Appendix B: Omitted Proofs from "Bayesian Learning in Social Networks" (Not for Publication)

Proof of Lemma 1

(a) By the definition of a private belief, we have for any $p_n \in (0,1)$,

$$\mathbb{P}(\theta = 1|s_n) = \mathbb{P}(\theta = 1|p_n).$$

Using Bayes' Rule, it follows that

$$p_n = \mathbb{P}_\sigma(\theta = 1|p_n) = \frac{d\mathbb{P}(p_n|\theta = 1)\mathbb{P}(\theta = 1)}{\sum_{j=0}^{1} d\mathbb{P}(p_n|\theta = j)\mathbb{P}(\theta = j)} = \frac{d\mathbb{P}(p_n|\theta = 1)}{\sum_{j=0}^{1} d\mathbb{P}(p_n|\theta = j)} = \frac{d\mathbb{G}_1(p_n)}{\sum_{j=0}^{1} d\mathbb{G}_j(p_n)}.$$

Because of the assumption that no signal is completely informative, i.e., $p_n \notin \{0,1\}$, we can rewrite this equation as

$$\frac{d\mathbb{G}_0}{d\mathbb{G}_1}(p_n) = \frac{1-p_n}{p_n},$$

completing the proof.

(b) For any $p \in (0,1)$,

$$\mathbb{G}_0(p) = \int_{r=0}^{p} d\mathbb{G}_0(r) = \int_{r=0}^{p} \frac{1-r}{r} d\mathbb{G}_1(r) = \left(\frac{1-p}{p}\right)\mathbb{G}_1(p) + \int_{r=0}^{p}\left(\frac{1}{r} - \frac{1}{p}\right) d\mathbb{G}_1(r),$$

where the second equality follows from part (a) of this lemma. We can provide a lower bound on the last integral as

$$\int_{r=0}^{p}\left(\frac{1}{r} - \frac{1}{p}\right) d\mathbb{G}_1(r) \geq \int_{r=0}^{z}\left(\frac{1}{r} - \frac{1}{p}\right) d\mathbb{G}_1(r) \geq \int_{r=0}^{z}\left(\frac{1}{z} - \frac{2}{z+p}\right) d\mathbb{G}_1(r) \geq \frac{p-z}{2}\mathbb{G}_1(z),$$

for any $z \in (0,p)$. Equivalently, the second relation is obtained by

$$\begin{aligned} 1 - \mathbb{G}_1(p) = \int_{r=p}^{1} d\mathbb{G}_1(r) &= \int_{r=p}^{1} \frac{r}{1-r} d\mathbb{G}_0(r) \\ &= (1 - \mathbb{G}_0(p))\left(\frac{p}{1-p}\right) + \int_{r=p}^{1}\left(\frac{r}{1-r} - \frac{p}{1-p}\right) d\mathbb{G}_0(r), \end{aligned}$$

where the following bound is valid for any $p < w < 1$,

$$\begin{aligned} \int_{r=p}^{1}\left(\frac{r}{1-r} - \frac{p}{1-p}\right) d\mathbb{G}_0(r) &\geq \int_{r=w}^{1}\left(\frac{r}{1-r} - \frac{p}{1-p}\right) d\mathbb{G}_0(r) \\ &\geq \int_{r=w}^{1}\left(\frac{w}{1-w} - \frac{p+w}{2-p-w}\right) d\mathbb{G}_0(r) \geq \frac{w-p}{2}(1 - \mathbb{G}_0(w)). \end{aligned}$$

(c) From part (a), we have for any $r \in (0,1)$,

$$\mathbb{G}_0(r) = \int_{x=0}^{r} d\mathbb{G}_0(x) = \int_{x=0}^{r}\left(\frac{1-x}{x}\right) d\mathbb{G}_1(x) \geq \int_{x=0}^{r}\left(\frac{1-r}{r}\right) d\mathbb{G}_1(x) = \left(\frac{1-r}{r}\right)\mathbb{G}_1(r). \tag{B1}$$

Using part (a) again,

$$d\left(\frac{\mathbb{G}_0(r)}{\mathbb{G}_1(r)}\right) = \frac{d\mathbb{G}_0(r)\mathbb{G}_1(r) - \mathbb{G}_0(r)d\mathbb{G}_1(r)}{(\mathbb{G}_1(r))^2}$$
$$= \frac{d\mathbb{G}_1(r)}{(\mathbb{G}_1(r))^2}\left[\left(\frac{1-r}{r}\right)\mathbb{G}_1(r) - \mathbb{G}_0(r)\right].$$

Since $\mathbb{G}_1(r) > 0$ for $r > \underline{\beta}$, $d\mathbb{G}_1(r) \geq 0$ and the term in brackets above is non-positive by Eq. (B1), we have

$$d\left(\frac{\mathbb{G}_0(r)}{\mathbb{G}_1(r)}\right) \leq 0,$$

thus proving the ratio $\mathbb{G}_0(r)/\mathbb{G}_1(r)$ is non-increasing.

We now show that

$$\mathbb{G}_0(r) \geq \mathbb{G}_1(r) \text{ for all } r \in [0,1]. \tag{B2}$$

From Eq. (B1), we obtain that Eq. (B2) is true for $r \leq 1/2$. For $r > 1/2$,

$$1 - \mathbb{G}_0(r) = \int_{x=r}^{1} d\mathbb{G}_0(x) = \int_{x=r}^{1}\left(\frac{1-x}{x}\right)d\mathbb{G}_1(x) \leq \int_{x=r}^{1} d\mathbb{G}_1(x) = 1 - \mathbb{G}_1(r),$$

thus proving Eq. (B2).

We proceed to prove the second part of the lemma. Suppose $\mathbb{G}_0(r)/\mathbb{G}_1(r) = 1$ for some $r < \overline{\beta}$. Suppose first $r \in (1/2, \overline{\beta})$. Then,

$$\begin{aligned}
\mathbb{G}_0(1) &= \mathbb{G}_0(r) + \int_{x=r}^{1} d\mathbb{G}_0(x) \\
&= \mathbb{G}_1(r) + \int_{x=r}^{1} d\mathbb{G}_0(x) \\
&= \mathbb{G}_1(r) + \int_{x=r}^{1}\left(\frac{1-x}{x}\right)d\mathbb{G}_1(x) \\
&\geq \mathbb{G}_1(r) + \left(\frac{1-r}{r}\right)\int_{x=r}^{1} d\mathbb{G}_1(x) \\
&\geq \mathbb{G}_1(r) + \left(\frac{1-r}{r}\right)[1 - \mathbb{G}_1(r)],
\end{aligned}$$

which yields a contradiction unless $\mathbb{G}_1(r) = 1$. However, $\mathbb{G}_1(r) = 1$ implies $r \geq \overline{\beta}$ – also a contradiction. Now, suppose $r \in (\underline{\beta}, 1/2]$. Since the ratio $\mathbb{G}_0(r)/\mathbb{G}_1(r)$ is non-increasing, this implies that for all $x \in (r, 1]$, $\mathbb{G}_0(x)/\mathbb{G}_1(x) \leq 1$. Combined with Eq. (B2), this yields $\mathbb{G}_0(x)/\mathbb{G}_1(x) = 1$ for all $x \in (r, 1]$, which yields a contradiction for $x \in (1/2, \overline{\beta})$. ■

Nonmonotonicity of Social Beliefs

In this subsection, we illustrate the difficulties involved in determining equilibrium learning in general social networks. In particular, we show that social beliefs, as defined in

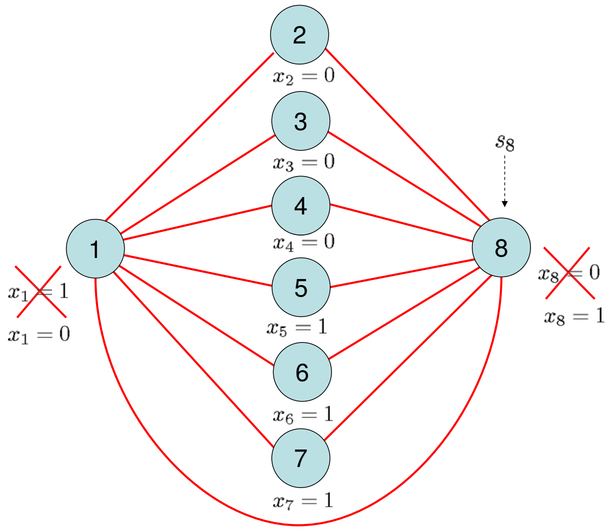

Figure 3: The figure illustrates a deterministic topology in which the social beliefs are nonmonotone.

Definition 3, may be nonmonotone, in the sense that additional observations of $x_n = 1$ in the neighborhood of an individual may reduce the social belief (i.e., the posterior derived from past observations that $x_n = 1$ is the correct action).

The following example establishes this point. Suppose the private signals are such that $\mathbb{G}_0(r) = 2r - r^2$ and $\mathbb{G}_1(r) = r^2$, which is a pair of private belief distributions $(\mathbb{G}_0, \mathbb{G}_1)$. Suppose the network topology is deterministic and for the first eight agents, it has the following structure: $B(1) = \emptyset, B(2) = ... = B(7) = \{1\}$ and $B(8) = \{1, ..., 7\}$ (see Figure 2).

For this social network, agent 1 has 3/4 probability of making a correct decision in either state of the world. If agent 1 chooses the action that yields a higher payoff (i.e., the correct decision), then agents 2 to 7 each have 15/16 probability of choosing the correct decision. However, if agent 1 fails to choose the correct decision, then agents 2 to 7 have a 7/16 probability of choosing the correct decision. Now suppose agents 1 to 4 choose action $x_n = 0$, while agents 5 to 7 choose $x_n = 1$. The probability of this event happening in each state of the world is:

$$\mathbb{P}_\sigma(x_1 = ... = x_4 = 0, x_5 = x_6 = x_7 = 1 | \theta = 0) = \frac{3}{4}\left(\frac{15}{16}\right)^3\left(\frac{1}{16}\right)^3 = \frac{10125}{2^{26}},$$

$$\mathbb{P}_\sigma(x_1 = ... = x_4 = 0, x_5 = x_6 = x_7 = 1 | \theta = 1) = \frac{1}{4}\left(\frac{9}{16}\right)^3\left(\frac{7}{16}\right)^3 = \frac{250047}{2^{26}}.$$

Using Bayes' Rule, the social belief of agent 8 is given by

$$\left[1 + \frac{10125}{250047}\right]^{-1} \backsimeq 0.961.$$

Now, consider a change in x_1 from 0 to 1, while keeping all decisions as they are. Then,

$$\mathbb{P}_\sigma(x_1 = 1, x_2 = x_3 = x_4 = 0, x_5 = x_6 = x_7 = 1|\theta = 0) = \frac{1}{4}\left(\frac{7}{16}\right)^3\left(\frac{9}{16}\right)^3 = \frac{250047}{2^{26}},$$

$$\mathbb{P}_\sigma(x_1 = 1, x_2 = x_3 = x_4 = 0, x_5 = x_6 = x_7 = 1|\theta = 1) = \frac{1}{4}\left(\frac{1}{16}\right)^3\left(\frac{16}{16}\right)^3 = \frac{10125}{2^{26}}.$$

This leads to a social belief of agent 8 given by

$$\left[1 + \frac{250047}{10125}\right]^{-1} \backsimeq 0.039.$$

Therefore, this example has established that when x_1 changes from 0 to 1, agent 8's social belief declines from 0.961 to 0.039. That is, while the agent strongly believes the state is 1 when $x_1 = 0$, he equally strongly believes the state is 0 when $x_1 = 1$. This happens because when half of the agents in $\{2, \ldots, 7\}$ choose action 0 and the other half choose action 1, agent n places a high probability to the event that $x_1 \neq \theta$. This leads to a nonmonotonicity in social beliefs.

Proof of Lemma 2. Let $h : \{(n, B(n)) : n \in N, B(n) \subseteq \{1, 2, ..., n-1\}\} \to \mathbb{N}$ be an arbitrary function that maps an agent and a neighborhood of the agent into an element of the neighborhood, i.e., $h(n, B(n)) \in B(n)$. In view of the characterization of the equilibrium decision x_n [cf. Eq. (2)], it follows that for any private signal s_n, neighborhood $B(n) \subseteq \{1, 2, ..., n-1\}$, and decisions x_k, $k \in B(n)$, we have

$$\mathbb{P}_\sigma(x_n = \theta \mid s_n, B(n), x_k,\ k \in B(n)) \geq \mathbb{P}_\sigma(x_{h(n,B(n))} = \theta \mid s_n, B(n), x_k,\ k \in B(n)).$$

By integrating over all possible private signals and decisions of agents in the neighborhood, we obtain that for any n and any $B(n) = \mathfrak{B}$,

$$\mathbb{P}_\sigma(x_n = \theta \mid B(n) = \mathfrak{B}) \geq \mathbb{P}_\sigma(x_{h(n,B(n))} = \theta \mid B(n) = \mathfrak{B}) = \mathbb{P}_\sigma(x_{h(n,\mathfrak{B})} = \theta),$$

where the equality follows by the assumption that each neighborhood is generated independently from all other neighborhoods. By taking the maximum over all functions h, we obtain

$$\mathbb{P}_\sigma(x_n = \theta \mid B(n) = \mathfrak{B}) \geq \max_{b \in \mathfrak{B}} \mathbb{P}_\sigma(x_b = \theta),$$

showing the desired relation. ■

Proof of Theorem 3

Proof of part (a): The proof consists of two steps. We first show that the lower and upper supports of the social belief $q_n = \mathbb{P}_\sigma(\theta = 1|x_1, ..., x_{n-1})$ are bounded away from 0 and 1. We next show that this implies that x_n does not converge to θ in probability.

Let $x^{n-1} = (x_1, ..., x_{n-1})$ denote the sequence of decisions up to and including $n-1$. Let $\varphi_{\sigma,x^{n-1}}(q_n, x_n)$ represent the social belief q_{n+1} given the social belief q_n and the decision x_n, for a given strategy σ and decisions x^{n-1}. We use Bayes' Rule to determine the dynamics of the social belief. For any x^{n-1} compatible with q_n, and $x_n = \overline{x}$ with $\overline{x} \in \{0,1\}$, we have

$$\begin{aligned}
\varphi_{\sigma,x^{n-1}}(q_n,\overline{x}) &= \mathbb{P}_\sigma(\theta = 1 \mid x_n = \overline{x}, q_n, x^{n-1}) \\
&= \left[1 + \frac{\mathbb{P}_\sigma(x_n = \overline{x}, q_n, x^{n-1}, \theta = 0)}{\mathbb{P}_\sigma(x_n = \overline{x}, q_n, x^{n-1}, \theta = 1)}\right]^{-1} \\
&= \left[1 + \frac{\mathbb{P}_\sigma(q_n, x^{n-1} \mid \theta = 0)}{\mathbb{P}_\sigma(q_n, x^{n-1} \mid \theta = 1)} \frac{\mathbb{P}_\sigma(x_n = \overline{x} \mid q_n, x^{n-1}, \theta = 0)}{\mathbb{P}_\sigma(x_n = \overline{x} \mid q_n, x^{n-1}, \theta = 1)}\right]^{-1} \\
&= \left[1 + \left(\frac{1}{q_n} - 1\right) \frac{\mathbb{P}_\sigma(x_n = \overline{x} \mid q_n, x^{n-1}, \theta = 0)}{\mathbb{P}_\sigma(x_n = \overline{x} \mid q_n, x^{n-1}, \theta = 1)}\right]^{-1}. \qquad \text{(B3)}
\end{aligned}$$

Let $\alpha_{\sigma,x^{n-1}}$ denote the probability that agent n chooses $x = 0$ in equilibrium σ when he observes history x^{n-1} and is indifferent between the two actions. Let

$$\mathbb{G}_j^-(r) = \lim_{s \uparrow r} \mathbb{G}_j(s),$$

for any $r \in [0,1]$ and any $j \in \{0,1\}$. Then, for any $j \in \{0,1\}$,

$$\begin{aligned}
\mathbb{P}_\sigma(x_n = 0 \mid q_n, x^{n-1}, \theta = j) &= \mathbb{P}_\sigma(p_n < 1 - q_n \mid q_n, \theta = j) \\
&\quad + \alpha_{\sigma,x^{n-1}} \mathbb{P}_\sigma(p_n = 1 - q_n \mid q_n, \theta = j) \\
&= \mathbb{G}_j^-(1 - q_n) + \alpha_{\sigma,x^{n-1}} \left[\mathbb{G}_j(1 - q_n) - \mathbb{G}_j^-(1 - q_n)\right].
\end{aligned}$$

From Lemma 1(a), $d\mathbb{G}_0/d\mathbb{G}_1(r) = (1-r)/r$ for all $r \in [0,1]$. Therefore,

$$\frac{1-r}{r} \le \frac{\mathbb{G}_0(r)}{\mathbb{G}_1(r)}, \quad \text{and} \quad \frac{\mathbb{G}_0^-(r)}{\mathbb{G}_1^-(r)} \le \frac{1-\underline{\beta}}{\underline{\beta}}.$$

Hence, for any $\alpha_{\sigma,x^{n-1}}$,

$$\begin{aligned}
\frac{\mathbb{P}_\sigma(x_n = 0 \mid q_n, x^{n-1}, \theta = 0)}{\mathbb{P}_\sigma(x_n = 0 \mid q_n, x^{n-1}, \theta = 1)} &= \frac{\mathbb{G}_0^-(1-q_n) + \alpha_{\sigma,x^{n-1}} \left[\mathbb{G}_0(1-q_n) - \mathbb{G}_0^-(1-q_n)\right]}{\mathbb{G}_1^-(1-q_n) + \alpha_{\sigma,x^{n-1}} \left[\mathbb{G}_1(1-q_n) - \mathbb{G}_0^-(1-q_n)\right]} \\
&\in \left[\frac{q_n}{1-q_n}, \frac{1-\underline{\beta}}{\underline{\beta}}\right].
\end{aligned}$$

Combining this with Eq. (B3), we obtain

$$\begin{aligned}
\varphi_{\sigma,x^{n-1}}(q_n,0) &\in \left[\left(1 + \left(\frac{1}{q_n} - 1\right)\left(\frac{1-\underline{\beta}}{\underline{\beta}}\right)\right)^{-1}, \left(1 + \left(\frac{1}{q_n} - 1\right)\left(\frac{q_n}{1-q_n}\right)\right)^{-1}\right] \\
&= \left[\frac{\underline{\beta} q_n}{1 - \underline{\beta} - q_n + 2\underline{\beta} q_n}, \frac{1}{2}\right]
\end{aligned}$$

Note that $\frac{\underline{\beta} q_n}{1-\underline{\beta}-q_n+2\underline{\beta} q_n}$ is an increasing function of q_n and if $q_n \in [1-\overline{\beta}, 1-\underline{\beta}]$, then this function is minimized at $1 - \overline{\beta}$. This implies that

$$\varphi_{\sigma,x^{n-1}}(q_n,0) \in \left[\frac{\underline{\beta}(1-\overline{\beta})}{-\underline{\beta} + \overline{\beta} + 2\underline{\beta}(1-\overline{\beta})}, \frac{1}{2}\right] = \left[\underline{\Delta}, \frac{1}{2}\right],$$

where $\underline{\Delta}$ is a constant strictly greater than 0. An analogous argument for $x_n = 1$ establishes that there exists some $\overline{\Delta} < 1$ such that if $q_n \in [1-\overline{\beta}, 1-\underline{\beta}]$, then

$$\varphi_{\sigma,x^{n-1}}(q_n, 1) \in \left[\frac{1}{2}, \overline{\Delta}\right].$$

We next show that $q_n \in [\underline{\Delta}, \overline{\Delta}]$ for all n. Suppose this is not true. Let N be the first agent such that

$$q_N \in [0, \underline{\Delta}) \cup (\overline{\Delta}, 1] \tag{B4}$$

in some equilibrium and some realized history. Then, $q_{N-1} \in [0, 1-\overline{\beta}) \cup (1-\underline{\beta}, 1]$ because otherwise, the dynamics of q_n implies a violation of Eq. (B4) for any x_{N-1}. But note that if $q_{N-1} < 1-\overline{\beta}$, then by Proposition 2 agent $N-1$ chooses action $x_{N-1} = 0$ and, thus by Eq. (B3),

$$\begin{aligned} q_N &= \left[1 + \left(\frac{1}{q_{N-1}} - 1\right) \frac{\mathbb{P}_\sigma(x_{N-1} = 0 \mid q_{N-1}, x^{N-2}, \theta = 0)}{\mathbb{P}_\sigma(x_{N-1} = 0 \mid q_{N-1}, x^{N-2}, \theta = 1)}\right]^{-1} \\ &= \left[1 + \left(\frac{1}{q_{N-1}} - 1\right) \frac{1}{1}\right]^{-1} \\ &= q_{N-1}. \end{aligned}$$

By the same argument, if $q_{N-1} > 1-\underline{\beta}$, we have that $q_N = q_{N-1}$. Therefore, $q_N = q_{N-1}$, which contradicts the fact that N is the first agent that satisfies Eq. (B4).

We next show that $q_n \in [\underline{\Delta}, \overline{\Delta}]$ for all n implies that x_n does not converge in probability to θ. Let y_k denote a realization of x_k. Then, for any n and any sequence of y_k's, we have

$$\mathbb{P}_\sigma(\theta = 1, x_k = y_k \text{ for all } k \le n) \le \overline{\Delta}\mathbb{P}_\sigma(x_k = y_k \text{ for all } k \le n),$$
$$\mathbb{P}_\sigma(\theta = 0, x_k = y_k \text{ for all } k \le n) \le (1-\underline{\Delta})\mathbb{P}_\sigma(x_k = y_k \text{ for all } k \le n).$$

By summing the preceding relations over all y_k for $k < n$, we obtain

$$\mathbb{P}_\sigma(\theta = 1, x_n = 1) \le \overline{\Delta}\mathbb{P}_\sigma(x_n = 1) \text{ and } \mathbb{P}_\sigma(\theta = 0, x_n = 0) \le (1-\underline{\Delta})\mathbb{P}_\sigma(x_n = 0).$$

Therefore, for any n, we have

$$\mathbb{P}_\sigma(x_n = \theta) \le \overline{\Delta}\mathbb{P}_\sigma(x_n = 1) + (1-\underline{\Delta})\mathbb{P}_\sigma(x_n = 0) \le \max\{\overline{\Delta}, 1-\underline{\Delta}\} < 1,$$

which completes the proof. ∎

Proof of part (b): The first step is the following lemma.

Lemma 11 *Let $B(n) = \{b\}$ for some n. We define*

$$f(\underline{\beta}, \overline{\beta}) = \max\left\{1 - \frac{\underline{\beta}}{2(1-\underline{\beta})}, \frac{3}{2} - \frac{1}{2\overline{\beta}}\right\}, \tag{B5}$$

where $\overline{\beta}$ and $\underline{\beta}$ are the lower and upper supports of the private beliefs (cf. Definition 4). Let σ be an equilibrium. Assume that $\mathbb{P}_\sigma(x_b = \theta) \le f(\underline{\beta}, \overline{\beta})$. Then, we have

$$\mathbb{P}_\sigma(x_n = \theta \mid B(n) = \{b\}) \le f(\underline{\beta}, \overline{\beta}).$$

Proof. We first assume that $\mathbb{P}_\sigma(x_b = \theta) = f(\underline{\beta}, \overline{\beta})$ and show that this implies

$$U_b^\sigma \geq \overline{\beta}, \qquad \text{and} \qquad L_b^\sigma \leq \underline{\beta}, \tag{B6}$$

where U_b^σ and L_b^σ are defined in Eq. (A14). We can rewrite U_b^σ as

$$U_b^\sigma = \frac{N_b^\sigma}{1 - 2\mathbb{P}_\sigma(x_b = \theta) + 2N_b^\sigma} = \frac{N_b^\sigma}{1 - 2f(\underline{\beta}, \overline{\beta}) + 2N_b^\sigma}.$$

This is a decreasing function of N_b^σ and, therefore,

$$U_b^\sigma \geq \frac{1}{1 - 2f(\underline{\beta}, \overline{\beta}) + 2}.$$

Using $f(\underline{\beta}, \overline{\beta}) \geq \frac{3}{2} - \frac{1}{2\overline{\beta}}$, the preceding relation implies $U_b^\sigma \geq \overline{\beta}$. An analogous argument shows that $L_b^\sigma \leq \underline{\beta}$.

Since the support of the private beliefs is $[\underline{\beta}, \overline{\beta}]$, using Lemma 3 and Eq. (B6), there exists an equilibrium $\sigma' = (\sigma_n', \sigma_{-n})$ such that $x_n = x_b$ with probability one (with respect to measure $P_{\sigma'}$). Since this gives an expected payoff $\mathbb{P}_{\sigma'}(x_n = \theta \mid B(n) = b) = \mathbb{P}_\sigma(x_b = \theta)$, it follows that, $\mathbb{P}_\sigma(x_n = \theta \mid B(n) = b) = \mathbb{P}_\sigma(x_b = \theta)$. This establishes the claim that

$$\text{if } \mathbb{P}_\sigma(x_b = \theta) = f(\underline{\beta}, \overline{\beta}), \text{ then } \mathbb{P}_\sigma(x_n = \theta \mid B(n) = \{b\}) = f(\underline{\beta}, \overline{\beta}). \tag{B7}$$

We next assume that $\mathbb{P}_\sigma(x_b = \theta) < f(\underline{\beta}, \overline{\beta})$. To arrive at a contradiction, suppose that

$$\mathbb{P}_\sigma(x_n = \theta \mid B(n) = \{b\}) > f(\underline{\beta}, \overline{\beta}). \tag{B8}$$

Now consider a hypothetical situation where agent n observes a private signal generated with conditional probabilities $(\mathbb{F}_0, \mathbb{F}_1)$ and a coarser version of the observation x_b, i.e., the random variable $\tilde{x}_b$ distributed according to

$$\mathbb{P}(\tilde{x}_b = 1 \mid \theta = 1) = 1 - Y_b^\sigma \left[\frac{1 - f(\underline{\beta}, \overline{\beta})}{\mathbb{P}_\sigma(x_b = \theta)}\right] \quad \text{and} \quad \mathbb{P}(\tilde{x}_b = 0 \mid \theta = 0) = 1 - N_b^\sigma \left[\frac{1 - f(\underline{\beta}, \overline{\beta})}{\mathbb{P}_\sigma(x_b = \theta)}\right].$$

It follows from the preceding conditional probabilities that $\mathbb{P}(\tilde{x}_b = \theta) = f(\underline{\beta}, \overline{\beta})$. We assume that agent n uses the equilibrium strategy σ_n. Using a similar argument as in the proof of Eq. (B7), this implies that

$$\mathbb{P}(x_n = \theta \mid B(n) = \{b\}) = f(\underline{\beta}, \overline{\beta}). \tag{B9}$$

Let z be a binary random variable with values $\{0, 1\}$ and is generated independent of θ with probabilities

$$\mathbb{P}(z = 1) = 1 - \frac{2Y_b^\sigma}{\mathbb{P}_\sigma(x_b = \theta)} \quad \text{and} \quad \mathbb{P}(z = 0) = 1 - \frac{2N_b^\sigma}{\mathbb{P}_\sigma(x_b = \theta)}.$$

This implies that $\mathbb{P}(z = j \mid \theta = j) = \mathbb{P}(z = j)$ for $j \in \{0, 1\}$. Using $\tilde{x}_b$ with probability $\frac{1}{1 + f(\underline{\beta}, \overline{\beta})}\left[2 + \frac{(Y_b^\sigma - 1)\mathbb{P}_\sigma(x_b = \theta)}{Y_b^\sigma}\right]$ and z otherwise generates the original observation (random variable) x_b. Therefore, from Eq. (B8), $\mathbb{P}(x_n = \theta | B(n) = \{b\}) > f(\underline{\beta}, \overline{\beta})$, which contradicts Eq. (B9), and completes the proof. ■

Let f be defined in Eq. (B5). We show by induction that

$$\mathbb{P}_\sigma(x_n = \theta) \le f(\underline{\beta}, \overline{\beta}) \qquad \text{for all } n. \tag{B10}$$

Suppose that for all agents up to $n-1$ the preceding inequality holds. Since $|B(n)| \le 1$, we have

$$\begin{aligned} \mathbb{P}_\sigma(x_n = \theta) &= \mathbb{P}_\sigma(x_n = \theta \mid B(n) = \emptyset)\mathbb{Q}_n(B(n) = \emptyset) \\ &\quad + \sum_{b=1}^{n-1} \mathbb{P}_\sigma(x_n = \theta \mid B(n) = b)\mathbb{Q}_n(B(n) = \{b\}) \\ &\le \mathbb{P}_\sigma(x_n = \theta \mid B(n) = \emptyset)\mathbb{Q}_n(B(n) = \emptyset) + \sum_{b=1}^{n-1} f(\underline{\beta}, \overline{\beta})\mathbb{Q}_n(B(n) = \{b\}), \end{aligned} \tag{B11}$$

where the inequality follows from the induction hypothesis and Lemma 11. Note that having $B(n) = \emptyset$ is equivalent to observing a decision b such that $\mathbb{P}_\sigma(x_b = \theta) = 1/2$. Since $1/2 \le f(\underline{\beta}, \overline{\beta})$, Lemma 11 implies that $\mathbb{P}_\sigma(x_n = \theta | B(n) = \emptyset) \le f(\underline{\beta}, \overline{\beta})$. Combined with Eq. (B11), this completes the induction.

Since the private beliefs are bounded, i.e., $\underline{\beta} > 0$ and $\overline{\beta} < 1$, we have $f(\underline{\beta}, \overline{\beta}) < 1$ [cf. Eq. (B5)]. Combined with Eq. (B10), this establishes that $\liminf_{n\to\infty} \mathbb{P}_\sigma(x_n = \theta) < 1$, showing that asymptotic learning does not occur at any equilibrium σ. ■

Proof of part (c): We start with the following lemma, which will be used subsequently in the proof.

Proof.

Lemma 12 *Assume that asymptotic learning occurs in some equilibrium σ, i.e., we have $\lim_{n\to\infty} \mathbb{P}_\sigma(x_n = \theta) = 1$. For some constant K, let $\mathcal{D}$ be the set of all subsets of $\{1, ..., K\}$. Then,*

$$\lim_{n\to\infty} \min_{D\in\mathcal{D}} \mathbb{P}_\sigma(x_n = \theta \mid x_k = 1,\ k \in D) = 1.$$

Proof. First note that since the event $x_k = 1$ for all $k \le K$ is the intersection of events $x_k = 1$ for each $k \le K$,

$$\min_{D\in\mathcal{D}} \mathbb{P}_\sigma(x_k = 1,\ k \in D) = \mathbb{P}_\sigma(x_k = 1,\ k \le K).$$

Let $\Delta = \mathbb{P}_\sigma(x_k = 1,\ k \le K)$. Fix some $\tilde{D} \in \mathcal{D}$. Then,

$$\mathbb{P}_\sigma(x_k = 1,\ k \in \tilde{D}) \ge \Delta > 0,$$

where the second inequality follows from the fact that there is a positive probability of the first K agents choosing $x_k = 1$. Let $\mathcal{A} = \{0, 1\}^{|\tilde{D}|}$, i.e., $\mathcal{A}$ is the set of all possible actions for the set of agents $\tilde{D}$. Then,

$$\mathbb{P}_\sigma(x_n = \theta) = \sum_{a_k \in \mathcal{A}} \mathbb{P}_\sigma(x_n = \theta \mid x_k = a_k,\ k \in \tilde{D})\mathbb{P}_\sigma(x_k = a_k,\ k \in \tilde{D}).$$

Since $\mathbb{P}_\sigma(x_n = \theta)$ converges to 1 and all elements in the set $\mathbb{P}_\sigma(x_k = 1,\ k \in \tilde{D})\}_{\tilde{D}\in\mathcal{A}}$ are greater than or equal to $\Delta > 0$, it follows that the sequence $\{\mathbb{P}_\sigma(x_n = \theta \mid x_k = 1,\ k \in \tilde{D})$ also converges to 1. Hence, for each $\epsilon > 0$, there exists some $N_\epsilon(\tilde{D})$ such that for all $n \geq N_\epsilon(\tilde{D})$,

$$\mathbb{P}_\sigma(x_n = \theta \mid x_k = 1,\ k \in \tilde{D}) \geq 1 - \epsilon.$$

Therefore, for any $\epsilon > 0$,

$$\min_{D\in\mathcal{D}} \mathbb{P}_\sigma(x_n = \theta \mid x_k = 1,\ k \in D) \geq 1 - \epsilon \quad \text{for all} \quad n \geq \max_{D\in\mathcal{D}} N_\epsilon(D),$$

thus completing the proof. ■ ■

We prove the claim in this part by contradiction. To arrive at a contradiction, we assume that in some equilibrium $\sigma \in \Sigma^*$, $\lim_{n\to\infty} \mathbb{P}_\sigma(x_n = \theta) = 1$. The key part of the proof is to show that this implies

$$\lim_{n\to\infty} \mathbb{P}_\sigma(\theta = 1 \mid x_k = 1,\ k \in B(n)) = 1. \tag{B12}$$

To prove this claim, we show that for any $\epsilon > 0$, there exists some $\tilde{K}(\epsilon)$ such that for any neighborhood $\mathfrak{B}$ with $|\mathfrak{B}| \leq M$ and $\max_{b\in\mathfrak{B}} b \geq \tilde{K}(\epsilon)$ we have

$$\mathbb{P}_\sigma(\theta = 1 \mid x_k = 1,\ k \in \mathfrak{B}) \geq 1 - \epsilon. \tag{B13}$$

In view of the assumption that $\max_{b\in B(n)} b$ converges to infinity with probability 1, this implies the desired claim (B12).

For a fixed $\epsilon > 0$, we define $\tilde{K}(\epsilon)$ as follows: We recursively construct M thresholds $K_0 < ... < K_{M-1}$ and let $\tilde{K}(\epsilon) = K_{M-1}$. We consider an arbitrary neighborhood $\mathfrak{B}$ with $|\mathfrak{B}| \leq M$ and $\max_{b\in\mathfrak{B}} b \geq K_{M-1}$, and for each $d \in \{0, ..., M-1\}$, define the sets

$$\mathfrak{B}_d = \{b \in \mathfrak{B} : b \geq K_d\} \text{ and } \mathfrak{C}_d = \{b \in \mathfrak{B} : b < K_{d-1}\},$$

where $\mathfrak{C}_0 = \emptyset$. With this construction, it follows that there exists at least one $d \in \{0, ..., M-1\}$ such that $\mathfrak{B} = \mathfrak{B}_d \cup \mathfrak{C}_d$, in which case we say $\mathfrak{B}$ is of type d. We show below that for any $\mathfrak{B}$ of type d, we have

$$\mathbb{P}_\sigma(\theta = 1 \mid x_k = 1,\ k \in \mathfrak{B}_d \cup \mathfrak{C}_d) \geq 1 - \epsilon, \tag{B14}$$

which implies the relation in (B13).

We first define K_0 and show that for any $\mathfrak{B}$ of type 0, relation (B14) holds. Since $\lim_{n\to\infty} \mathbb{P}_\sigma(x_n = \theta) = 1$ by assumption, there exists some N_0 such that for all $n \geq N_0$,

$$\mathbb{P}_\sigma(x_n = \theta) \geq 1 - \frac{\epsilon}{2M}.$$

Let $K_0 = N_0$. Let $\mathfrak{B}$ be a neighborhood of type 0, implying that $\mathfrak{B} = \mathfrak{B}_0$ and all elements $b \in \mathfrak{B}_0$ satisfy $b \geq K_0$. By using a union bound, the preceding inequality implies

$$\mathbb{P}_\sigma(x_k = \theta,\ k \in \mathfrak{B}_0) \geq 1 - \sum_{k\in\mathfrak{B}_0} \mathbb{P}_\sigma(x_k \neq \theta) \geq 1 - \frac{\epsilon}{2}.$$

Hence, we have

$$\mathbb{P}_\sigma(x_k = \theta,\ k \in \mathfrak{B}_0 \mid \theta = 1)\,\frac{1}{2} + \mathbb{P}_\sigma(x_k = \theta,\ k \in \mathfrak{B}_0 \mid \theta = 0)\,\frac{1}{2} \geq 1 - \frac{\epsilon}{2},$$

and for any $j \in \{0, 1\}$,

$$\mathbb{P}_\sigma(x_k = \theta,\ k \in \mathfrak{B}_0 \mid \theta = j) \geq 1 - \epsilon. \tag{B15}$$

Therefore, for any such $\mathfrak{B}_0$,

$$\begin{aligned}\mathbb{P}_\sigma(\theta = 1 \mid x_k = 1,\ k \in \mathfrak{B}_0) &= \left[1 + \frac{\mathbb{P}_\sigma(x_k = 1,\ k \in \mathfrak{B}_0 \mid \theta = 0)\mathbb{P}_\sigma(\theta = 0)}{\mathbb{P}_\sigma(x_k = 1,\ k \in \mathfrak{B}_0 \mid \theta = 1)\mathbb{P}_\sigma(\theta = 1)}\right]^{-1} \\ &\geq \left[1 + \frac{\epsilon}{1-\epsilon}\right]^{-1} = 1 - \epsilon,\end{aligned}$$

showing that relation (B14) holds for any $\mathfrak{B}$ of type 0.

We proceed recursively, i.e., given K_{d-1} we define K_d and show that relation (B14) holds for any neighborhood $\mathfrak{B}$ of type d. Lemma 12 implies that

$$\lim_{n\to\infty} \min_{D\subseteq\{1,\dots,K_{d-1}-1\}} \mathbb{P}_\sigma(x_n = \theta \mid x_k = 1,\ k \in D) = 1.$$

Therefore, for any $\delta > 0$, there exists some K_d such that for all $n \geq K_d$,

$$\min_{D\subseteq\{1,\dots,K_{d-1}-1\}} \mathbb{P}_\sigma(x_n = \theta \mid x_k = 1,\ k \in D) \geq 1 - \delta\epsilon.$$

From the equation above and definition of $\mathfrak{C}_d$ it follows that for any $\mathfrak{C}_d$,

$$\mathbb{P}_\sigma(x_n = \theta \mid x_k = 1,\ k \in \mathfrak{C}_d) \geq 1 - \delta\epsilon.$$

By a union bound,

$$\begin{aligned}\mathbb{P}_\sigma(x_k = \theta,\ k \in \mathfrak{B}_d \mid x_k = 1,\ k \in \mathfrak{C}_d) &\geq 1 - \sum_{k\in\mathfrak{B}_d} \mathbb{P}_\sigma(x_k \neq \theta \mid x_k = 1,\ k \in \mathfrak{C}_d) \\ &\geq 1 - (M - d)\delta\epsilon.\end{aligned}$$

Repeating the argument from Eq. (B15), for any $j \in \{0, 1\}$,

$$\mathbb{P}_\sigma(x_k = \theta,\ k \in \mathfrak{B}_d \mid \theta = j, x_k = 1,\ k \in \mathfrak{C}_d) \geq 1 - \frac{(M - d)\delta\epsilon}{\mathbb{P}_\sigma(\theta = j \mid x_k = 1,\ k \in \mathfrak{C}_d)}.$$

Hence, for any such $\mathfrak{B}_d$,

$$\begin{aligned}&\mathbb{P}_\sigma(\theta = 1 \mid x_k = 1,\ k \in \mathfrak{B}_d \cup \mathfrak{C}_d) \\ &= \left[1 + \frac{\mathbb{P}_\sigma(x_k = 1,\ k \in \mathfrak{B}_d \mid \theta = 0, x_k = 1,\ k \in \mathfrak{C}_d)\mathbb{P}_\sigma(\theta = 0, x_k = 1,\ k \in \mathfrak{C}_d)}{\mathbb{P}_\sigma(x_k = 1,\ k \in \mathfrak{B}_d \mid \theta = 1, x_k = 1,\ k \in \mathfrak{C}_d)\mathbb{P}_\sigma(\theta = 1, x_k = 1,\ k \in \mathfrak{C}_d)}\right]^{-1} \\ &\geq \left[1 + \frac{\frac{(M-d)\delta\epsilon}{\mathbb{P}_\sigma(\theta=0 \mid x_k=1,\ k\in\mathfrak{C}_d)}\mathbb{P}_\sigma(\theta = 0, x_k = 1,\ k \in \mathfrak{C}_d)}{\left(1 - \frac{(M-d)\delta\epsilon}{\mathbb{P}_\sigma(\theta=1 \mid x_k=1,\ k\in\mathfrak{C}_d)}\right)\mathbb{P}_\sigma(\theta = 1, x_k = 1,\ k \in \mathfrak{C}_d)}\right]^{-1} \\ &= 1 - \frac{(M - d)\delta\epsilon}{\mathbb{P}_\sigma(\theta = 1 \mid x_k = 1,\ k \in \mathfrak{C}_d)}.\end{aligned}$$

Choosing

$$\delta = \left(\frac{1}{M-d}\right) \min_{D\subseteq\{1,\ldots,K_{d-1}-1\}} \mathbb{P}_\sigma(\theta = 1 \mid x_k = 1,\ k \in D),$$

we obtain that for any neighborhood $\mathfrak{B}$ of type d

$$\mathbb{P}_\sigma(\theta = 1 \mid x_k = 1,\ k \in \mathfrak{B}_d \cup \mathfrak{C}_d) \geq 1 - \epsilon.$$

This proves that Eq. (B14) holds for any neighborhood $\mathfrak{B}$ of type d, and completing the proof of Eq. (B13) and therefore of Eq. (B12).

Since the private beliefs are bounded, we have $\underline{\beta} > 0$. By Eq. (B12), there exists some $\overline{N}$ such that

$$\mathbb{P}_\sigma(\theta = 1 | x_k = 1,\ k \in B(n)) \geq 1 - \frac{\underline{\beta}}{2} \qquad \text{for all } n \geq \overline{N}.$$

Suppose the first $\overline{N}$ agents choose 1, i.e., $x_k = 1$ for all $k \leq \overline{N}$, which is an event with positive probability for any state of the world θ. We now prove inductively that this event implies that $x_n = 1$ for all $n \in \mathbb{N}$. Suppose it holds for some $n \geq \overline{N}$. Then, by Eq. (B13),

$$\mathbb{P}_\sigma(\theta = 1 \mid s_{n+1}) + \mathbb{P}_\sigma(\theta = 1 \mid x_k = 1,\ \in B(n+1)) \geq \underline{\beta} + 1 - \frac{\underline{\beta}}{2} > 1.$$

By Proposition 2, this implies that $x_{n+1} = 1$. Hence, we conclude there is a positive probability $x_n = 1$ for all $n \in \mathbb{N}$ in any state of the world, contradicting $\lim_{n\to\infty} \mathbb{P}_\sigma(x_n = \theta) = 1$, and completing the proof. ■

Proof of Corollary 3

For $M = 1$, the result follows from Theorem 3 part (b). For $M \geq 2$, we show that, under the assumption on the network topology, $\max_{b\in B(n)} b$ goes to infinity with probability one. To arrive at a contradiction, suppose this is not true. Then, there exists some $K \in \mathbb{N}$ and scalar $\epsilon > 0$ such that

$$\mathbb{Q}\left(\max_{b\in B(n)} b \leq K \text{ for infinitely many } n\right) \geq \epsilon.$$

By the Borel-Cantelli Lemma (see, e.g., Breiman, Lemma 3.14, p. 41), this implies that

$$\sum_{n=1}^{\infty} \mathbb{Q}_n\left(\max_{b\in B(n)} b \leq K\right) = \infty.$$

Since the samples are all uniformly drawn and independent, for all $n \geq 2$,

$$\mathbb{Q}_n\left(\max_{b\in B(n)} b \leq K\right) = \left(\frac{\min\{K, n-1\}}{n-1}\right)^M.$$

Therefore,

$$\sum_{n=1}^{\infty} \mathbb{Q}_n\left(\max_{b\in B(n)} b \leq K\right) = 1 + \sum_{n=1}^{\infty}\left(\frac{\min\{K, n-1\}}{n}\right)^M \leq 1 + \sum_{n=1}^{\infty}\left(\frac{K}{n}\right)^M < \infty,$$

where the last inequality holds since $M \geq 2$. Hence, we obtain a contradiction. The result follows by using Theorem 3 part (c). ■