Appendix B: Omitted Proofs from “Bayesian Learning in Social Networks” (Not for Publication)

Proof of Lemma 1

(a) By the definition of a private belief, we have for any \( p_n \in (0, 1) \),

\[
\mathbb{P}(\theta = 1|s_n) = \mathbb{P}(\theta = 1|p_n).
\]

Using Bayes’ Rule, it follows that

\[
p_n = \mathbb{P}_\sigma(\theta = 1|p_n) = \frac{d\mathbb{P}(p_n|\theta = 1)\mathbb{P}(\theta = 1)}{\sum_{j=0}^1 d\mathbb{P}(p_n|\theta = j)\mathbb{P}(\theta = j)} = \frac{d\mathbb{P}(p_n|\theta = 1)}{\sum_{j=0}^1 d\mathbb{P}(p_n|\theta = j)} = \frac{d\mathbb{G}_1(p_n)}{\sum_{j=0}^1 d\mathbb{G}_j(p_n)}.
\]

Because of the assumption that no signal is completely informative, i.e., \( p_n \notin \{0, 1\} \), we can rewrite this equation as

\[
\frac{d\mathbb{G}_0(p_n)}{d\mathbb{G}_1(p_n)} = \frac{1-p_n}{p_n},
\]

completing the proof.

(b) For any \( p \in (0, 1) \),

\[
\mathbb{G}_0(p) = \int_{r=0}^p d\mathbb{G}_0(r) = \int_{r=0}^p \frac{1-r}{r} d\mathbb{G}_1(r) = \left(1 - \frac{p}{p}\right) \mathbb{G}_1(p) + \int_{r=0}^p \left(\frac{1}{p} - \frac{1}{r}\right) d\mathbb{G}_1(r),
\]

where the second equality follows from part (a) of this lemma. We can provide a lower bound on the last integral as

\[
\int_{r=0}^p \left(\frac{1}{r} - \frac{1}{p}\right) d\mathbb{G}_1(r) \geq \int_{r=0}^z \left(\frac{1}{r} - \frac{1}{p}\right) d\mathbb{G}_1(r) \geq \int_{r=0}^z \left(\frac{1}{r} - \frac{2}{z+p}\right) d\mathbb{G}_1(r) \geq \frac{p-z}{2} \mathbb{G}_1(z),
\]

for any \( z \in (0, p) \). Equivalently, the second relation is obtained by

\[
1 - \mathbb{G}_1(p) = \int_{r=p}^1 d\mathbb{G}_1(r) = \int_{r=p}^1 \frac{r}{1-r} d\mathbb{G}_0(r) = (1 - \mathbb{G}_1(p)) \left(\frac{p}{1-p}\right) + \int_{r=p}^1 \left(\frac{r}{1-r} - \frac{p}{1-p}\right) d\mathbb{G}_0(r),
\]

where the following bound is valid for any \( p < w < 1 \),

\[
\int_{r=p}^1 \left(\frac{r}{1-r} - \frac{p}{1-p}\right) d\mathbb{G}_0(r) \geq \int_{r=w}^1 \left(\frac{r}{1-r} - \frac{p}{1-p}\right) d\mathbb{G}_0(r) \geq \int_{r=w}^1 \left(\frac{w}{1-w} - \frac{w+p}{2-p-w}\right) d\mathbb{G}_0(r) \geq \frac{w-p}{2} (1 - \mathbb{G}_0(w)).
\]

(c) From part (a), we have for any \( r \in (0, 1) \),

\[
\mathbb{G}_0(r) = \int_{x=0}^r d\mathbb{G}_0(x) = \int_{x=0}^r \left(1 - \frac{x}{r}\right) d\mathbb{G}_1(x) \geq \int_{x=0}^r \left(\frac{1-r}{r}\right) d\mathbb{G}_1(x) = \left(\frac{1-r}{r}\right) \mathbb{G}_1(r).
\]

(B1)
Using part (a) again,

\[
\frac{d \left[ \frac{G_0(r)}{G_1(r)} \right]}{G_1(r)} = \frac{dG_0(r)G_1(r) - G_0(r)dG_1(r)}{(G_1(r))^2} = \frac{dG_1(r)}{(G_1(r))^2} \left[ \left( \frac{1-r}{r} \right) G_1(r) - G_0(r) \right].
\]

Since \( G_1(r) > 0 \) for \( r > \beta \), \( dG_1(r) \geq 0 \) and the term in brackets above is non-positive by Eq. (B1), we have

\[
\frac{d}{G_1(r)} \left[ \frac{G_0(r)}{G_1(r)} \right] \leq 0,
\]

thus proving the ratio \( G_0(r)/G_1(r) \) is non-increasing.

We now show that \( G_0(r) \geq G_1(r) \) for all \( r \in [0, 1] \).

From Eq. (B1), we obtain that Eq. (B2) is true for \( r \leq 1/2 \). For \( r > 1/2 \),

\[
1 - G_0(r) = \int_{x=r}^{1} dG_0(x) = \int_{x=r}^{1} \left( \frac{1-x}{x} \right) dG_1(x) \leq \int_{x=r}^{1} dG_1(x) = 1 - G_1(r),
\]

thus proving Eq. (B2).

We proceed to prove the second part of the lemma. Suppose \( G_0(r)/G_1(r) = 1 \) for some \( r < \beta \). Suppose first \( r \in (1/2, \beta) \). Then,

\[
G_0(1) = G_0(r) + \int_{x=r}^{1} dG_0(x)
\]

\[
= G_1(r) + \int_{x=r}^{1} dG_0(x)
\]

\[
= G_1(r) + \int_{x=r}^{1} \left( \frac{1-x}{x} \right) dG_1(x)
\]

\[
\geq G_1(r) + \left( \frac{1-r}{r} \right) \int_{x=r}^{1} dG_1(x)
\]

\[
\geq G_1(r) + \left( \frac{1-r}{r} \right) \left[ 1 - G_1(r) \right],
\]

which yields a contradiction unless \( G_1(r) = 1 \). However, \( G_1(r) = 1 \) implies \( r \geq \beta \) – also a contradiction. Now, suppose \( r \in (\beta, 1/2] \). Since the ratio \( G_0(r)/G_1(r) \) is non-increasing, this implies that for all \( x \in (r, 1] \), \( G_0(x)/G_1(x) \leq 1 \). Combined with Eq. (B2), this yields \( G_0(x)/G_1(x) = 1 \) for all \( x \in (r, 1] \), which yields a contradiction for \( x \in (1/2, \beta) \).

**Nonmonotonicity of Social Beliefs**

In this subsection, we illustrate the difficulties involved in determining equilibrium learning in general social networks. In particular, we show that social beliefs, as defined in
Figure 3: The figure illustrates a deterministic topology in which the social beliefs are nonmonotone.

Definition 3, may be nonmonotone, in the sense that additional observations of $x_n = 1$ in the neighborhood of an individual may reduce the social belief (i.e., the posterior derived from past observations that $x_n = 1$ is the correct action).

The following example establishes this point. Suppose the private signals are such that $G_0(r) = 2r - r^2$ and $G_1(r) = r^2$, which is a pair of private belief distributions $(G_0, G_1)$. Suppose the network topology is deterministic and for the first eight agents, it has the following structure: $B(1) = \emptyset, B(2) = \ldots = B(7) = \{1\}$ and $B(8) = \{1, \ldots, 7\}$ (see Figure 2).

For this social network, agent 1 has $3/4$ probability of making a correct decision in either state of the world. If agent 1 chooses the action that yields a higher payoff (i.e., the correct decision), then agents 2 to 7 each have $15/16$ probability of choosing the correct decision. However, if agent 1 fails to choose the correct decision, then agents 2 to 7 have a $7/16$ probability of choosing the correct decision. Now suppose agents 1 to 4 choose action $x_n = 0$, while agents 5 to 7 choose $x_n = 1$. The probability of this event happening in each state of the world is:

$$P\sigma(x_1 = \ldots = x_4 = 0, x_5 = x_6 = x_7 = 1 | \theta = 0) = \frac{3}{4} \left(\frac{15}{16}\right)^3 \left(\frac{1}{16}\right)^3 = \frac{10125}{2^{26}},$$

$$P\sigma(x_1 = \ldots = x_4 = 0, x_5 = x_6 = x_7 = 1 | \theta = 1) = \frac{1}{4} \left(\frac{9}{16}\right)^3 \left(\frac{7}{16}\right)^3 = \frac{250047}{2^{26}}.$$

Using Bayes’ Rule, the social belief of agent 8 is given by

$$\left[1 + \frac{10125}{250047}\right]^{-1} \approx 0.961.$$
Now, consider a change in $x_1$ from 0 to 1, while keeping all decisions as they are. Then,
\[
\mathbb{P}_\sigma(x_1 = 1, x_2 = x_3 = x_4 = 0, x_5 = x_6 = x_7 = 1 | \theta = 0) = \frac{1}{4} \left( \frac{7}{16} \right)^3 \left( \frac{9}{16} \right)^3 = \frac{250047}{2^{26}}.
\]
\[
\mathbb{P}_\sigma(x_1 = 1, x_2 = x_3 = x_4 = 0, x_5 = x_6 = x_7 = 1 | \theta = 1) = \frac{1}{4} \left( \frac{1}{16} \right)^3 \left( \frac{16}{16} \right)^3 = \frac{10125}{2^{26}}.
\]
This leads to a social belief of agent 8 given by
\[
\left[ 1 + \frac{250047}{10125} \right]^{-1} \approx 0.039.
\]

Therefore, this example has established that when $x_1$ changes from 0 to 1, agent 8’s social belief declines from 0.961 to 0.039. That is, while the agent strongly believes the state is 1 when $x_1 = 0$, he equally strongly believes the state is 0 when $x_1 = 1$. This happens because when half of the agents in \{2, ..., 7\} choose action 0 and the other half choose action 1, agent $n$ places a high probability to the event that $x_1 \neq \theta$. This leads to a nonmonotonicity in social beliefs.

**Proof of Lemma 2.** Let $h : \{(n, B(n)) : n \in N, B(n) \subseteq \{1, 2, ..., n-1\}\} \to \mathbb{N}$ be an arbitrary function that maps an agent and a neighborhood of the agent into an element of the neighborhood, i.e., $h(n, B(n)) \in B(n)$. In view of the characterization of the equilibrium decision $x_n$ [cf. Eq. (2)], it follows that for any private signal $s_n$, neighborhood $B(n) \subseteq \{1, 2, ..., n-1\}$, and decisions $x_k, k \in B(n)$, we have
\[
\mathbb{P}_\sigma(x_n = \theta | s_n, B(n), x_k, k \in B(n)) \geq \mathbb{P}_\sigma(x_{h(n,B(n))} = \theta | s_n, B(n), x_k, k \in B(n)).
\]
By integrating over all possible private signals and decisions of agents in the neighborhood, we obtain that for any $n$ and any $B(n) = \mathfrak{B}$,
\[
\mathbb{P}_\sigma(x_n = \theta | B(n) = \mathfrak{B}) \geq \mathbb{P}_\sigma(x_{h(n,B(n))} = \theta | B(n) = \mathfrak{B}) = \mathbb{P}_\sigma(x_{h(n,\mathfrak{B})} = \theta),
\]
where the equality follows by the assumption that each neighborhood is generated independently from all other neighborhoods. By taking the maximum over all functions $h$, we obtain
\[
\mathbb{P}_\sigma(x_n = \theta | B(n) = \mathfrak{B}) \geq \max_{b \in \mathfrak{B}} \mathbb{P}_\sigma(x_b = \theta),
\]
showing the desired relation. ■

**Proof of Theorem 3**

**Proof of part (a):** The proof consists of two steps. We first show that the lower and upper supports of the social belief $q_n = \mathbb{P}_\sigma(\theta = 1 | x_1, ..., x_{n-1})$ are bounded away from 0 and 1. We next show that this implies that $x_n$ does not converge to $\theta$ in probability.

Let $x^{n-1} = (x_1, ..., x_{n-1})$ denote the sequence of decisions up to and including $n - 1$. Let $\varphi_{\sigma,x^{n-1}}(q_n, x_n)$ represent the social belief $q_{n+1}$ given the social belief $q_n$ and the
decision $x_n$, for a given strategy $\sigma$ and decisions $x^{n-1}$. We use Bayes’ Rule to determine the dynamics of the social belief. For any $x^{n-1}$ compatible with $q_n$, and $x_n = \bar{x}$ with $\bar{x} \in \{0,1\}$, we have

$$\varphi_{\sigma,x^{n-1}}(q_n, \bar{x}) = \mathbb{P}_\sigma(\theta = 1 \mid x_n = \bar{x}, q_n, x^{n-1})$$

$$= \left[ 1 + \frac{\mathbb{P}_\sigma(x_n = \bar{x}, q_n, x^{n-1}, \theta = 0)}{\mathbb{P}_\sigma(x_n = \bar{x}, q_n, x^{n-1}, \theta = 1)} \right]^{-1}$$

$$= \left[ 1 + \frac{\mathbb{P}_\sigma(q_n, x^{n-1} \mid \theta = 0) \mathbb{P}_\sigma(x_n = \bar{x} \mid q_n, x^{n-1}, \theta = 0)}{\mathbb{P}_\sigma(q_n, x^{n-1} \mid \theta = 1) \mathbb{P}_\sigma(x_n = \bar{x} \mid q_n, x^{n-1}, \theta = 1)} \right]^{-1}$$

$$= \left[ 1 + \left( \frac{1}{q_n} - 1 \right) \frac{\mathbb{P}_\sigma(x_n = \bar{x} \mid q_n, x^{n-1}, \theta = 0)}{\mathbb{P}_\sigma(x_n = \bar{x} \mid q_n, x^{n-1}, \theta = 1)} \right]^{-1}.$$  \hspace{1cm} (B3)

Let $\alpha_{\sigma,x^{n-1}}$ denote the probability that agent $n$ chooses $x = 0$ in equilibrium $\sigma$ when he observes history $x^{n-1}$ and is indifferent between the two actions. Let $G_j(r) = \lim_{s \uparrow r} G_j(s)$, for any $r \in [0,1]$ and any $j \in \{0,1\}$. Then, for any $j \in \{0,1\}$,

$$\mathbb{P}_\sigma(x_n = 0 \mid q_n, x^{n-1}, \theta = j) = \mathbb{P}_\sigma(p_n < 1 - q_n \mid q_n, \theta = j) + \alpha_{\sigma,x^{n-1}} \mathbb{P}_\sigma(p_n = 1 - q_n \mid q_n, \theta = j)$$

$$G_j(1 - q_n) + \alpha_{\sigma,x^{n-1}} \left[ G_j(1 - q_n) - G_j(1 - q_n) \right].$$

From Lemma 1(a), $dG_0/dG_1(r) = (1 - r)/r$ for all $r \in [0,1]$. Therefore,

$$\frac{1 - r}{r} \leq \frac{G_0(r)}{G_1(r)}, \quad \text{and} \quad \frac{G_0(r)}{G_1(r)} \leq \frac{1 - \beta}{\beta}.$$  \hspace{1cm} 

Hence, for any $\alpha_{\sigma,x^{n-1}},$

$$\frac{\mathbb{P}_\sigma(x_n = 0 \mid q_n, x^{n-1}, \theta = 0)}{\mathbb{P}_\sigma(x_n = 0 \mid q_n, x^{n-1}, \theta = 1)} = \frac{G_0(1 - q_n) + \alpha_{\sigma,x^{n-1}} \left[ G_0(1 - q_n) - G_0(1 - q_n) \right]}{G_1(1 - q_n) + \alpha_{\sigma,x^{n-1}} \left[ G_1(1 - q_n) - G_0(1 - q_n) \right]}$$

$$\in \left[ \frac{q_n}{1 - q_n}, \frac{1 - \beta}{\beta} \right].$$

Combining this with Eq. (B3), we obtain

$$\varphi_{\sigma,x^{n-1}}(q_n, 0) \in \left[ \left( 1 + \left( \frac{1}{q_n} - 1 \right) \left( \frac{1 - \beta}{\beta} \right) \right)^{-1}, \left( 1 + \left( \frac{1}{q_n} - 1 \right) \left( \frac{q_n}{1 - q_n} \right) \right) \right]^{-1}$$

$$= \left[ \frac{\beta q_n}{1 - \beta - q_n + 2\beta q_n}, \frac{1}{2} \right].$$

Note that $\frac{\beta q_n}{1 - \beta - q_n + 2\beta q_n}$ is an increasing function of $q_n$ and if $q_n \in [1 - \beta, 1 - \beta]$, then this function is minimized at $1 - \beta$. This implies that

$$\varphi_{\sigma,x^{n-1}}(q_n, 0) \in \left[ \frac{\beta(1 - \beta)}{-\beta + \beta + 2\beta(1 - \beta)}, \frac{1}{2} \right] = \left[ \frac{\Delta}{2} \right].$$
where $\Delta$ is a constant strictly greater than 0. An analogous argument for $x_n = 1$ establishes that there exists some $\overline{\Delta} < 1$ such that if $q_n \in [1 - \overline{\beta}, 1 - \beta]$, then

$$\varphi_{\sigma,x^{n-1}}(q_n, 1) \in \left[\frac{1}{2}, \overline{\Delta}\right].$$

We next show that $q_n \in [\Delta, \overline{\Delta}]$ for all $n$. Suppose this is not true. Let $N$ be the first agent such that

$$q_N \in [0, \Delta) \cup (\overline{\Delta}, 1] \quad \text{(B4)}$$

in some equilibrium and some realized history. Then, $q_{N-1} \in [0, 1 - \overline{\beta}) \cup (1 - \beta, 1]$ because otherwise, the dynamics of $q_n$ implies a violation of Eq. (B4) for any $x_{N-1}$. But note that if $q_{N-1} < 1 - \overline{\beta}$, then by Proposition 2 agent $N - 1$ chooses action $x_{N-1} = 0$ and, thus by Eq. (B3),

$$q_N = \left[1 + \left(\frac{1}{q_{N-1}} - 1\right)\frac{\overline{\beta}}{\overline{\beta}}\right]^{-1} = q_{N-1}.\]$$

By the same argument, if $q_{N-1} > 1 - \beta$, we have that $q_N = q_{N-1}$. Therefore, $q_N = q_{N-1}$, which contradicts the fact that $N$ is the first agent that satisfies Eq. (B4).

We next show that $q_n \in [\Delta, \overline{\Delta}]$ for all $n$ implies that $x_n$ does not converge in probability to $\theta$. Let $y_k$ denote a realization of $x_k$. Then, for any $n$ and any sequence of $y_k$’s, we have

$$\mathbb{P}_\sigma(\theta = 1, x_k = y_k \text{ for all } k \leq n) \leq \overline{\Delta} q_{N-1},$$

$$\mathbb{P}_\sigma(\theta = 0, x_k = y_k \text{ for all } k \leq n) \leq (1 - \overline{\Delta}) q_{N-1}.\]$$

By summing the preceding relations over all $y_k$ for $k < n$, we obtain

$$\mathbb{P}_\sigma(\theta = 1, x_n = 1) \leq \overline{\Delta} q_{N-1} q_{n} = 1\]$$

and $\mathbb{P}_\sigma(\theta = 0, x_n = 0) \leq (1 - \overline{\Delta}) q_{N-1} q_{n} = 0$. Therefore, for any $n$, we have

$$\mathbb{P}_\sigma(x_n = \theta) \leq \overline{\Delta} q_{N-1} q_{n} + (1 - \overline{\Delta}) q_{N-1} q_{n} \leq \max\{\overline{\Delta}, 1 - \overline{\Delta}\} < 1,$$

which completes the proof. ■

**Proof of part (b):** The first step is the following lemma.

**Lemma 11** Let $B(n) = \{b\}$ for some $n$. We define

$$f(\overline{\beta}, \overline{\beta}) = \max \left\{1 - \frac{\beta}{2(1 - \overline{\beta})}, \frac{1}{2} - \frac{1}{2\overline{\beta}} \right\},$$

where $\overline{\beta}$ and $\beta$ are the lower and upper supports of the private beliefs (cf. Definition 4). Let $\sigma$ be an equilibrium. Assume that $\mathbb{P}_\sigma(x_b = \theta) \leq f(\overline{\beta}, \overline{\beta})$. Then, we have

$$\mathbb{P}_\sigma(x_n = \theta \mid B(n) = \{b\}) \leq f(\overline{\beta}, \overline{\beta}).$$

6
Proof. We first assume that $P_\sigma(x_b = \theta) = f(\beta, \beta)$ and show that this implies

$$U_\sigma^b \geq \beta, \quad \text{and} \quad L_\sigma^b \leq \beta,$$

where $U_\sigma^b$ and $L_\sigma^b$ are defined in Eq. (A14). We can rewrite $U_\sigma^b$ as

$$U_\sigma^b = \frac{N_\sigma^b}{1 - 2P_\sigma(x_b = \theta) + 2N_\sigma^b} = \frac{N_\sigma^b}{1 - 2f(\beta, \beta) + 2N_\sigma^b}.$$  

This is a decreasing function of $N_\sigma^b$ and, therefore,

$$U_\sigma^b \geq \frac{1}{1 - 2f(\beta, \beta) + 2}.$$  

Using $f(\beta, \beta) \geq \frac{3}{2} - \frac{1}{2\beta}$, the preceding relation implies $U_\sigma^b \geq \beta$. An analogous argument shows that $L_\sigma^b \leq \beta$.

Since the support of the private beliefs is $[\beta, \beta]$, using Lemma 3 and Eq. (B6), there exists an equilibrium $\sigma^* = (\sigma^*_n, \sigma^*_{-n})$ such that $x_b = x_b$ with probability one (with respect to measure $P_{\sigma^*}$). Since this gives an expected payoff $P_{\sigma^*}(x_n = \theta \mid B(n) = b) = P_\sigma(x_b = \theta)$, it follows that, $P_{\sigma}(x_n = \theta \mid B(n) = b) = P_{\sigma}(x_b = \theta)$. This establishes the claim that

$$\text{if } P_{\sigma}(x_b = \theta) = f(\beta, \beta), \text{ then } P_{\sigma}(x_n = \theta \mid B(n) = \{b\}) = f(\beta, \beta). \quad (B7)$$

We next assume that $P_{\sigma}(x_b = \theta) < f(\beta, \beta)$. To arrive at a contradiction, suppose that

$$P_{\sigma}(x_n = \theta \mid B(n) = \{b\}) > f(\beta, \beta). \quad (B8)$$

Now consider a hypothetical situation where agent $n$ observes a private signal generated with conditional probabilities $(P_0, P_1)$ and a coarser version of the observation $x_b$, i.e., the random variable $\tilde{x}_b$ distributed according to

$$\mathbb{P}(\tilde{x}_b = 1 \mid \theta = 1) = 1 - Y_\sigma^b \left[ \frac{1 - f(\beta, \beta)}{P_\sigma(x_b = \theta)} \right] \quad \text{and} \quad \mathbb{P}(\tilde{x}_b = 0 \mid \theta = 0) = 1 - N_\sigma^b \left[ \frac{1 - f(\beta, \beta)}{P_\sigma(x_b = \theta)} \right].$$

It follows from the preceding conditional probabilities that $\mathbb{P}(\tilde{x}_b = \theta) = f(\beta, \beta)$. We assume that agent $n$ uses the equilibrium strategy $\sigma_n$. Using a similar argument as in the proof of Eq. (B7), this implies that

$$\mathbb{P}(x_n = \theta \mid B(n) = \{b\}) = f(\beta, \beta). \quad (B9)$$

Let $z$ be a binary random variable with values $\{0, 1\}$ and is generated independent of $\theta$ with probabilities

$$\mathbb{P}(z = 1) = 1 - \frac{2Y_\sigma^b}{P_\sigma(x_b = \theta)} \quad \text{and} \quad \mathbb{P}(z = 0) = 1 - \frac{2N_\sigma^b}{P_\sigma(x_b = \theta)}.$$  

This implies that $\mathbb{P}(z = j \mid \theta = j) = \mathbb{P}(z = j)$ for $j \in \{0, 1\}$. Using $\tilde{x}_b$ with probability $\frac{1}{1 + f(\beta, \beta)} \left[ 2 + \frac{(Y_\sigma^b - 1)P_\sigma(x_b = \theta)}{Y_\sigma^b} \right]$ and $z$ otherwise generates the original observation (random
variable) \( x_b \). Therefore, from Eq. (B8), \( \mathbb{P}(x_n = \theta | B(n) = \{b\}) > f(\beta, \bar{\beta}) \), which contradicts Eq. (B9), and completes the proof. \( \blacksquare \)

Let \( f \) be defined in Eq. (B5). We show by induction that

\[
\mathbb{P}_\sigma(x_n = \theta) \leq f(\beta, \bar{\beta}) \quad \text{for all } n. \tag{B10}
\]

Suppose that for all agents up to \( n - 1 \) the preceding inequality holds. Since \( |B(n)| \leq 1 \), we have

\[
\mathbb{P}_\sigma(x_n = \theta) = \mathbb{P}_\sigma(x_n = \theta | B(n) = \emptyset)\mathbb{Q}_n(B(n) = \emptyset) + \sum_{b=1}^{n-1} \mathbb{P}_\sigma(x_n = \theta | B(n) = b)\mathbb{Q}_n(B(n) = \{b\})
\]

\[
\leq \mathbb{P}_\sigma(x_n = \theta | B(n) = \emptyset)\mathbb{Q}_n(B(n) = \emptyset) + \sum_{b=1}^{n-1} f(\beta, \bar{\beta})\mathbb{Q}_n(B(n) = \{b\}),
\]

where the inequality follows from the induction hypothesis and Lemma 11. Note that having \( B(n) = \emptyset \) is equivalent to observing a decision \( b \) such that \( \mathbb{P}_\sigma(x_b = \theta) = 1/2 \). Since \( 1/2 \leq f(\beta, \bar{\beta}) \), Lemma 11 implies that \( \mathbb{P}_\sigma(x_n = \theta | B(n) = \emptyset) \leq f(\beta, \bar{\beta}) \). Combined with Eq. (B11), this completes the induction.

Since the private beliefs are bounded, i.e., \( \beta > 0 \) and \( \bar{\beta} < 1 \), we have \( f(\beta, \bar{\beta}) < 1 \) [cf. Eq. (B5)]. Combined with Eq. (B10), this establishes that \( \lim_{n \to \infty} \mathbb{P}_\sigma(\bar{a}_n = \theta) < 1 \), showing that asymptotic learning does not occur at any equilibrium \( \sigma \). \( \blacksquare \)

**Proof of part (c):** We start with the following lemma, which will be used subsequently in the proof.

**Proof.**

**Lemma 12** Assume that asymptotic learning occurs in some equilibrium \( \sigma \), i.e., we have \( \lim_{n \to \infty} \mathbb{P}_\sigma(\bar{a}_n = \theta) = 1 \). For some constant \( K \), let \( \mathcal{D} \) be the set of all subsets of \( \{1, \ldots, K\} \). Then,

\[
\lim_{n \to \infty} \min_{D \in \mathcal{D}} \mathbb{P}_\sigma(x_n = \theta | x_k = 1, \ k \in D) = 1.
\]

**Proof.** First note that since the event \( x_k = 1 \) for all \( k \leq K \) is the intersection of events \( x_k = 1 \) for each \( k \leq K \),

\[
\min_{D \in \mathcal{D}} \mathbb{P}_\sigma(x_k = 1, \ k \in D) = \mathbb{P}_\sigma(x_k = 1, \ k \leq K).
\]

Let \( \Delta = \mathbb{P}_\sigma(x_k = 1, \ k \leq K) \). Fix some \( \tilde{D} \in \mathcal{D} \). Then,

\[
\mathbb{P}_\sigma(x_k = 1, \ k \in \tilde{D}) \geq \Delta > 0,
\]

where the second inequality follows from the fact that there is a positive probability of the first \( K \) agents choosing \( x_k = 1 \). Let \( \mathcal{A} = \{0, 1\}^{|\tilde{D}|} \), i.e., \( \mathcal{A} \) is the set of all possible actions for the set of agents \( \tilde{D} \). Then,

\[
\mathbb{P}_\sigma(x_n = \theta) = \sum_{a_k \in \mathcal{A}} \mathbb{P}_\sigma(x_n = \theta | x_k = a_k, \ k \in \tilde{D})\mathbb{P}_\sigma(x_k = a_k, \ k \in \tilde{D}).
\]
Since $\mathbb{P}_\sigma(x_n = \theta)$ converges to 1 and all elements in the set $\{\mathbb{P}_\sigma(x_k = 1, k \in \hat{D})\}_{D \in A}$ greater than or equal to $\Delta > 0$, it follows that the sequence $\{\mathbb{P}_\sigma(x_n = \theta | x_k = 1, k \in \hat{D})\}$ also converges to 1. Hence, for each $\epsilon > 0$, there exists some $N_\epsilon(\hat{D})$ such that for all $n \geq N_\epsilon(\hat{D})$,
\[
\mathbb{P}_\sigma(x_n = \theta | x_k = 1, k \in \hat{D}) \geq 1 - \epsilon.
\]
Therefore, for any $\epsilon > 0$,
\[
\min_{D \in \mathcal{D}} \mathbb{P}_\sigma(x_n = \theta | x_k = 1, k \in D) \geq 1 - \epsilon \quad \text{for all} \quad n \geq \max_{D \in \mathcal{D}} N_\epsilon(D),
\]
thus completing the proof. \hfill \blacksquare\blacksquare

We prove the claim in this part by contradiction. To arrive at a contradiction, we assume that in some equilibrium $\sigma \in \Sigma^*$, $\lim_{n \to \infty} \mathbb{P}_\sigma(x_n = \theta) = 1$. The key part of the proof is to show that this implies
\[
\lim_{n \to \infty} \mathbb{P}_\sigma(\theta = 1 | x_k = 1, k \in B(n)) = 1. \tag{B12}
\]
To prove this claim, we show that for any $\epsilon > 0$, there exists some $\tilde{K}(\epsilon)$ such that for any neighborhood $\mathfrak{B}$ with $|\mathfrak{B}| \leq M$ and $\max_{b \in \mathfrak{B}} b \geq \tilde{K}(\epsilon)$ we have
\[
\mathbb{P}_\sigma(\theta = 1 | x_k = 1, k \in \mathfrak{B}) \geq 1 - \epsilon. \tag{B13}
\]
In view of the assumption that $\max_{b \in \mathcal{B}(n)} b$ converges to infinity with probability 1, this implies the desired claim (B12).

For a fixed $\epsilon > 0$, we define $\tilde{K}(\epsilon)$ as follows: We recursively construct $M$ thresholds $K_0 < \ldots < K_{M-1}$ and let $\tilde{K}(\epsilon) = K_{M-1}$. We consider an arbitrary neighborhood $\mathfrak{B}$ with $|\mathfrak{B}| \leq M$ and $\max_{b \in \mathfrak{B}} b \geq K_{M-1}$, and for each $d \in \{0, \ldots, M-1\}$, define the sets
\[
\mathfrak{B}_d = \{b \in \mathfrak{B} : b \geq K_d\} \quad \text{and} \quad \mathfrak{C}_d = \{b \in \mathfrak{B} : b < K_{d-1}\},
\]
where $\mathfrak{C}_0 = \emptyset$. With this construction, it follows that there exists at least one $d \in \{0, \ldots, M-1\}$ such that $\mathfrak{B} = \mathfrak{B}_d \cup \mathfrak{C}_d$, in which case we say $\mathfrak{B}$ is of type $d$. We show below that for any $\mathfrak{B}$ of type $d$, we have
\[
\mathbb{P}_\sigma(\theta = 1 | x_k = 1, k \in \mathfrak{B}_d \cup \mathfrak{C}_d) \geq 1 - \epsilon, \tag{B14}
\]
which implies the relation in (B13).

We first define $K_0$ and show that for any $\mathfrak{B}$ of type 0, relation (B14) holds. Since $\lim_{n \to \infty} \mathbb{P}_\sigma(x_n = \theta) = 1$ by assumption, there exists some $N_0$ such that for all $n \geq N_0$,
\[
\mathbb{P}_\sigma(x_n = \theta) \geq 1 - \frac{\epsilon}{2M}.
\]
Let $K_0 = N_0$. Let $\mathfrak{B}$ be a neighborhood of type 0, implying that $\mathfrak{B} = \mathfrak{B}_0$ and all elements $b \in \mathfrak{B}_0$ satisfy $b \geq K_0$. By using a union bound, the preceding inequality implies
\[
\mathbb{P}_\sigma(x_k = \theta, k \in \mathfrak{B}_0) \geq 1 - \sum_{k \in \mathfrak{B}_0} \mathbb{P}_\sigma(x_k \neq \theta) \geq 1 - \frac{\epsilon}{2}.
\]
Hence, we have
\[ P_\sigma(x_k = \theta, \ k \in \mathcal{B}_0 \mid \theta = 1) \frac{1}{2} + P_\sigma(x_k = \theta, \ k \in \mathcal{B}_0 \mid \theta = 0) \frac{1}{2} \geq 1 - \epsilon, \]
and for any \( j \in \{0, 1\}, \)
\[ P_\sigma(x_k = \theta, \ k \in \mathcal{B}_0 \mid \theta = j) \geq 1 - \epsilon. \] (B15)

Therefore, for any such \( \mathcal{B}_0, \)
\[ P_\sigma(\theta = 1 \mid x_k = 1, \ k \in \mathcal{B}_0) = \left[ 1 + \frac{P_\sigma(x_k = 1, \ k \in \mathcal{B}_0 \mid \theta = 0)P_\sigma(\theta = 0)}{P_\sigma(x_k = 1, \ k \in \mathcal{B}_0 \mid \theta = 1)} \right]^{-1} \geq \left[ 1 + \frac{\epsilon}{1 - \epsilon} \right]^{-1} = 1 - \epsilon, \]
showing that relation (B14) holds for any \( \mathcal{B} \) of type 0.

We proceed recursively, i.e., given \( K_{d-1} \) we define \( K_d \) and show that relation (B14) holds for any neighborhood \( \mathcal{B} \) of type \( d \). Lemma 12 implies that
\[ \lim_{n \to \infty} \min_{D \subseteq \{1, \ldots, K_{d-1}\}} P_\sigma(x_n = \theta \mid x_k = 1, \ k \in D) = 1. \]

Therefore, for any \( \delta > 0 \), there exists some \( K_d \) such that for all \( n \geq K_d, \)
\[ \min_{D \subseteq \{1, \ldots, K_{d-1}\}} P_\sigma(x_n = \theta \mid x_k = 1, \ k \in D) \geq 1 - \delta \epsilon. \]

From the equation above and definition of \( \mathcal{C}_d \) it follows that for any \( \mathcal{C}_d, \)
\[ P_\sigma(x_n = \theta \mid x_k = 1, \ k \in \mathcal{C}_d) \geq 1 - \delta \epsilon. \]

By a union bound,
\[ P_\sigma(x_k = \theta, \ k \in \mathcal{B}_d \mid x_k = 1, \ k \in \mathcal{C}_d) \geq 1 - \sum_{k \in \mathcal{B}_d} P_\sigma(x_k \neq \theta \mid x_k = 1, \ k \in \mathcal{C}_d) \geq 1 - (M - d) \delta \epsilon. \]

Repeating the argument from Eq. (B15), for any \( j \in \{0, 1\}, \)
\[ P_\sigma(x_k = \theta, \ k \in \mathcal{B}_d \mid \theta = j, x_k = 1, \ k \in \mathcal{C}_d) \geq 1 - \frac{(M - d) \delta \epsilon}{P_\sigma(\theta = j \mid x_k = 1, \ k \in \mathcal{C}_d)}. \]

Hence, for any such \( \mathcal{B}_d, \)
\[ P_\sigma(\theta = 1 \mid x_k = 1, \ k \in \mathcal{B}_d \cup \mathcal{C}_d) = \left[ 1 + \frac{P_\sigma(x_k = 1, \ k \in \mathcal{B}_d \mid \theta = 0, x_k = 1, \ k \in \mathcal{C}_d)P_\sigma(\theta = 0, x_k = 1, \ k \in \mathcal{C}_d)}{P_\sigma(x_k = 1, \ k \in \mathcal{B}_d \mid \theta = 1, x_k = 1, \ k \in \mathcal{C}_d)P_\sigma(\theta = 1, x_k = 1, \ k \in \mathcal{C}_d)} \right]^{-1} \geq \left[ 1 + \frac{(M - d) \delta \epsilon}{P_\sigma(\theta = 0 \mid x_k = 1, \ k \in \mathcal{C}_d)}P_\sigma(\theta = 0, x_k = 1, \ k \in \mathcal{C}_d) \right]^{-1} \geq 1 - \frac{(M - d) \delta \epsilon}{P_\sigma(\theta = 1 \mid x_k = 1, \ k \in \mathcal{C}_d)}. \]
Choosing
\[ \delta = \left( \frac{1}{M - d} \right) \min_{D \subseteq \{1, \ldots, K_{d-1}, \ldots, d-1\}} \mathbb{P}_\sigma(\theta = 1 \mid x_k = 1, k \in D), \]
we obtain that for any neighborhood \( \mathcal{B} \) of type \( d \)
\[ \mathbb{P}_\sigma(\theta = 1 \mid x_k = 1, k \in \mathcal{B}_d \cup \mathcal{C}_d) \geq 1 - \epsilon. \]
This proves that Eq. (B14) holds for any neighborhood \( \mathcal{B} \) of type \( d \), and completing the proof of Eq. (B13) and therefore of Eq. (B12).

Since the private beliefs are bounded, we have \( \beta > 0 \). By Eq. (B12), there exists some \( N \) such that
\[ \mathbb{P}_\sigma(\theta = 1 \mid x_k = 1, k \in B(n)) \geq 1 - \frac{\beta}{2} \quad \text{for all } n \geq N. \]
Suppose the first \( N \) agents choose 1, i.e., \( x_k = 1 \) for all \( k \leq N \), which is an event with positive probability for any state of the world \( \theta \). We now prove inductively that this event implies that \( x_n = 1 \) for all \( n \in \mathbb{N} \). Suppose it holds for some \( n \geq N \). Then, by Eq. (B13),
\[ \mathbb{P}_\sigma(\theta = 1 \mid s_{n+1}) + \mathbb{P}_\sigma(\theta = 1 \mid x_k = 1, \in B(n+1)) \geq \beta + 1 - \frac{\beta}{2} > 1. \]
By Proposition 2, this implies that \( x_{n+1} = 1 \). Hence, we conclude there is a positive probability \( x_n = 1 \) for all \( n \in \mathbb{N} \) in any state of the world, contradicting \( \lim_{n \to \infty} \mathbb{P}_\sigma(x_n = \theta) = 1 \), and completing the proof.

**Proof of Corollary 3**

For \( M = 1 \), the result follows from Theorem 3 part (b). For \( M \geq 2 \), we show that, under the assumption on the network topology, \( \max_{b \in B(n)} b \) goes to infinity with probability one. To arrive at a contradiction, suppose this is not true. Then, there exists some \( K \in \mathbb{N} \) and scalar \( \epsilon > 0 \) such that
\[ \mathbb{Q} \left( \max_{b \in B(n)} b \leq K \text{ for infinitely many } n \right) \geq \epsilon. \]
By the Borel-Cantelli Lemma (see, e.g., Breiman, Lemma 3.14, p. 41), this implies that
\[ \sum_{n=1}^{\infty} \mathbb{Q}_n \left( \max_{b \in B(n)} b \leq K \right) = \infty. \]
Since the samples are all uniformly drawn and independent, for all \( n \geq 2 \),
\[ \mathbb{Q}_n \left( \max_{b \in B(n)} b \leq K \right) = \left( \frac{\min\{K, n - 1\}}{n - 1} \right)^M. \]
Therefore,
\[ \sum_{n=1}^{\infty} \mathbb{Q}_n \left( \max_{b \in B(n)} b \leq K \right) = 1 + \sum_{n=1}^{\infty} \left( \frac{\min\{K, n - 1\}}{n} \right)^M \leq 1 + \sum_{n=1}^{\infty} \left( \frac{K}{n} \right)^M < \infty, \]
where the last inequality holds since \( M \geq 2 \). Hence, we obtain a contradiction. The result follows by using Theorem 3 part (c).