A. Online Appendix of "Optimal Multiperiod Pricing with Service Guarantees"

A.1. Appendix to Section 3

Proof of Lemma 1 First note that we can add the constraint $0 \le \mathbf{p} \le 1$ to the problem without loss of optimality, since customer valuations are bounded by 1. Consequently, it follows that for a given fixed ranking R, the set of consistent and feasible prices defines a closed and bounded set. Since the objective function is continuous in prices (for a fixed ranking), we conclude that optimal prices exist for any given ranking R. By maximizing over the finitely many possible rankings, we conclude that an optimal solution of OPT-2 exists.

For the second claim, observe that if $\mathbf{p}$ is a feasible solution of OPT-1, then $(\mathbf{p}, R^C(\mathbf{p}))$ is a feasible solution of OPT-2 with the same objective value. Thus, the maximum of OPT-2 is an upper bound on the supremum of OPT-1. Given an optimal solution $(\mathbf{p}^\star, R^\star)$ of OPT-2, and any $\epsilon > 0$, $\mathbf{p}^\star + \epsilon R^\star$ is a feasible vector of prices that is consistent with the ranking $R^\star$, and hence $(\mathbf{p}^\star + \epsilon R^\star, R^\star)$ is a feasible solution of OPT-2. This is because, if $R_t^\star < R_{t'}^\star$, then $p_t^\star \le p_{t'}^\star$, and consequently $p_t^\star + \epsilon R_t^\star < p_{t'}^\star + \epsilon R_{t'}^\star$. Moreover, this inequality also implies that in $\mathbf{p}^\star + \epsilon R^\star$ no price is repeated, and hence the only consistent ranking with this price vector is $R^\star$. This implies that $R^\star$ is the customer-preferred ranking corresponding to $\mathbf{p}^\star + \epsilon R^\star$, and thus this price vector is feasible in OPT-1 with the same objective value. Since the objective of OPT-2 is continuous in prices for a fixed ranking $R^\star$, the value of $(\mathbf{p}^\star + \epsilon R^\star, R^\star)$ approaches to that of $(\mathbf{p}^\star, R^\star)$, as ϵ goes to 0. Thus for $\epsilon > 0$, $\mathbf{p}^\star + \epsilon R^\star$ is a feasible solution of OPT-1, value of which converges to maximum of OPT-2 as ϵ goes to 0. Since maximum of OPT-2 is an upper bound on the supremum of OPT-1, it follows that these values are equal, and $\mathbf{p}^\star + \epsilon R^\star$ converges to the supremum of OPT-1, as claimed.

If $\mathbf{p}$ is an optimal solution of OPT-1, then its value equals to the supremum value. However, as explained earlier this value equals to the maximum of OPT-2, and $(\mathbf{p}, R^C(\mathbf{p}))$ is a feasible solution of this problem with the same value. Thus, the claim follows.

A.2. Appendix to Section 4

Proof of Proposition 1 Since the valuations are bounded by 1, it is not beneficial to set a price above 1. Now, suppose $(\mathbf{p}, R)$ is a feasible and consistent price ranking. Let $\mathbf{p}'$ be the price vector such that $p'_t = \max\{p_M, p_t\}$. We claim that $(\mathbf{p}', R)$ is both consistent and feasible. For consistency, note that if $R_t < R_{t'}$ then $p_t \le p_{t'}$. Hence, $\max\{p_M, p_t\} \le \max\{p_M, p_{t'}\}$. Therefore, $(\mathbf{p}', R)$ is consistent. Moreover, because we have (weakly) increased the prices, it is a feasible solution. Finally, observe that the revenue obtained from $(\mathbf{p}', R)$ is at least equal to the revenue of $(\mathbf{p}, R)$. The reason is $\rho_t(R)$ does not change, but the uncapacitated revenue function, $p(1-F(p))$, increases. Namely,

$$\sum_t p_t(1-F(p_t))\rho_t(R) \le \sum_t p'_t(1-F(p'_t))\rho_t(R).$$

since by definition p_M maximizes $p(1-F(p))$. Therefore, starting from a feasible solution, we can construct another one with weakly better objective value, where all prices are weakly above p_M, thus the claim follows.

A.3. Appendix to Section 6

LEMMA 6. *Let A^L be the population matrix with elements $a^L_{i,j}$ and A^U be the population matrix with elements $a^U_{i,j}$. Then, OPT-4 is equivalent to:*

$$\begin{aligned} \max_{\mathbf{p} \ge 0, R \in \mathcal{P}(T)} \quad & \sum_{t=1}^{T} p_t D_t(p_t, R, A^L) \\ s.t. \quad & D_t(p_t, R, A^U) \le c_t^L \qquad \textit{for all } t \in \{1, \dots, T\} \\ & R_t < R_{t'} \Rightarrow p_t \le p_{t'} \qquad \textit{for all } t, t' \in \{1, \dots, T\}, \end{aligned} \tag{OPT-R}$$

Proof of Lemma 6: The function $D_t(p_t,R,A)$ is weakly increasing in all the elements of the matrix A. Therefore, for all $A\in\mathcal{A}$, the tightest constraint among all of constraints of the form $M\le\sum_{t=1}^T p_tD_t(p_t,R,A)$ is the one given by A^L. At optimal solutions of OPT-4, M should be replaced by the maximum value it can attain, which is $\sum_{t=1}^T p_tD_t(p_t,R,A^L)$. Similarly, the tightest constraint among of the constraints of the form $D_t(p_t,R,A)\le c_t$ is the one given by A^U and c_t^L. Thus, the claim follows replacing constraints of this form by $D_t(p_t,R,A^U)\le c_t^L$.

Proof of Proposition 4: The constraint set of OPT-R is identical to that of an instance of OPT-2 with parameters $(\mathbf{c}^L,A^U)$. Additionally, the objective functions of both problems are nonincreasing for all $p\ge p_M$. Since, Proposition 3 relied on the monotonicity of revenue in prices, and the properties of constraint sets, it follows that for OPT-R, a set L with $O(T^3)$ prices that contains all candidate optimal prices can be constructed (using parameters $(\mathbf{c}^L,A^U)$). Thus, we can still use the recursion in (7) to find the optimal sequence of prices. However, $\gamma_k^{ij}(p)$ needs to be modified slightly since in OPT-R the feasibility constraints involve A^U, whereas, the revenue function involves A^L. Therefore, the recursion in (7) solves OPT-R (again in $O(T^6)$), using the following modified definition of $\gamma_k^{ij}(p)$:

$$\gamma_k^{ij}(p)=\begin{cases}\left(\sum_{l=i+1}^{k}\sum_{m=k}^{j-1}a_{lm}^L\right)(1-F(p))p & \text{if } \left(\sum_{l=i+1}^{k}\sum_{m=k}^{j-1}a_{lm}^U\right)(1-F(p))\le c_k^L\\ -\infty & \text{otherwise.}\end{cases}$$

Proof of Proposition 5 Let $\mathbf{c}^*$, A^* denote the capacity and arrivals in a given problem instance. We will show that $V(\mathbf{c}^*,A^*)-V^{ROB}(\mathcal{C},\mathcal{A},\mathbf{c}^*,A^*)\le\theta(2H+1)P(A^U)$. Since, $\mathbf{c}^*$, and A^* are arbitrary, the claim then follows from taking supremum over all $\mathbf{c}^*$ and A^*.

Let $V_R(\mathbf{c},A,A^*)$ denote the revenue obtained by (i) offering a price vector consistent with ranking R, (ii) ensuring that prices are feasible for arrival matrix A and capacity vector $\mathbf{c}$, (iii) having arrival realization A^*, i.e.,

$$\begin{aligned}V_R(\mathbf{c},A,A^*)=\max_{\mathbf{p}\ge0}\quad&\sum_{t=1}^T p_tD_t(p_t,R,A^*)\\ s.t.\quad&D_t(p_t,R,A)\le c_t &&\text{for all } t\in\{1,...,T\}\\ &p_t\le p_{t'}\text{ if } R_t<R_{t'} &&\text{for all } t,t'\in\{1,...,T\}.\end{aligned}$$

Note that imposing the constraint $p_t\le p_{t'}$ if $R_{t'}=R_t+1$ (for all t,t') is equivalent to imposing the constraint $p_t\le p_{t'}$ if $R_t<R_{t'}$ in the above optimization problem, due to the transitivity of the inequalities. Thus, we conclude

$$\begin{aligned}V_R(\mathbf{c},A,A^*)=\max_{\mathbf{p}\ge0}\quad&\sum_{t=1}^T p_tD_t(p_t,R,A^*)\\ s.t.\quad&D_t(p_t,R,A)\le c_t &&\text{for all } t\in\{1,...,T\}\\ &p_t\le p_{t'}\text{ if } R_{t'}=R_t+1 &&\text{for all } t,t'\in\{1,...,T\}.\end{aligned}\tag{16}$$

Let $\lambda_t\ge0$ denote the Lagrange multiplier corresponding to the capacity constraint associated with time t, and $\mu_{t,t'}\ge0$ be the Lagrange multiplier associated with the ranking constraint $p_t\le p_{t'}$, assuming $R_{t'}=R_t+1$. The KKT conditions (see, for example, Bertsekas (1999)) imply that for all t, the optimal prices satisfy:

$$D_t(p_t,R,A^*)+p_t\frac{\partial D_t(p_t,R,A^*)}{\partial p_t}-\lambda_t\frac{\partial D_t(p_t,R,A)}{\partial p_t}+\mu_{t'',t}-\mu_{t,t'}=0,\tag{17}$$

where t, t', and t'' are such that $R_{t'}=R_t+1$ and $R_t=R_{t''}+1$. By the complementary slackness conditions, if two prices p_t and $p_{t'}$ are different, then $\mu_{t,t'}=0$. Thus, summing the KKT conditions for all periods that have the same price p (and noting that ranking of such periods are necessarily consecutive), μ_t terms cancel, and we obtain:

$$\sum_{t:\ p_t=p}\left[D_t(p,R,A^*)+p\frac{\partial D_t(p,R,A^*)}{\partial p}-\lambda_t\frac{\partial D_t(p,R,A)}{\partial p}\right]=0.$$

By definition $D_t(p,R,A) = \rho_t(R,A)(1-F(p))$, where $\rho_t(R,A)$ is the R-induced potential demand for population matrix A. Hence, using the notation $F'(p) = f(p)$, we obtain

$$\sum_{t:\ p_t=p} [\rho_t(R,A^*)(1-F(p)) - p\ \rho_t(R,A^*)f(p) + \lambda_t \rho_t(R,A)f(p)] = 0.$$

Rearranging terms, this equation leads to

$$\sum_{t:\ p_t=p} \rho_t(R,A)\lambda_t = \sum_{t:\ p_t=p} \rho_t(R,A^*)\left[p - \frac{1-F(p)}{f(p)}\right] \leq \sum_{t:\ p_t=p} \rho_t(R,A^*),$$

where the inequality follows from the fact that optimal prices are bounded by 1, and $\frac{1-F(p)}{f(p)} \geq 0$. Thus, summing the above equality over all periods t (or all different price levels p that appear in an optimal solution) we obtain

$$\sum_{t=1}^{T} \rho_t(R,A)\lambda_t \leq \sum_{t=1}^{T} \rho_t(R,A^*) = P(A^*) \leq P(A^U), \tag{18}$$

where $P(A) = \sum_{i,j} a_{i,j}$. By the complementary slackness conditions, $c_t = D_t(p_t,R,A) = \rho_t(R,A)(1-F(p_t)) \leq \rho_t(R,A)$ whenever the Lagrange multiplier $\lambda_t \neq 0$. Hence, the above inequality also implies

$$\sum_{t=1}^{T} c_t \lambda_t \leq P(A^U). \tag{19}$$

We next consider how $V_R(\mathbf{c},A,A^*)$ changes as $\mathbf{c}$ increases and A decreases. The Envelope Theorem (see Kimball (1952)) suggests that the derivatives $\frac{\partial V_R(\mathbf{c},A,A^*)}{\partial c_t}$ and $\frac{\partial V_R(\mathbf{c},A,A^*)}{\partial a_{i,j}}$ are equal to

$$\frac{\partial V_R(\mathbf{c},A,A^*)}{\partial c_t} = \lambda_t \quad \text{and} \quad \frac{\partial V_R(\mathbf{c},A,A^*)}{\partial a_{i,j}} = -\lambda_{t'(i,j,R)}(1-F(p_{t'(i,j,R)})), \tag{20}$$

where $t'(i,j,R)$ represents the period t' that has minimum ranking in R within $\{i,...,j\}$, i.e., the time period population $a_{i,j}$ receives service.

Observe that by definition V_R is increasing in $\mathbf{c}$ and decreasing in A. Since $\frac{c_t^U}{c_t^L}$ and $\frac{a_{i,j}^U}{a_{i,j}^L} \leq 1+\theta$, it follows that

$$\begin{aligned} 0 \leq V_R(\mathbf{c}^*,A^*,A^*) - V_R(\mathbf{c}_L,A^U,A^*) &\leq V_R(\mathbf{c}_U,A^L,A^*) - V_R(\mathbf{c}_L,A^U,A^*) \\ &\leq V_R((1+\theta)\mathbf{c}_L,A^L,A^*) - V_R(\mathbf{c}_L,A^L(1+\theta),A^*) \end{aligned} \tag{21}$$

Using the Fundamental Theorem of Calculus (and the notation $g_R(x) = V_R((1+x)\mathbf{c}^L,A^L,A^*)$) it follows that

$$\begin{aligned} 0 \leq V_R((1+\theta)\mathbf{c}_L,A^L,A^*) - V_R(\mathbf{c}_L,A^L,A^*) &= \int_{x=0}^{\theta} \frac{dg_R(x)}{dx}dx = \int_{x=0}^{\theta} \sum_{t=1}^{T} \frac{\partial V_R}{\partial c_t}((1+x)\mathbf{c}_L,A^L,A^*)c_t^L dx \\ &\leq \int_{x=0}^{\theta} \sum_{t=1}^{T} \frac{\partial V_R}{\partial c_t}((1+x)\mathbf{c}_L,A^L,A^*)(1+x)c_t^L dx. \end{aligned} \tag{22}$$

Observing from (20) that $\frac{\partial V_R}{\partial c_t}((1+x)\mathbf{c}_L,A^L,A^*)$ equals to the Lagrange multiplier λ_t for the problem instance with capacity vector $(1+x)\mathbf{c}_L$, and using (19) and (22), we obtain

$$V_R((1+\theta)\mathbf{c}_L,A^L,A^*) - V_R(\mathbf{c}_L,A^L,A^*) \leq \int_{x=0}^{\theta} P(A^U)dx = \theta P(A^U). \tag{23}$$

Following a similar approach, we also obtain

$$\begin{aligned} 0 \leq V_R(\mathbf{c}_L,A^L,A^*) - V_R(\mathbf{c}_L,(1+\theta)A^L,A^*) &= -\int_{x=0}^{\theta} \sum_{i,j} \frac{\partial V_R}{\partial a_{i,j}}(\mathbf{c}_L,(1+x)A^L,A^*)a_{i,j}dx \\ &\leq -\int_{x=0}^{\theta} \sum_{i,j} \frac{\partial V_R}{\partial a_{i,j}}(\mathbf{c}_L,(1+x)A^L,A^*)(1+x)a_{i,j}dx \end{aligned} \tag{24}$$

Using (20), it follows that $-\frac{\partial V_R}{\partial a_{i,j}}(\mathbf{c}_L,(1+x)A^L,A^*) = \lambda_{t'(i,j,R)}(1-F(p_{t'(i,j,R)})) \leq \lambda_{t'(i,j,R)}$, where λ_t denotes the Lagrange multiplier in a problem instance with parameters $\mathbf{c}_L,(1+x)A^L,A^*$. Thus, using (24) and noting from the definition of $t'(i,j,R)$ that $\rho_t(R,A) = \sum_{i,j:t'(i,j,R)=t} a_{i,j}$, we obtain,

$$V_R(\mathbf{c}_L,A^L,A^*) - V_R(\mathbf{c}_L,(1+\theta)A^L,A^*) \leq \int_{x=0}^{\theta} \sum_{t=1}^{T} \lambda_t \rho_t(R,A^L(1+x))dx. \tag{25}$$

Thus it follows from (18) that

$$V_R(\mathbf{c}_L,A^L,A^*) - V_R(\mathbf{c}_L,(1+\theta)A^L,A^*) \leq \int_{x=0}^{\theta} P(A^U)dx = \theta P(A^U). \tag{26}$$

Adding (23) and (26), and using it in the right hand side of (21) it follows that

$$V_R(\mathbf{c}^*,A^*,A^*) - V_R(\mathbf{c}_L,A^U,A^*) \leq 2\theta P(A^U). \tag{27}$$

Note that by linearity of the objective of (16) in its third argument, and the fact that $a^L_{i,j} \leq a^*_{i,j} \leq a^U_{i,j} \leq (1+\theta)a^L_{i,j}$, it follows that $V_R(\mathbf{c}_L,A^U,A^*) \leq V_R(\mathbf{c}_L,A^U,A^L)(1+\theta)$. On the other hand, since maximum price customers can pay for service is 1, it follows from the definition of $P(A)$ that $V_R(\mathbf{c}_L,A^U,A^L) \leq P(A^L) \leq P(A^U)$. Thus, we conclude $V_R(\mathbf{c}_L,A^U,A^*) - V_R(\mathbf{c}_L,A^U,A^L) \leq \theta P(A^U)$. Combining this with (27) we obtain

$$V_R(\mathbf{c}^*,A^*,A^*) \leq V_R(\mathbf{c}_L,A^U,A^L) + 3\theta P(A^U). \tag{28}$$

Maximizing both sides of this inequality over R and noting that $\max_R V_R(\mathbf{c}^*,A^*,A^*) = V(\mathbf{c}^*,A^*)$, we conclude $V(\mathbf{c}^*,A^*) \leq \max_R V_R(\mathbf{c}_L,A^U,A^L) + 3\theta P(A^U)$. Note that by definition $\max_R V_R(\mathbf{c}_L,A^U,A^L)$ equals the solution of OPT-R and $V^{ROB}(\mathcal{C},\mathcal{A},\mathbf{c}^*,A^*)$ is larger than this solution (OPT-R gives the worst case profits for optimal prices that are feasible for all capacities in $\mathcal{C}$, and arrivals in $\mathcal{A}$, whereas $V^{ROB}(\mathcal{C},\mathcal{A},\mathbf{c}^*,A^*)$ is the realized profit). Thus, we conclude $V(\mathbf{c}^*,A^*) \leq V^{ROB}(\mathcal{C},\mathcal{A},\mathbf{c}^*,A^*) + 3\theta P(A^U)$, and the claim follows.

A.4. Appendix to Section 7

Proof of Lemma 2 Let $\mathbf{p}^\star$ and $R^\star$ denote an optimal solution of OPT-5. Observe that for all t, the set $\mathbf{P}_\epsilon \cap [p^\star_t, p^\star_t + \epsilon)$ contains a single element. Denote this element by $\hat{p}_t$.

We first show that $\hat{\mathbf{p}}$ is consistent with ranking $R^\star$. Note that if $R^\star_t < R^\star_{t'}$ then $p^\star_{t'} \geq p^\star_t$. Moreover, since we have $p^\star_{t'} + \epsilon \geq p^\star_t + \epsilon$, and $\hat{p}_k$ is characterized by intersection of $[p^\star_k, p^\star_k + \epsilon)$ with $\mathbf{P}_\epsilon$ for all k, it follows that $\hat{p}_{t'} \geq \hat{p}_t$, and hence the consistency claim.

By Assumption 2, $h_t(p,R^\star)$ is decreasing in p, for any R. Therefore, $(\{\hat{p}_t\},R^\star)$ is a feasible solution of OPT-5. By Assumption 2 again, and the fact that $\hat{p}_t \in [p^\star_t, p^\star_t + \epsilon)$ for all t, it follows that

$$v = \sum_t g_t(p^\star_t,R^\star) \leq \sum_t \Big(g_t(\hat{p}_t,R^\star) + l_t\epsilon\Big). \tag{29}$$

On the other hand, by construction $\hat{p}_t \in \mathbf{P}_\epsilon$ for all t, thus $(\{\hat{p}_t\},R^\star)$ is a feasible solution of OPT-6. Hence $v_\epsilon \geq \sum_t g_t(\hat{p}_t,R^\star)$, and together with (29), this implies that $v_\epsilon \geq v - \epsilon\sum_t l_t$.

Proof of Lemma 3 Construction of optimal prices and ranking, using the dynamic programming recursion in (10) is identical to the construction given in Theorem 3, and is omitted. In the rest of the proof we characterize the computational complexity of this construction.

In order to solve the recursion in (10) we compute all values of $\omega(i,j,p)$ by solving $O(T^2|\mathbf{P}_\epsilon|)$ subproblems. At each step of the recursion we solve for the optimal k and p. Finding these requires at most $O(T|P_\epsilon|)$ trials. Given a value of p and k, we need to evaluate $\hat{\gamma}_k^{ij}(p)$. This requires checking if constraints are satisfied in the subproblem (hence computing $h_k(p,R)$), and evaluating the corresponding objective value ($g_k(p,R)$) in the relevant subproblem. Thus, computation of $\hat{\gamma}_k^{ij}(p)$ can be completed in $O(s(T))$ time, and the overall complexity is $O\left(\frac{T^3 s(T)}{\epsilon^2}\right)$.

A.5. Appendix to Section 8

Proof of Lemma 4: Let S_1 be a revenue maximizing vector in $\mathcal{S}(i,j,\underline{p},k)$. Suppose there exits a price vector $S_2 \in \mathcal{S}(i,j,\underline{p},k)$ and period t such that $p_t(S_2) = \underline{p} < p_t(S_1)$. We show that no such price vector exists, hence proving the lemma.

Define S' to be the price vector such that

$$p_t(S') = \begin{cases} \underline{p} & p_t(S_1) = \underline{p} \text{ or } p_t(S_2) = \underline{p} \\ p_t(S_1) & \text{Otherwise} \end{cases}$$

To prove the claim, first consider the assignment of the populations to the periods when the price vector is S_1. Let A_1 be the set of periods that have price $\underline{p}$ under S_1 and A_2 be the set of such periods under S_2. Note that in S', we update S_1 by decreasing the prices of periods in $A_2 \setminus A_1$ to $\underline{p}$. Observe that the price change only matters for populations that are present in the system in a period in $A_2 \setminus A_1$, but not $A_2 \cap A_1$, since it is possible to schedule all the remaining populations exactly as we did under S_1. Since $S_2 \in \mathcal{S}(i,j,\underline{p},k)$ is feasible, it follows that for S', there exists a feasible schedule that assigns all populations that are present in a time instant in $A_2 \setminus A_1$, but not $A_2 \cap A_1$, to time instants in $A_2 \setminus A_1$, even when the price offered at these periods equals to $\underline{p}$. Moreover, because $\underline{p} \geq p_M$, reducing the prices can only increase the revenue, implying that S' leads to higher revenues than S_1, and contradicting with the assumption that S_1 is a revenue maximizer. Hence, the lemma follows.

Proof of Lemma 5: We say that interval $[l,k]$ satisfies the "minimum requirements" for having price p if this interval can serve all customers who can receive service only in this interval at price p. Namely, $\sum_{u=l}^{k} c_u \geq (1 - F(p)) \sum_{u=l}^{k} \sum_{v=u}^{k} a_{uv}$. This is a necessary condition for time instants in this interval to have price equal to p.

We claim that the algorithm described in Figure 6 finds set $\hat{L}$ in polynomial time. To prove the correctness of the algorithm, first observe that if interval $[i+1,k]$ satisfies the minimum requirements, then we can serve all the populations who arrive at the system after time i, and can wait up to (and including time) k, to this interval at price p. Hence, $\hat{L} = \{i+1, i+2, \cdots, k\}$. We also show at step 3 of the algorithm that if interval $[\ell,k]$ does not satisfy the minimum requirements, then ℓ does not belong to $\hat{L}(i,j,p,k)$. To show this consider the largest ℓ such that $[\ell,k]$ does not satisfy the minimum requirements, and assume that ℓ belongs to $\hat{L}$. Note that because we chose the largest such ℓ, all the populations that arrive at time ℓ or after that need to be scheduled to periods in $[\ell,k]$ for service. However, this period does not satisfy the minimum requirements, and we obtain a contradiction.

Finally, we note that the algorithm is polynomial; the number of recursions is bounded by $k-i$, and we can verify the minimum requirements in polynomial time.

FindL(i,k,p,A,c):

1. If interval $[k,k]$ does not satisfy the minimum requirements:
It is infeasible to assign time instant k price p.
Terminate and Return $S=\emptyset$.
2. If interval $[i+1,k]$ satisfies the minimum requirements.
Terminate and Return $S=\{i+1,\cdots,k\}$.
3. Let ℓ be the largest element in $[i+1,k]$ such that $[\ell,k]$ does not satisfy the minimum requirements.
(Time instant ℓ cannot receive price p)
   (a) Define a new problem instance where period ℓ is removed, and search for the maximal set which receives price p in this new problem instance.
   (b) Label periods in the new problem as $i',(i+1)',\dots(k-1)'$. Construct a population matrix $A'=\{a'_{i,j}\}$ and capacity vector c' for the new problem.

$$a'_{i',j'} = \begin{cases} a_{i',j'} & j' < \ell - 1 \\ a_{i',j'} + a_{i',\ell} & i' < \ell, j' = \ell - 1 \\ a_{i',j'+1} & i' < \ell, j' > \ell - 1 \\ a_{\ell,j'+1} + a_{\ell+1,j'+1} & i' = \ell \\ a_{i'+1,j'+1} & \ell < i' \end{cases}$$

$$c'_{j'} = \begin{cases} c'_j & j' < \ell \\ c_{j'+1} & \ell \le j' < k \end{cases}$$

   (c) Let $S' = FindL(i',k',p,A',c')$
      i. If $S'=\emptyset$
It is infeasible to assign price p to any subset of $[i,k]$.
Terminate and Return $S=\emptyset$.
      ii. If $S'\neq\emptyset$
Construct the solution from S'.

$$S=\{s|s\in S' \text{ and } s<\ell, \text{ or } s\geq l \text{ and } s-1\in S'\}.$$

Terminate and Return S.

Figure 6 Algorithm *FindL(i,k,p,A,c)*: Finds the maximal subset of $(i,k]$ (excluding time i but including time k) which can receive price equal to p , where the population matrix and capacities are given by A and c.