Optimizing Product Launches in the Presence of Strategic Consumers

Appendix

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A1. Technical proofs

Proof of Lemma 1

The lowest threshold z such that the customer is still willing to purchase every product launched is, by the one-stage deviation principle, the one that makes the customer indifferent to buying at a given product launch. Since the game is stationary, we can verify whether the customer is indifferent to purchasing the first launch assuming he will buy at all future launches. From Eq. (1), if customer c is indifferent to the first purchase then from Eq. (1):

$$\mathbb{E}[q_1^c(e^{-\delta\kappa_1^c} - e^{-\delta\kappa_2^c})\frac{v}{\delta} - pe^{-\delta\kappa_1^c}|\kappa_1^c] = 0.$$

By assumption, the technology at this launch is $q_1^c = z$. We can thus simplify the equation above to

$$z(1 - \mathbb{E}[e^{-\delta X(z)}]) = \frac{p\delta}{v},$$

where $X(z) = \kappa_2^c - \kappa_1^c$ and is the random variable that represents the time it takes for a Brownian motion of drift μ and variance σ^2 to hit level z. The moment generating function $\mathbb{E}[e^{-\delta X(z)}]$ of the hitting time $X(z)$ of a Brownian motion is well known to be $e^{-\Delta z}$, where Δ is given in Eq. (4) — e.g., see Karatzas and Shreve [22].

The uniqueness of the solution is guaranteed by the fact that the function $f(z) = z(1 - e^{-\Delta z})$ is strictly increasing, with $\lim_{z\to 0} f(z) = 0$, and $\lim_{z\to\infty} f(z) = \infty$. ∎

Proof of Theorem 1

We first argue there exists an MPE where the firm releases a new product whenever the gap between the technology on hand and the technology in the market is greater than or equal to z^*, i.e., $Z(t) - w(t) \geq z^*$, and all (or almost all) consumers buy at all releases when the technology gap is at least z^*. By Lemma 1, consumers do not have a profitable deviation. We now show the firm neither has a profitable deviation. Let U^* denote the firm's equilibrium expected utility. Consider a state of the game at time t where $Z(t) - w(t) = z'$ for some $z' \geq z^*$. By launching, the firm would earn a continuation payoff of $e^{-\delta t}(p - c - K + U^*)$, where U^* comes from the fact that consumers would buy the product anticipating the next launch to occur after a new technology gain of z^*, leading the game back to its starting state $M(t) = M(0) = (0, (0, 1))$. By delaying a launch to $t' > t$, the continuation payoff would be reduced to $e^{-\delta t'}(p - c - K + U^*)$, thus eliminating this deviation. Now, consider the state $M(t) = (z', (0, 1))$ for some $z' < z^*$. If the firm were to deviate and launch a new product, consumers would anticipate that another launch would occur when hitting technology level z^* and so would choose not to purchase. Therefore, this deviation is also not profitable.

We now argue that no other path is possible in an MPE. For any Markovian policy of the firm s^f, consider the value $m(s^f) = \inf\{m : s^f(m,(0,1)) = 1\}$ — the lowest value of the difference $Z(t)-w(t)$ at which the firm would launch a new product when all consumers own the latest technology in the market $w(t)$. If $m(s^f) > z^*$, the firm would have an incentive to deviate whenever $Z(t) - w(t) = \frac{z^*+m(s^f)}{2}$, as the continuation value of the deviation at time t would be $e^{-\delta t}(p - c - K + U^*)$, while the continuation value of the original strategy would be $e^{-\delta t'}(p-c-K+U^*)$ where $t' > t$ is the time the technology gap would reach $m(s^f)$. Now, consider the case where $m(s^f) < z^*$. Anticipating that another product would be released when hitting technology level z^*, consumers would choose not to purchase. Therefore, in all MPE $m(s^f) = z^*$.

The only remaining possible deviation is for some of the consumers not to purchase at a given launch. This would seem plausible by Lemma 1, as consumers are indifferent on making any one purchase when products are released every time the technology gain reaches z^*. However, if that were to happen, the firm would deviate and delay the product launch until reaching a technology level $z^* + \epsilon$, with $\epsilon > 0$. The marginal cost due to the delay associated with this strategy is continuous in ϵ (through the hitting time distribution of a Brownian motion with drift), but the gain is discontinuous in ϵ since the consumers would no longer be indifferent between buying and not buying, and would all purchase immediately. Therefore, this profitable deviation rules out MPEs that involve asymmetric consumer behavior, except among a set of consumers of measure zero. ∎

Proof of Lemma 2

We first argue that we can focus on symmetric consumer equilibria without loss of optimality. If there are multiple different strategies employed by the consumers in equilibrium, they must be indifferent between them since they are homogeneous in their valuations. We assumed the consumers adopt at most finitely many different strategies, so let $s^{c_1}, s^{c_2}, ..., s^{c_N}$ be consumer strategies adopted by respective fractions $\pi_1, \pi_2, ..., \pi_N$ of the consumer market. For any firm policy $\boldsymbol{z}$, the firm's profit is a weighted average of the profit from each strategy cohort, i.e.,

$$U^f(\boldsymbol{z}, (s^{c_1}, ..., s^{c_N})) = \sum_{i=1}^{N} \pi_i U^f(\boldsymbol{z}, s^{c_i}).$$

By selecting the cohort i with the highest $U^f(\boldsymbol{z}, s^{c_i})$, we find a symmetric equilibrium with a (weakly) higher profit for the firm than under the original asymmetric equilibrium.

We now argue that in a symmetric equilibrium consumers do not need to delay purchases, thus buying as soon as products are released. Consider a firm policy $\boldsymbol{z}$ and a consumer policy s^c where the consumers delay purchases with positive probability. Let $i^* \in \mathbb{N}$ be the first launch where the set of realizations $\omega \in \Omega$ that lead to a purchase has positive measure, but less than 1. The problem faced by the consumers at launch i^* is independent of the realization ω since they all purchased at launch $i^* - 1$, buying technology z_{i^*-1} and the technology is now exactly at z_{i^*}. Therefore, without loss of utility, the consumers can ignore the realization ω and either buy at launch i^* with probability 0 or with probability 1.

If consumers do not buy at launch i^* under any realization $\omega \in \Omega$, the firm should delay this introduction. Both the firm and the consumer would be better off if the firm delayed the launch until a time when all consumers would buy. Thus, the set of purchases is identical to the set of launches at optimality, and consumers need only consider deviations to buying at a subset of the launches when considering their best responses. ∎

Proof of Lemma 3

It is sufficient to show that $U^c(\boldsymbol{z}, \boldsymbol{q})$ is submodular pathwise. Consider technology level $z_i \in \boldsymbol{z}$ with corresponding launch time τ_i and any $\boldsymbol{q} \subseteq \boldsymbol{z} \backslash \{z_i\}$. Let k' be the technology level of the product owned by the consumer at time τ_i and k'' be the technology level of the next product purchased after time τ_i, with κ'' being the purchasing time of technology k''. The difference in consumer utility between policies $\boldsymbol{q}$ and $\boldsymbol{q} \cup \{z_i\}$ is

$$U^c(\boldsymbol{z}, \boldsymbol{q} \cup \{z_i\}) - U^c(\boldsymbol{z}, \boldsymbol{q}) = \frac{v}{\delta}(z_i - k')(e^{-\delta\tau_i} - e^{-\delta\kappa''}) - pe^{-\delta\tau_i},$$

with $z_i > k'$. Note that the difference in utilities is decreasing in k' and increasing in k''. Thus, if we consider two different consumer policies, $\boldsymbol{q} \subseteq \bar{\boldsymbol{q}}$, we obtain that

$$U^c(\boldsymbol{z}, \boldsymbol{q} \cup \{z_i\}) - U^c(\boldsymbol{z}, \boldsymbol{q}) \geq U^c(\boldsymbol{z}, \bar{\boldsymbol{q}} \cup \{z_i\}) - U^c(\boldsymbol{z}, \bar{\boldsymbol{q}}),$$

which is a characterization of a submodular function. ∎

Proof of Proposition 1

The purchasing policy $\boldsymbol{z}$ being a best response immediately implies that $\mathbb{E}[U^c(\boldsymbol{z}, \boldsymbol{z})] \geq \mathbb{E}[U^c(\boldsymbol{z}, \boldsymbol{z} \backslash \{z_i\})]$ for all $i \in \mathbb{N}$. Consider a consumer policy $\boldsymbol{q} \subset \boldsymbol{z}$ and represent the elements in the set $\boldsymbol{z} \backslash \boldsymbol{q}$ by $\{l_1, l_2, ...\}$. The consumer's expected utility from purchasing according to $\boldsymbol{z}$ is equal to the utility from purchasing according to $\boldsymbol{q}$ plus the differences in utility from adding each element in $\boldsymbol{z} \backslash \boldsymbol{q}$, i.e.,

$$\mathbb{E}[U^c(\boldsymbol{z}, \boldsymbol{z})] = \mathbb{E}\left[U^c(\boldsymbol{z}, \boldsymbol{q}) + \sum_{i=1}^{|\boldsymbol{z}\backslash\boldsymbol{q}|} \left(U^c(\boldsymbol{z}, \boldsymbol{q} \cup \{l_1, ..., l_i\}) - U^c(\boldsymbol{z}, \boldsymbol{q} \cup \{l_1, ..., l_{i-1}\})\right)\right].$$

Since $\boldsymbol{q} \cup \{l_1, ..., l_{i-1}\} \subset \boldsymbol{z}$, using the submodularity result from Lemma 3 on each of the terms inside the summation above, we obtain that

$$\mathbb{E}[U^c(\boldsymbol{z}, \boldsymbol{z})] \geq \mathbb{E}[U^c(\boldsymbol{z}, \boldsymbol{q})] + \sum_{i=1}^{|\boldsymbol{z}\backslash\boldsymbol{q}|} \mathbb{E}[(U^c(\boldsymbol{z}, \boldsymbol{z}) - U^c(\boldsymbol{z}, \boldsymbol{z} \backslash \{l_i\}))].$$

Therefore, if $\mathbb{E}[U^c(\boldsymbol{z}, \boldsymbol{z}) - U^c(\boldsymbol{z}, \boldsymbol{z} \backslash \{l_j\})] \geq 0$ for all $i \in \boldsymbol{z} \backslash \boldsymbol{q}$, then $\mathbb{E}[U^c(\boldsymbol{z}, \boldsymbol{z})] \geq \mathbb{E}[U^c(\boldsymbol{z}, \boldsymbol{q})]$, which characterizes the purchasing policy $\boldsymbol{z}$ as the best response to the launch policy $\boldsymbol{z}$. ∎

Proof of Theorem 2

To prove this theorem, we relax the firm's optimization problem and explicitly find an optimal solution for this relaxed counterpart. We then show that the solution generated is feasible for the original problem and thus, optimal.

We relax the problem OPT-3 by ignoring all constraints where i is an even number. That is, we consider the relaxed problem

$$\max_{r_i \in [0,\infty]} \quad \sum_{i=1}^{\infty} e^{-\Delta \sum_{j=1}^{i} r_j} \qquad \text{(RELAX-1)}$$
$$\text{s.t.} \quad f(r_i, r_{i+1}) \geq 0 \quad \text{for all } i = 1, 3, 5, ...$$

where we define $f(x,y) = \frac{v}{\delta}x(1-e^{-\Delta y}) - p$.

This relaxation is tractable because the constraints are *disconnected* in the following sense: each decision variable r_i appears in exactly one constraint. Note that the objective is decreasing in all r_i's and, for any i, the function $f(r_i, r_{i+1})$ is increasing in both r_i and r_{i+1}. Therefore, all the constraints in the relaxed problem must be binding since otherwise we could decrease the value of r_i and increase the firm's profit without violating any constraint. Therefore, problem RELAX-1 is equivalent to

$$\begin{aligned} \max_{r_i \in [0,\infty]} \quad & \sum_{i=1}^{\infty} e^{-\Delta \sum_{j=1}^{i} r_j} \\ \text{s.t.} \quad & f(r_i, r_{i+1}) = 0 \quad \text{for all } i = 1,3,5,... \end{aligned} \tag{A1}$$

Define the constant

$$\beta = \frac{\delta p}{v}.$$

The constraint $f(r_1, r_2) = 0$ holds and the decision variables r_1 and r_2 are non-negative reals whenever $r_1 \geq \beta$ and

$$e^{-\Delta r_2} = 1 - \frac{\beta}{r_1}.$$

We can use this relationship to eliminate the variable r_2 from (A1), obtaining

$$\begin{aligned} \max_{\substack{r_1 \geq \beta \\ r_i \in [0,\infty] \text{ for } i \geq 3}} \quad & e^{-\Delta r_1}\left(2 - \frac{\beta}{r_1}\right) + e^{-\Delta r_1}\left(1 - \frac{\beta}{r_1}\right)\sum_{i=3}^{\infty} e^{-\Delta \sum_{j=3}^{i} r_j} \\ \text{s.t.} \quad & f(r_i, r_{i+1}) = 0 \quad \text{for all } i = 3,5,7,... \end{aligned} \tag{A2}$$

Let $K_1(r_1) = e^{-\Delta r_1}\left(2 - \frac{\beta}{r_1}\right)$ and $K_2(r_1) = e^{-\Delta r_1}\left(1 - \frac{\beta}{r_1}\right)$. Consider now the optimization problem holding constant some $r_1 > \beta$, i.e.,

$$\begin{aligned} \max_{r_i \in [0,\infty] \text{ for } i \geq 3} \quad & K_1(r_1) + K_2(r_1)\sum_{i=3}^{\infty} e^{-\Delta \sum_{j=3}^{i} r_j} \\ \text{s.t.} \quad & f(r_i, r_{i+1}) = 0 \quad \text{for all } i = 3,5,7,... \end{aligned}$$

The set of optimal solutions of this problem is independent of r_1 and is identical to the set of optimal solutions of (A1) since the two problems are identical except for a linear scaling of the objective function. In order to characterize the optimal solution, there are two cases to consider based on the relationship between the decision variable r_1 and β.

- As long as $r_1 > \beta$, any value that optimizes r_1 must also optimize r_3. By induction, the same values must also optimize r_5, $r_7, \ldots$. Equivalently, any value that optimizes r_2 also optimizes r_4, $r_6, \ldots$.
- In case $r_1 = \beta$, then $r_2 = \infty$ and $K_2(r_1) = 0$, and thus none of the other decision variables matter.

Let $\hat{\alpha}$ be the optimal value of (A1). Note that $\hat{\alpha}$ is finite: We have constraints $f(r_i, r_{i+1}) = 0$, for $i \geq 1$, i.e. $r_i = \frac{\beta}{1-\exp(-\Delta r_{i+1})}$. This implies that $r_i \geq \beta$, hence trivially, $r_i + r_{i+1} \geq \beta$. We use these two bounds in the objective function so that for i even, $\sum_{j=1}^{i} r_i \geq (i/2)\beta$; and for i odd, $\sum_{j=1}^{i} r_i \geq ((i-1)/2)\beta + \beta = ((i+1)/2)\beta$. This yields that the objective function is bounded above by

$$2 \times \sum_{i=1}^{\infty} \exp(-i\Delta\beta) = \frac{2}{1-\exp(-\Delta\beta)} < \infty,$$

which completes the argument.

From (A2), $\hat{\alpha}$ must meet the necessary optimality condition

$$\hat{\alpha} = \max_{r_1 \geq \beta} \quad e^{-\Delta r_1}\left[2 - \frac{\beta}{r_1} + \left(1 - \frac{\beta}{r_1}\right)\hat{\alpha}\right],$$

or equivalently,

$$\hat{\alpha} = \max_{r_1 \geq \beta} \quad e^{-\Delta r_1}\left[(\hat{\alpha}+2) - \frac{(\hat{\alpha}+1)\beta}{r_1}\right]. \tag{A3}$$

The function in the RHS to be maximized is of the form

$$g(x) \triangleq e^{-\Delta x}\left[c_1 - \frac{1}{x}\right],$$

where $c_1 = \frac{\hat{\alpha}+2}{(\hat{\alpha}+1)\beta}$. The function $g(\cdot)$ is unimodal in $\mathbb{R}_+$ for any positive c_1, as $g'(x) = \frac{e^{-\Delta x}}{x^2}(-c_1\Delta x^2 + \Delta x + 1)$ has a unique positive root in $\mathbb{R}_+$. Therefore, there is a unique r_1 that satisfies (A3) for the optimal value $\hat{\alpha}$.

This observation implies that an optimal solution for (A1) necessarily satisfies $r_1 = r_3 = r_5 = \ldots = r_{2i+1}$, and $r_2 = r_4 = r_6 = \ldots = r_{2i}$, for all i; which leads to the following problem equivalent to (A1), but defined over just two variables:

$$\begin{aligned} &\max_{r_1 \geq \beta, r_2 \geq 0} \quad \sum_{i=1}^{\infty} e^{-i\Delta(r_1+r_2)} + e^{-\Delta r_1}\sum_{i=0}^{\infty} e^{-i\delta(r_1+r_2)} \\ &\text{s.t.:} \\ &f(r_1, r_2) = 0. \end{aligned} \tag{A4}$$

Using the geometric progression formula, we get:

$$\begin{aligned} &\max_{r_1 \geq \beta, r_2 \geq 0} \quad \frac{e^{-\Delta r_1} + e^{-\Delta(r_1+r_2)}}{1 - e^{-\Delta(r_1+r_2)}} \\ &\text{s.t.:} \\ &f(r_1, r_2) = 0. \end{aligned}$$

Using the expression $f(r_1, r_2) = 0$, we replace $\exp(-\Delta r_2)$ by $1 - \beta/r_1$ in the objective function and drop the constraint. This reduces the problem to the following formulation:

$$\max_{r_1 \geq \beta} \quad \frac{e^{-\Delta r_1} + e^{-\Delta r_1}\left(1 - \frac{\beta}{r_1}\right)}{1 - e^{-\Delta r_1}\left(1 - \frac{\beta}{r_1}\right)}.$$

Multiplying above and below by r_1, and then adding and subtracting r_1 in the numerator, we get

$$\max_{r_1 \geq \beta} \quad \frac{r_1 e^{-\Delta r_1} + r_1}{r_1 - e^{-\Delta r_1}(r_1 - \beta)} - 1.$$

From the function we wish to maximize, we define

$$r(x) = \frac{x e^{-\Delta x} + x}{x - e^{-\Delta x}(x - \beta)}.$$

Lemma A1. *The function $r : \mathbb{R}_{\geq \beta} \to \mathbb{R}$ has a unique maximizer $\bar{x}$. In particular, $\bar{x} = \max\{\bar{r}_1, \beta\}$, where $\bar{r}_1$ is the unique unconstrained maximizer over the extended domain $\mathbb{R}_+$.*

Proof: We have

$$r'(x) = \frac{-2\Delta e^{\Delta x} x^2 + \beta(1 + e^{\Delta x}(1 + \Delta x))}{\left[\beta + (e^{\Delta x} - 1)x\right]^2} = 0, \tag{A5}$$

where $\beta = \frac{\delta p}{v}$. The function $r(\cdot)$ defined over the extended domain $\mathbb{R}_+$ is increasing in the interval $(0, \bar{r_1})$ and decreasing in $(\bar{r_1}, \infty)$, where $\bar{r_1}$ is the unique positive solution to $r'(x) = 0$. Thus, $\bar{x} \triangleq \arg\max_{x \geq \beta} r(x) = \max\{\bar{r_1}, \beta\}$. ∎

In terms of our original problem, we get $\hat{r}_1 = \max\{\bar{r_1}, \beta\}$. Note that the expression for $r(\cdot)$ is also continuous in β and Δ. In particular, when $\bar{r_1} = \beta$, condition (A5) reduces to $r'(\beta) = 0$, i.e.,

$$2\Delta\beta e^{\Delta\beta} = 1 + e^{\Delta\beta}(1 + \Delta\beta). \tag{A6}$$

Using the Lambert W function, defined by the equation

$$z = W(z)\exp(W(z)),$$

where z is any complex number, we can rewrite equation (A6) as $\Delta\beta = W\left(\frac{1}{e}\right) + 1 \approx 1.27846$.

The unique solution to equation (A6) is $\Delta\beta \triangleq \lambda$. Recalling that $\beta = p\delta/v$, we have for $p/v = \lambda/(\delta\Delta)$, $\hat{r}_1 = \beta = \bar{r_1}$. The relation between p/v and problem parameters splits the plane in two parts:

$$\hat{r}_1 = \begin{cases} p\delta/v & \text{if } p/v \geq \lambda/(\delta\Delta), \\ \text{determined by (A5)} & \text{otherwise.} \end{cases}$$

The optimal $\hat{r}_2$ will be determined from $f(\hat{r}_1, \hat{r}_2) = 0$, i.e., from the equation

$$\hat{r}_1[1 - e^{-\Delta \hat{r}_2}] = \beta. \tag{A7}$$

In case $p/v \geq \lambda/(\delta\Delta)$, $\hat{r}_2 = \infty$. Otherwise, $\hat{r}_1 > \beta$ and $\hat{r}_2 < \infty$, and the optimal policy is to offer products in cycles alternating between $\hat{r}_1$ and $\hat{r}_2$.

We are left with proving that our solution to the relaxed problem (RELAX-1) is also feasible for the original problem (OPT-3). Since we dropped even numbered constraints and we have $\hat{r}_1 = r_{2i+1}$ and $\hat{r}_2 = r_{2i}$ for all i; we are just left with showing that $f(\hat{r}_2, \hat{r}_1) \geq 0$. In fact, if we could prove that $\hat{r}_1 < \hat{r}_2$, then from Eq. (A7), we could use the following sequence of equivalences to argue that $f(\hat{r}_2, \hat{r}_1) \geq 0$:

$$\begin{aligned} \hat{r}_1 < \hat{r}_2 \qquad & \Leftrightarrow \frac{1 - e^{-\Delta \hat{r}_1}}{\hat{r}_1} \geq \frac{1 - e^{-\Delta \hat{r}_2}}{\hat{r}_2} \quad \text{as } g_1(x) \triangleq \frac{1 - e^{-\Delta x}}{x} \text{ is decreasing in } x \\ & \Leftrightarrow \hat{r}_2[1 - e^{-\Delta \hat{r}_1}] \geq \hat{r}_1[1 - e^{-\Delta \hat{r}_2}] \\ & \Leftrightarrow \hat{r}_2[1 - e^{-\Delta \hat{r}_1}] \geq \beta \\ & \Leftrightarrow f(\hat{r}_2, \hat{r}_1) \geq 0, \quad \text{by definition of } f. \end{aligned}$$

In what follows, we show that indeed $\hat{r}_1 < \hat{r}_2$ holds. Note that there exists a unique value z^* such that $f(z^*, z^*) = 0$, i.e.

$$z^*(1 - e^{-\Delta z^*}) = \beta. \tag{A8}$$

Since f is strictly increasing in both arguments and $f(\hat{r}_1, \hat{r}_2) = 0$, we can see that if $\hat{r}_1 \neq \hat{r}_2 \neq z^*$, then $\hat{r}_1$ and $\hat{r}_2$ lie on different sides with respect to z^*. Formally, if $\hat{r}_1 \neq \hat{r}_2 \neq z^*$, then either $\hat{r}_1 < z^* < \hat{r}_2$ or $\hat{r}_2 < z^* < \hat{r}_1$. We prove that the former is always true. Observe that the point (z^*, z^*) is in the feasible region of problem (A4). Since the objective function r is unimodal with unconstrained maximizer $\bar{r_1}$, it is enough to check $r'(z^*) < 0$, to show that $\bar{r_1} < z^*$.
The derivative of $r(\cdot)$ is given by

$$r'(x) = \frac{-2\Delta e^{\Delta x} x^2 + \beta(1 + e^{\Delta x}(1 + \Delta x))}{[\beta + (e^{\Delta x} - 1)x]^2}$$

It is indeed enough to show that the numerator $-2\Delta e^{\Delta x} x^2 + \beta + \beta e^{\Delta x} + \Delta x \beta e^{\Delta x}$ evaluated at z^* is negative. So, we evaluate:

$$\begin{aligned}
-2\Delta e^{\Delta z^*}(z^*)^2 + \beta + \beta e^{\Delta z^*} + \Delta z^* \beta e^{\Delta z^*} &= -2\Delta e^{\Delta z^*}(z^*)^2 + z^*[1 - e^{-\Delta z^*}] + z^*[1 - e^{-\Delta z^*}]e^{\Delta z^*} \\
&\quad + \Delta(z^*)^2[1 - e^{-\Delta z^*}]e^{\Delta z^*} \quad \text{(by replacing } \beta \text{ using (A8))} \\
&= -2\Delta(z^*)^2 e^{\Delta z^*} + (\Delta(z^*)^2 + z^*)e^{\Delta z^*}[1 - e^{-\Delta z^*}] + z^*[1 - e^{-\Delta z^*}] \\
&= z^*(1 + e^{\Delta z^*})[1 - \Delta z^* - e^{-\Delta z^*}] \\
&= z^*(1 + e^{\Delta z^*})[1 - \Delta z^* - [1 - \Delta z^* + \frac{(\Delta z^*)^2}{2!} - \frac{(\Delta z^*)^3}{3!} \cdots]] \\
&= z^*(1 + e^{\Delta z^*})[-\frac{(\Delta z^*)^2}{2!} + \frac{(\Delta z^*)^3}{3!} \cdots] < 0
\end{aligned}$$

We can also check that $\beta < z^*$. To see this, define $h(x) = x(1 - e^{-\Delta x})$, where from (A8), $h(z^*) = \beta$. It can be verified that $h(x)$ is increasing, and since $h(\beta) < \beta$, we can assert that $\beta < z^*$. To conclude, $\hat{r}_1 \triangleq \max\{\bar{r_1}, \beta\} < z^*$, so that $\hat{r}_1 < \hat{r}_2$, and the two cycle policy $(\hat{r}_1, \hat{r}_2)$ is feasible and optimal for the original problem (OPT-3). ∎

Proof of Proposition 2

In this proof, we show that the firm's expected utility is unimodal in z^*, which implies the desired result given that the price p is increasing in z^* (see Eq. (8)). We begin from Eq. (9) and perform a change of variable, introducing $x = e^{-\Delta z^*}$ for $x \in (0, 1)$, and rewriting the firm's utility as

$$\begin{aligned}
\mathbb{E}[U^f(x, x)] &= \left(\frac{-v \log x(1 - x)}{\delta \Delta} - c - K \right) \frac{x}{1 - x} \\
&= \frac{-v}{\delta \Delta} x \log x - (c + K)\frac{x}{1 - x}.
\end{aligned}$$

The above function is of the form $g(x) = x\left[-a \log x - \frac{b}{1-x}\right]$ where a and b are positive constants. The above function is unimodal as follows. Taking the derivative with respect to x, we get

$$g'(x) = \left[-a \log x - \frac{b}{1 - x}\right] + x\left[\frac{-a}{x} - \frac{b}{(1 - x)^2}\right]$$

We have $\lim_{x\to 0^+} g'(x) = \infty$ and $\lim_{x\to 1^-} g'(x) = -\infty$. Also, it is easy to see $g'(x)$ is strictly decreasing function; hence there is a unique point that satisfies the first order condition that is a maximum of $g(\cdot)$ function. ∎

Proof of Theorem 3

Let r_1 be the unique launch technology level (or the optimal short cycle), and r_2 be the corresponding optimal long cycle derived in Theorem 2 for a given price p. First, we will argue that the joint launching and pricing optimization problem, which is in principle a two-dimensional search on p and r_1, reduces to a single dimensional search (recall that r_2 is uniquely determined by r_1 through Eq. (A7)). In fact, the relation between p and r_1 is given by the following expression:

$$r_1(p) = \begin{cases} \text{determined by (A5)} & \text{if } p/v < \lambda/(\delta\Delta), \\ p\delta/v & \text{otherwise,} \end{cases} \tag{A9}$$

where β in Eq. (A5) is given by $\beta = \delta p/v$. Recall that $r_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous in p (c.f., Lemma A1 and thereafter in the proof of Theorem 2). It is also strictly increasing in p, for $p > 0$; which implies that there is one-to-one correspondence between p and r_1. In addition, from (A5), $\lim_{p\to 0} r_1(p) = 0$ (see Figure 4(right) for an illustration)[1].

Recall that λ is the unique solution to the equation

$$\frac{2\lambda e^{\lambda}}{1 + e^{\lambda}(1+\lambda)} = 1, \text{ or equivalently }, 1 + e^{\lambda} = \lambda e^{\lambda}, \tag{A10}$$

giving $\lambda \approx 1.27846$. Let us also define the function $L(x)$, to be extensively used in the following analysis:

$$L(x) = \frac{2xe^x}{1 + e^x(1+x)}, \text{ with } L(\lambda) = 1. \tag{A11}$$

Thus, using equation (A9), we can define $p(r_1)$, i.e., the inverse function of $r_1(p)$, as follows:

$$p(r_1) = \begin{cases} \frac{2v\Delta(r_1)^2 e^{\delta r_1}}{\delta(1+e^{\delta r_1}(1+\delta r_1))} & \text{if } 0 \le r_1 \le \lambda/\Delta, \\ r_1 v/\delta & \text{otherwise,} \end{cases}$$

or equivalently, by using the L function in (A11),

$$p(r_1) = \begin{cases} \frac{r_1 v L(\delta r_1)}{\delta} & \text{if } 0 \le r_1 \le \lambda/\Delta, \\ \frac{r_1 v}{\delta} & \text{otherwise.} \end{cases} \tag{A12}$$

Note that $p(r_1)$ is well defined in the domain $\mathbb{R}_+$ and strictly increasing. Hence, in order to prove that the firm's utility is unimodal in p conditionally on the firm using the optimal launch policy $r_1(p)$ in (A9), it is sufficient to prove that the firm's utility is unimodal in r_1.

In fact, the firm's expected utility as a function of r_1, when using the optimal price $p(r_1)$ in (A12), is given by:

$$\begin{aligned} \mathbb{E}[U_p^f(r_1)] &= (p(r_1) - c - K)\left[\frac{r_1 e^{-\Delta r_1} + r_1}{r_1 - e^{-\delta r_1}(r_1 - p(r_1)\delta/v)} - 1\right] \\ &= (p(r_1) - c - K)\left[\frac{r_1 e^{-\Delta r_1} + r_1}{r_1 - e^{-\delta r_1}(r_1 - p(r_1)\delta/v)} - 1\right]. \end{aligned}$$

[1] For sake of the argument in this analysis, we are extending the range of p. Recall that by assumption, $p > c+K$. However, in order to simplify notation and w.l.o.g., we assume in this proof that $p > 0$, even though it could lead to a firm's negative utility.

Let $K_1 \triangleq c + K$ be the effective cost per launch. The firm's expected utility function becomes

$$\mathbb{E}[U_p^f(r_1)] = (p(r_1) - K_1)\left[\frac{r_1 e^{-\Delta r_1} + r_1}{r_1 - e^{-\delta r_1}(r_1 - p(r_1)\delta/v)} - 1\right].$$

Substituting $p(r_1)$ by its expression in (A12), we get:

$$\mathbb{E}[U_p^f(r_1)] = \begin{cases} \left(\frac{r_1 v L(\Delta r_1)}{\delta} - K_1\right)\left[\frac{e^{\Delta r_1}+1}{e^{\Delta r_1}-1+L(\Delta r_1)} - 1\right], & \text{if } 0 \leq r_1 \leq \lambda/\Delta, \\ \left(\frac{r_1 v}{\delta} - K_1\right) e^{-\Delta r_1}, & \text{otherwise.} \end{cases}$$

Note that, $\mathbb{E}[U_p^f(r_1)]$ is continuous; it is clearly piecewise continuous and at $r_1 = \lambda/\Delta$, $E[U_p^f(\lambda/\Delta)] = \left(\frac{\lambda v}{\delta\Delta} - K_1\right) e^{-\lambda}$ for both parts, as we know from (A11) that $L(\lambda) = 1$.
Define separate functions for each part in the definition of $\mathbb{E}[U_p^f(r_1)]$:

$$M(r_1) = \left(\frac{r_1 v L(\Delta r_1)}{\delta} - K_1\right)\left[\frac{e^{\Delta r_1} + 1}{e^{\Delta r_1} - 1 + L(\Delta r_1)} - 1\right], \tag{A13}$$

and

$$S(r_1) = \left(\frac{r_1 v}{\delta} - K_1\right) e^{-\Delta r_1}, \tag{A14}$$

so that

$$\mathbb{E}[U_p^f(r_1)] = \begin{cases} M(r_1) & \text{if } 0 \leq r_1 \leq \lambda/\Delta, \\ S(r_1) & \text{otherwise.} \end{cases}$$

Next, we show that $\mathbb{E}[U_p^f(r_1)]$ is unimodal in r_1 in three steps (see Figure A1):

1. Show that $S(r_1)$ is a unimodal function of r_1.
2. Show that $M(r_1)$ is a unimodal function of r_1.
3. Show that the derivatives of $S(r_1)$ and $M(r_1)$ at the split value $r_1 = \lambda/\Delta$ have the same sign.

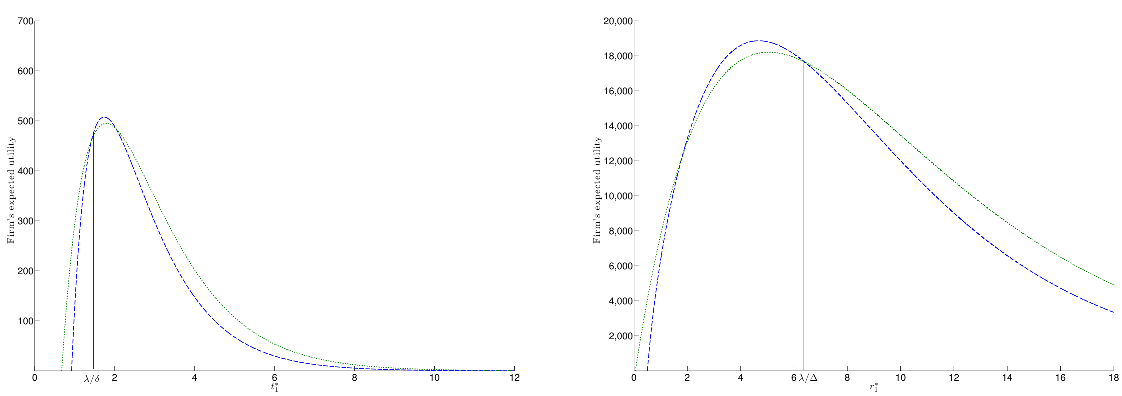

Figure A1: Unimodality of $U_p^f(r_1)$. Each part of the function $U_p^f(r_1)$ is unimodal, and they cross either when they are both increasing or decreasing, which warrants the global unimodality. Left: Parameters are $\delta = 0.9, K = 1500, v = 2000, c = 0, \mu = 1, \sigma = 0$. Right: Parameters are $\delta = 0.2, K = 500, v = 2000, c = 0, \mu = 1, \sigma = 0$.

Step 1: From (A14), we have

$$S'(r_1) = \frac{e^{-\Delta r_1}(\delta \Delta K_1 - \Delta v r_1 + v)}{\delta}. \tag{A15}$$

Observe that $S'(0) > 0$ and that $\lim_{r_1 \to \infty} S'(r_1) = -\infty$. Also, $S'(r_1)$ has only one root at $r_1 = \delta K_1/v + 1/\Delta$. So, $S(r_1)$ is increasing over $0 \le r_1 \le \delta K_1/v + 1/\Delta$, and decreasing over $\delta K_1/v + 1/\Delta \le r_1 < \infty$. Hence, $S(r_1)$ is unimodal function. ∎

Step 2: From (A13), we have

$$M'(r_1) = Q_1(r_1) Q_2(r_1), \tag{A16}$$

where

$$Q_1(r_1) \triangleq \frac{2(r_1)^2 e^{3\Delta r_1}(\Delta r_1 + 2)}{\left(e^{\Delta r_1}(\Delta r_1 + 1) - 1\right)^2 \left(e^{\Delta r_1}(\Delta r_1 + 1) + 1\right)^2}$$

and

$$Q_2(r_1) \triangleq \Delta^3 K_1 + \frac{2\Delta^2 K_1 \left(e^{-\Delta r_1} + 1\right)}{r_1} + \frac{\Delta K_1 \left(e^{-\Delta r_1} + 1\right)^2}{(r_1)^2} + \frac{2v\Delta \left(1 - e^{-2\Delta r_1}\right)}{\delta r_1} - \frac{2\Delta^3 v r_1}{\delta}$$

It can be easily verified that $Q_1(r_1)$ is positive in $(0, \infty)$. It can also be checked that $Q_2(r_1)$ is strictly decreasing with:

$$\lim_{r_1 \to 0} Q_2(r_1) = \infty \quad \text{and} \quad \lim_{r_1 \to \infty} Q_2(r_1) = -\infty.$$

Thus, $Q_2(r_1)$ has a single root that we denote α. Given the definition in (A16), $M(r_1)$ turns out to be strictly increasing in $0 < r_1 \le \alpha$ and strictly decreasing in $\alpha < r_1 < \infty$. Hence, it is unimodal in r_1. ∎

Before going over Step 3, we need the following auxiliary result:

Claim 1. $L'(\lambda) = \Delta/2$
Proof: From (A11), we have

$$L'(\Delta r_1) = \frac{2\Delta e^{\Delta r_1}(\Delta r_1 + e^{\Delta r_1} + 1)}{(e^{\Delta r_1}(\Delta r_1 + 1) + 1)^2}.$$

Hence,

$$L'(\lambda) = \frac{2\Delta e^{\lambda}(\lambda + e^{\lambda} + 1)}{(e^{\lambda}(\lambda + 1) + 1)^2}. \tag{A17}$$

Using Eq. (A10) and substituting into the numerator of (A17), we get

$$L'(\lambda) = \frac{2\Delta e^{\lambda}(\lambda + \lambda e^{\lambda})}{(e^{\lambda}(\lambda + 1) + 1)^2}. \tag{A18}$$

Substituting again Eq. (A10), now into the numerator of (A18), we have

$$\begin{aligned} L'(\lambda) =& \frac{\Delta}{2}\left[\frac{2\lambda e^{\lambda}}{e^{\lambda}(\lambda + 1) + 1}\right]^2 \\ =& \frac{\Delta}{2} \times 1 = \frac{\Delta}{2}, \end{aligned}$$

and the claim is proved ∎

Step 3: We want to prove that both $S'(\lambda/\Delta)$ and $M'(\lambda/\Delta)$ have the same sign.
From (A15), we have:

$$S'(\lambda/\Delta) = \frac{e^{-\lambda}(K_1\delta\Delta - \lambda v + v)}{\delta}. \tag{A19}$$

From (A13), we can write the derivative of $M(r_1)$ as:

$$\begin{aligned} M'(r_1) =& \frac{v}{\delta}\left(L(\Delta r_1) + r_1 L'(\Delta r_1)\right)\left[\frac{e^{\Delta r_1}+1}{e^{\Delta r_1}-1+L(\Delta r_1)} - 1\right] + \\ & \left(\frac{r_1 v L(\Delta r_1)}{\delta} - K_1\right)\left[\frac{\Delta e^{\Delta r_1}(e^{\Delta r_1}-1+L(\Delta r_1)) - (e^{\Delta r_1}+1)(\Delta e^{\Delta r_1} + L'(\Delta r_1))}{(e^{\Delta r_1}-1+L(\Delta r_1))^2}\right]. \end{aligned}$$

So,

$$\begin{aligned} M'(\lambda/\Delta) =& \frac{v}{\delta}\left(L(\lambda) + \lambda L'(\lambda)/\Delta\right)\left[\frac{e^{\lambda}+1}{e^{\lambda}-1+L(\lambda)} - 1\right] + \\ & \left(\frac{\lambda v L(\lambda)}{\delta\Delta} - K_1\right)\left[\frac{\Delta e^{\lambda}(e^{\lambda}-1+L(\lambda)) - (e^{\lambda}+1)(\Delta e^{\lambda} + L'(\lambda))}{(e^{\lambda}-1+L(\lambda))^2}\right]. \end{aligned}$$

Substituting $L(\lambda) = 1$ and $L'(\lambda) = \Delta/2$ in the equation above, we get:

$$\begin{aligned} M'(\lambda/\Delta) &= \frac{v}{\delta}\left[1 + \frac{\lambda}{2}\right]e^{-\lambda} + \left(\frac{\lambda v}{\delta\Delta} - K_1\right)\left[\frac{\Delta e^{2\lambda} - (e^{\lambda}+1)(\Delta e^{\lambda} + \frac{\Delta}{2})}{e^{2\lambda}}\right] \\ &= \frac{v}{\delta}\left[1 + \frac{\lambda}{2}\right]e^{-\lambda} + \left(\frac{\lambda v - K_1\delta\Delta}{\delta}\right)\left[1 - (1+e^{-\lambda})(1+e^{-\lambda}/2)\right] \\ &= \left[\frac{2v + \lambda v}{2\delta}\right]e^{-\lambda} - \left[\frac{\lambda v - K_1\delta\Delta}{2\delta}\right]e^{-\lambda}(3 + e^{-\lambda}) \end{aligned}$$

From Eq. (A10), using that $1 + e^{-\lambda} = \lambda$, and substituting above, we get:

$$\begin{aligned} M'(\lambda/\Delta) &= \left[\frac{2v + \lambda v}{2\delta}\right]e^{-\lambda} - \left[\frac{\lambda v - K_1\delta\Delta}{2\delta}\right]e^{-\lambda}(2 + \lambda) \\ &= \left(\frac{e^{-\lambda}}{2\delta}\right)\left[2v + 2K_1\delta\Delta - \lambda v + \lambda(K_1\delta\Delta - \lambda v)\right] \\ &= \left(\frac{e^{-\lambda}}{2\delta}\right)(\lambda + 2)\left[K_1\delta\Delta - \lambda v + v\right] \end{aligned} \tag{A20}$$

We can see from (A19) and (A20) that both $S'(\lambda/\Delta)$ and $M'(\lambda/\Delta)$ have the same sign as $K_1\delta\Delta - \lambda v + v$. This completes the proof of Step 3, and of Theorem 3. ∎