the filtration generated by \( \{(t_n, B(n))\}_{n \in \mathbb{N}} \) that satisfy

\[
\lim_{i \to \infty} \mathbb{Q}(\alpha_k \in B(\alpha_i)) = 1 \quad \text{for all } k \in \mathbb{N}.
\]

A cluster is a generalization of the concept of a (randomly generated) clique. Any complete subnetwork—a clique—with infinitely many members is a cluster. A subset \( \mathcal{C} \) of the agents such that any member of \( \mathcal{C} \) is observed with probability approaching one by later members of \( \mathcal{C} \) is also a cluster. Clusters may exist deterministically, with the indices \( \alpha_i \) degenerate random variables as in the complete network, or they may arise stochastically. For instance, suppose types are i.i.d. and all agents with types above .8 are connected with each other through some correlation between neighborhoods and types. In this case, the stopping time \( \alpha_i \) would refer to the \( i \)th agent with type above .8 and this group would form a cluster. Note that a cluster potentially has far fewer edges than a clique; the ratio of the number of edges in a cluster to the number of edges in a clique can approach zero as \( n \) grows.

As in Section 3, we introduce an independence assumption for tractability.
Assumption 4. For each stopping time $\alpha_i$ in the cluster $C$, conditional on the event $\alpha_i = n$, the type $t_n$ is generated according to the distribution $H_{\alpha_i}$ independently of the history $t_1, \ldots, t_{n-1}$ and $B(1), \ldots, B(n)$.

The assumption above represents a non-trivial technical restriction on the random agents that form a cluster. If the types are i.i.d., then any stopping time will satisfy the assumption, but if the types are correlated, the condition needs to be verified. We still have a great deal of flexibility to represent different network structures. Examples 2 and 3 demonstrate two ways to represent essentially the same network structure, both of which comply with Assumption 4.

Example 2. Suppose types are i.i.d., and suppose the network topology is such that any agent $n$ with type $t_n \geq \frac{1}{2}$ observes any previous agent $m$ with $t_m \geq \frac{1}{2}$ and any agent $n$ with type $t_n < \frac{1}{2}$ similarly observes any previous agent $m$ with $t_m < \frac{1}{2}$. The sequence of stopping times $\{\alpha_i\}_{i \in \mathbb{N}}$, where $\alpha_i$ is the $i$th agent with type at least $\frac{1}{2}$, forms a cluster, and the agents with types below $\frac{1}{2}$ form another.

In the example above, a cluster of agents forms according to realized types; neighborhood realizations are correlated such that high type agents link to all other high type agents. A similar clustering of types can be achieved in a network with deterministic neighborhoods and correlated types instead of correlated neighborhoods.

Example 3. Let $\mathbb{H}_-$ and $\mathbb{H}_+$ denote the distribution $\mathbb{H}$ conditional on a type realization less than or at least $\frac{1}{2}$ respectively. Suppose agents are partitioned into two disjoint fixed subsequences $C = \{c_i\}_{i \in \mathbb{N}}$ and $D = \{d_i\}_{i \in \mathbb{N}}$ with $c_1 = 1$. Conditional on $t_1 < \frac{1}{2}$, agents in $C$ realize types independently according to $\mathbb{H}_-$ with agents in $D$ realizing types independently from $\mathbb{H}_+$; if $t_1 \geq \frac{1}{2}$, the type distributions for the subsequences are switched. Let the network structure be deterministic, with all agents in $C$ observing all previous agents in $C$, and likewise for the subsequence $D$. The members of $C$ form one cluster, and the members of $D$ form another.

This section’s main finding is a sufficient condition for learning within a cluster $\{\alpha_i\}_{i \in \mathbb{N}}$. We say that asymptotic learning occurs within a cluster $\{\alpha_i\}_{i \in \mathbb{N}}$ if we have

$$\lim_{i \to \infty} \mathbb{P}_\alpha(x_{\alpha_i} = \theta) = 1.$$ 

We require the cluster satisfy two properties.

Definition 8. A cluster is identified if there exists a family of neighbor choice functions $\{\gamma_i^k\}_{i,k \in \mathbb{N}}$ such that for each $k$, we have

$$\lim_{i \to \infty} \mathbb{Q}(\gamma_i^k(B(\alpha_i)) = \alpha_k) = 1.$$ 

That is, a cluster is identified only if, in the limit as the network grows large, agents in a given cluster can identify which other individuals belong to the same cluster.

A cluster is uniformly diverse if the following two conditions hold.
(a) For any interval $I \subseteq (0,1)$ there exists a constant $\epsilon_I > 0$ such that $\mathbb{P}(t_{\alpha_i} \in I) \geq \epsilon_I$ for infinitely many $i$.

(b) There exists a finite measure $\mu$ such that the Radon-Nykodim derivative $\frac{d\mu}{dH_{\alpha_i}} \geq 1$ almost surely for all $i$.

The first part of the uniform diversity condition generalizes the notion of unbounded preferences. It is unnecessary for any particular member of a cluster to have a type drawn from a distribution with full support; we simply need infinitely many members of the cluster to fall in any given interval of types. The second part of the condition is technical, requiring the existence of a finite measure on $(0,1)$ that dominates the type distribution of all agents in the cluster. Without this condition, it would be possible to construct pathological situations by having the type distributions $H_{\alpha_i}$ concentrate increasing mass near the endpoints of $(0,1)$ as $i$ grows. Together, these conditions are sufficient for asymptotic learning in clusters regardless of the signal structure or the structure of the rest of the network.

**Theorem 4.** Let Assumption 4 hold. Suppose $\{\alpha_i\}_{i \in \mathbb{N}}$ is an identified, uniformly diverse cluster. Asymptotic learning obtains within the cluster.

**Proof.** See Appendix.

With enough preference diversity, there is sufficient experimentation within a cluster to allow full information aggregation, even with bounded private beliefs. A key and novel element in our characterization is the concept of identification. Agents need to be able to tell with high probability who the other members of their cluster are to ensure asymptotic learning within the cluster. We can immediately apply this result to any network without homophily where the entire network forms a cluster.

**Corollary 1.** Suppose preferences are unbounded, and the types $\{t_n\}_{n \in \mathbb{N}}$ form an i.i.d. sequence that is also independent of the neighborhoods. If for each $m$ we have
\[
\lim_{n \to \infty} Q(m \in B(n)) = 1,
\]
Eq. (3) asymptotic learning obtains.

**Proof.** Define the stopping indices $\alpha_i = i$, and define the neighbor choice functions
\[
\gamma^i_n(B(n)) = \begin{cases} i & \text{if } i \in B(n) \\ \emptyset & \text{otherwise.} \end{cases}
\]

Theorem 4 applies.

This corollary substantially generalizes the asymptotic learning result of Goeree et al. [2006] for the complete network; it obtains a positive result that applies even to networks with correlated neighborhoods and gaps, as long as Eq. (3) is satisfied. Both identification and uniform diversity play a vital role in Theorem 4. Without identification, agents may be unable to interpret the information available to them, and without uniform diversity, herd behavior may appear.
Example 4. Suppose the types \(\{t_n\}_{n \in \mathbb{N}}\) form an i.i.d. sequence, \(\mathbb{H}\) is symmetric around \(\frac{1}{2}\), and private beliefs are bounded. Consider a network in which agents sort themselves into three clusters \(C_i\) for \(i \in \{1, 2, 3\}\). Let \(1 \in C_1\), and let \(c_n\) denote the number of agents in \(C_1\) with index less than \(n\). Each agent \(n\) is contained in \(C_1\) with probability \(\frac{1}{2^n}\), and these agents observe the entire history of action. Otherwise, if \(t_n < \frac{1}{2}\) we have \(n \in C_2\), and \(n\) observes all prior agents in \(C_2\) and only those agents for sure. Likewise, if \(t_n \geq \frac{1}{2}\) we have \(n \in C_3\), and \(n\) observes all prior agents in \(C_3\) and only those agents for sure.

The cluster \(C_1\) is uniformly diverse, but unidentified, and learning fails. The clusters \(C_2\) and \(C_3\) are identified, but the lack of diversity means herding occurs with positive probability; asymptotic learning fails here too.

In Example 4, agents sort themselves into three clusters: a small uniformly diverse cluster, a large cluster of low types, and a large cluster of high types. Since private beliefs are bounded, if the low type cluster approaches a social belief close to one, no agent will ever deviate from action 1. Likewise, if the high type cluster approaches a social belief close to zero, no agent will deviate from action 0. Both of these clusters have a positive probability of herding. If this occurs, the agents in \(C_1\) observe a history in which roughly half of the agents choose each action, and this is uninformative. Since the other members of \(C_1\) make up a negligible share of the population and cannot be identified, their experimentation fails to provide any useful information to agents in \(C_1\).

This example clearly suggests a negative role for homophily in a dense network. Densely connected networks offer an opportunity to overcome the difficulty presented by bounded private beliefs, but if homophily is strong enough that no identified cluster is uniformly diverse, the benefits of increased connectivity are lost. However, an inefficient herding outcome requires extreme isolation from types with different preferences, and even slight exposure is enough to recover a positive long-run result.

Consider the network introduced in Example 3. For a given marginal type distribution \(\mathbb{H}\) with unbounded preferences, let \(\mathbb{H}_-\) denote the distribution \(\mathbb{H}\) conditional on a type realization less than \(\frac{1}{2}\), and \(\mathbb{H}_+\) the distribution \(\mathbb{H}\) conditional on a type realization of at least \(\frac{1}{2}\). Agents are partitioned into two disjoint deterministic subsequences \(C = \{c_i\}_{i \in \mathbb{N}}\) and \(D = \{d_i\}_{i \in \mathbb{N}}\) with \(r_1 = 1\). Conditional on \(t_1 < \frac{1}{2}\), agents in \(C\) realize types independently according to \(\mathbb{H}_-\), while agents in \(D\) realize types independently according to \(\mathbb{H}_+\), with the distributions switched conditional on \(t_1 \geq \frac{1}{2}\). We assume that \(c_i \in B(c_j)\) and \(d_i \in B(d_j)\) for all \(i < j\), so clearly \(C\) and \(D\) are identified clusters. If the two clusters are totally isolated from one another, meaning agents in \(C\) never observe an agent in \(D\) and vice versa, herd behavior occurs if the private beliefs are bounded. However, the outcome changes drastically if agents in each cluster occasionally observe a member of the other cluster. Only a small amount of bidirectional communication between the clusters is needed to disrupt herding, leading to asymptotic learning in the entire network.

Proposition 4. Assume private beliefs are bounded. Consider a network topology \(\mathbb{Q}\) with deterministic neighborhoods that is partitioned into two clusters as described above. Asymptotic learning occurs in both clusters if and only if

\[
\sup\{i : \exists j, c_i \in B(d_j)\} = \sup\{i : \exists j, d_i \in B(c_j)\} = \infty.
\]
Proof. See Appendix.

The key requirement in this statement is another type of minimal connectivity. For each cluster, the observations of the other cluster cannot be confined to a finite set. This means that periodically there is some agent in each cluster who makes a new observation of the other. This single observation serves as an aggregate statistic for an entire cluster’s accumulated social information, so it holds substantial weight even though it is generated by a cluster of agents with significantly different preferences. This suggests that herding in two cluster networks is difficult to sustain indefinitely, even in the presence of homophily, though if observations across groups are very rare, convergence may be quite slow.

5 Conclusions

Preference heterogeneity and homophily are pervasive in real-world social networks, so understanding their effects is crucial if we want to bring social learning theory closer to how people make decisions in practice. Preference diversity, homophily, and network structure impact learning in complex ways, and the results presented in this paper provide insight on how these three phenomena interact. Underlying our analysis is the idea that learning occurs through at least two basic mechanisms, one based on the improvement principle and the other based on a law-of-large-numbers effect. The structure of the network dictates which mechanisms are available to the agents, and the two mechanisms are affected differently by preference heterogeneity and homophily.

The improvement principle is one way agents can learn about the world. An agent can often combine her private information with that provided by her neighbor’s action to arrive at a slightly better decision than her neighbor. In sparsely connected networks, the improvement principle is the primary means through which agents learn. However, this mechanism is, to some extent, quite fragile; it requires the possibility of extremely strong signals and a precise understanding of the decisions made by individual neighbors. Preference heterogeneity introduces noise in the chain of observations, and our results show this noise challenges the operation of the improvement principle. The problems that arise are due both to uncertainty regarding how a neighbor makes her decision and to different tradeoffs neighbors face. Homophily ameliorates both issues, making it unambiguously helpful to this type of learning.

Dense connectivity allows a far more robust learning mechanism to operate. Having strong information from individual neighbors becomes less important when one can observe the actions of many neighbors; as long as each observation provides some new information, the law of large numbers ensures that learning succeeds. Diverse preferences provide a degree of independence to the observations, facilitating this means of learning. Homophily, since it reduces preference diversity within a neighborhood, has the potential to interfere, but our results suggest this learning mechanism is resilient enough to be unaffected unless homophily is particularly extreme.
Overall, we find that preference heterogeneity and homophily play positive, complementary roles in social learning as typical complex networks include both sparse and dense components. If private signals are of bounded strength, preference diversity is necessary to ensure learning in the dense parts of the network via the law-of-large-numbers mechanism. If homophily is also present, the information accumulated in the dense parts of the network can spread to the sparse parts via the improvement principle mechanism. The combination of preference heterogeneity and homophily should generally benefit information aggregation in complex, realistic networks.

References


A Appendix

We first prove part (b) of Theorem 2 from section 3.1 and establish a version of the improvement principle to prove the results in section 3.2. We conclude by applying the theory of martingales to establish results from section 4.

Proof of Theorem 2 part (b)

Our first task is to show the assumption on \( G_0 \) implies a similar condition on \( G_1 \):

\[
\lim_{r \to 0} \frac{G_0(r)}{r^{K-1}} = c \quad \Rightarrow \quad \lim_{r \to 0} \frac{G_1(r)}{r^K} = c \left( 1 - \frac{1}{K} \right).
\]

An application of Lemma 1 and integration by parts gives

\[
G_1(r) = \int_0^r dG_1(s) = \int_0^r \frac{s}{1-s} dG_0(s)
\]

\[
= \frac{r}{1-r} G_0(r) - \int_0^r \frac{1}{(1-s)^2} G_0(s) ds.
\]

Using our assumption on \( G_0 \), given any \( \epsilon > 0 \), we may find \( r_\epsilon \) such that for all \( r \leq r_\epsilon \),

\[
c(1-\epsilon) r^{K-1} \leq G_0(r) \leq c(1+\epsilon) r^{K-1}.
\]

Thus, for any \( r \leq r_\epsilon \) we have

\[
\int_0^r \frac{c(1-\epsilon)s^{K-1}}{(1-s)^2} ds \leq \int_0^r \frac{1}{(1-s)^2} G_0(s) ds \leq \int_0^r \frac{c(1+\epsilon)s^{K-1}}{(1-s)^2} ds.
\]

Now compute,

\[
\int_0^r \frac{s^{K-1}}{(1-s)^2} ds = \frac{r^K}{1-r} - (K-1) \int_0^r \frac{s^{K-1}}{(1-s)} ds
\]

\[
= \frac{r^K}{1-r} - (K-1) \int_0^r \sum_{i=0}^{\infty} s^{K-1+i} ds
\]

\[
= \frac{r^K}{1-r} - (K-1) \frac{r^K}{K} + O(r^{K+1}).
\]

It follows that for any \( r \leq r_\epsilon \),

\[
\frac{r}{1-r} G_0(r) - c(1+\epsilon) \left( \frac{r^K}{1-r} - (K-1) \frac{r^K}{K} \right) + O(r^{K+1}) \leq G_1(r), \quad \text{and}
\]

\[
G_1(r) \leq \frac{r}{1-r} G_0(r) - c(1-\epsilon) \left( \frac{r^K}{1-r} - (K-1) \frac{r^K}{K} \right) + O(r^{K+1}).
\]
Dividing through by $r^K$ and letting $r$ go to zero, we have

$$c \left(1 - \frac{1 + \epsilon}{K}\right) \leq \lim_{r \to 0} \frac{G_1(r)}{r^K} \leq c \left(1 - \frac{1 - \epsilon}{K}\right)$$

for any $\epsilon$, proving the result.

We now show that

$$\int_0^1 t^{K-1} \frac{(K - (2K-1)t)}{(1-t)^K} d\mathbb{H}(t) < 0,$$

implies the existence of $\epsilon > 0$ such that $Z(y) < y$ for all $y \in [1 - \epsilon, 1)$. Choose a function $h(\epsilon)$ such that $h(\epsilon) > 0$ whenever $\epsilon > 0$, and

$$\lim_{\epsilon \to 0} \int_0^{1-\epsilon} \frac{h(\epsilon)}{(1-t)^K} d\mathbb{H}(t) = 0.$$

For a given $\epsilon > 0$, the argument $R^y_{1-\epsilon}$ goes to zero as $y$ approaches 1, so for sufficiently large $y$ and all $t \leq 1 - \epsilon$,

$$(1 - y)G_0(R^y_{1-\epsilon}) - yG_1(R^y_{1-\epsilon}) \leq c \left[ (1 - y)(1 + h(\epsilon))(R^y_{1-\epsilon})^{K-1} - y(1 - h(\epsilon))\left(\frac{K-1}{K}(R^y_{1-\epsilon})^K\right) \right]
= (1 - y)K^{ctK-1} \left[ (1 + h(\epsilon))(t(1-y) + y(1-t)) - yt(1 - h(\epsilon))\left(\frac{K-1}{K}\right) \right].$$

For $y$ sufficiently close to 1, the portion of the improvement term

$$\int_0^{1-\epsilon} (1 - y)G_0(R^y_{1-\epsilon}) - yG_1(R^y_{1-\epsilon}) d\mathbb{H}(t)$$

is then bounded above by

$$\frac{c}{K} (1 - y)^K \int_0^{1-\epsilon} t^{K-1} \frac{[K(1 + h(\epsilon))(t(1-y) + y(1-t)) - (K-1)(1-h(\epsilon))ty]}{(t(1-y) + y(1-t))^K} d\mathbb{H}(t). \quad (5)$$

For $t \in [0, 1 - \epsilon]$, the integrand in Eq. (5) converges uniformly as $y$ approaches 1 to

$$t^{K-1} \frac{(K - (2K-1)t + h(\epsilon)(K-t))}{(1-t)^K}.$$

Therefore, for any $\epsilon' > 0$ and $y$ sufficiently close to 1,

$$\int_0^{1-\epsilon} t^{K-1} \frac{[K(1 + h(\epsilon))(t(1-y) + y(1-t)) - (K-1)(1-h(\epsilon))ty]}{(t(1-y) + y(1-t))^K} d\mathbb{H}(t)
\leq \epsilon' + \int_0^{1-\epsilon} t^{K-1} \frac{(K - (2K-1)t + h(\epsilon)(K-t))}{(1-t)^K} d\mathbb{H}(t).$$
As $\epsilon$ approaches zero, this converges to
\[ \epsilon' + \int_0^1 \frac{t^{K-1} (K - (2K - 1)t)}{(1-t)^K} \frac{1}{d\mathcal{H}(t)}. \] (6)

If the integral in Eq. (6) is negative, then we can choose some $\epsilon' > 0$, $\epsilon > 0$, and a corresponding $y^*$ such that for any $y \in [y^*, 1)$, the integral in Eq. (5) is negative. Therefore,
\[ \int_0^{1-\epsilon} (1 - y)G_0(R_y^\epsilon) - yG_1(R_y^\epsilon) \, d\mathcal{H}(t) < 0 \]
for any $y \in [y^*, 1)$.

To complete the proof, we show that for a sufficiently small choice of $\epsilon$, there exists $y_\epsilon < 1$ such that
\[ \int_{1-\epsilon}^1 (1 - y)G_0(R_y^\epsilon) - yG_1(R_y^\epsilon) \, d\mathcal{H}(t) < 0 \]
for all $y \in [y_\epsilon, 1)$. Thus, for $y \in [\max(y_\epsilon, y^*), 1)$, the entire improvement term is negative. Again using Lemma 1 and integration by parts we have
\[
(1 - y)G_0(R_y^\epsilon) - yG_1(R_y^\epsilon) \\
= \int_0^{R_y^\epsilon} (1 - y)G_0(r) - yG_1(r) \, dr \\
= \int_0^{R_y^\epsilon} \frac{1 - y - r}{r} \, dr \\
= \frac{y(1 - 2t)}{t} \frac{G_1(R_y^\epsilon)}{R_y^\epsilon} + \int_0^{R_y^\epsilon} \frac{(1 - y)G_1(r)}{r^2} \, dr.
\]

Since $G_1$ is increasing and bounded, there exist constants $0 < \zeta < \tau$, such that
\[ G_1^K \leq G_1(r) \leq \tau r^K \]
for all $r \in [0, 1]$. Therefore, for any $t > \frac{1}{2}$ we have
\[ \frac{y(1 - 2t)}{t} \frac{G_1(R_y^\epsilon)}{R_y^\epsilon} + \int_0^{R_y^\epsilon} \frac{(1 - y)G_1(r)}{r^2} \, dr \leq \frac{\zeta y(1 - 2t)}{t} [R_y^\epsilon]^K + \frac{\tau(1 - y)}{K - 1} [R_y^\epsilon]^{K-1}. \]

The right hand side is negative whenever $t > \frac{y(\zeta(K-1) + \pi)}{2y\zeta(K-1) + (2y-1)\pi}$. This threshold is decreasing in $y$, with a limiting value that is strictly less than 1. Therefore, for any $\epsilon < 1 - \frac{\zeta(K-1) + \pi}{2\zeta(K-1) + \pi}$, we can find $y_\epsilon$ such that
\[ (1 - y)G_0(R_y^\epsilon) - yG_1(R_y^\epsilon) < 0 \]
for all $t \geq 1 - \epsilon$ and all $y \in [y_\epsilon, 1)$, completing the proof. \[\square\]

**Proof of Proposition 1**
Fixing an $\epsilon > 0$, we can bound the improvement term as
\[
\int_0^1 (1-y)G_0(R_i^y) - yG_1(R_i^y) \, dH(t) \geq \int_0^{1-\epsilon} (1-y)G_0(R_i^y) - yG_1(R_i^y) \, dH(t) - \int_{1-\epsilon}^1 yG_1(R_i^y) \, dH(t).
\]

Carrying out a similar exercise as in the previous proof, we can show that for any $\epsilon'$, we can find $y^*$ such that
\[
\int_0^{1-\epsilon} (1-y)G_0(R_i^y) - yG_1(R_i^y) \, dH(t) \geq \frac{c}{K}(1-y)^K \left( \int_0^{tK-1} \frac{(K - (2K - 1)t)}{(1-t)^K} \, dH(t) - \epsilon' \right)
\]
for all $y \in [y^*, 1)$. There exists a constant $\bar{c}$ such that $G_1(r) \leq \bar{c}r^K$ for all $r$, so we have
\[
\int_1^{1-\epsilon} yG_1(R_i^y) \, dH(t) \leq \bar{c}(1-y)^K \int_1^{1-\epsilon} y(t(1-y) + y(1-t)) \, dH(t)
\]
\[
\leq \bar{c}(1-y)^K \int_1^{1-\epsilon} (1-t)^K \, dH(t)
\]
for all $\epsilon < \frac{1}{2}$. Since $\int_0^1 \frac{t^{K-1}-(2K-1)t}{(1-t)^K} \, dH(t)$ is positive, we know that $\int_0^1 \frac{t^K}{(1-t)^K} \, dH(t)$ is finite. Hence, for any $\epsilon'$, we can choose $\epsilon$ such that $\frac{K^2}{c} \int_1^{1-\epsilon} \frac{t^K}{(1-t)^K} \, dH(t) < \epsilon'$.

Take $\epsilon' < \frac{1}{3} \int_0^1 \frac{t^{K-1}-(2K-1)t}{(1-t)^K} \, dH(t)$, choose $\epsilon$ so that $\frac{K^2}{c} \int_1^{1-\epsilon} \frac{t^K}{(1-t)^K} \, dH(t) < \epsilon'$, and find $y^*$ such that
\[
\int_0^{1-\epsilon} (1-y)G_0(R_i^y) - yG_1(R_i^y) \, dH(t) \geq \frac{c}{K}(1-y)^K \left( \int_0^{tK-1} \frac{(K - (2K - 1)t)}{(1-t)^K} \, dH(t) - \epsilon' \right)
\]
for all $y \in [y^*, 1)$. Then for all $y \in [y^*, 1)$ there is a constant $C$ such that $Z(y) > y+C(1-y)^K$. The analysis of Lobel et al. [2009] now implies the result. $\square$

We use the following version of the improvement principle to establish positive learning results.

**Lemma 4 (Improvement Principle).** Let Assumption 3 hold. Suppose there exists a sequence of neighbor choice functions $\{\gamma_n\}_{n \in \mathbb{N}}$ and a continuous, increasing function $Z : [1,3/2] \to [1,3/2]$ with the following properties:

(a) The trusted network topology $Q_\gamma$ features expanding subnetworks,

(b) We have $Z(y) > y$ for any $y < \frac{3}{2}$,

(c) For any $\epsilon > 0$, we have
\[
\lim_{n \to \infty} \mathbb{P}_\sigma \left\{ \mathbb{E}_\sigma \left[ u(t_n, x_n, \theta) | \gamma_n(B(n)) \right] < Z \left( \mathbb{E}_\sigma \left[ u(t_{\gamma_n(B(n))}, x_{\gamma_n(B(n))}, \theta) \right] \right) \right\} = 0.
\]
Asymptotic learning obtains.

Proof. Note that asymptotic learning is equivalent to
\[
\lim_{n \to \infty} \mathbb{E}_{\sigma} [u(t_n, x_n, \theta)] = \frac{3}{2}.
\]

We construct two sequences \(\{\eta_k\}\) and \(\{\phi_k\}\) with the property that for all \(k \geq 1\) and \(n \geq \eta_k\), \(\mathbb{E}_{\sigma} [u(t_n, x_n, \theta)] \geq \phi_k\). Upon showing that \(\lim_{k \to \infty} \phi_k = \frac{3}{2}\), we shall have our result. Using our assumptions, for any integer \(K\) and \(\varepsilon > 0\), we can find a positive integer \(N(K, \varepsilon)\) such that
\[
Q(\gamma_n(B(n)) = \{m\}, m < K) < \frac{\varepsilon}{2}, \quad \text{and}
\]
\[
\mathbb{P}_\sigma \{ \mathbb{E}_{\sigma} [u(t_n, x_n, \theta) | \gamma_n(B(n))] < Z(\mathbb{E}_{\sigma} [u(t_{\gamma_n(B(n))}, x_{\gamma_n(B(n))}, \theta)]) - \varepsilon \} < \frac{\varepsilon}{2}
\]
for all \(n \geq N(K, \varepsilon)\). Set \(\eta_1 = 1\) and \(\phi_1 = 1\), and let \(\epsilon_k \equiv \frac{1}{2} \left(1 + Z(\phi_k) - \sqrt{1 + 2\phi_k + Z(\phi_k)^2}\right)\); we define the rest of the sequences recursively by
\[
\eta_{k+1} = N(\eta_k, \epsilon_k), \quad \phi_{k+1} = \frac{Z(\phi_k) + \phi_k}{2}.
\]

Given the assumptions on \(Z(\phi_k)\), these sequences are well-defined.

Proceed by induction to show that \(\mathbb{E}_{\sigma} [u(t_n, x_n, \theta)] \geq \phi_k\) for all \(n \geq \eta_k\). The base case \(k = 1\) trivially holds because an agent may always choose the action preferred a priori according to her type, obtaining expected utility at least 1. Considering \(n \geq \eta_{k+1}\) we have
\[
\mathbb{E}_{\sigma} [u(t_n, x_n, \theta)] \geq \sum_{m<n} \mathbb{E}_{\sigma} [u(t_n, x_n, \theta) | \gamma_n(B(n)) = m] Q(\gamma_n(B(n)) = m)
\]
\[
\geq (1 - \epsilon_k) (Z(\phi_k) - \epsilon_k) = \phi_{k+1}.
\]

To see that \(\phi_k\) converges to \(\frac{3}{2}\), note the definition implies \(\{\phi_k\}\) is a non-decreasing, bounded sequence, so it has a limit \(\phi^*\). Since \(Z\) is continuous, the limit must be a fixed point of \(Z\). The only fixed point is \(\frac{3}{2}\), finishing the proof.

\[\Box\]

Proof of Theorem 3

Our goal is to construct a function \(Z\) on which to apply Lemma 4. We begin by characterizing the decision \(\tilde{x}_n\), defined as
\[
\tilde{x}_n = \arg\max_{y \in \{0,1\}} \mathbb{E}_{\sigma} [u(t_n, y, \theta) | \gamma_n(B(n)) = \gamma_n(B(n))].
\]

The decision \(\tilde{x}_n\) is a decision based on a coarser information set than what agent \(n\) actually has access to. Therefore, the utility derived from this decision provides a lower bound on agent \(n\)’s utility in equilibrium.
Suppose $\gamma_n(B(n)) = m$. To simplify notation, we define $P_{it} = \mathbb{P}_\theta(x_m = i \mid \theta = i, t_m = t)$ and $P_i = \int_0^1 P_{it}d\mathbb{H}_{tm|tn}(t)$, where $\mathbb{H}_{tm|tn}$ denotes the distribution of $t_m$ conditioned on the realized $t_n$ and the event $\gamma_n(B(n)) = m$. We further define $E_{it}$ as the expected utility of agent $m$ given that $t_m = t$ and $E_i$ analogously to $P_i$. These quantities are related by

$$E_{0t} = 2tP_{0t} + 1 - t, \quad E_{1t} = 2(1 - t)P_{1t} + t. \quad (7)$$

Note that $P_{it}$ and $E_{it}$ are constants, independent of the realization of $t_n$, while $P_i$ and $E_i$ are random variables as functions of $t_n$ through the conditional distribution $\mathbb{H}_{tm|tn}$. Define the thresholds

$$L_{tn} = \frac{t_n(1 - P_0)}{t_n(1 - P_0) + (1 - t_n)P_1}, \quad U_{tn} = \frac{t_nP_0}{t_nP_0 + (1 - t_n)(1 - P_1)}.$$

For the remainder of the proof we suppress the subscript $t_n$ to simplify notation. An application of Bayes’ rule shows that

$$\bar{x}_n = \begin{cases} 0 & \text{if } p_n < L \\ x_m & \text{if } p_n \in (L, U) \\ 1 & \text{if } p_n > U. \end{cases} \quad (8)$$

Fixing $t_n$, the expected payoff from the decision $\bar{x}_n$ is easily computed as

$$\frac{1}{2}\left[ (\mathbb{G}_0(L)(1 + t_n) + (\mathbb{G}_0(U) - \mathbb{G}_0(L))(2t_nP_0 + 1 - t_n) + (1 - \mathbb{G}_0(U))(1 - t_n) \\ + (1 - \mathbb{G}_1(U))(2 - t_n) + (\mathbb{G}_1(U) - \mathbb{G}_1(L))(2(1 - t_n)P_1 + t_n) + \mathbb{G}_1(L)t_n \right]. \quad (9)$$

with the first line corresponding to the payoff when $\theta = 0$ and the second to the payoff when $\theta = 1$. An application of Lemma 1 provides the following inequalities.

$$\mathbb{G}_0(L) \geq \frac{1 - L}{L} \mathbb{G}_1(L) + \frac{L}{4} \mathbb{G}_1\left(\frac{L}{2}\right),$$

$$1 - \mathbb{G}_1(U) \geq \frac{U}{1 - U}(1 - \mathbb{G}_0(U)) + \frac{1 - U}{4}\left(1 - \mathbb{G}_1\left(\frac{1 + U}{2}\right)\right).$$

Substituting into Eq. (9), we find

$$\mathbb{E}_\sigma[u(t_n, \bar{x}_n, \theta) \mid t_n, \gamma_n(B(n)) = m] \geq \frac{1}{2} + t_nP_0 + (1 - t_n)P_1 + t_n(1 - P_0)\frac{L}{4} \mathbb{G}_1\left(\frac{L}{2}\right) \\ + (1 - t_n)(1 - P_1)\frac{1 - U}{4}\left(1 - \mathbb{G}_0\left(\frac{1 + U}{2}\right)\right). \quad (10)$$

We collectively refer to the first three terms above as the “base” terms, and the last two as the “improvement” terms. We focus first on the base terms. Using Eq. (7) the base terms
can be written as
\[
\frac{E_0 + E_1}{2} + \int_0^1 (t_n - t)(P_{0t} - P_{1t})d\mathbb{H}_{t_n|t_n}(t)
= \mathbb{E}_\sigma [u(t_m, x_m, \theta) | \gamma_n (B(n)) = m, t_n] + \int_0^1 (t_n - t)(P_{0t} - P_{1t})d\mathbb{H}_{t_n|t_n}(t). \tag{11}
\]

For a given \( \epsilon > 0 \), define \( p_\epsilon = \mathbb{P} (|t_m - t_n| > \epsilon | \gamma_n (B(n)) = m) \). Integrating over \( t_n \) and using Assumption 3, for any \( \epsilon > 0 \) we can bound the integrated base terms from below by
\[
\mathbb{E}_\sigma [u(t_m, x_m, \theta)] + \int_0^1 \int_0^1 (s - t)(P_{0t} - P_{1t})d\mathbb{H}_{t_m|t_n}(t)d\mathbb{H}_{t_n}(s)
\geq \mathbb{E}_\sigma [u(t_m, x_m, \theta)] - \epsilon - p_\epsilon. \tag{12}
\]

Moving to the improvement terms, note that
\[
P_0 = \int_0^1 P_{0t}d\mathbb{H}_{t_m|t_n}(t) = \int_0^1 \frac{E_{0t} - (1 - t)}{2t}d\mathbb{H}_{t_m|t_n}(t).
\]
The last integrand is Lipschitz continuous in \( t \) for \( t \) bounded away from zero. Therefore, for any \( \epsilon \) with \( 0 < 2\epsilon < t_n \), we can find a constant \( c \) such that
\[
\int_0^1 \frac{E_{0t} - (1 - t)}{2t_n}d\mathbb{H}_{t_m|t_n}(t) - c\epsilon - \frac{p_\epsilon}{t_n} \leq \int_0^1 \frac{E_{0t} - (1 - t)}{2t_n}d\mathbb{H}_{t_m|t_n}(t)
\leq \int_0^1 \frac{E_{0t} - (1 - t)}{2t_n}d\mathbb{H}_{t_m|t_n}(t) + c\epsilon + \frac{p_\epsilon}{t_n}.
\]
This is equivalent to
\[
\frac{E_0 - (1 - t_n)}{2t_n} - c\epsilon - \frac{p_\epsilon}{t_n} \leq P_0 \leq \frac{E_0 - (1 - t_n)}{2t_n} + c\epsilon + \frac{p_\epsilon}{t_n}.
\]

Similarly, we can find a constant \( c \) such that
\[
\frac{E_1 - t_n}{2(1 - t_n)} - c\epsilon - \frac{p_\epsilon}{1 - t_n} \leq P_1 \leq \frac{E_1 - t_n}{2(1 - t_n)} + c\epsilon + \frac{p_\epsilon}{1 - t_n}.
\]
Consider a modification of the improvement terms in Eq. (10) where we replace \( P_0 \) by \( \frac{E_0 - (1 - t_n)}{2t_n} \) and \( P_1 \) by \( \frac{E_1 - t_n}{2(1 - t_n)} \), including in the definitions of \( L \) and \( U \). Our work above, together with the continuity of the belief distributions, implies that the modified terms differ from the original improvement terms by no more than some function \( \delta(\epsilon, p_\epsilon, t_n) \), where \( \delta \) converges to zero as \( \epsilon \) and \( p_\epsilon \) approach zero together, and the convergence is uniform in \( t_n \) for \( t_n \) bounded away from 0 and 1. We can then bound the improvement terms by
\[
t_n(1 - P_0)\frac{L}{4} \mathbb{G}_1 \left( \frac{L}{2} \right) \geq \frac{1}{8} \frac{(1 + t_n - E_0)^2}{1 - E_0 + E_1} \mathbb{G}_1 \left( \frac{1 + t_n - E_0}{2(1 - E_0 + E_1)} \right) - \delta(\epsilon, p_\epsilon, t_n), \tag{13}
\]
\[
(1 - t_n)(1 - P_1) \frac{1 - U}{4} \left( 1 - G_0 \left( \frac{1 + U}{2} \right) \right) \\
\geq \frac{1}{8} \frac{(2 - t_n - E_1)^2}{1 + E_0 - E_1} \left( 1 - G_0 \left( 1 - \frac{2 - t_n - E_1}{2(1 + E_0 - E_1)} \right) \right) - \delta(\epsilon, p_e, t_n). \tag{14}
\]

Let \( y^* = \frac{E_0 + E_1}{2} \); we must have either \( E_0 \leq \frac{2}{3} y^* + t_n \) or \( E_1 \leq \frac{2}{3} y^* + 1 - t_n \) since \( y^* \leq \frac{3}{2} \). The first term on the right hand side of Eq. (13) can be rewritten as

\[
\frac{1}{16} \left( 1 - \frac{2}{3} y^* \right)^2 \left( 1 - G_0 \left( 1 - \frac{1}{4} \right) \right) - \delta(\epsilon, p_e, t_n). \tag{15}
\]

Similarly, if \( E_1 \leq \frac{2}{3} y^* + 1 - t_n \), then Eq. (13) is bounded below by

\[
\frac{1}{16} \left( 1 - \frac{2}{3} y^* \right)^2 \left( 1 - G_0 \left( 1 - \frac{1}{4} \right) \right) - \delta(\epsilon, p_e, t_n). \tag{16}
\]

To simplify notation, define

\[
Z(y^*) = \frac{1}{16} \left( 1 - \frac{2}{3} y^* \right)^2 \left( 1 - G_0 \left( 1 - \frac{1}{4} \right) \right). \tag{17}
\]

Recall that \( y^* = y^*(t_n) \) is a function of \( t_n \) through \( E_0 \) and \( E_1 \). Using Eqs. (15) and (16), we can integrate over \( t_n \) to bound the contribution of the improvement terms to agent \( n \)'s utility. Since the improvement terms are non-negative, we can choose any \( \epsilon' > 0 \) and restrict the range of integration to obtain a lower bound of

\[
\int_{\epsilon'}^{1 - \epsilon'} Z(y^*(t)) - 2\delta(\epsilon, p_e, t) d\mathcal{H}(t).
\]

Now define \( y = \mathbb{E}_{\sigma}[u(t_m, x_m, \theta)] \) and note that \( y = \int_{0}^{1} y^*(t) d\mathcal{H}(t) \). Observe since \( y^* \geq \frac{1}{2} \), this implies \( \mathbb{P}_\sigma \left( y^* \leq \frac{3}{4} + \frac{y}{2} \right) \geq \frac{3 - 2y}{1 + 2y} \). Choosing \( \epsilon' \) sufficiently small we can bound the improvement terms below by

\[
\frac{3 - 2y}{2(1 + 2y)} Z(y) - \int_{\epsilon'}^{1 - \epsilon'} 2\delta(\epsilon, p_e, t) d\mathcal{H}(t). \tag{17}
\]

Finally, define \( \mathcal{Z}(y) = y + \frac{3 - 2y}{2(1 + 2y)} Z \left( \frac{3}{4} + \frac{y}{2} \right) \). Combining Eqs. (10), (12), and (17), and using that \( Z \) is non-increasing, we have

\[
\mathbb{E}_{\sigma}[u(t_n, x_n, \theta) \mid \gamma_n(B(n)) = m] \geq \mathcal{Z}(y) - \epsilon - p_e - 2 \int_{\epsilon'}^{1 - \epsilon'} 2\delta(\epsilon, p_e, t) d\mathcal{H}(t).
\]
for any $\epsilon > 0$ and some fixed $\epsilon' > 0$. The hypothesis of the theorem implies that for any $\epsilon > 0$, $p_\epsilon$ approaches zero as $n$ grows without bound. Using the uniform convergence to 0 of $\delta$ for $t_n \in [\epsilon', 1 - \epsilon']$, we see that $Z$ satisfies the hypothesis of Lemma 4, completing the proof. □

Proof of Proposition 3

Since $Q'$ is strongly homophilous, there exist neighbor choice functions $\{\gamma_n\}_{n \in \mathbb{N}}$ for which $\lim_{n \to \infty} Q'(|t_n - t_{\gamma_n(B(n))}| > \epsilon) = 0$, and the improvement principle of Lemma 4 applies to establish learning along the chains of observations in $Q'$. We shall see the same neighbor choice functions can be used to establish learning in $Q$. Considering an arbitrary agent $n$ with $\gamma_n(B(n)) = m$, recall the decision thresholds $L_{t_n}$ and $U_{t_n}$ defined in the proof of Theorem 3. Here we let $L_{t_n}$ and $U_{t_n}$ denote the corresponding thresholds for the network $Q$, and let $L'_{t_n}$ and $U'_{t_n}$ denote the thresholds for the network $Q'$. As in the proof of Theorem 3, we suppress the subscript $t_n$ from here on. A lower bound on the improvement agent $n$ makes in network $Q'$ can be obtained by using the suboptimal decision thresholds $L$ and $U$. Since the last of the base terms (see Eq. (11)) does not depend on the decision thresholds, the proof of Theorem 3 may be followed with essentially no changes to obtain an improvement function $Z'$ satisfying the assumptions of Lemma 4. Crucially, this improvement function applies for decisions made via the suboptimal decision thresholds that are appropriate to the network $Q$ as opposed to $Q'$. By assumption we have an improvement function $Z$ for the network $Q$. Considering an arbitrary $\lambda$-mixture of $Q$ and $Q'$, we suppose that $n$ uses the decision thresholds $L$ and $U$, instead of the optimal thresholds for $Q$. It is immediately clear that

$$Z^{(\lambda)} = \min_{\lambda \in [0,1]} \lambda Z + (1 - \lambda) Z'$$

is an improvement function for the suboptimal decision rule. This provides a lower bound on the true improvement, so $Z^{(\lambda)}$ is an improvement function for $Q^{(\lambda)}$. Lemma 4 applies. □

The following lemmas provide the essential machinery for the proof of Theorem 4. The first lemma shows that if a subsequence of decisions in a cluster provides information that converges to full knowledge of the state, the entire cluster must learn. The essence of the proof is that the expected social belief of agents in an identified cluster, conditional on the decisions of this subsequence, must also converge on the state $\theta$. Since social beliefs are bounded between 0 and 1, the realized social belief of each agent in the cluster deviates from this expectation only with very small probability.

The second lemma establishes a lower bound on the amount of new information provided by the decision of an agent within a uniformly diverse cluster. Given a subset of the information available to that agent, we bound how much information the agent’s decision conveys on top of that. Under the assumption of uniform diversity, we can show that either the agent acts on her own signal with positive probability, or she has access to significant additional information not contained in the subset we considered. This is a key technical innovation.
that allows us to apply martingale convergence arguments even when agents within a cluster do not have nested information sets.

**Lemma 5.** Let Assumption 4 hold; suppose \{α_i\}_{i \in N} is a uniformly diverse, identified cluster. If there exists a subsequence \{α_{i(j)}\}_{j \in N} such that the social beliefs \(\hat{q}_k = \mathbb{P}_\sigma(θ = 1 \mid x_α)\), \(j \leq k\) converge to \(θ\) almost surely, then asymptotic learning obtains within the cluster.

**Proof.** Consider the case where \(θ = 0\); the case \(θ = 1\) is analogous. Fixing any \(ε > 0\), by assumption there exists some random integer \(K_ε\), which is finite with probability one, such that \(\hat{q}_k \leq \frac{ε}{3}\) for all \(k \geq K_ε\). Fix a \(k\) large enough so that \(\mathbb{P}(K_ε > k) \leq \frac{ε}{3}\). Since the cluster is identified, for large enough \(n\) there are neighbor choice functions \(\gamma_{α_n}^{(1)}, \ldots, \gamma_{α_n}^{(k)}\) such that

\[
\mathbb{P}_σ (\exists i \leq k \mid \gamma_{α_n}^{(i)}(B(α_n)) \neq α_i) \leq \frac{ε}{3}.
\]

We then have

\[
\mathbb{E}_σ [q_{α_n} \mid X_{γ_{α_n}^{(i)}(B(α_n))}, i \leq k, θ = 0] \leq (1 - \frac{ε}{3}) \mathbb{E}_σ [q_k \mid θ = 0] + \frac{ε}{3} \\
\leq (1 - \frac{ε}{3})^2 \mathbb{E}_σ [q_k \mid k \geq K_ε, θ = 0] + \frac{2ε}{3} \\
\leq ε.
\]

Since the social belief \(q_{α_n}\) is bounded below by 0, we necessarily have

\[
\mathbb{P}_σ(q_{α_n} > \sqrt{ε} \mid θ = 0) \leq \sqrt{ε}.
\]

Therefore,

\[
\mathbb{P}_σ(x_{α_n} = 0 \mid θ = 0) \geq (1 - \sqrt{ε}) \int_0^1 \mathbb{P}_σ(x_{α_n} = 0 \mid θ = 0, q_{α_n} = \sqrt{ε}, t_{α_n} = t) \, dH_{α_n}(t) \\
= (1 - \sqrt{ε}) \int_0^1 G_0 \left(R_0^{1/4}\right) \, dH_{α_n}(t).
\]

Condition (b) of uniform diversity ensures this last integral approaches 1 as \(ε\) approaches zero, completing the proof.

\[\square\]

**Lemma 6.** Let Assumption 4 hold. Let \(α\) be a stopping time within a cluster, and suppose \(I\) is a random variable taking values contained in agent \(α\)'s information set with probability one. For a given realization \(I\) of \(I\), define \(\hat{q}_α = \mathbb{P}_σ(θ = 1 \mid I = I)\), and suppose there exist \(p,d > 0\) such that

\[
H_α(R_1^{1 - \hat{q}_α} - d) - H_α(R_2^{1 - \hat{q}_α} + d) \geq p.
\]

If for some \(ε > 0\) we have \(\hat{q}_α \geq ε\), then there exists some \(δ > 0\) such that

\[
\frac{\mathbb{P}_σ(x_α = 0 \mid I = I, θ = 0)}{\mathbb{P}_σ(x_α = 0 \mid I = I, θ = 1)} \geq 1 + δ.
\]
Similarly, if $\hat{q}_\alpha \leq 1 - \epsilon$, then we can find $\delta > 0$ such that
\[
\frac{\mathbb{P}_\sigma(x_\alpha = 1 | \mathcal{I} = I, \theta = 1)}{\mathbb{P}_\sigma(x_\alpha = 1 | \mathcal{I} = I, \theta = 0)} \geq 1 + \delta.
\]

Proof. Fix a realization $I$ of $\mathcal{I}$, and define
\[
P_{qi} = d\mathbb{P}_\sigma(q_\alpha = q | I, \theta = i).
\]
We can rewrite the ratio
\[
\frac{\mathbb{P}_\sigma(x_\alpha = 0 | I, \theta = 0)}{\mathbb{P}_\sigma(x_\alpha = 0 | I, \theta = 1)} = \frac{\int_0^1 \mathbb{P}_\sigma(x_\alpha = 0 | I, \theta = 0, t_\alpha = t)d\mathbb{H}_\alpha(t)}{\int_0^1 \mathbb{P}_\sigma(x_\alpha = 0 | I, \theta = 1, t_\alpha = t)d\mathbb{H}_\alpha(t)}
\]
\[
= \frac{\int_0^1 \int_0^1 P_{q_0}G_0(R_t^q) \, dq \, d\mathbb{H}_\alpha(t)}{\int_0^1 \int_0^1 P_{q_1}G_1(R_t^q) \, dq \, d\mathbb{H}_\alpha(t)}
\]
\[
= \frac{\int_0^1 \int_0^1 P_{q_1}G_0(R_t^q) + (P_{q_0} - P_{q_1})G_0(R_t^q) \, dq \, d\mathbb{H}_\alpha(t)}{\int_0^1 \int_0^1 P_{q_1}G_1(R_t^q) \, dq \, d\mathbb{H}_\alpha(t)}
\]
\[
= \frac{\int_0^1 \int_0^1 P_{q_1}G_0(R_t^q) \, dq \, d\mathbb{H}_\alpha(t)}{\int_0^1 \int_0^1 P_{q_1}G_1(R_t^q) \, dq \, d\mathbb{H}_\alpha(t)}.
\]
(18)

It follows from the definition of the social belief and an application of Bayes’ rule that for $\mathbb{P}_\sigma$ almost all $q$ we have
\[
q = \left[1 + \left(\frac{1}{\hat{q}_\alpha} - 1\right) \frac{P_{q_0}}{P_{q_1}}\right]^{-1}.
\]
In particular, if $q < \hat{q}_\alpha$, then $P_{q_0} > P_{q_1}$, and the same holds with the inequalities reversed.

Now for any given $\epsilon' > 0$, define $\hat{q}_\epsilon < \hat{q}_\epsilon^+$ as the values given above when we take $\frac{P_{q_1}}{P_{q_0}}$ equal to $1 - \epsilon'$ and $1 + \epsilon'$ respectively.

We shall consider three cases. First, suppose $\mathbb{P}_\sigma(q_\epsilon < q_\alpha | I) \geq 1/3$. For $q \leq \hat{q}_\epsilon^-$ we have
\[
(P_{q_0} - P_{q_1}) \left(G_0(R_t^q) - G_0(R_t^q)\right) = P_{q_0} \left(1 - \frac{P_{q_1}}{P_{q_0}}\right) \left(G_0(R_t^q) - G_0(R_t^q)\right)
\]
\[
\geq \epsilon' P_{q_0} \left(G_0(R_t^q) - G_0(R_t^q)\right).
\]

Integrating over all such $q$ and choosing $\epsilon'$ small enough we have
\[
\int_0^1 \int_0^1 (P_{q_0} - P_{q_1})G_0(R_t^q) \, dq \, d\mathbb{H}_\alpha(t) \geq \frac{\epsilon'}{6} \int_0^1 \left(G_0(R_t^q) - G_0(R_t^q)\right) \, d\mathbb{H}_\alpha(t)
\]
\[
\geq \frac{pe'}{6} \min_{t \in [R_t^{q_\epsilon - q_\alpha + d}, R_t^{q_\epsilon - q_\alpha - d}]} \left(G_0(R_t^q) - G_0(R_t^q)\right)
\]
\[
> 0,
\]

\[\xi\]
where the last inequality follows because $R_{t}^{q_{a}}$ is bounded strictly between $\beta$ and $\bar{\beta}$ for the given range, and $G_{0}$ has full support in $(\beta, \bar{\beta})$. Since the denominator in Eq. (18) is bounded by 1, and the first term of the numerator is at least as large by Lemma 1 part (d), the inequality above gives us our desired $\delta$. For the second case, if $P_{\sigma}(q_{c}^{+} \leq q_{a} | I = I) \geq 1/3$, then a similar exercise gives

$$\int_{0}^{1} \int_{0}^{1} (P_{q_{0}} - P_{q_{1}})G_{0}(R_{t}^{q_{0}})dq dH_{a}(t) \geq \frac{\epsilon}{6} \int_{0}^{1} \left( G_{0}(R_{t}^{q_{a}}) - G_{0}(R_{t}^{q_{a}^{+}}) \right) dH_{a}(t).$$

For small $\epsilon'$, the required $\delta$ is forthcoming by the same argument.

Finally, we assume that $P_{\sigma}(q_{c}^{-} \leq q_{a} \leq q_{c}^{+} | I = I) \geq 1/3$. We can then bound Eq. (18) below by

$$\int_{0}^{1} \int_{0}^{1} P_{q_{1}}G_{0}(R_{t}^{q_{0}})dq dH_{a}(t) \geq \int_{0}^{1} \int_{0}^{1} P_{q_{1}}G_{0}(R_{t}^{q_{0}})dq dH_{a}(t) + 1.$$

For $q$ in the given range, Lemma 1 part (d) gives

$$\frac{\int_{0}^{1} G_{0}(R_{t}^{q})dH_{a}(t)}{\int_{0}^{1} G_{1}(R_{t}^{q})dH_{a}(t)} = \frac{\int_{0}^{1} G_{0}(R_{t}^{q})dH_{a}(t) + \int_{0}^{1} G_{0}(R_{t}^{q})dH_{a}(t)}{\int_{0}^{1} G_{1}(R_{t}^{q})dH_{a}(t)} \geq \frac{\int_{0}^{1} G_{0}(R_{t}^{q})dH_{a}(t) + \int_{0}^{1} G_{1}(R_{t}^{q})dH_{a}(t)}{\int_{0}^{1} G_{1}(R_{t}^{q})dH_{a}(t)} + 1 + \left( \frac{G_{0}(1/2)}{G_{1}(1/2)} - 1 \right) \int_{0}^{1} G_{1}(R_{t}^{q})dH_{a}(t) \geq 1 + \left( \frac{G_{0}(1/2)}{G_{1}(1/2)} - 1 \right) \int_{0}^{1} G_{1}(R_{t}^{q})dH_{a}(t).$$

The denominator of the last expression is at most 1, and the numerator is non-negative. Define $\bar{b} = \frac{2\beta + 1}{4}$. Observe that for any $q \in (0, 1)$, we have $q > R_{0}^{\bar{b}}$, so we may fix $\epsilon'$ small enough so that $q_{c}^{-} > R_{q_{c}^{-}}^{\bar{b}}$. Restricting the range of integration, Eq. (19) is bounded below by

$$1 + \left( \frac{G_{0}(1/2)}{G_{1}(1/2)} - 1 \right) \int_{R_{q_{c}^{-}}^{\bar{b}}} G_{1}(R_{t}^{q})dH_{a}(t) \geq 1 + \left( \frac{G_{0}(1/2)}{G_{1}(1/2)} - 1 \right) \int_{R_{q_{c}^{-}}^{\bar{b}}} G_{1}(b)dH_{a}(t) \geq 1 + \left( \frac{G_{0}(1/2)}{G_{1}(1/2)} - 1 \right) G_{1}(\bar{b}) \left( H_{a}(q_{c}^{-}) - H_{a}(R_{q_{c}^{-}}^{\bar{b}}) \right)$$

for small enough $\epsilon'$. Choosing $\epsilon'$ small enough for all three cases to satisfy the corresponding inequalities finishes the proof of the first statement; the second statement is analogous.
Proof of Theorem 4

We first construct a subsequence on which we can apply Lemma 5. Define the indices 
\(i(j)\) recursively, beginning with \(i(1) = 1\). For \(j > 1\), let

\[
\epsilon_j = \frac{1}{2} \min_{x} \min \left\{ \mathbb{P}_\sigma (\theta = 1 \mid x_{\alpha_{i(j)}}, l < j), 1 - \mathbb{P}_\sigma (\theta = 1 \mid x_{\alpha_{i(j)}}, l < j) \right\}.
\]

Since the cluster is identified, we can choose \(i(j)\) large enough so that

\[
\mathbb{P}_\sigma \left( \exists l < j \mid r_{\alpha_{i(j)}} (B(\alpha_{i(j)})) \neq \alpha_{i(l)} \right) \leq \epsilon_j.
\]

Since the cluster is uniformly diverse, we can choose the indices so that the resulting subsequence is also uniformly diverse.

For the subsequence \(\{\alpha_{i(j)}\}_{j \in \mathbb{N}}\), define \(\hat{q}_k = \mathbb{P}_\sigma (\theta = 1 \mid x_{\alpha_{i(j)}}, j \leq k)\). The sequence \(\hat{q}_k\) is clearly a bounded martingale, so the sequence converges almost surely to a random variable \(q^*\) by the martingale convergence theorem. Conditional on \(\theta = 1\), the likelihood ratio \(\frac{1-\hat{q}_k}{\hat{q}_k}\) is also a non-negative martingale [Doob, 1953, Eq. (7.12)]. Therefore, conditional on \(\theta = 1\), the ratio \(\frac{1-\hat{q}_k}{\hat{q}_k}\) converges with probability 1 to a limiting random variable. In particular,

\[
\mathbb{E}_\sigma \left[ \frac{1 - q^*}{q^*} \mid \theta = 1 \right] < \infty,
\]

[Breiman, 1968, Theorem 5.14], and therefore, \(q^* > 0\) with probability 1 when \(\theta = 1\). Similarly, \(\frac{\hat{q}_k}{1-\hat{q}_k}\) is a non-negative martingale conditional on \(\theta = 0\), and by a similar argument we have \(q^* < 1\) with probability 1 when \(\theta = 0\).

We shall see that the random variable \(q^*\) equals \(\theta\) with probability 1. Let \(x^{(k)}\) denote the vector comprised of the actions \(\{x_{\alpha_{i(j)}}\}_{j \leq k}\), and suppose without loss of generality that \(x_{\alpha_{i(k+1)}} = 0\) for infinitely many \(k\). Using Bayes’ Rule twice,

\[
\hat{q}_{k+1} = \mathbb{P} (\theta = 1 \mid x_{\alpha_{i(k+1)}} = 0, x^{(k)}) = \left[ 1 + \frac{\mathbb{P}(x^{(k)} \mid \theta = 0)\mathbb{P}(x_{\alpha_{i(k+1)}} = 0 \mid x^{(k)}, \theta = 0)}{\mathbb{P}(x^{(k)} \mid \theta = 1)\mathbb{P}(x_{\alpha_{i(k+1)}} = 0 \mid x^{(k)}, \theta = 1)} \right]^{-1}
\]

\[
= \left[ 1 + \left( \frac{1}{\hat{q}_k} - 1 \right) \frac{\mathbb{P}(x_{\alpha_{i(k+1)}} = 0 \mid x^{(k)}, \theta = 0)}{\mathbb{P}(x_{\alpha_{i(k+1)}} = 0 \mid x^{(k)}, \theta = 1)} \right]^{-1}.
\]

To simplify notation, let \(f(x^{(k)}) = \frac{\mathbb{P}(x_{\alpha_{i(k+1)}} = 0 \mid x^{(k)}, \theta = 0)}{\mathbb{P}(x_{\alpha_{i(k+1)}} = 0 \mid x^{(k)}, \theta = 1)}\). Thus,

\[
\hat{q}_{k+1} = \left[ 1 + \left( \frac{1}{\hat{q}_k} - 1 \right) f(x^{(k)}) \right]^{-1}.
\]

Suppose \(\hat{q}_k \in [\epsilon, 1 - \epsilon]\) for all sufficiently large \(k\) and some \(\epsilon > 0\). From our construction of the subsequence \(\alpha_{i(j)}\), this implies

\[
\hat{q}_k \equiv \mathbb{P}_\sigma (\theta = 1 \mid x_m, m = r_{\alpha_{i(k+1)}} (B(\alpha_{i(k+1)})), l \leq k) \in \left[ \frac{\epsilon}{2}, 1 - \frac{\epsilon}{2} \right]
\]
for all such \( k \). By Lemma 6 together with the uniform diversity property, there exists \( \delta_\epsilon > 0 \) such that \( f(x^{(k)}) \geq 1 + \delta_\epsilon \). Given this \( \delta_\epsilon \), we can bound the difference \( \hat{q}_k - \hat{q}_{k+1} \). If \( x_{\alpha_{i(k+1)}} = 0 \), we have

\[
\hat{q}_k - \hat{q}_{k+1} \geq \hat{q}_k - \left[ 1 + \left( \frac{1}{\hat{q}_k} - 1 \right) (1 + \delta_\epsilon) \right]^{-1} = \frac{\hat{q}_k(1 - \hat{q}_k)\delta_\epsilon}{1 + \delta_\epsilon(1 - \hat{q}_k)} \geq \frac{\epsilon (1 - \epsilon) \delta_\epsilon}{1 + (1 - \epsilon)\delta_\epsilon} = K_0(\epsilon) > 0.
\]

(22)

If \( \hat{q}_k \in [\epsilon, 1 - \epsilon] \) for all sufficiently large \( k \), this implies the sequence \{\( \hat{q}_k \)\} is not Cauchy. This contradicts the almost sure convergence of the sequence, so the support of \( q^* \) cannot contain \([\epsilon, 1 - \epsilon] \) for any \( \epsilon > 0 \). Lemma 5 completes the proof. \( \Box \)

Proof of Proposition 4

The negative implication follows similar reasoning as Example 4. If one cluster fails to observe the other beyond a certain point, then once social beliefs become strong enough in favor of the cluster’s preferred state, the cluster will herd. Since this happens with positive probability, the other cluster obtains bounded information from this cluster, and herds with positive probability as well. We now prove the positive implication.

Define \( \hat{q}_k^C = \mathbb{P}_\sigma(\theta = 1 \mid x_{c_i}, i \leq k) \) and \( \hat{q}_k^D = \mathbb{P}_\sigma(\theta = 1 \mid x_{d_i}, i \leq k) \). Following the same argument as in the proof of Theorem 4, the sequences \( \hat{q}_k^C \) and \( \hat{q}_k^D \) converge almost surely to random variables \( q^C \) and \( q^D \) respectively. We further have for each \( i \in \{0, 1\} \) that

\[
\mathbb{P}_\sigma(q^C = 1 - i \mid \theta = i) = \mathbb{P}_\sigma(q^D = 1 - i \mid \theta = i) = 0.
\]

Without loss of generality, assume \( t_{c_1} \in (0, 1/2) \); following the argument from Theorem 4, and applying Lemma 6, we immediately find that the support of \( q^C \) is contained in \([0, 1 - \beta]\) and the support of \( q^D \) is contained in \([\overline{1 - \beta}, 1]\).

Define the constants \( \delta^C \) and \( \delta^D \) by

\[
\delta^C = \inf\{\delta \mid \mathbb{P}_\sigma(q^C \in [0, 1 - \delta, 1]) = 1\} \quad \text{and} \quad \delta^D = \inf\{\delta \mid \mathbb{P}_\sigma(q^D \in [0, \delta, 1]) = 1\}.
\]

Our first task is to show that if \( q^C \in [1 - \delta^C, 1) \), then only finitely many agents in \( C \) select action 0; similarly if \( q^D \in (0, \delta^D] \), only finitely many agents in \( D \) select action 1. We will analyze only the cluster \( C \) since the other result is analogous. The argument should be familiar by now; we establish a contradiction with almost sure convergence of the martingale.

Suppose \( x_{c_{k+1}} = 0 \) with positive probability, and recall the computation leading to Eq. (18) in the proof of Lemma 6. Consider the corresponding result with \( i = \{x_{c_i}\}_{i \leq k} \),

\[
\mathbb{P}_\sigma(x_{c_{k+1}} = 0 \mid x_{c_i}, i \leq k, \theta = 0) + \mathbb{P}_\sigma(x_{c_{k+1}} = 0 \mid x_{c_i}, i \leq k, \theta = 1) = \frac{\int_0^1 \int_0^1 P_{q_1} G_0\left(R^D_t\right) + (P_{q_0} - P_{q_1})\left(G_0\left(R^D_t\right) - G_0\left(R^D_{t_{c_1}}\right)\right) \, dq \, d\mathbb{H}_-(t)}{\int_0^1 \int_0^1 P_{q_1} G_1\left(R^D_t\right) \, dq \, d\mathbb{H}_-(t)}.
\]

xv
If \( q^C \in [1 - \delta^C, 1 - \epsilon] \) for some fixed \( \epsilon > 0 \), then by almost sure convergence, for all \( k \) large enough we must have \( \hat{q}_k^C \geq 1 - \delta^C - \epsilon \geq 1 - \beta - \epsilon \). Since \( t_{ck+1} \leq \frac{1}{2} \) with probability 1, we have \( \mathbb{G}_0(R_q^t) = \mathbb{G}_1(R_q^t) = 0 \) for any \( q > 1 - \beta \), so these values of \( q \) do not contribute to the integrals; from here on we consider the densities \( P_q \) above to be conditional on this event. We analyze two cases; first suppose that conditional on \( q \leq 1 - \beta \), we have \( q \leq 1 - \beta - 2\epsilon \) with probability at least \( 1/2 \). Following the argument of Lemma 6, this gives us a lower bound on the second term in the numerator, and therefore bounds the ratio strictly above one.

Alternatively, suppose that conditional on \( q \leq 1 - \beta \) we have \( q \in [1 - \beta - 2\epsilon, 1 - \beta] \) with probability at least \( 1/2 \). For any \( t \) sufficiently close to \( 1/2 \), we have

\[
\frac{\mathbb{G}_0(R_q^t)}{\mathbb{G}_1(R_q^t)} \geq \frac{\mathbb{G}_0(\beta + 2\epsilon)}{\mathbb{G}_1(\beta + 2\epsilon)} > 1.
\]

Since \( \mathbb{H}_- \) is supported in any interval \([1/2 - \delta, 1/2] \), positive measure is assigned to these values of \( t \), and together with Lemma 1 part (d) this again allows us to bound the ratio strictly above one. Following the argument of Theorem 4, infinitely many agents in \( C \) selecting action 0 would contradict almost sure convergence of \( \hat{q}_k^C \) if \( q^C \in [1 - \delta^C, 1 - \epsilon] \). Since \( \epsilon \) was arbitrary, the result follows.

Our next task is to show for a fixed \( q \in [1 - \delta^C, 1) \) we have

\[
\mathbb{P}_\sigma(q^D \in (0, \delta^D) \mid q^C = q, \theta = 1) \geq \frac{(1 - \beta)(1 - q)}{\beta q}.
\]

Similarly, for a fixed \( q \in (0, \delta^D] \) we have

\[
\mathbb{P}_\sigma(q^C \in [\delta^C, 1) \mid q^C = q, \theta = 0) \geq \frac{\beta q}{(1 - \beta)(1 - q)}.
\]

Let \( N^C \) and \( N^D \) denote the smallest integer valued random variables such that \( x_{ck} = 1 \) for all \( k \geq N^C \) and \( x_{dk} = 0 \) for all \( k \geq N^D \). From the above work, we have that \( N^C \) is finite with probability one conditional on \( q^C \in [1 - \delta^C, 1) \), and \( N^D \) is finite with probability one if \( q^D \in (0, \delta^D] \). Consider a particular sequence of decisions \( \{x_{ck}\}_{k \in \mathbb{N}} \) such that \( q^C = q \in [1 - \delta^C, 1) \). This immediately implies

\[
\lim_{k \to \infty} \frac{\mathbb{P}_\sigma(x_{dk} = 0 \mid \{x_{ck}\}_{k \in \mathbb{N}}, \theta = 1)}{\mathbb{P}_\sigma(x_{dk} = 0 \mid \{x_{ck}\}_{k \in \mathbb{N}}, \theta = 0)} = \frac{\mathbb{P}_\sigma(q^D \in (0, \delta^D) \mid \{x_{ck}\}_{k \in \mathbb{N}}, \theta = 1)}{\mathbb{P}_\sigma(q^D \in [0, \delta^D) \mid \{x_{ck}\}_{k \in \mathbb{N}}, \theta = 0)} = \frac{\mathbb{P}_\sigma(q^D \in (0, \delta^D) \mid \{x_{ck}\}_{k \in \mathbb{N}}, \theta = 1)}{\mathbb{P}_\sigma(q^D \in (0, \delta^D) \mid \{x_{ck}\}_{k \in \mathbb{N}}, \theta = 1)}.
\]

Given an \( \epsilon > 0 \) we can choose \( N_\epsilon \) large enough so that \( \mathbb{P}_\sigma(N_C \leq N_\epsilon \mid \{x_{ck}\}_{k \leq N_\epsilon}) \geq 1 - \epsilon \). If further we have \( N_\epsilon \) larger than the realized \( N^C \) for our fixed sequence, we have the bounds

\[
\frac{\mathbb{P}_\sigma(x_{dj} = 0 \mid \{x_{ck}\}_{k \in \mathbb{N}}, \theta = 1)}{\mathbb{P}_\sigma(x_{dj} = 0 \mid \{x_{ck}\}_{k \in \mathbb{N}}, \theta = 0)} \leq \frac{\mathbb{P}_\sigma(x_{dj} = 0 \mid \{x_{ck}\}_{k \leq n}, \theta = 1) + \epsilon}{\mathbb{P}_\sigma(x_{dj} = 0 \mid \{x_{ck}\}_{k \leq n}, \theta = 0) - \epsilon},
\]

and
\[ \frac{\mathbb{P}_\sigma(x_{d_j} = 0 \mid \{x_{c_k}\}_{k \in \mathbb{N}}, \theta = 1)}{\mathbb{P}_\sigma(x_{d_j} = 0 \mid \{x_{c_k}\}_{k \in \mathbb{N}}, \theta = 0)} \geq \frac{\mathbb{P}_\sigma(x_{d_j} = 0 \mid \{x_{c_k}\}_{k \leq n}, \theta = 1) - \epsilon}{\mathbb{P}_\sigma(x_{d_j} = 0 \mid \{x_{c_k}\}_{k \leq n}, \theta = 0) + \epsilon} \]

for any \( n \geq N_\epsilon \) and all \( j \).

Suppose \( \mathbb{P}_\sigma(q^D \in (0, \delta^D) \mid q^C = q, \theta = 1) < \frac{(1 - \beta)(1 - q)}{2q} \). From the above work, we can find a collection of sequences \( \{x_{c_k}\}_{k \in \mathbb{N}} \) with positive measure conditional on \( q^C = q \), an \( \epsilon > 0 \), and an integer \( N'_\epsilon \) such that

\[ \frac{\mathbb{P}_\sigma(x_{d_m} = 0 \mid \{x_{c_k}\}_{k \leq n}, \theta = 1)}{\mathbb{P}_\sigma(x_{d_m} = 0 \mid \{x_{c_k}\}_{k \leq n}, \theta = 0)} \leq \frac{(1 - \beta)(1 - q)}{\beta q} - \epsilon \]

for all \( n, m \geq N'_\epsilon \) when one of the sequences \( \{x_{c_k}\}_{k \in \mathbb{N}} \) is realized. For all sufficiently large \( n \), \( \hat{q}_n^C \) is within \( \frac{\epsilon}{2} \) of \( q \); this in turn implies that for all such \( n \) and \( m \geq N'_\epsilon \), we have an \( \epsilon' > 0 \) such that

\[ \mathbb{P}_\sigma(\theta = 1 \mid x_{d_m} = 0, x_{c_k}, k \leq n) \leq 1 - \beta - \epsilon'. \]

Note that for any subsequence of the cluster \( D \), since types are conditionally independent and arbitrarily close to 1, infinitely many members of the subsequence will select action 0. If \( d_m \in B(c_n) \) for \( n, m \) as above, we have \( \mathbb{E}_\sigma[q_{c_n} \mid x_{d_m} = 0, x_{c_k}, k \leq n] \leq 1 - \beta - \epsilon' \). Since social beliefs are bounded between 0 and 1, there is a positive lower bound on the probability that \( q_{c_n} \leq 1 - \beta - \epsilon' \) conditional on these observations. Since infinitely many agents \( c_n \) observe such \( d_m \) and have types arbitrarily close to 1/2, with probability one infinitely many agents in \( C \) choose action 0. We conclude that either \( \delta^C = 0 \) or

\[ \lim_{n \to \infty} \mathbb{P}_\sigma(q^D \in (0, \delta^D) \mid x_{c_k}, k \leq n, \theta = 1) = \mathbb{P}_\sigma(q^D \in (0, \delta^D) \mid q^C = q, \theta = 1) \geq \frac{(1 - \beta)(1 - q)}{\beta q} \]

as claimed. In the former case, decisions of agents in \( C \) become fully informative, and asymptotic learning clearly obtains. On the other hand, if Eqs. (23) and (24) hold,

\[ \mathbb{P}_\sigma(q^C \in [1 - \delta^C, 1] \mid \theta = 0) = \int_0^{\delta^D} \mathbb{P}_\sigma(q^C \in [1 - \delta^C, 1] \mid q^D = q, \theta = 0) d\mathbb{P}_\sigma(q^D = q \mid \theta = 0) \]

\[ \geq \int_0^{\delta^D} \mathbb{P}_\sigma(q^D = q \mid \theta = 0) d\mathbb{P}_\sigma(q^D = q \mid \theta = 0) \]

\[ = \int_0^{\delta^D} \frac{\beta q}{(1 - \beta)(1 - q)} d\mathbb{P}_\sigma(q^D = q \mid \theta = 1) \]

Now, observe the definition of \( q^D \) implies

\[ \frac{d\mathbb{P}_\sigma(q^D = q \mid \theta = 0)}{d\mathbb{P}_\sigma(q^D = q \mid \theta = 1)} = \frac{\mathbb{P}_\sigma(\theta = 0 \mid q^D = q)}{\mathbb{P}_\sigma(\theta = 1 \mid q^D = q)} = \frac{1 - q}{q}. \]

xvi
Substituting into Eq. (25) gives

\[
\mathbb{P}_\sigma(q^C \in [1 - \delta, 1] \mid \theta = 0) \geq \frac{\beta}{1 - \beta} \int_0^{\delta^D} d\mathbb{P}_\sigma(q^D = q \mid \theta = 1) = \frac{\beta}{1 - \beta} \mathbb{P}_\sigma(q^D \in [0, \delta^D] \mid \theta = 1).
\]

A similar calculation for \( q^D \) gives

\[
\mathbb{P}_\sigma(q^D \in [0, \delta^D] \mid \theta = 1) \geq \frac{1 - \beta}{\beta} \mathbb{P}_\sigma(q^C \in [1 - \delta^C, 1] \mid \theta = 0).
\]

Combining the two gives

\[
\mathbb{P}_\sigma(q^C \in [1 - \delta^C, 1] \mid \theta = 0) \geq \frac{\beta(1 - \beta)}{(1 - \beta)\beta} \mathbb{P}_\sigma(q^C \in [1 - \delta^C, 1] \mid \theta = 0).
\]

Since the ratio on the right hand side is strictly larger than one, this leads to a contradiction if the probability is larger than zero. We conclude that both probabilities are equal to zero, and consequently, asymptotic learning obtains. □