Customer Referral Incentives and Social Media

Ilan Lobel, Evan Sadler and Lav R. Varshney

Online Appendix

Proof of Proposition 3

Fix all parameters of the model except y and δ. Holding the net referral value $y - \delta$ fixed, the optimal sequence of discounts $\{I_d\}_d \in \mathbb{N}$ is independent of δ, and hence P_σ in the corresponding consumer equilibrium is independent of δ. The payment function that implements the optimal sequence of referral and discount pairs takes the form

$$\mathbf{w}(d) = \mathbf{w}_0(d) + \frac{\delta d}{P_\sigma}, \tag{14}$$

where $\mathbf{w}_0(n)$ is the optimal payment function when $\delta = 0$. The rightmost term exactly compensates consumers for making referrals, so this claim is immediate from the definition of equilibrium.

From (14), it is sufficient to show that there exists d such that $\mathbf{w}_0(d+1) < \mathbf{w}_0(d)$. Let d^* be the smallest degree d such that $I_d = p$ in the optimal policy, and suppose that $\mathbf{w}_0$ is non-decreasing up to d^*. We clearly have

$$\mathbb{E}\left[\mathbf{w}_0\left(B_{P_\sigma,\mathbf{w}_0}(d^*+1)\right)\right] \geq \mathbb{E}\left[\mathbf{w}_0\left(B_{P_\sigma,\mathbf{w}_0}(d^*)\right)\right].$$

If $P_\sigma > 0$, then $B_{P_\sigma,\mathbf{w}_0}(d^*+1)$ strictly dominates $B_{P_\sigma,\mathbf{w}_0}(d^*)$, and we must have $\mathbf{w}_0(d^*+1) \leq \mathbf{w}_0(d^*)$. If additionally $P_\sigma < 1$, then the number of successful referrals takes all feasible values with positive probability, implying that either $\mathbf{w}_0(d^*+1) < \mathbf{w}_0(d^*)$ or $\mathbf{w}_0(d^*+1) = \mathbf{w}_0(0) = 0$. The latter condition is inconsistent with a positive price, so we conclude that $\mathbf{w}_0(d^*+1) < \mathbf{w}_0(d^*)$. Furthermore, we can see from (14) that with sufficiently high δ, we can implement optimal payments with a monotone $\mathbf{w}$ because the linear component will dominate. $\square$

Proof of Proposition 4

Set $I = \frac{1}{2\mu^{1-\alpha/2}}$. For sufficiently large μ, the optimal policy gives $I = p$ to all consumers with degree higher than $\frac{\mu^{1-\alpha/2}}{2}$, and the loss from over payments to high-degree consumers is no more than $\frac{\mu}{2\mu^{1-\alpha/2}} = \frac{\mu^{\alpha/2}}{2}$. Given our constraint on the tail of the degree distribution, we can find a constant C' such that the probability that a consumer has degree less than $\frac{\mu^{1-\alpha/2}}{2}$ is no more than

$\frac{C'}{\mu^{1-\alpha}}$. Consequently, the loss from consumers with degree less than $\frac{\mu^{1-\alpha/2}}{2}$ is at most proportional to $\frac{\mu^{1-\alpha/2}}{2}\frac{C'}{\mu^{1-\alpha}} = \frac{C'\mu^{\alpha/2}}{2}$, completing the proof. □

Proof of Proposition 6

Existence follows the same argument as in Proposition 1, applying Brouwer's fixed point theorem instead of the intermediate value theorem. For the second claim, note that whenever $d \geq d'$, we can couple the neighborhood realizations for a random degree d consumer and a random degree d' consumer such that the degree d consumer always has the option to mimic the degree d' consumer. To construct the coupling, realize the first d' neighbors and signals for the degree d consumer. The distribution of these d' neighbors and signals exactly matches the neighborhood and signal distribution for a random degree d' consumer. We can match degree d' consumers with h high signals to the degree d consumers with h high signals among their first d' neighbors, and then we can realize the remaining $d - d'$ neighbors and signals. For each matching, the degree d consumer can mimic the corresponding degree d' consumer, and hence must derive at least as much value from the referral program. Hence, each degree d consumer purchases at a (weakly) lower valuation threshold than the corresponding degree d' consumer in the coupling. Averaging over the neighbor and signal distribution for the degree d' consumer implies the second claim. □

Proof of Proposition 7

In equilibrium, we can characterize the behavior of neighbors via the pair of probabilities (P^-, P^+), denoting the probability that a low-degree and a high-degree neighbor respectively will purchase. Since $p = \frac{1}{2}$ and valuations are uniform, we know that each of these probabilities is at least $\frac{1}{2}$. The expected payment from referring a low-degree neighbor is at least

$$\frac{(y+\delta)P^-}{2P^*} \geq \frac{(y+\delta)}{4} \geq \delta,$$

so all neighbors will be given referrals in equilibrium.

Consider the equilibrium of the game without additional information. In this game, let $\hat{P}^-$ denote the probability that a random neighbor with degree no more than τ will purchase, and let $\hat{P}^+$ denote the corresponding probability for a random neighbor with degree greater than τ. We show that $P^- = \hat{P}^-$ and $P^+ = \hat{P}^+$ in equilibrium, and in fact, a random neighbor of a given degree has the same equilibrium purchase probability in both games.

We can check this using the best reply mapping. Suppose that neighbors follow a strategy profile such that $P^- = \hat{P}^-$ and $P^+ = \hat{P}^+$. Given a consumer with degree d, we can couple neighborhood realizations with h high signals and $d-h$ high signals, which are equally likely events. The expected benefit from a referral to a low-degree neighbor is

$$\frac{(y+\delta)\hat{P}^-}{2P^*} - \delta \equiv I^-,$$

and the expected benefit from a referral to a high-degree neighbor is

$$\frac{(y+\delta)\hat{P}^+}{2P^*} - \delta \equiv I^+.$$

The consumer with h high signals will then purchase with probability $\frac{1}{2} + hI^+ + (d-h)I^-$, while the consumer with $d-h$ high signals will purchase with probability $\frac{1}{2} + (d-h)I^+ + hI^-$. The average purchase probability of a degree d consumer is then

$$\frac{1}{2} + d\frac{I^- + I^+}{2} = \frac{1}{2} + d\frac{y-\delta}{2}$$

as in the model without additional information.

While the number of purchasing consumers, as well as the number of referrals, remains the same, we can plainly see that the value of referrals increases relative to the game without additional information. The average consumer who purchases the product will have more high-degree neighbors than low-degree neighbors, yielding a higher average referral value. □