Surge Pricing and its Spatial Supply Response

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Abstract

We consider the pricing problem faced by a revenue maximizing platform matching price-sensitive customers to flexible supply units within a geographic area. This can be interpreted as the problem faced in the short-term by a ride-hailing platform. We propose a two-dimensional framework in which a platform selects prices for different locations, and drivers respond by choosing where to relocate in equilibrium based on prices, travel costs and driver congestion levels. The platform's problem is an infinite-dimensional optimization problem with equilibrium constraints. We elucidate structural properties of supply equilibria and the corresponding utilities that emerge and establish a form of spatial decomposition, which allows us to localize the analysis to regions of movement. In turn, uncovering an appropriate knapsack structure to the platform's problem, we establish a crisp local characterization of the optimal prices and the corresponding supply response. In the optimal solution the platform applies different treatments to different locations. In some locations, prices are set so that supply and demand are perfectly matched; over-congestion is induced in other locations, and some less profitable locations are indirectly priced out. To obtain insights on the global structure of an optimal solution, we derive in quasi-closed form the optimal solution for a family of models characterized by a demand shock. The optimal solution, while better balancing supply and demand around the shock, quite interestingly, also ends up inducing movement away from it.

Keywords: spatial pricing, revenue management, ride-hailing, strategic supply, market design.

1 Introduction

Pricing and revenue management have seen significant developments over the years in both practice and the literature. At a high level, the main focus has been to investigate tactical pricing decisions given the dynamic evolution of inventories, with prototypical examples coming from the airline, hospitality and retail industries (Talluri & Van Ryzin (2006)). With the emergence and multiplication of two-sided marketplaces, a new question has emerged: how to price when capacity/supply units are strategic and can decide when and where to participate. This is particularly relevant for ride-hailing platforms such as Uber and Lyft. In these platforms, drivers are independent contractors who have the ability to relocate strategically within their cities to boost their own profits. While this operating model leads to a more flexible supply, it also restricts the platform from reallocating supply across locations at will. Instead, a platform must ensure that incentives are in place for drivers to select to reallocate themselves. Consider the spatial pricing problem within a city faced by a platform that shares its revenues with drivers. Suppose there are different demand and supply conditions across the city. The platform may want to increase prices at locations with high demand and low supply. Such an increase would have two effects. The first effect is a local demand response, which pushes the riders who are not willing to pay a higher price away from the system. The second effect is global in nature, as drivers throughout the city may find the locations with high prices more attractive than the ones where they are currently located and may decide to relocate. In turn, this may create a deficit of drivers at some locations. In other words, prices set in *one region* of a city impact demand and supply at this region, but also potentially impact supply in *other regions*. This brings to the foreground the question of how to price in space when supply units are strategic.

The central focus of this paper is to understand the interplay between spatial pricing and supply response. In particular, we aim to understand how to optimally set prices across locations in a city, and what the impact of those prices is on the strategic repositioning of drivers. To that end, we consider a short-term model over a given timeframe where overall supply is constant. That is, drivers respond to pricing and congestion by moving to other locations, but not by entering or exiting the system. In our short-term framework, the platform's only tool for increasing the supply of drivers at a given location is to encourage drivers to relocate from other places. In turn, this time scale permits us to isolate the spatial implications on the different agents' strategic behavior. In this sense, our model can be thought of as a building block to better understand richer temporal-dynamic environments.

In more detail, we consider a revenue-maximizing platform that sets prices to match price-sensitive riders (demand) to strategic drivers (supply) who receive a fixed commission. In making their decisions, drivers take into account prices, supply levels across the city, and transportation costs. More formally, we consider a measure-theoretical Stackelberg game with three groups of players: a platform, drivers and potential customers. Supply and demand are composed of non-atomic agents, who are initially arbitrarily positioned throughout the city. We use non-negative measures to model how these agents are distributed in the city. The players interact with each other in a two-dimensional city. Every location can admit different levels of supply and demand. For example, Figure 1 showcases an instance of a rectangular city in which the measure or distribution of supply has two peaks, while the distribution of demand has one peak in the center. The platform moves first, selecting prices for the different locations around the city. Once prices are set, the mass of customers willing to pay such prices is determined. Then, drivers move in equilibrium in a simultaneous-move game, choosing where to reposition based on prices, supply levels and driving costs. In fact, besides prices and transportation costs, supply levels across the city are a key consideration for drivers when optimizing their repositioning. If too many other drivers are at a given location, a driver relocating there will be less likely to be matched to a rider, negatively affecting that driver's utility. The platform's optimization problem consists of finding prices for all locations given that drivers move in equilibrium.

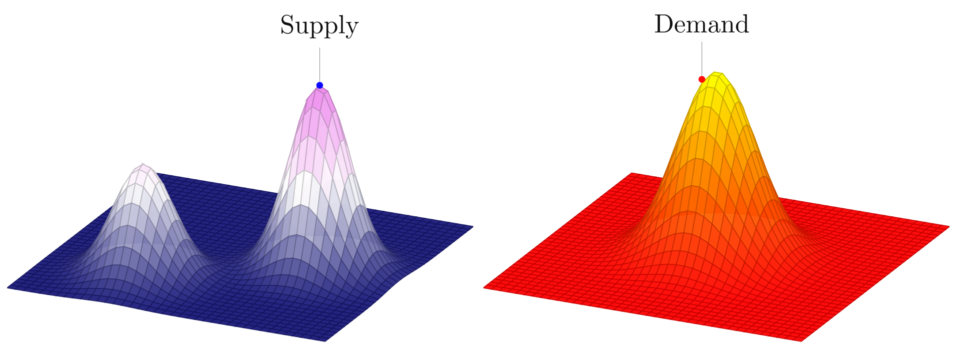


Figure 1: **Two-dimensional setting.** Distribution of initial supply and potential demand across locations.

Main contributions. Our first set of contributions is at the modeling level. We propose a general measure-theoretical framework that encompasses a wide range of environments. Our setup can be used to study spatial interactions in both discrete and continuous locations settings.

The platform's problem is an infinite-dimensional optimization problem with equilibrium constraints. This is a notably "hard" class of problems. In our second set of contributions we develop a methodology to study the platform's problem. Our main result provides a structural characterization of the optimal prices, and resulting drivers' equilibrium in regions of the city where drivers relocate. Our approach relies on a series of transformations, localization and relaxations. In particular, we first establish that the platform's objective can be reformulated as a function of only the equilibrium utilities of drivers and their equilibrium post-relocation distribution. In turn, we establish structural properties of these two objects. We first characterize properties of the drivers' equilibrium utilities and prove that the city admits a form of spatial decomposition into regions where movement may emerge in equilibrium, "attraction regions," and the rest of the city. Furthermore, we establish that the equilibrium utility of drivers and the local equilibrium post-relocation supply are linked through a congestion bound. The former admits a fundamental upper bound parametrized by the latter. Based on these properties, we derive a relaxation of the platform's problem that takes the form of *coupled continuous bounded knapsack* problems. Notably, we establish that this relaxation is tight and, leveraging the knapsack structure, use it to obtain a crisp *local* structural characterization of an optimal pricing solution and its supply response.

While the framework above provide local structural properties of optimal pricing policies in arbitrary two-dimensional regions of movement, in our third set of contributions, we shed light on the scope of prices as an incentive mechanism and provide insights into the *global* structure of an optimal policy. To that end, we focus on a family of one-dimensional instances (which could be interpreted as a cut of a symmetric two dimensional city) that are rich enough to capture the core interactions among supply agents, but also confined enough to derive *quasi-closed form solutions* that allow to crisply identify some key features of an optimal solution. In particular, we study a family of cases in which a central location in the city, the origin, experiences a shock of demand.

Leveraging our earlier methodological results in conjunction with the derivation of new results, we characterize in quasi-closed form the optimal pricing policy and its corresponding supply response. Strikingly, the optimal pricing policy induces movement toward the demand shock but potentially also *away* from the demand shock. The platform may create *damaged regions* through both prices and congestion to steer the flow of drivers toward more profitable regions. Compared to a heuristic that would only adjust the price at the shock location, the optimal solution incentivizes more drivers to travel toward the demand shock.

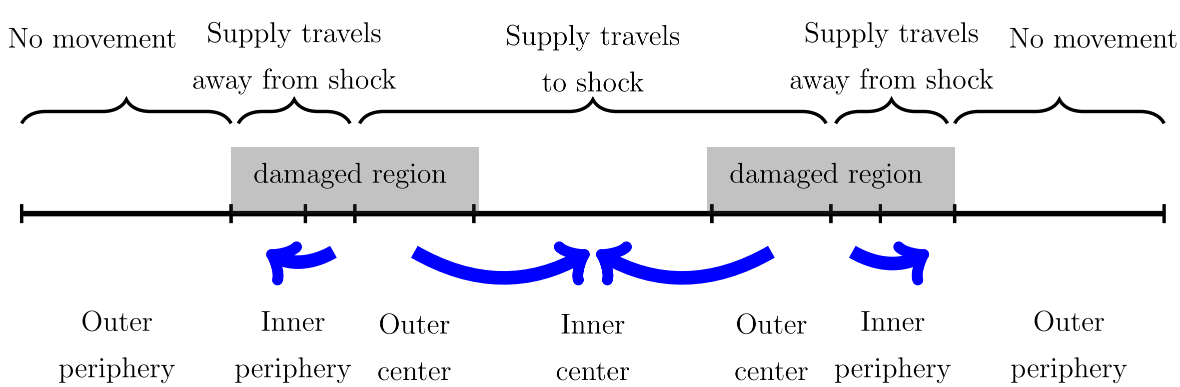


Figure 2: **The optimal solution splits the city into multiple regions.**

The optimal pricing policy splits the city into multiple regions around the origin (Figure 2). The mass of customers needing rides at the location of the shock is serviced by three subregions around it: the origin, the inner center and the outer center. The origin is the most profitable location and so the platform surges its price, encouraging the movement of a mass of drivers to meet its high levels of demand. These drivers come from both the inner and the outer center. In the former, locations are positively affected by the shock, and some drivers choose to stay in them while others travel toward the origin. In the latter, drivers are too far from the demand shock and so the platform has to *deliberately damage* this region through prices to create incentives for drivers to relocate toward the origin. However, drivers in this region have an option: instead of driving toward the demand shock at the origin, they could drive away from it. This gives rise to the next region, the inner periphery. Consider the marginal driver, i.e., the furthest driver willing to travel to the origin. To incentivize the marginal driver to move to the origin, the platform is obligated to also damage conditions in the inner periphery. The optimal solution creates two subregions within the inner periphery. In the first, conditions are degraded through prices that make it unattractive for drivers. Drivers in this region leave toward the second region. That is, they drive in the direction opposite to the demand shock. The action of the platform in the second region is more subtle. Here, the platform does not need to play with prices. The mere fact that drivers from the first region run away to this area creates congestion, and this is sufficient degradation to make the region unattractive for the marginal driver. The final region is the outer periphery, which is too far from the origin to be affected by its demand shock.

We complement our analysis with a set of numerics that highlights that the optimal policy can generate significantly more revenues than a heuristic that would simply respond locally to a shock in demand. In other words, anticipating the global supply response and taking advantage of the full flexibility of spatial pricing plays a key role in revenue optimization.

2 Related Literature

Several recent papers examine the operations of ride-hailing platforms from diverse perspectives. We first review works that do not take spatial considerations into account. There is a recent but significant body of work on the impact of incentive schemes on agents' participation decisions. Gurvich et al. (2016) study the cost of self-scheduling capacity in a newsvendor-like model in which the firm chooses the number of agents it recruits and, in each period, selects a compensation level as well as a cap on the number of available workers. Cachon et al. (2017) analyze various compensation schemes in a setting in which the platform takes into account drivers' long-term and short-term incentives. They establish that in high-demand periods all stakeholders can benefit from dynamic pricing, and that fixed commission contracts can be nearly optimal. The performance of such contracts in two-sided markets is analyzed by Hu & Zhou (2017) who derive performance guarantees. Taylor (2017) considers how uncertainty affects the price and wage decisions of on-demand platforms when facing delay-sensitive customers and autonomous capacity. Nikzad (2018) focuses on the effect of market thickness and competition on wages, prices and welfare and shows that, in some circumstances, more supply could lead to higher wages, and that competition across platforms could lead to high prices and low consumer welfare.

In the context of matching in ride-hailing without pricing, Feng et al. (2017) compare the waiting time performance, in a circular city, of on-demand matching versus traditional street-hailing matching. Hu & Zhou (2016) analyze a dynamic matching problem as well as the structure of optimal policies. Relatedly, Ozkan & Ward (2016) develop a heuristic based on a continuous linear program to maximize the number of matches in a network. Afèche et al. (2017) study demand admission controls and drivers' repositioning in a two-location network, without pricing, and show that the value of the controls is large when both capacity is moderate and demand is imbalanced.

Most closely related to our work are papers that study pricing with spatial considerations. Castillo et al. (2017) take space into account, but only in reduced form through the shape of the supply curve. This paper points out that surge pricing can help to avoid an inefficient situation termed the "wild goose chase" in which drivers' earnings are low due to long pick-up times. Banerjee et al. (2015) consider a queueing network where drivers do not make decisions in the short-term (no repositioning decisions) but they do care about their long-term earning. They prove that a localized static policy is optimal as long as the system parameters are constant, but that a dynamic pricing policy is more robust to changes in these parameters. Banerjee et al. (2016) find approximation methods to find source-destination prices in a network to maximize various long-run average metrics. In Banerjee et al. (2016), customers have a destination and react to prices, but supply units do not behave strategically. An important contribution to the field is Bimpikis et al. (2019). That paper studies pricing under steady-state conditions in a network in which drivers behave in equilibrium and decide wether and when to provide service as well as where to reposition. They are able to isolate an interesting "balance" property of the network and establish its implications for prices, profits and consumer surplus. Buchholz (2017) structurally estimates a spatial model to understand the welfare costs of taxi fare regulations. These papers investigate long-term implications of spatial pricing. In contrast, our work examines how the platform should respond to short-term supply-demand imbalances given that the supply units are strategic. Relatedly, Guda & Subramanian (2019) study a two location setting and show the benefits of using strategic pricing in a short-term time scale. From an empirical perspective, short-term strategic repositioning decisions of drivers to changes in heat maps of prices have been demonstrated using Uber data in Lu et al. (2018). Our work complements such empirical and theoretical studies by providing a general multi-location framework for pricing while accounting for short-term strategic relocation decisions.

From a methodological point of view, our work borrows tools from the literature on non-atomic congestion games. Our equilibrium concept is similar to the one used by Roughgarden & Tardos (2002) and Cole et al. (2003) to analyze selfish routing under congestion in discrete settings: in equilibrium, drivers only depart for locations that yield the largest earnings. We consider a more general measure-theoretical environment that can be traced back to Schmeidler (1973), Mas-Colell (1984) and, more recently, to Blanchet & Carlier (2015) . The latter builds on the work of Mas-Colell by introducing an equilibrium notion that accounts for local congestion effects and relates it to the theory of optimal transport (see e.g., Villani (2008)). Our equilibrium concept relates to the one introduced by these authors in that it can be applied to general measure-theoretical frameworks and that it captures local congestion effects. Once the platform sets prices, drivers must decide where to relocate. This creates a "flow" or a "transport plan" in the city from initial supply (initial measure) to post-relocation supply (final measure). A fundamental difference is that in our case the final measure is endogenous and most of our work is focused on determining this measure.

Finally, some of our insights relate back to the damaged goods literature. Deneckere & McAfee (1996) explain that a firm can strategically degrade a good in order to price discriminate. In our setting the platform can damage some regions in the city through prices and congestion to steer drivers toward more profitable locations and thus increase revenues.

3 Problem Formulation

We will use measure-theoretic objects to represent supply, demand and related concepts. This level of generality will enable us to capture the rich interactions that arise in the system through a spatial model that subsumes continuous and discrete settings. Our general framework allows us to focus on the central "physical" quantities that are not tailored to the nature of the model, but also allows for quasi-closed form solutions in special cases of interest. We now introduce some basic preliminaries to make the exposition rigorous.

Preliminaries. For an arbitrary metric set $\mathcal{X}$ equipped with a norm $\|\cdot\|$ and the Borel $\sigma-$algebra, we let $\mathcal{M}(\mathcal{X})$ denote the set of non-negative finite measures on $\mathcal{X}$. The notation $\mathcal{T} \ll \mathcal{T}'$ represents measure $\mathcal{T}$ being absolutely continuous with respect to measure $\mathcal{T}'$. The notation $\operatorname{ess\,sup}_{\mathcal{B}}$ corresponds to the essential supremum, which is the measure-theoretical version of a supremum that does not take into account sets of measure zero. The notation $\mathcal{T}-a.e.$ represents almost everywhere with respect to measure $\mathcal{T}$. For any measure $\mathcal{T}$ in a product space $\mathcal{B} \times \mathcal{B}$, $\mathcal{T}_1$ and $\mathcal{T}_2$ will denote, respectively, the first and second marginals of $\mathcal{T}$. We use $\mathbf{1}_{\{\cdot\}}$ to denote the indicator function and S^c to represent the complement of a set S. We denote the closed and open line segment between any two points by $[x, y]$ and (x, y), respectively.

3.1 Model elements

Our model contains four fundamental elements: a city, a platform, drivers and potential customers. For consistency, we use masculine pronouns to refer to drivers and feminine ones to refer to customers. We represent the city by a convex, compact subset $\mathcal{C}$ of $\mathbb{R}^2$, and a measure $\mathit{\Gamma}$ in $\mathcal{M}(\mathcal{C})$. We refer to this measure as the city measure and it characterizes the "size" of every location of the city. For example, if $\mathit{\Gamma}$ has a point mass at some location then that location is large enough to admit a point mass of supply and demand.

Demand (potential customers) and supply (drivers) are assumed to be infinitesimal and initially distributed on $\mathcal{C}$. We denote the initial demand measure by $\mathit{\Lambda}(\cdot)$ and the supply measure by $\mathit{\Theta}(\cdot)$, with both measures belonging to $\mathcal{M}(\mathcal{C})$. For example, if $\mathit{\Theta}$ is the Lebesgue measure on $\mathcal{C}$, then drivers are uniformly distributed over the city. Both the demand and supply measures are assumed to be absolutely continuous with respect to the city measure, i.e., $\mathit{\Lambda}, \mathit{\Theta} \ll \mathit{\Gamma}$. The proportion of customers at location $y \in \mathcal{C}$ with willingness to pay below q is given by $F_y(q)$. We let $\overline{F}_y(q) = 1 - F_y(q)$. For all $y \in \mathcal{C}$, we assume the revenue function $q \mapsto q \cdot \overline{F}_y(q)$ is continuous and unimodular and that F_y is strictly increasing over its support $\left[0, \overline{V}\right]$, for some finite positive $\overline{V}$.

We model the interactions between platform, customers and supply as a game. The first player to act in this game is the platform. The platform selects fares across locations and facilitates the matching of drivers and customers. Specifically, the platform selects a measurable price mapping $p : \mathcal{C} \to [0, \overline{V}]$ so as to maximize its citywide revenues.

After prices are chosen, drivers select *whether* to relocate and *where* to do so. The relocation of drivers generates a flow/transport of mass from the initial measure of drivers $\mathit{\Theta}$ to some final endogenous measure of drivers. This final measure corresponds to the supply of drivers in the city after they have traveled to their chosen destinations. The movement of drivers across the city is modeled as a measure on $\mathcal{C} \times \mathcal{C}$, which we denote by $\mathcal{T}$. Any feasible flow has to preserve the initial mass of drivers in $\mathcal{C}$. That is, the first marginal of $\mathcal{T}$ should equal $\mathit{\Theta}$. Moreover, $\mathcal{T}$ generates a new (after relocation) distribution of drivers in the city, which corresponds to the second marginal of $\mathcal{T}$, $\mathcal{T}_2$. Formally, the set of feasible flows is defined as follows

$$\mathcal{F}(\mathit{\Theta}) = \{\mathcal{T} \in \mathcal{M}(\mathcal{C} \times \mathcal{C}) : \mathcal{T}_1 = \mathit{\Theta}, \quad \mathcal{T}_2 \ll \mathit{\Gamma}\}.$$

The first condition ensures consistency with the initial positioning of drivers, the second condition ensures that there is no mass of relocated supply at locations where the city itself has measure zero. In particular, given the latter, the Radon-Nikodym derivatives of $\mathcal{T}_2$ and Λ with respect to Γ, $d\mathcal{T}_2(y)/d\Gamma$ and $d\Lambda(y)/d\Gamma$, are well defined and for ease of notation we let, for any y in $\mathcal{C}$,

$$s^{\mathcal{T}}(y) \triangleq \frac{d\mathcal{T}_2}{d\Gamma}(y), \text{ and } \lambda(y) \triangleq \frac{d\Lambda}{d\Gamma}(y).$$

Physically, $s^{\mathcal{T}}(y)$ represents the *post-relocation supply* at location y normalized by the size of location y, and $\lambda(y)$ corresponds to the potential demand at location y also normalized by the same size of such location.[1] Here and in what follows, we will refer to $s^{\mathcal{T}}(y)$ and $\lambda(y)$ as the *post-relocation supply* and potential demand at y, respectively. In order to lighten the exposition, and without loss of generality, we assume that $\lambda(y) > 0$ $\Gamma-$almost everywhere in $\mathcal{C}$.

Given the prices in place, the effective demand at a location y is given by $\lambda(y) \cdot \overline{F}_y(p(y))$, as at location y, only the fraction $\overline{F}_y(p(y))$ is willing to purchase at price $p(y)$. At the same time, the supply at y is given by $s^{\mathcal{T}}(y)$. Therefore, the ratio of effective (as opposed to potential) demand to supply at y is given by

$$\frac{\lambda(y) \cdot \overline{F}_y(p(y))}{s^{\mathcal{T}}(y)},$$

assuming $s^{\mathcal{T}}(y) > 0$. Since a driver can pick up at most one customer within the time frame of our game, a driver relocating to y will face a utilization rate of $\min\left\{1, \lambda(y) \cdot \overline{F}_y(p(y))/s^{\mathcal{T}}(y)\right\}$, again assuming $s^{\mathcal{T}}(y) > 0$. The effective utilization can be interpreted as the probability that a driver who relocated to y will be matched to a customer within the time frame of our game. In particular, if $s^{\mathcal{T}}(y) > \lambda(y) \cdot \overline{F}_y(p(y))$, there is driver congestion at location y, and not all drivers will be matched to a customer. If $s^{\mathcal{T}}(y) = 0$ at location y, we say the utilization rate is one if the effective demand at y is positive and zero if the effective demand is zero. Formally, the utilization rate at location y is given by

$$R\Big(y, p(y), s^{\mathcal{T}}(y)\Big) \triangleq \begin{cases} \min\left\{1, \frac{\lambda(y) \cdot \overline{F}_y(p(y))}{s^{\mathcal{T}}(y)}\right\} & \text{if } s^{\mathcal{T}}(y) > 0; \\ 1 & \text{if } s^{\mathcal{T}}(y) = 0,\ \lambda(y) \cdot \overline{F}_y(p(y)) > 0; \\ 0 & \text{if } \lambda(y) \cdot \overline{F}_y(p(y)) = 0. \end{cases}$$

When deciding whether to relocate, drivers take three effects into account: prices, travel distance and congestion. The driver congestion effect (or utilization rate) is the one described in the paragraph above. We assume that the platform uses a commission model and transfers a fraction α in $(0, 1)$ of the fare to the driver. As a result, a driver who starts in location y and chooses to

[1] The Radon-Nikodym derivate can also be interpreted as a measure of the units of demand or supply per unit of area (e.g., square mile).

remain there earns utility equal to

$$U\Big(y, p(y), s^{\mathcal{T}}(y)\Big) \triangleq \alpha \cdot p(y) \cdot R\Big(y, p(y), s^{\mathcal{T}}(y)\Big). \tag{1}$$

That is, the utility is given by the compensation per ride times the probability of a match. We model the cost for drivers of repositioning from location x to location y through the distance between the locations, $\|y - x\|$. Therefore, a driver originating in x who repositions to y earns utility

$$\Pi\Big(x, y, p(y), s^{\mathcal{T}}(y)\Big) \triangleq U\Big(y, p(y), s^{\mathcal{T}}(y)\Big) - \|y - x\|. \tag{2}$$

When clear from context, and with some abuse of notation, we omit the dependence on price and the supply-demand ratio, writing $U(y)$ and $\Pi(x, y)$. Thus far, we have introduced the objects to study a classic *Cournot-Nash equilibrium* (see e.g., Blanchet & Carlier (2015)). The space of players' types is $\mathcal{C}$ endowed with the type distribution $\Theta \in \mathcal{M}(\mathcal{C})$. The action space is the set of possible destinations $\mathcal{C}$ and there is an action distribution $\mathcal{V} \in \mathcal{M}(\mathcal{C})$ with density ν. Given an action distribution $\mathcal{V}$, an agent with type x choosing action y receives a total utility given by $\Pi\Big(x, y, p(y), \nu(y)\Big)$. We are now ready to define the notion of a supply equilibrium. An equilibrium specifying type-action pairs is defined through a flow $\mathcal{T} \in \mathcal{M}(\mathcal{C} \times \mathcal{C})$. This distribution should be consistent with the type and action distributions, $\mathcal{T}_1 = \Theta$ and $\mathcal{T}_2 = \mathcal{V}$, and the actions of agents should be optimal given all other agents' actions. We next define precisely a supply equilibrium.

Definition 1 (Supply Equilibrium)**.** *A flow $\mathcal{T} \in \mathcal{F}(\Theta)$ is an equilibrium if it satisfies*

$$\mathcal{T}\left(\Big\{(x, y) \in \mathcal{C} \times \mathcal{C} : \Pi(x, y, p(y), s^{\mathcal{T}}(y)) = \operatorname*{ess\,sup}_{\mathcal{C}} \Pi\Big(x, \cdot, p(\cdot), s^{\mathcal{T}}(\cdot)\Big)\Big\}\right) = \Theta(\mathcal{C}),$$

where the essential supremum is taken with respect to the city measure Γ.

That is, an equilibrium flow of supply is a feasible flow such that essentially no driver wishes to unilaterally change his destination. As a result, the mass of drivers selecting the best location for themselves has to equal the original mass of drivers in the system.

The platform's objective is to maximize the revenues it garners across all locations in $\mathcal{C}$. With the assumed commission structure in place, from a given location y, it earns $(1 - \alpha) \cdot p(y) \cdot \min\{s^{\mathcal{T}}(y), \lambda(y) \cdot \overline{F}_y(p(y))\}$. The term $(1 - \alpha) \cdot p(y)$ corresponds to the platform's share of each fare at location y, and the term $\min\{s^{\mathcal{T}}(y), \lambda(y) \cdot \overline{F}_y(p(y))\}$ denotes the quantity of matches of potential customers to drivers at location y. The platform's price optimization problem can in turn be written as

$$\sup_{p(\cdot),\, \mathcal{T}\in\mathcal{F}(\Theta)} \ (1-\alpha)\int_{\mathcal{C}} p(y)\cdot \min\{s^{\mathcal{T}}(y), \lambda(y)\cdot \overline{F}_y(p(y))\}\, d\Gamma(y) \qquad (\mathcal{P}_1)$$

$$s.t. \quad \mathcal{T} \text{ is a supply equilibrium,}$$

$$s^{\mathcal{T}} = \frac{d\mathcal{T}_2}{d\Gamma}.$$

Remark. Our model may be interpreted as a basic model to understand the short-term operations of a ride-hailing company. In particular, each driver completes at most one customer pickup within the time frame of our game and there is not enough time for the entry of new drivers into the system. In the present model, we do not account explicitly for the destinations of the rides. We do so in order to isolate the interplay of supply incentives and pricing. In that regard, one could view our model as capturing origin-based pricing, a common practice in the ride-hailing industry. Note that our framework could be modified to partially account for origin-destination pricing by incorporating the fact that different locations lead to different average lengths of rides (such a factor would multiply the price on the right hand side of Eq. (1)). With such a modification, drivers, before repositioning to a location, would consider the corresponding average price for the total trip.

3.2 Solution approach

Problem $(\mathcal{P}_1)$ can be classified as an infinite dimensional mathematical program with equilibrium constraints. This is a notably difficult class of problems to solve in general, even numerically. For the particular problem we study, we will show that significant structure exists and that one can obtain a crisp local characterization of an optimal solution. We next lay out our approach.

In Section 4 we reformulate the platform's objective. The new objective is "well-behaved" and showcases the fundamental structure of the problem. In particular, if we relaxed all equilibrium constraints the problem would be a continuous knapsack problem in which the limited initial budget of drivers $\Theta(\mathcal{C})$ must be allocated across locations. However, the solution to the knapsack relaxation may not satisfy the equilibrium constraints. To address this, in Sections 5 and 6 we derive equilibrium properties that we later add as constraints to the aforementioned relaxation. More precisely, in Section 5 we prove a fundamental upper bound on the amount of post relocation supply that there could be at any location in equilibrium. In Section 6 we identify and localize the analysis to the regions in the city where drivers move. We study their properties and show these regions, subject to appropriate constraints, can be optimized in isolation. In Section 7 we add these developments as constraints to problem $(\mathcal{P}_1)$ and relax the equilibrium constraints. The resulting problem is a relaxation of $(\mathcal{P}_1)$ in a region of potential movement that has the structure of a continuous bounded knapsack problem. We then show that this local relaxation is tight thereby providing a characterization of the optimal solution within regions of potential movement. In Section 8, we combine these results, together with new results and a flow mimicking technique (Appendix D.2), to solve in quasi-closed form for the optimal solution in a family of instances. Figure 3 provides a summary of our solution approach.

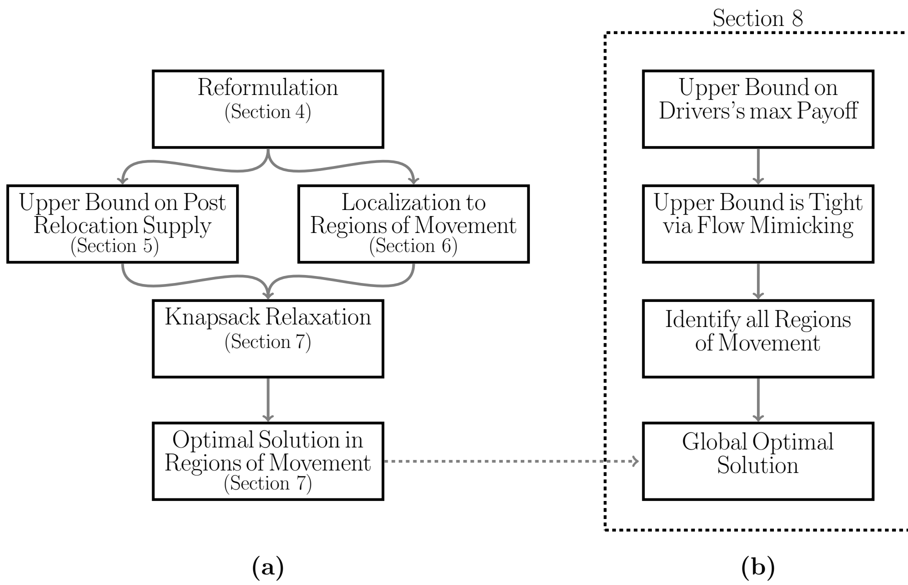


Figure 3: **Solution approach. (a)** General solution approach to characterize the optimal solution within regions of movement. **(b)** Application of general results to obtain optimal solution in a family of instances with a shock of demand in the center of the city (Section 8).

4 Structural Properties and Reformulation

A key challenge in solving the optimization problem presented in $(\mathcal{P}_1)$ is that the decision variables, the flow $\mathcal{T}$ and the price function $p(\cdot)$, are complicated objects. The flow $\mathcal{T}$, being a measure over movements on a two-dimensional space, is obviously a complex object to manipulate. The price function will turn out to be a difficult object to manipulate as well. In order to analyze our problem, we will need to introduce a better-behaved object. This object, which will be central to our analysis, is the (after movement) driver equilibrium utility.

Drivers' utilities. For a given price function p and flow $\mathcal{T}$, we denote by $V_{\mathcal{B}}(x|\, p, \mathcal{T})$ the essential maximum utility that a driver departing from location x can garner by going anywhere within a measurable region $\mathcal{B} \subseteq \mathcal{C}$. In particular, the mapping $V_{\mathcal{B}}(\cdot|\, p, \mathcal{T}) : \mathcal{C} \to \mathbb{R}$ is defined as

$$V_{\mathcal{B}}(x|\, p, \mathcal{T}) \triangleq \operatorname*{ess\,sup}_{\mathcal{B}} \Pi\left(x, \cdot, p(\cdot), s^{\mathcal{T}}(\cdot)\right). \tag{3}$$

When $\mathcal{B} = \mathcal{C}$, we use V instead of $V_{\mathcal{C}}$. Note that in the essential supremum $\mathcal{B}$ enters Π when we evaluate the second argument of Π, $p(\cdot)$ and $s^{\mathcal{T}}(\cdot)$. By the definition of a supply equilibrium, essentially all drivers departing from location x earn $V(x|\, p, \mathcal{T})$ utility in equilibrium.

We now show that the equilibrium utility $V_{\mathcal{B}}(\cdot|p, \mathcal{T})$ must be 1-Lipschitz continuous. Intuitively, drivers from two different locations x and y that consider relocating to $\mathcal{B}$ see exactly the same potential destinations. Hence, the largest utility drivers departing from x can garner must be greater or equal to that of the drivers departing from y minus the disutility stemming from relocating from x to y, that is, $V_{\mathcal{B}}(x) \geq V_{\mathcal{B}}(y) - \|x - y\|$. Since this argument is symmetric, we deduce the 1-Lipschitz property.

Lemma 1. *(Lipschitz) Consider a measurable set $\mathcal{B} \subseteq \mathcal{C}$ such that $\Gamma(\mathcal{B}) > 0$. Let p be a measurable mapping $p : \mathcal{B} \to \mathbb{R}_+$, and let $\mathcal{T} \in \mathcal{F}(\Theta)$. Then, the function $V_{\mathcal{B}}(\cdot|\, p, \mathcal{T})$ is 1-Lipschitz continuous.*

Reformulating the platform's problem. We now introduce a reformulation of $(\mathcal{P}_1)$ that focuses on the equilibrium utility V and the post-relocation supply $s^{\mathcal{T}}$ as the central elements. The next result plays a key role in our solution approach and is what later motivates the development of the upper bound on $s^{\mathcal{T}}$ and the localization of the analysis to regions of movement (see Figure 3). In what follows, we define $\gamma \triangleq (1 - \alpha)/\alpha$.

Proposition 1 (Problem Reformulation)**.** *The following problem*

$$\sup_{p(\cdot)\,,\, \mathcal{T} \in \mathcal{F}(\Theta)} \quad \gamma \cdot \int_{\mathcal{C}} V(x|\, p, \mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x) \tag{$\mathcal{P}_2$}$$

$$\text{s.t.} \quad \mathcal{T} \text{ is an equilibrium flow,}$$

$$V(x|\, p, \mathcal{T}) = \operatorname*{ess\,sup}_{\mathcal{C}} \Pi\Big(x, \cdot, p(\cdot), s^{\mathcal{T}}(\cdot)\Big), \quad s^{\mathcal{T}} = \frac{d\mathcal{T}_2}{d\Gamma},$$

admits the same value as the platform's optimization problem $(\mathcal{P}_1)$, and a pair $(p, \mathcal{T})$ that solves $(\mathcal{P}_2)$ also solves $(\mathcal{P}_1)$.

The first step in the proof of the proposition above is to rewrite the platform's objective in terms of the post-relocation supply $s^{\mathcal{T}}(x)$ and the pre-movement utility function $U\left(x, p(x), s^{\mathcal{T}}(x)\right)$ (see Eq. (1)). We note that the former step is driven by the commission structure of the contract offered by the platform which aligns in some way the platform's objective with the drivers' utility. This transformation is not particularly useful per se, since the function $U\left(x, p(x), s^{\mathcal{T}}(x)\right)$ is not necessarily well-behaved. The next step consists of establishing that $U\left(x, p(x), s^{\mathcal{T}}(x)\right)$ coincides with $V(x|p, \mathcal{T})$ whenever a location has positive post-movement equilibrium supply (see Lemma A-2 in the Appendix). Indeed, whenever the equilibrium outcome is such that a location has positive supply, the utility generated by staying at that location has to be equal to the best utility one could obtain by traveling to any other location. In turn, one can effectively replace $U\left(x, p(x), s^{\mathcal{T}}(x)\right)$ with $V(x|\, p, \mathcal{T})$ in the objective, which yields the alternative formulation.

The new formulation ($\mathcal{P}_2$) offers a new perspective on the platform's problem. The key objects that drive the platform's revenue are $s^{\mathcal{T}}(\cdot)$ and $V(\cdot|\, p, \mathcal{T})$. Intuitively, for each unit of supply allocated to some x the platform obtains $V(x|p, \mathcal{T})$. Observe that there is also a "budget" constraint that limits how much $s^{\mathcal{T}}(\cdot)$ can be allocated across locations because the total supply in the system is $\Theta(\mathcal{C})$. In turn, if we relaxed the equilibrium constraints in ($\mathcal{P}_2$), the problem would correspond to a *knapsack* problem. In this case we would set $s^{\mathcal{T}}(\cdot)$ as high as possible in the location where $V(\cdot|p, \mathcal{T})$ is the highest. However, doing so would violate the equilibrium constraints as that location would experience high levels of congestion which, as a result, would deter drivers from relocating to it in equilibrium. In order to deal with this, in the next section we develop a congestion bound which controls for the effect on drivers' utilities introduced by congestion.

Connection to Optimal Transport. Our equilibrium concept is closely related to the notion of transport plan in the theory of optimal transport. For example, it is possible to establish that in any equilibrium $\mathcal{T}$, the total mass of drivers repositions in the most efficient way as to minimize the total transportation cost (see, e.g., Blanchet & Carlier (2015) for a related result). In contrast with optimal transport, in our case, the destination measure $\mathcal{T}_2$ is an endogenous object, and one of the central tasks is to find it via optimization by solving ($\mathcal{P}_2$).

5 Congestion Bound

We first introduce some quantities that emerge from a classical capacitated monopoly pricing problem. Let us consider any location $x \in \mathcal{C}$ and ignore all other locations in the city. The problem that a monopolist faces when supply at x is s and demand is $\lambda(x)$ can be cast as

$$R^{loc}_x(s) \triangleq \max_{q \in [0, \overline{V}]} \; q \cdot \min\{s, \lambda(x) \cdot \overline{F}_x(q)\}, \tag{4}$$

with the price $\rho^{loc}_x(s)$ being defined as the argument that maximizes the equation above. Since $q \cdot \overline{F}_x(q)$ is assumed to be unimodular in q, the optimal price $\rho^{loc}_x(s)$ is uniquely determined and is characterized as follows

$$\rho^{loc}_x(s) = \max\{\rho^{bal}_x(s), \rho^u_x\}, \qquad \text{where } s = \lambda(x) \cdot \overline{F}_x(\rho^{bal}_x(s)), \quad \rho^u_x \in \arg\max_{\rho \in [0, \overline{V}]} \{\rho \cdot \overline{F}_x(\rho)\}. \tag{5}$$

That is, the optimal local price either balances supply and demand or maximizes the unconstrained local revenue. For a given local supply s, the maximum revenue that can be generated at location x is $R^{loc}_x(s)$, with a fraction α of that revenue being paid to the drivers. Therefore, $\alpha \cdot R^{loc}_x(s)/s$ is the maximum revenue a driver staying at this location can earn. To capture this notion, we introduce for every location x the supply *congestion* function $\psi_x : \mathbb{R}_+ \to [0, \alpha \cdot \overline{V}]$, which is defined as:

$$\psi_x(s) \triangleq \begin{cases} \alpha \cdot R_x^{loc}(s)/s & \text{if } s > 0; \\ \alpha \cdot \overline{V} & \text{if } s = 0. \end{cases}$$

In line with intuition, more drivers (in a single location problem) imply lower revenues per driver, and it is possible to show that the congestion function ψ_x is decreasing (see Lemma A-3). Crucially, the congestion function ψ_x yields an upper bound for the utility of drivers at almost any location with respect to the city measure.

Proposition 2 (Congestion Bound)**.** *Let $(p, \mathcal{T})$ be a feasible solution of $(\mathcal{P}_2)$. Then, the equilibrium driver utility function is bounded as follows:*

$$V(x|\, p, \mathcal{T}) \;\; \leq \;\; \psi_x\left(s^{\mathcal{T}}(x)\right) \qquad \Gamma - a.e. \;\, x \;\; in \; \mathcal{C}.$$

When there is a single location, the inequality above is an equality by the definition of ψ_x. For multiple locations, drivers may travel to any location and there is no a priori connection between the utility that drivers originating from x can garner, $V(x|\, p, \mathcal{T})$, and $\psi_x(s^{\mathcal{T}}(x))$. The result above establishes that the latter upper bounds the former. The bound captures the structural property that as equilibrium supply increases at a location, and hence driver congestion increases, the drivers originating from that location will earn less utility.

As discussed in the previous section, $(\mathcal{P}_2)$ admits a relaxation (not necessarily tight) that can be mapped to a knapsack problem. Proposition 2 provides a capacity constraint that can be added to such a relaxation. More precisely, since $\psi_x(\cdot)$ is strictly decreasing the congestion bound delivers

$$s^{\mathcal{T}}(x) \leq \psi_x^{-1}(V(x|\, p, \mathcal{T})) \qquad \Gamma - a.e. \;\, x \;\; \text{in } \mathcal{C}. \tag{6}$$

That is, given $V(\cdot|\, p, \mathcal{T})$, for almost every location in the city the amount of post relocation supply has a structural upper bound. Equation (6) can be plugged into the aforementioned relaxation to make it a bounded knapsack problem. Nevertheless, a solution to this new problem might still violate the equilibrium constraints and not be tight. The reason is that the resulting allocation may prescribe movement of drivers from far away locations. To address this, in the next section we localize the analysis to regions where drivers are willing to travel to and study their properties.

6 Localization to Regions of Potential Movement

A key feature of the problem at hand is that, in equilibrium, conditions at different locations are inherently linked as drivers select their destination among all locations. An important object that will help capture the link across various locations is the *indifference region* of a driver departing location x. The indifference region of x represents all the destinations to which drivers from x are willing to travel to. Formally, the indifference region for a driver departing from $x \in \mathcal{C}$ under prices p and flow $\mathcal{T}$ is given by

$$\mathcal{IR}(x|\, p, \mathcal{T}) \triangleq \Big\{ y \in \mathcal{C} : V(y|p, \mathcal{T}) - \|y - x\| = V(x|p, \mathcal{T}) \Big\},$$

Intuitively, the definition above says that if $y \in \mathcal{IR}(x|p, \mathcal{T})$, then drivers departing from x maximize their utility by relocating to y. The converse concept, which will turn out to be fundamental in our analysis is the *attraction region* of a location z. The attraction region of z represents the set of all possible sources for which location z is their best option, as we formally define next.

Definition 2 (Attraction Region)**.** *Let $(p, \mathcal{T})$ be a feasible solution of ($\mathcal{P}_2$). For any $z \in \mathcal{C}$, its* attraction region $A(z|\, p, \mathcal{T})$ *is the set of locations from which drivers are willing to relocate to z:*

$$A(z|\, p, \mathcal{T}) \triangleq \{x \in \mathcal{C} : z \in \mathcal{IR}(x|\, p, \mathcal{T})\}.$$

We call a location $z \in \mathcal{C}$ a sink *if its attraction region $A(z|\, p, \mathcal{T})$ is non-empty and $z \notin A(z'|\, p, \mathcal{T})$ for all $z' \neq z$.*

Note that, within an attraction region, the equilibrium utility of drivers $V(\cdot|\, p, \mathcal{T})$ is fully determined up to its value at the sink $V(z|\, p, \mathcal{T})$. Importantly, sinks and corresponding attraction regions emerge as soon as drivers move in the city, as formalized in the next proposition.

Proposition 3 (Existence of Attraction Regions)**.** *Let $(p, \mathcal{T})$ be a feasible solution of ($\mathcal{P}_2$).*

(i) Any flow in the city gives rise to an associated attraction region, i.e., for any $\mathcal{L} \subseteq \mathcal{C} \times \mathcal{C}$ such that $\mathcal{T}(\mathcal{L}) > 0$, there exists $(x, y) \in \mathcal{L}$ such that $y \in \mathcal{IR}(x|\, p, \mathcal{T})$.

(ii) Moreover, if $y \in \mathcal{IR}(x|\, p, \mathcal{T})$ for some $x \neq y$ then there exists a sink $z \in \mathcal{C}$ such that $x, y \in A(z|\, p, \mathcal{T})$ and x, y, z are collinear points.

Representation and properties of an attraction region. We will anchor the coming discussion around Figure 4, where we depict an illustration of an attraction region (shaded region), as well as various of its structural properties.

In line with the literature on optimal transport, see e.g., Ambrosio & Pratelli (2003), it will be useful in our analysis to study the behavior of drivers along rays around a particular location z. We use R_z to denote the set of all rays originating from z (excluding z) and index the elements of R_z by a. With such a representation, for a feasible solution $(p, \mathcal{T})$, and for any sink $z \in \mathcal{C}$, one can derive various structural properties.

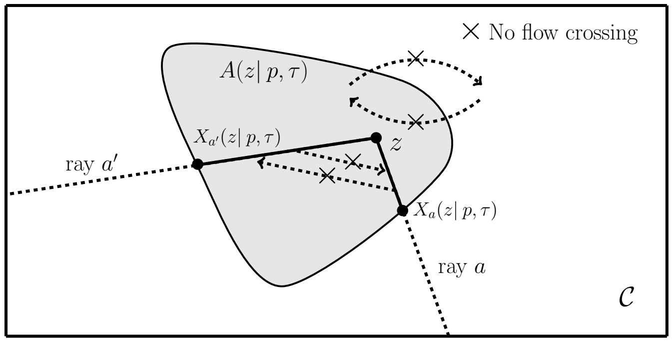


Figure 4: **Attraction region.** Illustration of structural properties of attraction regions. No flow crosses the boundaries of $A(z|\,p,\mathcal{T})$ and there is no flow traveling from one ray to another ray.

- *Property 1 (Closed Attraction Region).* The attraction region $A(z|\,p,\mathcal{T})$ is closed and can be represented as a collection of segments of the form $[z, X_a(z|\,p,\mathcal{T})]$, where $X_a(z|\,p,\mathcal{T})$ is the last point in the attraction region on ray a (see Figure 4). We provide a formal statement in Lemma B-1 in the Appendix.

- *Property 2 (Flow Separation).* An attraction region does not receive external supply and supply units within such a region do not travel travel outside. Furthermore, any movement takes place along the rays originating from the sink z. We provide a formal statement in Proposition B-1 in the Appendix.

- *Property 3 (Pasting).* Informally, this result states the following "pasting" property. Suppose we start from a price-equilibrium pair $(p,\mathcal{T})$ and a sink z and its attraction region $A(z|\,p,\mathcal{T})$. Then, we can replace the price-flow within $A(z|\,p,\mathcal{T})$ with any other local price-equilibrium, say $(\tilde{p},\tilde{\mathcal{T}})$, within that attraction region as long as we do not change $V(x|\,p,\mathcal{T})$ for $x \in A(z|\,p,\mathcal{T})$ that have initial supply. One can obtain a new feasible solution $(\hat{p},\hat{\mathcal{T}})$ in $\mathcal{C}$ by merging the old solution $(p,\mathcal{T})$ in the complement of $A(z|p,\mathcal{T})$ with the modified solution $(\tilde{p},\tilde{\mathcal{T}})$ in the attraction region $A(z|p,\mathcal{T})$. We illustrate this property in Figure 5. We provide a formal statement in Proposition B-2 in the Appendix.

These properties have important implications. Attraction regions lead to a natural decoupling of the platform's problem, as they provide a natural way of segmenting the city. In particular, these regions can be optimized in isolation. The new solution within these regions can then be pasted to the old solution outside these regions to obtain a new global feasible solution to the platform's problem. In the next section we introduce a local relaxation of $(\mathcal{P}_2)$, derive its solution, and prove its tightness. We construct this relaxation by leveraging our reformulation, congestion bound and localization results.

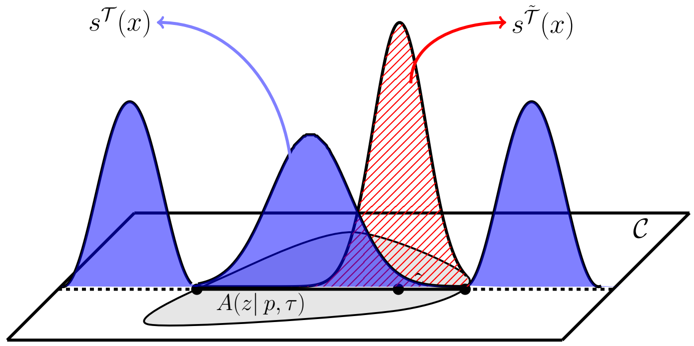


Figure 5: **Illustration of the pasting property (Property 3).** In $A(z|p,\mathcal{T})$ we can modify flows, post relocation supply, and prices in a feasible manner. We show the pre and post modification supply along a line segment. The initial supply corresponds to $s^{\mathcal{T}}(x)$ (solid shaded). The supply after modification is $s^{\tilde{\mathcal{T}}}(x)$ inside $A(z|p,\mathcal{T})$ (striped), and coincides with $s^{\mathcal{T}}(x)$ outside $A(z|p,\mathcal{T})$.

7 Structure of Optimal Solution Within an Attraction Region

In this section, we leverage our previous results to derive a structural characterization of the optimal prices and post-relocation supply of drivers within any attraction region.

Consider an arbitrary feasible solution $(p,\mathcal{T})$ of $(\mathcal{P}_2)$. Let $z \in \mathcal{C}$ be a sink and $A(z|\, p,\mathcal{T})$ its corresponding attraction region. The next result establishes that one can construct a second feasible solution of $(\mathcal{P}_2)$ with (weakly) greater revenue, and in turn uncovers the structure of prices and supply in an optimal solution.

Theorem 1. *(Optimal Supply and Prices within an Attraction Region) Consider a feasible solution $(p,\mathcal{T})$ of $(\mathcal{P}_2)$, and let $z \in \mathcal{C}$ be a sink. Then, there exists another feasible solution $(\hat{p},\hat{\mathcal{T}})$ that weakly revenue dominates $(p,\mathcal{T})$, and is such that its supply $s^{\hat{\mathcal{T}}}$ in $A(z|\, p,\mathcal{T})$ is given by:*

$$s^{\hat{\mathcal{T}}}(x) = \begin{cases} \psi_x^{-1}(V(z|\, p,\mathcal{T}) - \|x-z\|) & \text{if } x \in \bigcup_{a\in R_z}[z, r_a); \\ s_a & \text{if } x = r_a,\ a \in R_z; \\ 0 & \text{otherwise,} \end{cases}$$

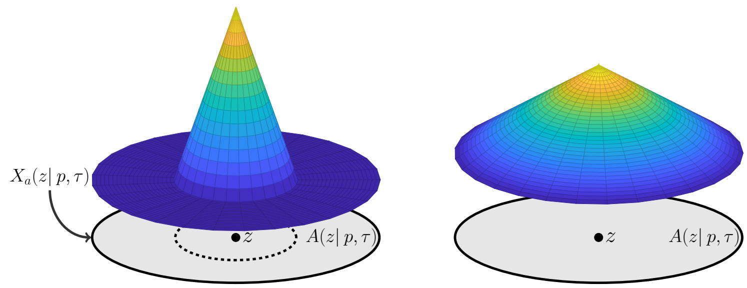
for a set of values $\{r_a\}$ such that $r_a \in [z, X_a(z|\, p,\mathcal{T})]$ and $s_a \geq 0, a \in R_z$. Furthermore,

$$\hat{p}(x) = \begin{cases} \rho_x^{loc}(s^{\hat{\mathcal{T}}}(x)) & \text{if } x \in A(z|\, p,\mathcal{T}) \setminus \bigcup_{a\in R_z}\{r_a\}; \\ p_a & \text{if } x = r_a,\ a \in R_z, \end{cases}$$

where p_a is such that $U(r_a, p_a, s_a) = V(r_a|\, p,\mathcal{T})$ for $a \in R_z$.

The theorem above characterizes the structure of an optimal solution, including both prices and flows, within an attraction region.

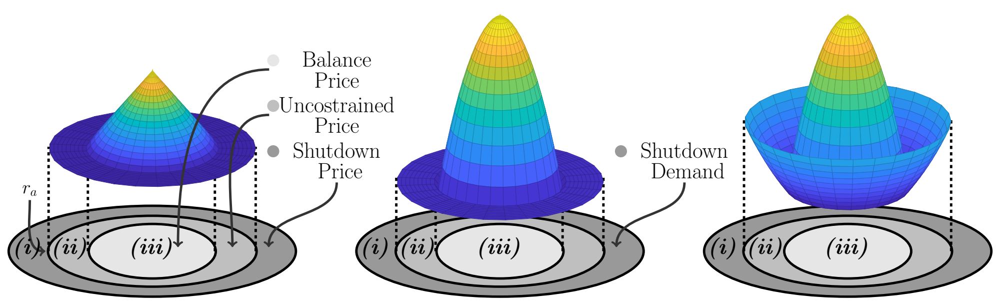
Illustration of Theorem 1. It is useful to consider a prototypical example to illustrate the structure of an optimal solution. In Figure 6, we consider a disc-shaped attraction region with a sink at its center z. We assume the demand density is a cone centered at z that flattens out after a certain distance from z (see Figure 6(a)). Given that the region plotted is an attraction region, the equilibrium utility must be a cone centered at z (see Figure 6(b)).



(a) Potential Demand **(b)** Equilibrium Utility

Figure 6: **Illustration of inputs for Theorem 1.** We consider a disc-shaped attraction region with sink z (light gray region). In **(b)** we plot the potential demand for rides, $\lambda(x)$. It increases towards the sink, and it is positive but constant in the region between the dashed and solid lines. In **(a)** we depict the equilibrium utility of drivers, $V(x|\,p,\tau) = V(z|\,p,\tau) - \|z - x\|$. We consider $\alpha = 0.8, V(z|\,p,\tau) = 0.56$, $X_a(z|\,p,\tau) = 0.35$ for all $a \in R_z$ and $\lambda(x) = \max\{0.9 - 4\|z - x\|, 0.3\}$. The norm is the Euclidean distance.

In Figure 7, we display the structure of an optimal solution within this attraction region. Figure 7 depicts the optimal prices (a), the effective demand (b) and the post-relocation equilibrium supply (c). Three subregions emerge. In the central subregion (subregion *(iii)*), the optimal prices



(a) Optimal Prices **(b)** Effective Demand **(c)** Post-Relocation Supply

Figure 7: **Illustration Theorem 1's solution.** Given $A(z|\,p,\tau)$, the figure depicts the structure of an optimal solution for the conditions associated with Figure 6. There are three subregions that emerge. **(a)** The optimal prices. In *(iii)* optimal prices perfectly match supply and demand, ρ_x^{bal}. In *(ii)*, optimal prices correspond to the unconstrained optimal price, ρ_x^u; in *(i)* prices are such that demand is shutdown. **(b)** The effective demand: $\lambda(x) \cdot \bar{F}_x(p(x))$. **(c)** The post-relocation supply. There is no supply in *(i)*, in *(ii)* there is more supply than effective demand and in *(iii)* supply and demand are matched. We consider $r_a = 0.25$ for all $a \in R_z$ and $F_x(q) = q$ for all x.

correspond to the prices that balance induced supply and demand, that is, $s^{\hat{\mathcal{T}}}(x) = \lambda(x) \cdot (1 - \hat{p}(x))$. In this subregion there is positive supply and, thus, from the equilibrium condition we must have that $\Pi(x, x) = V(x)$. In turn, since the utilization is one, $\alpha \hat{p}(x) = V(x)$ which implies prices grow linearly towards the sink. In this subregion, drivers in locations that are further from the sink earn lower profits as prices decrease. In the middle subregion (subregion *(ii)*), prices all equal the unconstrained optimal price. Here, there is more supply than effective demand. In this subregion, drivers' utility decreases as we move away from the sink due to congestion rather than due to price changes. Finally, in the outer subregion (subregion *(i)*), there is no post-relocation supply. In this subregion, prices are set in such a way so that drivers are better off repositioning towards the sink rather than staying in subregion *(i)*.

Theorem 1 establishes the general structure of an optimal solution within regions of movement. It also showcases the rich behavior that emerges, in the form of a menu of subregions, within an attraction region. In some location supply and demand are perfectly matched, other locations are over-congested, and some less profitable locations are priced out. In the next section, we will exploit Theorem 1 to study a prototypical family of instances, where we will characterize all attraction regions and, as a consequence, the optimal solution across the city in quasi-closed form. Before we do that, we discuss the main ideas that lead to the characterization of the optimal solution within attraction regions in Theorem 1.

Key ideas for Theorem 1. The key idea underlying the proof of the result is based on optimizing the contribution of the attraction region $A(z|\, p, \mathcal{T})$ to the overall objective by reallocating the supply around the sink, and then showing that this reallocation of supply constitutes an equilibrium flow in the original problem. In order to optimize the supply around the sink we consider the following optimization problem which, as explained below, is a relaxation of $(\mathcal{P}_2)$ within $A(z|\, p, \mathcal{T})$:

$$\max_{\tilde{s}(\cdot) \geq 0} \quad \int_{A(z|\, p, \mathcal{T})} V(x|\, p, \mathcal{T}) \cdot \tilde{s}(x)\, d\Gamma(x) \qquad (\mathcal{P}_{KP}(z))$$

$$\text{s.t} \quad \tilde{s}(x) \leq \psi_x^{-1}(V(x)) \qquad \Gamma - a.e.\ x \ \text{ in } \mathcal{C}, \qquad \text{(Congestion Bound)}$$

$$\int_{A(z|\, p, \mathcal{T})} \tilde{s}(x) d\Gamma(x) = \mathcal{T}_c, \qquad \text{(Flow Conservation)}$$

$$\int_{(z, X_a(z|\, p, \mathcal{T})]} \tilde{s}(x) d\Gamma_a(x) \leq \mathcal{T}_a, \qquad \Gamma^{\mathrm{p}} - a.e.\ a \in R_z, \qquad \text{(No Flow Crossing Rays)}$$

where $\mathcal{T}_c$ corresponds to the total flow that $\mathcal{T}$ transports from $A(z|\, p, \mathcal{T})$ to $A(z|\, p, \mathcal{T})$, and $\mathcal{T}_a$ correspond to the total flow in $A(z|p, \mathcal{T})$ that is transported along ray a, excluding z. The measures Γ_a correspond to the contribution of rays indexed by a on the total city measure, and Γ^{p} is a measure over rays that allows to integrate the contribution of the measures Γ_a to Γ (for more details see the proof of Theorem 1). Recall that given the post-relocation supply, $\tilde{s}$, the quantity $\int_{\mathcal{B}} \tilde{s}(x)\, d\Gamma(x)$, represents the post-relocation supply induced by $\tilde{s}$ in $\mathcal{B}$. Thus, the last two constraints in $(\mathcal{P}_{KP}(z))$ stand for consistency of the total post-relocation supply in each one of the relevant subregions of $A(z|\, p, \mathcal{T})$. The key is to observe that this is a relaxation of the original problem in the attraction region. In particular, the equilibrium constraint implies the conservation constraint (see Proposition B-1(i)), and the no-flow-crossing constraints (see Proposition B-1(ii)). The congestion bound is also a consequence of the equilibrium constraint (see Proposition 2). In words, in this formulation, we relax the equilibrium constraint but impose implications of it. We constrain the amount of mass that we can allocate on each direction around z but we fix the total amount of mass in $A(z|\, p, \mathcal{T})$.

In $(\mathcal{P}_{KP}(z))$, we fix the driver utilities and ask what should be the optimal allocation of drivers while satisfying flow balance in the regions $\{[z, X_a]\}_{z \in R_z}$ and imposing the congestion bound. Clearly selecting $\tilde{s} = s^{\mathcal{T}}$ is feasible for the problem above and hence the optimal value upper bounds the value generated by the initial price-equilibrium pair $(p, \mathcal{T})$ in the region $A(z|\, p, \mathcal{T})$. In the proof, we show that this relaxation is tight. Namely, it is possible to construct prices and equilibrium flows achieving the value of Problem $(\mathcal{P}_{KP}(z))$. The proof consists of two main steps: 1) verifying the structure of the optimal $\tilde{s}$ and 2) showing that the post-relocation supply that solves the relaxation can actually be obtained from appropriate prices and flows. For step 1), the main idea relies on recognizing that Problem $(\mathcal{P}_{KP}(z))$ is a measure-theoretical instance of a coupled collection of *Continuous Bounded Knapsack Problems*. The solution to $(\mathcal{P}_{KP}(z))$ is obtained by allocating as much as possible at locations where we can make the most revenue per unit of volume, i.e., we would like to make $\tilde{s}(x)$ as large as possible at locations where $V(x|\, p, \mathcal{T})$ is the largest. For step 2), we explicitly construct prices and flows that yield the same post-relocation supply and objective than the solution to $(\mathcal{P}_{KP}(z))$. The optimal prices correspond to the optimal local prices, $\rho_x^{loc}(s^{\hat{\mathcal{T}}}(x))$. We obtain the optimal flows by constructing a transport plan with two components. The first, transports drivers from all location in $A(z|p, \mathcal{T})$ to z; the second, transports drivers within rays, excluding z. To obtain the latter flows, along each segment $(z, X_a]$, we solve an optimal transport problem with cost function equal to the distance between any two points, initial measure equal to the reminder mass that was not sent to z, and final measure equal to the restriction of the solution of Problem $(\mathcal{P}_{KP}(z))$ in $(z, X_a]$. Finally, we apply the pasting result (Proposition B-2) to obtain a feasible price-equilibrium in the whole city $\mathcal{C}$.

8 A 1D Family of Instances: Global Optimal Solution and Insights

The results derived in the previous sections characterize the structure of an optimal pricing policy and the corresponding supply response *locally in attraction regions* for general demand and supply conditions in 2D. In this section, to further understand the interplay of spatial supply incentives and spatial pricing, we aim to develop insights into the global optimal policy and the interactions across attraction regions. To that end, we focus on a family of instances that will be rich enough to capture spatial supply-demand imbalances while isolating the interplay above. Leveraging previous results for the local form of optimal solutions, but also developing new results to characterize attraction regions, we determine the global optimal solution for this family of instances in quasi-closed form.

In particular, we focus on a one-dimensional city and a family of models that captures a potential local surge in demand. The restriction to a 1D setting is for simplicity and exposition. The results in this section can be viewed as the one-dimensional cut of the solution to a disc-shaped city in which the conditions along diameters are symmetric. More precisely, we specialize the model to the case where the city measure is supported on the interval $\mathcal{C} = [-H, H]$ and is given by

$$\mathit{\Gamma}(\mathcal{B}) = \mathbf{1}_{\{0 \in \mathcal{B}\}} + \int_{\mathcal{B}} dx, \quad \text{for any measurable set } \mathcal{B} \subseteq \mathcal{C},^{2}$$

that is, the origin may admit point masses of supply and demand while the rest of the locations in $\mathcal{C}$ only admit infinitesimal amounts of supply and demand. We fix the city measure throughout, but we parametrize the supply and demand measures. Supply is initially evenly distributed throughout the city, with a density of drivers equal to θ_1 everywhere. Potential demand will also be assumed to have a uniform density on the line interval, except potentially at the origin.

We analyze what happens when a potential demand shock at the origin (the potential high-demand location) materializes and, in particular, we investigate the optimal pricing policy in response to such a shock. We represent the demand shock by a Dirac delta at this location. Therefore, for any measurable set $B \subseteq \mathcal{C}$, the potential demand measure (after the shock) is given by

$$\mathit{\Lambda}(\mathcal{B}) = \lambda_0 \cdot \mathbf{1}_{\{0 \in \mathcal{B}\}} + \int_{\mathcal{B}} \lambda_1 dx,$$

where $\lambda_0 \geq 0$ and $\lambda_1 > 0$. In particular, we refer to the case $\lambda_0 = 0$ as the *pre-demand shock environment* and the case $\lambda_0 > 0$ as the *demand shock environment*. For this family of models, we assume that customer willingness to pay distribution is location-independent and denoted by $F(\cdot)$. This special structure will enable us to elucidate the spatial supply response induced by surge pricing and the structural insights on the optimal policies that emerge.

Throughout this section we will use short-hand notation to present the optimal solution in a streamlined fashion. Let $(p, \mathcal{T})$ be a price equilibrium pair we use $A(0), X_l$ and X_r to denote

[2] Observe that thanks to the generality of our measure-theoretic framework, all the structural results developed thus far apply to this one-dimensional setting.

$A(0|\, p, \mathcal{T})$, and the end points of the left and right rays around z, respectively. Moreover, when clear from context, we write $V(\cdot)$ instead of $V(\cdot|\, p, \mathcal{T})$.

The Pre-demand Shock Environment. We start by analyzing the pre-shock environment. Both demand and supply are uniformly distributed along the city, with respective densities λ_1 and θ_1. In line with intuition, in this highly symmetric setting, one can show that the optimal price policy does not induce any movement of supply, and the optimal price at each location is the same and simply that of a single location capacitated pricing problem. We denote the pre-demand shock optimal price by $\rho_1 = \rho_x^{loc}(\theta_1)$ (not location dependent), and use ψ_1 to denote $\psi_x(\theta_1)$ (also not location dependent). For completeness, this is formalized in Proposition D-1 in the Appendix.

8.1 Benchmark: Myopic Price Response to a Demand Shock

We next start our analysis of the demand shock environment. Before turning our attention to an optimal policy in Section 8.2, we first focus on a simple type of pricing heuristic which responds to changes in demand conditions through changes in prices *only* where these changes occur. In particular, in the context of the demand shock model, this corresponds to responding to a shock in demand at the origin by only adjusting the price at the origin; we call this policy the *myopic price response*. This provides a benchmark to better understand the structure and performance of an optimal policy. We next characterize the optimal myopic price response.

Proposition 4 (Myopic Price Response to a Demand Shock)**.** *Fix $\lambda_0 > 0$. Suppose that $p(x) = \rho_1$ for all x in $\mathcal{C} \setminus \{0\}$ and that the firm optimizes the price $p(0)$. Then,*

(i) (Prices) The optimal price at the origin is given by $p(0) = \rho_0^{loc}(s^{\mathcal{T}}(0))$, and $p(0) \geq \rho_1$.

(ii) (Movement) There exists two thresholds $X_r \geq X_r^0 \geq 0$, such that $X_r > 0$ and:

- *for all x in $[-X_r^0, X_r^0]$, all of the supply units move to the origin,*
- *for all x in $[-X_r, -X_r^0]$ and all x in $[X_r^0, X_r]$, a fraction of the supply units move to the origin and the other fraction does not move,*
- *for all x in $\mathcal{C} \setminus [-X_r, X_r]$, no supply unit moves.*

Furthermore, the platform's revenue is strictly larger than in the pre-demand shock environment.

The result above characterizes the structure of a myopic price response as well as the structure of the supply movement it induces. Figure 8 depicts the structure of the supply response. In particular, the myopic price response leads to a higher price at the origin to respond to the surge of demand at that location. In turn, this higher price attracts drivers from a symmetric region around the origin. In that region, for locations close to the origin, all supply units move to the origin. After

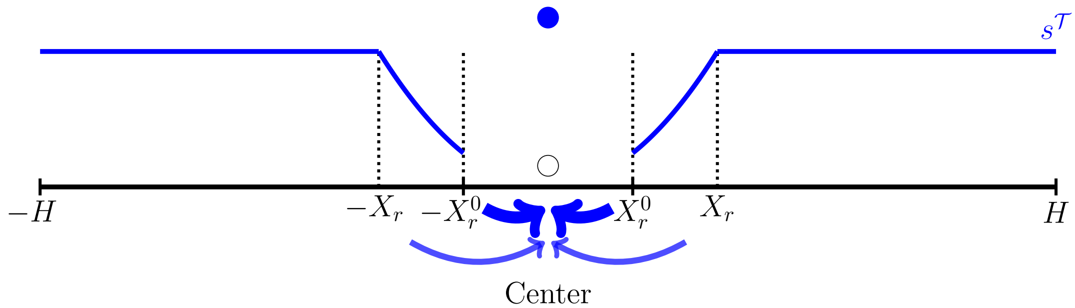


Figure 8: **Myopic price response: induced supply response for a case with $\theta_1 > \lambda_1 \cdot \overline{F}(\rho^u)$.**

a given threshold X_r^0, only a fraction of the drivers will move to the origin. Intuitively, as one gets further from the origin, traveling to it becomes a less attractive option compared to staying put or traveling elsewhere; and an increasingly smaller fraction of units travels to the origin. We also establish that supply units have no incentive to travel anywhere else in the city. As a result, units that do not travel to the origin stay put and serve local demand. Beyond the threshold X_r, no supply units move in the equilibrium induced by the myopic price response. In a supply constrained regime, $\theta_1 \leq \lambda_1 \cdot \overline{F}(\rho^u)$, all drivers within $[-X_r, X_r]$ drive to the origin, i.e., $X_r^0 = X_r$. However, in a supply unconstrained regime, $\theta_1 > \lambda_1 \cdot \overline{F}(\rho^u)$, the two thresholds are different, $X_r^0 < X_r$, as depicted in Figure 8. This occurs because in locations further from the origin but still within $[-X_r, X_r]$, as underutilized drivers drive toward the origin, conditions at the departing point improve and in equilibrium, staying put becomes competitive with driving to the origin.

8.2 Optimal Solution

In this subsection, we focus on the optimal *global* price response across all locations in the city. To that end, we will first develop results to identify the attraction regions in the city and then leverage the results developed for the general model to ultimately obtain a quasi-closed form solution to the platform's problem in this specialized setting. Our first result demonstrates that we can focus on price-equilibrium pairs such that the high demand location is a sink.

Lemma 2 (Origin is a Sink)**.** *Without loss of optimality, one can restrict attention to price-equilibrium pairs $(p, \mathcal{T})$ such that the origin is a sink such that $X_l < 0 < X_r$.*

The intuition behind this proposition harks back to the fact that the performance of the pre-shock environment is dominated by that of the myopic price response solution. Solutions for which the origin is not a sink have revenues capped by that of the pre-demand shock environment. At a high-level, in those solutions, there is no positive mass of drivers willing to travel to the demand shock location and, thus, the city resembles a city without a demand shock. However, the myopic price response solution incentivizes drivers from both sides to travel to the demand shock and has a strictly larger revenue. This implies that at optimality we must have drivers coming from both sides to the origin, that is, $X_l < 0 < X_r$. An important consequence of Lemma 2 is that the attraction region $A(0)$ is a well defined non-empty set. We could thus apply Theorem 1 to obtain a local characterization of the optimal solution within $A(0)$. However, our goal in this section is to obtain the full global optimal solution as opposed to just a solution in $A(0)$. Hence, before we use Theorem 1, in what follows we characterize all attraction regions in $\mathcal{C}$. To make our exposition clear and highlight the solution's spatial aspects, we call the interval $[X_l, X_r]$ the *center* region, and the region outside of it will be referred to as the *periphery*.

8.2.1 Equilibrium Utilities and Attraction Regions

In this subsection we characterize $V(\cdot)$ throughout $\mathcal{C}$. This characterization is key as it will enable us to identify all the attraction regions in $\mathcal{C}$.

Theorem 2. *(Equilibrium utilities) Under an optimal price-equilibrium pair $(p, \mathcal{T})$, the equilibrium utility function $V(\cdot)$ is fully parametrized by the three values $V(0)$ and X_l, X_r as follows:*

$$V(x) = \begin{cases} V(0) - |x| & \text{if } x \in [X_l, X_r], \\ \min\{V(0) - 2X_r + x, \psi_1\} & \text{if } x > X_r, \\ \min\{V(0) - 2|X_l| + |x|, \psi_1\} & \text{if } x < X_l. \end{cases}$$

Moreover, $V(0) > \psi_1$ and $V(X_l), V(X_r) \leq \psi_1$.

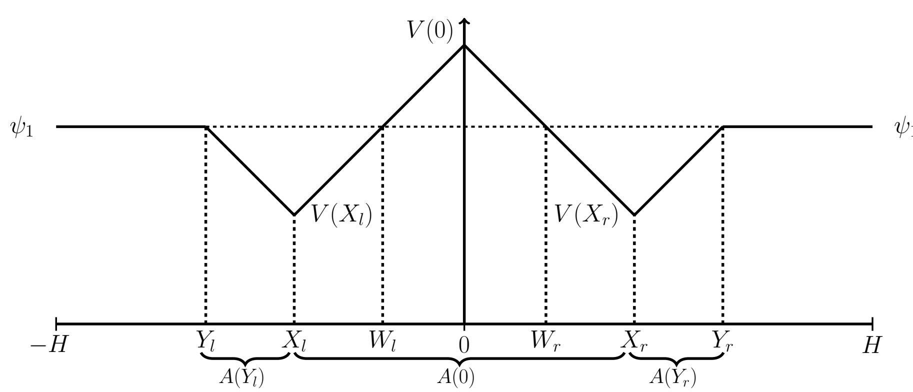


Figure 9: **Drivers' equilibrium utility under an optimal pricing policy.** The equilibrium utility is fully characterized up to $V(0)$, X_l and X_r; the intervals $[Y_l, X_l], [X_r, Y_r]$ and $[X_l, X_r]$ are attraction regions.

The first main implication of this result is that we know, up to $V(0)$, X_l and X_r, how much utility each supply unit garners under optimal prices throughout the entire city. Quite strikingly, the characterization of $V(\cdot)$ is "independent" of the flows. That is, in order to characterize the equilibrium utility we did not need to pin down the distribution of after-movement supply.

The second implication is that the city has at most three types of regions. Figure 9 depicts the equilibrium utility function. The center $[X_l, X_r]$ is by definition an attraction region. Let W_r and Y_r be defined as the points to the left and to the right of X_r where the driver's equilibrium utility function equals the pre-shock utility level ψ_1. To the right of the origin (and similarly to the left), we can observe three main regions. We first have the interval $[0, W_r]$, where drivers' utilities are above the pre-shock utility level. Drivers in this region are positively impacted by the shock of demand at the origin (and the global optimal prices). The second region $[W_r, Y_r]$ is notable. Here, drivers garner strictly less utility compared to the pre-shock environment. In $[W_r, X_r]$ drivers are "too far" from the origin so their utilities are negatively affected by the cost of driving to the origin. Drivers in $[X_r, Y_r]$ are outside the origin's attraction region and, thus, do not relocate to the origin. This interval forms an attraction region with sink Y_r, that is, Y_r belongs to the indifference region of any location in the interval and Y_r does not belong to the indifference region of any other location. In turn, besides $A(0)$ there are two other attraction regions, $A(Y_l)$ and $A(Y_r)$, in $\mathcal{C}$. Interestingly, drivers in $[X_r, Y_r]$ suffer because the platform has to make sure that drivers in $[0, X_r]$ stay within the attraction region of the origin. For the marginal drivers at X_r to be willing to travel to the origin, the conditions to the right of X_r should not be too attractive. The final region corresponds to $[Y_r, H]$; this region is not affected by the shock of demand as it is far from the origin.

Key ideas for the proof of Theorem 2. The proof of the result relies on leveraging structural properties of the equilibrium utility function, the congestion bound, and a novel flow-mimicking technique. At a high level, we focus on each region separately, center and periphery, and solve for $V(\cdot)$ in each of these regions.

We start by considering the center region, which is easy to analyze. Lemma 2 establishes that we can focus on solutions such that $A(0) = [X_l, X_r]$ is a non-empty interval that strictly contains the origin. Then by definition of $A(0)$, $V(x) = V(0) - |x|$, for all $x \in [X_l, X_r]$. Importantly, the characterization of $V(\cdot)$ in this region only depends on three parameters, namely, $V(0)$, X_l and X_r.

Switching attention to the periphery, consider the right periphery $(X_r, H]$ (the treatment for the left periphery is analogous). We first argue that, in this region, the drivers' equilibrium utility has a non-trivial upper bound

$$V(x) \leq \min\{V(X_r) + x - X_r, \psi_1\}, \qquad \text{for all } x \in (X_r, H]. \tag{7}$$

The upper bound above follows from two bounds. A first upper bound can be derived using the 1-Lipschitz property of V (Lemma 1), which limits the growth rate of V. Thus, $V(x)$ is bounded by

$V(X_r)+x-X_r$. A second bound may be obtained by leveraging the congestion bound (Proposition 2). One may show that drivers from almost any location that do not have an incentive to travel to the origin have their utilities capped by the pre-demand shock utility level ψ_1.

The core of the argument toward characterizing the equilibrium utilities in the periphery resides in establishing that the upper bound in Eq. (7) is always binding. We show this in two steps. We first establish that the value function has to be non-decreasing in $[X_r, H]$ (see Proposition D-2 in the Appendix), this implies that drivers only move right (or do not move) in the right peripheral region. Then, exploiting the monotonicity, we use a flow mimicking argument to establish that the upper bound is achieved under an optimal pricing policy (Proposition D-3 in the Appendix).

8.2.2 From Equilibrium Utilities to Supply Distribution and Optimal Prices

Given that we pinned down the equilibrium utility function across the city and all attraction regions, we next solve for prices and supply through the problem reformulation in Proposition 1. Leveraging Theorem 1 and a symmetry argument, one can solve for the optimal $s^{\mathcal{T}}$ and the corresponding prices in each attraction region. The solution for the no-movement regions reduces to the pre-shock environment. Then we use the pasting property (cf. Property 3 in Section 6) to paste the solution from each region and, in turn, obtain a quasi-closed form characterization of the optimal solution to the platform's problem as presented in Theorem 3.

Theorem 3. *(Optimal Prices and Flows) An optimal price-equilibrium pair $(p, \mathcal{T})$ is such that $V(\cdot)$ is as in Theorem 2, $X_r = -X_l$, and prices and flows are characterized as follows.*

1. *(Prices) The optimal prices are given by $p(x) = \rho_x^{loc}(s^{\mathcal{T}}(x))$, where $s^{\mathcal{T}}(x)$ is as below.*

2. *(Post-relocation Supply) There exists unique $\beta_c \in [0, W_r]$ and $\beta_p \in [X_r, Y_r]$ such that*

$$\int_{-\beta_c}^{\beta_c} \psi_x^{-1}(V(x))d\Gamma(x) = \theta_1 \cdot 2 \cdot X_r \quad \text{and} \quad \int_{\beta_p}^{Y_r} \psi_x^{-1}(V(x))d\Gamma(x) = \theta_1 \cdot (Y_r - X_r),$$

   and the optimal post-relocation supply is given by

$$s^{\mathcal{T}}(x) = \begin{cases} 0 & \text{if } x \in (\beta_c, \beta_p) \cup (-\beta_p, \beta_c), \\ \psi_x^{-1}(V(x)) & \text{otherwise.} \end{cases}$$

3. *(Movement)*

   - *for all x in $[-\beta_c, \beta_c]$, drivers move in the direction of the origin,*
   - *for all x in $[-X_r, -\beta_c) \cup (\beta_c, X_r]$, all drivers move to $[-\beta_c, \beta_c]$,*
   - *for all x in $[X_r, \beta_p)$ (resp. $(-\beta_p, -X_r]$), all drivers move to $[\beta_p, Y_r]$ (resp. $[-Y_r, -\beta_p]$),*

- *for all x in $[\beta_p, Y_r]$ (resp. $[-Y_r, -\beta_p]$), drivers move in the direction of Y_r (resp. $-Y_r$),*
- *for all x in $[-H, -Y_r) \cup (Y_r, H]$, drivers do not relocate.*

In other words, we have fully characterized the optimal solution across the city and it is fully parametrized by only on two values: $V(0)$ and X_r.

Discussion. We depict in Figure 10 the structure of the solution obtained in Theorem 3. The

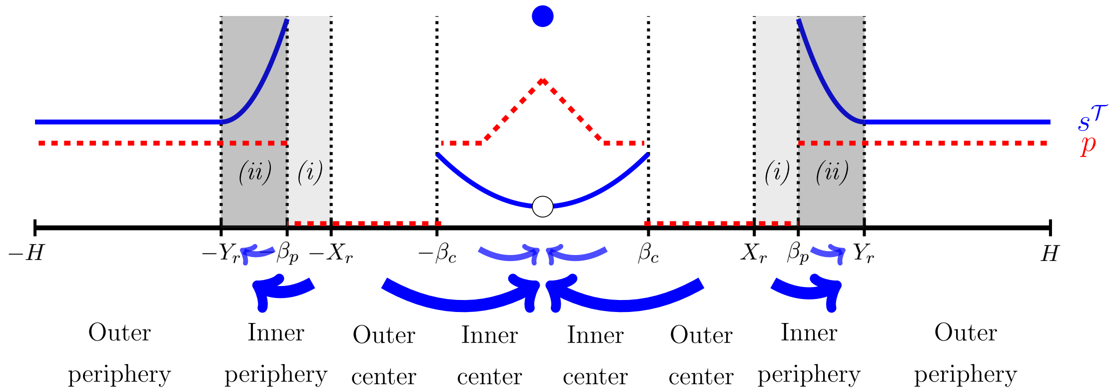


Figure 10: **Supply response (solid-blue line) induced by optimal prices (dashed-red line).**

main feature of the optimal solution is that it separates each side of the city, with respect to the origin, into multiple regions. For clarity, we focus our discussion on the right side of the city.

The origin receives a mass of supply equal to $\psi_0^{-1}(V(0))$. This mass of drivers comes from two regions, the inner and the outer center, which we now define. The first corresponds to the interval $(0, \beta_c]$. Some drivers in this region choose to stay put while others, attracted by the favorable conditions at the center of the city, choose to drive to the origin. In equilibrium, drivers staying or traveling to the origin garner the same utility. The outer center is the interval $(\beta_c, X_r]$. Here, the platform sets prices to $\overline{V}$ (or 0) and therefore supply is equal to zero. That is, the platform chooses prices to shut down demand, giving no incentive for drivers to stay there (or alternatively sets prices at zero to again give no incentive for drivers to stay there). In turn, this incentivizes all drivers in this region to move somewhere else. In order to incentivize these drivers to move towards the origin, the platform creates one more region: the inner periphery.

The inner periphery corresponds to the interval $(X_r, Y_r]$. The platforms "artificially" degrades the conditions for drivers in this interval in two different ways, leading to the two sub regions, *(i)* and *(ii)* in Figure 10. In region (i), the platform sets prices equal to $\overline{V}$ (or 0) in $(X_r, \beta_p]$, shutting down demand, so no drivers want to either travel to or stay in this region. As a result the interval $(\beta_p, Y_r]$ receives all drivers from $(X_r, \beta_p]$. This creates driver congestion and, thus, endogenously worsens driver conditions in the interval $(\beta_p, Y_r]$. The reason the platform selects these inner periphery prices is to discourage drivers in the outer center from driving towards the periphery. Quite strikingly, the optimal global price response to a demand shock at the origin induces supply movement *away* from the origin in the inner periphery. The final region is the outer periphery. All drivers in this region stay put, leading to $s^{\mathcal{T}}(x) = \theta_1$. Here, drivers collect the same utility they would make if there was no demand shock at the origin.

In sum, the optimal global price response to a demand shock, while correcting the supply-demand imbalance at the origin, also creates significant imbalances across the city. This is driven by the self-interested nature of capacity units and the need to incentivize them through spatial pricing. See Proposition 4 for how the optimal policy differs from the myopic best response.

8.3 Myopic price response versus the optimal solution

In this section, we will use the myopic price response solution as a benchmark for comparison to put the optimal solution in perspective. The objective is to illustrate through several metrics the different features of the optimal solution as well as its performance in terms of revenue maximization (we complement the comparison, including welfare performance, in Appendix E). Throughout this section, we use superscripts `my` and `opt` to label relevant quantities associated with the myopic price response and optimal solution, respectively (except when obvious from context).

We first observe that the attraction region around the origin of the demand shock location is always wider under the optimal solution than under the local best response. That is, $A^{\texttt{my}}(0) \subset A^{\texttt{opt}}(0)$. In particular, this means that more locations are affected by a demand shock in the optimal solution than under the myopic price response. Hence, the largest interval in which both solutions differ corresponds to $[-Y_r^{\texttt{opt}}, Y_r^{\texttt{opt}}]$. We denote this interval by $\mathcal{C}_{\text{diff}}$.

Next, we illustrate and discuss through a set of numerics the differences between the two policies. In order to obtain numerical solutions for the global optimal policy, we rely on Theorem 2 and Theorem 3. From Theorem 2 we know that $V(x)$ is characterize by three values: $V(0), X_r, X_l$. In Theorem 3, given $V(\cdot)$, we provide a full characterization of the optimal solution. Also, we establish that $X_l = -X_r$. In turn, to find the optimal solution we perform a grid search over $[0, H] \times [0, \overline{V}]$. For the myopic price response we proceed in a similar fashion by making use of the closed form expressions developed in the proof of Proposition 4. We consider a range of instances that includes various levels of supply availability. We fix the city to be characterized by $H = 1$ and assume that the demand is uniformly distributed across locations with $\lambda_1 = 4$. The origin experiences a shock of demand ranging from low to high: $\lambda_0 \in \{3, 6, 9\}$. We vary the initial supply $\theta_1 \in \{1, 1.5, 2, \ldots, 4.5, 5\}$ so that when low, the city (excluding the origin) is supply constrained, and when high, the city is supply unconstrained. Consumer valuation is uniformly distributed in the unit interval. Note that the city (excluding the origin) is supply constrained whenever $\theta_1 < \lambda_1 \cdot \overline{F}(p^u) = 2$. To eliminate any strong dependence on the choice of H, for each instance, we compare the myopic price response performance and optimal solution performances within the sub-region of the city corresponding to the largest interval in which both solutions differ, $\mathcal{C}_{\text{diff}}$.

Revenue Improvement. The revenue performance of the optimal solution with respect to our benchmark in $\mathcal{C}_{\text{diff}}$ is shown in Table 1. For any level of demand shock, we observe that the revenue

θ_1	1	1.5	2	2.5	3	3.5	4	4.5	5
$\lambda_0 = 3$	2.05	4.64	9.59	13.02	13.87	12.92	11.00	8.60	5.91
$\lambda_0 = 6$	2.17	3.11	4.99	8.73	9.96	10.01	9.56	8.92	8.21
$\lambda_0 = 9$	2.69	3.51	4.69	8.75	10.16	10.30	9.81	9.10	8.29

Table 1: Revenue improvement (in %) of optimal solution over myopic price response in $\mathcal{C}_{\text{diff}}$.

improvement reaches its maximum value for medium to high levels of supply, and can be significant, above 10%. In order to appreciate where the revenue gains stem from, consider Table 2 below, which summarizes some key quantities for the case $\theta_1 = 3$, $\lambda_0 = 9$ (so that ψ_1 equals 0.27). Let us analyze

$V^{\texttt{opt}}(0)$	$s^{\texttt{opt}}(0)$	$p^{\texttt{opt}}(0)$	$X_r^{\texttt{opt}}$	$Y_r^{\texttt{opt}}$	$V^{\texttt{my}}(0)$	$s^{\texttt{my}}(0)$	$p^{\texttt{my}}(0)$	$X_r^{\texttt{my}}$	$X_r^{0,\texttt{my}}$
0.62	1.97	0.78	0.46	0.57	0.65	1.66	0.81	0.38	0.25

Table 2: Metrics for the local response and optimal solution for the case $\theta_1 = 3$, $\lambda_0 = 9$.

the various contributions to revenues under both policies. We start by noticing that the drivers' equilibrium utility at the shock location is lower under the optimal solution than under the myopic price response, $V^{\texttt{opt}}(0) = 0.62$ and $V^{\texttt{my}}(0) = 0.65$. However, since $X_r^{\texttt{opt}} = 0.46$ and $X_r^{\texttt{my}} = 0.38$, the optimal solution is able to incentivize the movement of a larger mass of drivers towards the demand shock, leading to a mass $s^{\texttt{opt}}(0) = 1.97$ versus $s^{\texttt{my}}(0) = 1.66$. Focusing on the objective reformulation in Proposition 1, this extra mass of drivers delivers 0.14 units $(0.62 \times 1.97 - 0.65 \times 1.66)$ of extra revenue to the platform. The revenue difference is further increased by the fact that the remainder 0.79 units of drivers in the attraction region of zero $(2 \times 3 \times 0.46 - 1.97)$ in the optimal solution travel to locations nearby the demand shock, where $V(\cdot)$ is close to 0.62. In contrast, the benchmark solution has the remainder 0.62 drivers $(2 \times 3 \times 0.38 - 1.66)$ staying within $[X_r^{0,\texttt{my}}, X_r^{\texttt{my}}]$ where $V(\cdot)$ is below 0.37 $(V^{\texttt{my}}(0) - X_r^{0,\texttt{my}})$. Through these two mechanisms, the optimal policy garners more revenue than the benchmark solution in the region $[-X_r^{\texttt{opt}}, X_r^{\texttt{opt}}]$.

However, the benefits come at a cost. To induce the "right" incentives in the shock's attraction region, the platform has to alter conditions to the right of the attraction region. In order to incentivize the movement of drivers in $[-X_r^{\texttt{opt}}, X_r^{\texttt{opt}}]$ towards the demand shock, the region $[X_r^{\texttt{opt}}, Y_r^{\texttt{opt}}]$ is damaged by having the 0.22 units of drivers in it $(2 \times (0.57 - 0.46))$ contributing values strictly below $\psi_1 = 0.27$ to the platform's objective. The same units of drivers in the benchmark solution contribute exactly 0.27 per unit to the platform's revenue. This cost is offset by the proceeds that incentivizing the movement of a larger amount of drivers toward the demand shock generates.

9 Conclusion

The present study analyzes the short-term pricing problem faced by a ride-hailing platform. Given supply and demand conditions across a two-dimensional region, the platform sets prices at every location and supply units select where to reposition in equilibrium.

To analyze this problem we employ a measure-theoretic framework that subsumes both discrete and continuous settings. The resulting problem is a mathematical program with equilibrium constraints for which no standard solution approach is readily available. We provide two main contributions. First we establish a characterization of the optimal solution, prices and flows, within regions of potential movement (attraction regions). Our approach consists of relaxing some of the equilibrium constraints and identifying that our relaxation, localized to attraction regions, is tight and leads to coupled continuous bounded knapsack problems, which we solve to optimality.

Our second contribution is in terms of managerial insights. The general idea of our result is that both positive and negative incentives can be employed to induce the right movement of drivers. We can raise prices in profitable under-supplied regions, and we can damage (using prices) less profitable regions thus incentivizing the relocation of drivers towards regions that are more beneficial for the platform. These insights are illustrated through the development of a quasi-closed form solution for a family of instances.

There are several potential directions for future work. Our results may be used to study possible heuristics that approximate the global optimal solution in arbitrary instances. A direction would be to develop two-stage heuristics that first sets candidate locations and shapes for attraction regions, and then leverage the results developed to optimize within those. One approach is to first parametrize their shape, for example using circles or hexagons, and then do a search for regions of large supply-demand imbalances in order to identify sink locations. Each region then would be parametrized by the shape of its border and the utility at the sink location. We can then solve for the optimal solution within each of them and paste to obtain a candidate solution. A master global optimization would follow to tune the shape and the utilities at the sink locations across the city.

That is, the general methodology we developed may be leveraged to compute parametric global solutions to the platform's problem. Another important extension is the incorporation of dynamics. The model studied in the present paper can be regarded as a two-stage model. In the first stage drivers are initially positioned. In the second stage they reposition in equilibrium given prices and demand conditions. This does not consider that drivers that are closer to a given location might be more likely to be matched to riders in such location (they can get to that location faster than other drivers). It also does not consider the continuation game that emerges after drivers are matched. Studying these different settings and their variations are interesting avenues of future inquiry.

Finally, the framework and ideas developed here could be leveraged in settings beyond ride-hailing. Given the generality of our framework, all results before Section 8 can be extended to higher-dimensional settings. One could use our framework to study problems in which agents have different types characterized by high-dimensional vectors. Agents can modify their types by exerting some costly effort, but their potential earning when doing so will depend on how many other agents end up being of the same type.

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Online Appendix for:
Surge Pricing and its Spatial Supply Response

This appendix is divided into five sub-appendices, Appendix A covers the proofs in Section 4 and Section 5; Appendix B covers the proofs in Section 6; Appendix C covers the proofs in Section 7; Appendix D covers the proofs in Section 8; Appendix E provides additional numerical results for section 8.3.

A Proofs for Section 4 and Section 5

Proof of Lemma 1. Consider any $z, y \in \mathcal{C}$. Then, for essentially any $w \in \mathcal{B}$, we have

$$V_{\mathcal{B}}(y) \geq U(w) - \|w-y\| \; = \; U(w) - \|z-w\| + \|z-w\| - \|w-y\| \; \geq \; U(w) - \|z-w\| - \|z-y\|,$$

where the second inequality follows from the triangular inequality. This implies, by the definition of the essential supremum, that

$$V_{\mathcal{B}}(y) + \|z-y\| \geq V_{\mathcal{B}}(z).$$

Next, we would like to subtract $V_{\mathcal{B}}(y)$ from both sides of the previous inequality. This operation can be done only if $V_{\mathcal{B}}(y)$ is finite for any y in $\mathcal{C}$, but this is guaranteed by Lemma A-1 (stated and proved right after this proof). Hence, we obtain $V_{\mathcal{B}}(z) - V_{\mathcal{B}}(y) \leq \|z-y\|$. Since we can interchange the roles of z and y, we have proved that $|V_{\mathcal{B}}(z) - V_{\mathcal{B}}(y)| \leq \|z-y\|$, for all $z, y \in \mathcal{C}$. □

Lemma A-1. *Consider a measurable set $\mathcal{B} \subseteq \mathcal{C}$ such that $\Gamma(\mathcal{B}) > 0$, let p be a measurable mapping $p : \mathcal{B} \to \mathbb{R}_+$, and let $\mathcal{T} \in \mathcal{F}(\Theta)$. Then, $V_{\mathcal{B}}(x|\, p, \mathcal{T}) \in \left[-H, \alpha \cdot \overline{V}\right]$ for all $x \in \mathcal{C}$, where $H = \max_{x,y \in \mathcal{C}} \|x-y\|$. Furthermore, $V(x|\, p, \mathcal{T}) \geq 0$ for all $x \in \operatorname{supp}(\Gamma)$.*

Proof. Fix $x \in \mathcal{C}$, we show that $V_{\mathcal{B}}(x|\, p, \mathcal{T}) \in \left[-H, \alpha \cdot \overline{V}\right]$. For the lower bound, note that for any $y \in \mathcal{B}$, we have $U(y) - \|y-x\| \geq -H$. Since $\Gamma(\mathcal{B}) > 0$, the definition of essential supremum implies that $V_{\mathcal{B}}(x|\, p, \mathcal{T}) \geq -H$. Similarly, for the upper bound, note that for any $y \in \mathcal{B}$, $\alpha \cdot \overline{V} \geq U(y) - \|y-x\|$ and hence the definition of essential supremum yields $V_{\mathcal{B}}(x|\, p, \mathcal{T}) \leq \alpha \cdot \overline{V}$.

Finally, we show that $V(x|p, \mathcal{T}) \geq 0$ for all $x \in \operatorname{supp}(\Gamma)$. Since $x \in \operatorname{supp}(\Gamma)$ we have that $\Gamma(B(x,\delta)) > 0$ for all $\delta > 0$, where $B(x,\delta)$ is an open ball of radius δ. For any $y \in B(x,\delta)$ we have $U(y) - \|y-x\| > -\delta$, and since $\Gamma(B(x,\delta)) > 0$ we deduce that $V_{B(x,\delta)}(x|\, p, \mathcal{T}) > -\delta$ for all $\delta > 0$. In turn, we have $V(x|\, p, \mathcal{T}) \geq V_{B(x,\delta)}(x|\, p, \mathcal{T}) > -\delta$ for all $\delta > 0$ and, therefore, $V(x|\, p, \mathcal{T}) \geq 0$. □

Proof of Proposition 1. We show how to reformulate the platform's objective as in the statement of the proposition. The key step is to establish that

$$U\left(x, p(x), s^{\mathcal{T}}(x)\right) \;=\; V(x|\, p, \mathcal{T}) \qquad \mathcal{T}_2 - a.e.\ x \in \mathcal{C}, \tag{A-1}$$

namely, whenever there is post-relocation supply at a given location in equilibrium, the drivers originating at such a location can achieve maximum utility by staying at that location. We state and prove this result in Lemma A-2 (stated and proved following this proof). Note that this result holds $\mathcal{T}_2 - a.e$ so if we want to interchange $U\left(x, p(x), s^{\mathcal{T}}(x)\right)$ with $V(x|\, p, \mathcal{T})$ we have to do it under the measure $\mathcal{T}_2$. We next analyze the main term in the platform's objective function.

$$\int_{\mathcal{C}} p(y) \cdot \min\left\{s^{\mathcal{T}}(y), \overline{F}_y(p(y))\lambda(y)\right\} d\Gamma(y) \stackrel{(a)}{=} \int_{\mathcal{C}} p(y) \cdot \min\left\{s^{\mathcal{T}}(y), \overline{F}_y(p(y))\lambda(y)\right\} \mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}} d\Gamma(y)$$
$$= \frac{1}{\alpha} \int_{\mathcal{C}} \alpha\, p(y) \cdot \min\left\{1, \frac{\overline{F}_y(p(y))\lambda(y)}{s^{\mathcal{T}}(y)}\right\} \mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}} s^{\mathcal{T}}(y) d\Gamma(y)$$
$$= \frac{1}{\alpha} \int_{\mathcal{C}} U(y, p(y), s^{\mathcal{T}}(y)) \mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}} s^{\mathcal{T}}(y) d\Gamma(y)$$
$$\stackrel{(b)}{=} \frac{1}{\alpha} \int_{\mathcal{C}} U(y, p(y), s^{\mathcal{T}}(y)) \mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}} d\mathcal{T}_2(y)$$
$$\stackrel{(c)}{=} \frac{1}{\alpha} \int_{\mathcal{C}} V(y) \mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}} d\mathcal{T}_2(y),$$

where (a) holds because whenever $s^{\mathcal{T}}(y) = 0$, the minimum term in the integral becomes zero; (b) follows from the fact that $U(y, p(y), s^{\mathcal{T}}(y))\mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}}$ is a measurable function with values in $[0, \alpha \cdot \overline{V}]$ and from recalling that $s^{\mathcal{T}} = d\mathcal{T}_2/d\Gamma$; and (c) is a consequence of Eq. (A-1) since we are integrating over the measure $\mathcal{T}_2$. In turn, focusing on the platform's objective function, this yields

$$(1-\alpha) \int_{\mathcal{C}} p(y) \cdot \min\left\{s^{\mathcal{T}}(y), \overline{F}_y(p(y))\lambda(y)\right\} d\Gamma(y) = \gamma \int_{\mathcal{C}} V(y) \mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}} d\mathcal{T}_2(y)$$
$$\stackrel{(a)}{=} \gamma \int_{\mathcal{C}} V(y) \mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}} s^{\mathcal{T}}(y) d\Gamma(y)$$
$$= \gamma \int_{\mathcal{C}} V(y) s^{\mathcal{T}}(y) d\Gamma(y),$$

where (a) holds because $V(y)\mathbf{1}_{\{s^{\mathcal{T}}(y)>0\}}$ is measurable with values in $[0, \alpha \cdot \overline{V}]$ and we recall again that $s^{\mathcal{T}} = d\mathcal{T}_2/d\Gamma$. This completes the proof. □

Lemma A-2 (Equilibrium Utilities)**.** *For any price mapping p and corresponding equilibrium $\mathcal{T}$, let $\mathcal{B} \subseteq \mathcal{C}$ such that $\Gamma(\mathcal{B}) > 0$, then*

$$U\left(y, p(y), s^{\mathcal{T}}(y)\right) \;=\; V_{\mathcal{B}}(y|\, p, \mathcal{T}) = V(y|\, p, \mathcal{T}) \qquad \mathcal{T}_2 - a.e.\ y \in \mathcal{B}.$$

Furthermore,

$$U\left(y, p(y), s^{\mathcal{T}}(y)\right) \;\leq\; V_{\mathcal{B}}(y|\, p, \mathcal{T}) \qquad \Gamma - a.e.\ y \in \mathcal{B}.$$

Proof. We prove that

$$U\left(y, p(y), s^{\mathcal{T}}(y)\right) \;=\; V_{\mathcal{B}}(y|\, p, \mathcal{T}) \qquad \mathcal{T}_2 - a.e.\ y \in \mathcal{B}.$$

The proof for $V(y|\, p, \mathcal{T})$ instead of $V_{\mathcal{B}}(y|\, p, \mathcal{T})$ follows the same steps and is, thus, omitted. Let $A \subseteq \mathcal{B}$ be a set defined by

$$A \triangleq \{y \in \mathcal{B} : U(y) = V_{\mathcal{B}}(y)\}. \tag{A-2}$$

We want to prove $\mathcal{T}_2(A^c) = 0$, where the complement is taken with respect to $\mathcal{B}$. Consider the sets

$$A^- \triangleq \{y \in \mathcal{B} : U(y) < V_{\mathcal{B}}(y)\}, \quad A^+ \triangleq \{y \in \mathcal{B} : U(y) > V_{\mathcal{B}}(y)\}.$$

We will establish that $\mathcal{T}_2(A^-) = 0$ and $\mathcal{T}_2(A^+) = 0$. We begin with A^- and note that

$$\begin{aligned}\mathcal{T}_2(A^-) &= \mathcal{T}(\mathcal{C} \times A^-) \\ &\stackrel{(a)}{=} \mathcal{T}(\{(x,y) \in \mathcal{C} \times A^- : U(y) - \|y - x\| = V(x)) \\ &\stackrel{(b)}{\leq} \mathcal{T}(\{(x,y) \in \mathcal{C} \times A^- : U(y) \geq V(y)) \\ &\stackrel{(c)}{\leq} \mathcal{T}(\{(x,y) \in \mathcal{C} \times A^- : U(y) \geq V_{\mathcal{B}}(y)) \\ &\stackrel{(d)}{\leq} \mathcal{T}(\{(x,y) \in \mathcal{C} \times \mathcal{B} : V_{\mathcal{B}}(y) > U(y) \geq V_{\mathcal{B}}(y)) \\ &= 0,\end{aligned}$$

where (a) follows from the equilibrium definition, and (b) from the fact that $V(x) + \|x - y\| \geq V(y)$ (see Lemma 1). In (c) we have used $V(y) \geq V_{\mathcal{B}}(y)$, while (d) follows from $y \in A^-$ and $A^- \subseteq \mathcal{B}$.

To show that $\mathcal{T}_2(A^+) = 0$, it suffices to show that show $\Gamma(A^+) = 0$ (this will also show the last statement of the lemma). For any $n \in \mathbb{N}$ define the set $A_n^+ \triangleq \{y \in \mathcal{B} : U(y) \geq V_{\mathcal{B}}(y) + \frac{1}{n}\}$, and note that $A^+ = \bigcup_{n\in\mathbb{N}} A_n^+$. It is enough to show that $\Gamma(A_n^+) = 0$ for all $n \in \mathbb{N}$. We proceed by contradiction. Suppose there exists $n \in \mathbb{N}$ such that $\Gamma(A_n^+) > 0$. Let $\epsilon > 0$ be such that $\epsilon < \frac{1}{2n}$, and consider a finite partition $\{I_i^\epsilon\}_{i=1}^{K(\epsilon)}$ of $\mathcal{C}$, where for any $x, y \in I_i^\epsilon$ we have $\|x - y\| \leq \epsilon$. Observe that

$$0 < \Gamma(A_n^+) = \Gamma(A_n^+ \cap \bigcup_{i=1}^{K(\epsilon)} I_i^\epsilon) = \sum_{i=1}^{K(\epsilon)} \Gamma(A_n^+ \cap I_i^\epsilon),$$

therefore, there exists $i \in \{1, \ldots, K(\epsilon)\}$ such that $\Gamma(A_n^+ \cap I_i^\epsilon) > 0$. Take $x \in I_i^\epsilon$, then for any $y \in A_n^+ \cap I_i^\epsilon$

$$U(y) \geq V_{\mathcal{B}}(y) + \frac{1}{n} \geq V_{\mathcal{B}}(x) - \|y - x\| + \frac{1}{n} > V_{\mathcal{B}}(x) - \|y - x\| + 2\epsilon \geq V_{\mathcal{B}}(x) + \|y - x\|,$$

where the second inequality comes from the Lipschitz property (see Lemma 1). The last two inequalities hold because of our choice of ϵ and $x, y \in I_i^\epsilon$. We conclude that

$$A_n^+ \cap I_i^\epsilon \subseteq \{y \in \mathcal{B} : \Pi(x,y) > V_{\mathcal{B}}(x)\}.$$

This would therefore imply that $\Gamma(\{y \in \mathcal{B} : \Pi(x,y) > V_{\mathcal{B}}(x)\}) > 0$. However, this contradicts the definition of $V_{\mathcal{B}}(x)$. Hence we must have $\Gamma(A_n^+) = 0$ for all $n \in \mathbb{N}$, and in turn $\Gamma(A^+) = 0$. □

Lemma A-3. *The congestion function $\psi_x(\cdot)$ is a strictly decreasing function $\Gamma - a.e.\ x$ in $\mathcal{C}$.*

Proof of Lemma A-3. Recall that $\lambda(x) > 0$ $\Gamma - a.e.\ x$ in $\mathcal{C}$ and that the price achieving the maximum in the definition of $R_x^{loc}(s)$ is $\rho_x^{loc}(s) = \max\{\rho_x^{bal}(s), \rho_x^u\}$. Let s^u be equal to $\lambda(x) \cdot \overline{F}_x(\rho_x^u)$, that is, $\rho_x^{bal}(s^u) = \rho_x^u$ (here we are using that $q \mapsto q \cdot \overline{F}_y(q)$ is continuous and unimodular in q). Then, since $\rho_x^{bal}(\cdot)$ is decreasing we have that $\rho_x^{loc}(s) = \rho_x^{bal}(s)$ for all $0 < s \leq s^u$ and, therefore,

$$\frac{R_x^{loc}(s)}{s} = \rho_x^{bal}(s) = F^{-1}(1 - \frac{s}{\lambda(x)}), \quad \text{for all } 0 < s \leq s^u.$$

Since F is strictly increasing, the quotient above is strictly decreasing for $s \in (0, s^u]$. Moreover, since $F^{-1}(1) = \overline{V}$, the point just made also includes $s = 0$. Now, for $s > s^u$ we have $\rho_x^{loc}(s) = \rho_x^u$, thus

$$\frac{R_x^{loc}(s)}{s} = \rho_x^u \cdot \frac{\lambda(x) \cdot \overline{F}_x(\rho_x^u)}{s},$$

which is strictly decreasing. In any case, we conclude that $\psi_x(\cdot)$ is strictly decreasing $\Gamma - a.e.\ x$ in $\mathcal{C}$. □

Proof of Proposition 2. Define the set $B \triangleq \{x \in C : V(x) > \psi_x(s^{\mathcal{T}}(x))\}$. We want to show that $\Gamma(B) = 0$. First we argue that $B \subseteq \{x \in \mathcal{C} : U(x) \neq V(x)\}$, indeed, let $x \in B$ then

$$V(x) > \psi_x(s^{\mathcal{T}}(x)) \geq U\Big(x, p(x), s^{\mathcal{T}}(x)\Big),$$

that is, $V(x) > U(x)$ as desired. By Lemma A-2 we know that $\mathcal{T}_2(\{x \in \mathcal{C} : U(x) \neq V(x)\}) = 0$ and, therefore, $\mathcal{T}_2(B) = 0$. This yields,

$$0 = \mathcal{T}_2(B) = \int_B s^{\mathcal{T}}(x)\, d\Gamma(x). \tag{A-3}$$

If $\Gamma(B) = 0$ then we are done. Suppose $\Gamma(B) > 0$, from equation (A-3) we can conclude that $s^{\mathcal{T}}(x) = 0$, $\Gamma - a.e.$ $x \in B$. The definition of $\psi_x(s^{\mathcal{T}}(x))$ implies that $\Gamma - a.e$ in B we have that $\psi_x(s^{\mathcal{T}}(x))$ equals $\alpha \cdot \overline{V}$. Because $\alpha \cdot \overline{V}$ is the maximum value that $V(\cdot)$ can attain (see Lemma A-1), we conclude that

$$\alpha \cdot \overline{V} \geq V(x) > \psi_x(s^{\mathcal{T}}(x)) = \alpha \cdot \overline{V} \qquad \Gamma - a.e.\ x \in B.$$

But since we are assuming that $\Gamma(B) > 0$, this yields a contradiction. □

B Proofs for Section 6

When z is a sink, we represent the endpoints of its attraction region along a ray $a \in R_z$ by

$$X_a(z|\,p,\mathcal{T}) \triangleq \sup\{x \in A_a(z|\,p,\mathcal{T})\},$$

where $A_a(z|\,p,\mathcal{T})$ is the restriction of $A(z|\,p,\mathcal{T})$ in the direction of ray a. Note that in the definition of $X_a(z|\,p,\mathcal{T})$, as x moves away from z along $A_a(z|\,p,\mathcal{T})$, x increases. In turn, for every ray a the segment $[z, X_a(z|\,p,\mathcal{T})]$ represents the contribution of $A_a(z|\,p,\mathcal{T})$ to $A(z|\,p,\mathcal{T})$. The next result formalizes this and characterizes the shape of attraction regions.

Lemma B-1 (Attraction Region)**.** *Let $(p,\mathcal{T})$ be a feasible solution of $(\mathcal{P}_2)$. For any sink $z \in \mathcal{C}$, its attraction region $A(z|\,p,\mathcal{T})$ is a closed set containing z, $A_a(z|\,p,\mathcal{T}) = [z, X_a(z|\,p,\mathcal{T})]$ and*

$$A(z|\,p,\mathcal{T}) = \bigcup_{a\in R_z} A_a(z|\,p,\mathcal{T}).$$

Proof of Lemma B-1. For ease of notation let us use x_a to denote $X_a(z|\,p,\mathcal{T})$. We also denote $A(z|p,\mathcal{T})$ by $A(z)$.

Closure: Let the sequence $\{x^n\}_{n\in\mathbb{N}} \subset A(z)$ be such that $x^n \to x$. We show that $x \in A(z)$, that is, $V(x) = V(z) - \|x - z\|$. Indeed, since $x_n \in A(z)$ we have that $V(x_n) = V(z) - \|x_n - z\|$. Because $V(\cdot)$ is Lipschitz (see Lemma 1), it is continuous, and the desired conclusion follows.

Interval: We show that $A_a(z) = [z, x_a]$. The definition of x_a immediately implies that $A_a(z) \subseteq [z, x_a]$, so we only need to prove the reverse inclusion. First, since we can always construct a sequence $\{x^n\}_{n\in\mathbb{N}} \subset A(z)$, with $x^n \to x_a$, the closure property implies that $x_a \in A(z)$. Second, we make use of Lemma B-2 (stated and proved right after this proof). Consider $x \in [z, x_a]$ then Lemma B-2 implies that $z \in \mathcal{IR}(x|\,p,\mathcal{T})$ or, equivalently, $x \in A_a(z)$.

Union: Since for every $a \in R_z$ we have $A_a(z) \subset A(z)$, the same is true for the union. In the opposite direction, if we take $x \in A(z)$ then there exists $a \in R_z$ such that $x \in [z, x_a] = A_a(z)$.

□

Lemma B-2. *For any price mapping p and corresponding equilibrium $\mathcal{T}$, if $y \in \mathcal{IR}(x|\,p,\mathcal{T})$ then $y \in \mathcal{IR}(z|\,p,\mathcal{T})$ for all $z \in [x \wedge y, x \vee y]$.*

Proof. Let $y \in \mathcal{IR}(x|\,p,\mathcal{T})$. If $x = y$ there is nothing to prove. Without loss of generality suppose $x < y$, where we use the natural order in the segment $[x, y]$. Fix $z \in [x, y]$, we want to prove that $y \in \mathcal{IR}(z|\,p,\mathcal{T})$, i.e., $V(z) = V(y) - |z - y|$. Note that

$$\begin{aligned}\|z-y\| \overset{(a)}{\geq} V(y) - V(z) = V(y) - V(x) + V(x) - V(z) &\overset{(b)}{=} \|x-y\| + V(x) - V(z)\\ &\overset{(c)}{\geq} \|x-y\| - \|x-z\|\\ &\overset{(d)}{=} \|z-y\|,\end{aligned}$$

where (a) and (c) come from the Lipschitz property (see Lemma 1), (b) follows from $y \in \mathcal{IR}(x|\,p,\mathcal{T})$, and (d) holds because x, z, y are collinear points.

□

Proof of Proposition 3. Consider the segment $[x, y]$ (with the standard order) and define the set

$$L \triangleq \{y' \in \mathcal{C} : \exists t \geq 0 \text{ such that } y' = x + t\cdot(y - x)\},$$

that is L is the set of point along the ray that starts at x and contains the segment $[x, y]$. Since $y \in L$ and $y \in \mathcal{IR}(x|\,p,\mathcal{T})$ the following quantity is well defined

$$z \triangleq \sup\{y' \in L : y' \in \mathcal{IR}(x|\,p,\mathcal{T})\}.$$

We prove that z is a sink location such that $x, y \in A(z|\, p, \mathcal{T})$. First, we show that $z \in \mathcal{IR}(x|\, p, \mathcal{T})$. Consider a sequence $\{z_n\} \subset L$ such that $z_n \in \mathcal{IR}(x|p, \mathcal{T})$ and $z_n \to z$. Then, $V(z_n) - \|z_n - x\| = V(x)$, we can take the limit as $n \uparrow \infty$ and use the Lipschitz property (see Lemma 1) to conclude that $V(z) - \|z - x\| = V(x)$. That is, $z \in \mathcal{IR}(x|\, p, \mathcal{T})$ which also shows that $A(z) \neq \emptyset$.

Next, to show that z is a sink location we argue that we cannot have $z \in A(z')$ for some $z' \neq z$. If we did then $z' \in \mathcal{IR}(z|p, \mathcal{T})$ for some $z' \neq z$. First suppose that $z' \in L$. If $z' > z$ this would contradict the definition of z as being maximal. If $z' < z$ then by the definition of $\mathcal{IR}(z|\, p, \mathcal{T})$ the function $V(\cdot)$ would be decreasing in (z', z), but since $z \in \mathcal{IR}(x|\, p, \mathcal{T})$ we have that $V(\cdot)$ is increasing in (x, z). This is a contradiction.

Second, suppose that $z' \notin L$. That is the vectors $z' - x$ and $z - x$ are not collinear. Because $z' \in \mathcal{IR}(z|p, \mathcal{T})$ and $z \in \mathcal{IR}(x|\, p, \mathcal{T})$ we have

$$V(z') - \|z' - z\| = V(z) \ \text{ and } \ V(z) - \|z - x\| = V(x).$$

Combining these expression yields

$$V(z') - V(x) = \|z' - z\| + \|z - x\| > \|z' - x\|,$$

where the strict inequality comes from the fact that $z' - x$ and $z - x$ are not collinear. But this contradict the fact that $V(\cdot)$ is Lipschitz (see Lemma 1). We conclude that z is a sink location. Moreover, because $x \in A(z)$ ($z \in \mathcal{IR}(x|\, p, \mathcal{T})$) and $x < y \leq z$ (recall these three points are collinear) Lemma B-2 guarantees that $y \in A(z)$. □

Proposition B-1 (Flow Separation)**.** *Let $(p, \mathcal{T})$ be a feasible solution of $(\mathcal{P}_2)$, and let $z \in \mathcal{C}$ be a sink. Then, there is no flow crossing the endpoints of the attraction region, and there is no flow crossing the sink, z. Formally, with some abuse of notation, let $L(z|\, p, \mathcal{T})$ denote $\bigcup_{a \in R_z} \{X_a(z|\, p, \mathcal{T})\}$ then*

(i) $\mathcal{T}(A(z|\, p, \mathcal{T})^c \times A(z|\, p, \mathcal{T})) = 0$ *and* $\mathcal{T}(\bigcup_{a \in R_z} [z, X_a(z|\, p, \mathcal{T}) \times \Big(A(z|\, p, \mathcal{T})^c \cup L(z|\, p, \mathcal{T}) \setminus \{z\}\Big)) = 0.$

(ii) Let $R_1, R_2 \subset R_z$ with $R_1 \cap R_2 = \emptyset$ then $\mathcal{T}\Big(\bigcup_{a \in R_1} (z, X_a(z|\, p, \mathcal{T})] \times \bigcup_{a \in R_2} (z, X_a(z|\, p, \mathcal{T})]\Big) = 0.$

Proof of Proposition B-1. With some abuse of notation let

$$A^\circ(z|\, p, \mathcal{T}) = \bigcup_{a \in R_z} (z, X_a(z|\, p, \mathcal{T}).$$

This result is based on the following properties:

a) For all $(x, y) \in A(z|\, p, \mathcal{T})^c \times A(z|\, p, \mathcal{T})$, $y \notin \mathcal{IR}(x|\, p, \mathcal{T})$.

b) For all $(x, y) \in (A^\circ(z|\, p, \mathcal{T}) \cup \{z\}) \times \Big(A(z|\, p, \mathcal{T})^c \cup L(z|\, p, \mathcal{T}) \setminus \{z\}\Big)$, $y \notin \mathcal{IR}(x|\, p, \mathcal{T})$.

Before we provide a formal proof of these properties, we use them to show the statement of the proposition. We will also make use of Lemma B-3 which we prove and state after the present proof.

We begin with the first part of *(i)*, that is, we show that $\mathcal{T}(A(z|\, p, \mathcal{T})^c \times A(z|\, p, \mathcal{T})) = 0$. If this is not true then by Lemma B-3 we can find $(x, y) \in A(z|\, p, \mathcal{T})^c \times A(z|\, p, \mathcal{T})$ such that $y \in \mathcal{IR}(x|\, p, \mathcal{T})$. We obtain a contradiction with property a) above. Therefore it must be the case that $\mathcal{T}(A(z|\, p, \mathcal{T})^c \times A(z|\, p, \mathcal{T})) = 0$.

Next, we show the second part of *(i)*, namely, $\mathcal{T}((A^\circ(z|p, \mathcal{T}) \cup \{z\}) \times \Big(A(z|p, \mathcal{T})^c \cup L(z|p, \mathcal{T}) \setminus \{z\}\Big)) = 0$. If this is not true then by Lemma B-3 we can find $(x, y) \in (A^\circ(z|p, \mathcal{T}) \cup \{z\}) \times \Big(A(z|p, \mathcal{T})^c \cup L(z|p, \mathcal{T}) \setminus \{z\}\Big)$ such that $y \in \mathcal{IR}(x|\, p, \mathcal{T})$ but this contradicts property b) above. Therefore it must be the case that $\mathcal{T}((A^\circ(z|\, p, \mathcal{T}) \cup \{z\}) \times \Big(A(z|\, p, \mathcal{T})^c \cup L(z|\, p, \mathcal{T}) \setminus \{z\}\Big)) = 0$.

Now we provide a proof for *(ii)*. Let $R_1, R_2 \subset R_z$ with $R_1 \cap R_2 = \emptyset$ we show that

$$\mathcal{T}\Big(\bigcup_{a \in R_1} (z, X_a(z|\, p, \mathcal{T})] \times \bigcup_{a \in R_2} (z, X_a(z|\, p, \mathcal{T})]\Big) = 0.$$

Suppose by contradiction that this is not true then by Lemma B-3 we can find $(x, y) \in \bigcup_{a \in R_1}(z, X_a(z|p, \mathcal{T})] \times \bigcup_{a \in R_2}(z, X_a(z|\, p, \mathcal{T})]$ such that $y \in \mathcal{IR}(x|\, p, \mathcal{T})$. This implies that $x \in A(y|\, p, \mathcal{T})$. Moreover, since z is a sink location we have $x \in A(z|\, p, \mathcal{T})$ and $y \in A(z|\, p, \mathcal{T})$ thus

$$V(x) = V(y) - \|y - x\|, \quad V(x) = V(z) - \|z - x\|, \quad \text{and} \quad V(y) = V(z) - \|z - y\|.$$

In turn, we can use the first to equalities to obtain $V(y) = V(z) + \|y - x\| - \|z - x\|$. Plugging this into the last equality yields $\|z - x\| = \|y - x\| + \|z - y\|$; however, because $R_1 \cap R_2 = \emptyset$ we have that $x \in (z, X_{a_1}(z|\, p, \mathcal{T})]$ and $y \in (z, X_{a_2}(z|p, \mathcal{T})]$ with $a_1 \neq a_2$. In other words, x and y belong to different rays around z. In turn, the latter equality cannot hold and we must have that $\mathcal{T}\Big(\bigcup_{a \in R_1}(z, X_a(z|\, p, \mathcal{T})] \times \bigcup_{a \in R_2}(z, X_a(z|\, p, \mathcal{T})]\Big) = 0$.

Next we verify properties a) and b). We start with a). We argue by contradiction. Suppose there exists $x \in A(z|\, p, \mathcal{T})^c$ and $y \in A(z|\, p, \mathcal{T})$ such that $y \in \mathcal{IR}(x|\, p, \mathcal{T})$. Let a index the ray that contains the vector $(x - z)$. Recall that by Lemma B-1 we have that $A_a(z|p, \mathcal{T}) = [z, X_a(z|p, \mathcal{T})]$. Since $x \in A(z|\, p, \mathcal{T})^c$ we must have that $x \notin [z, X_a(z|p, \mathcal{T})]$. In particular $\|x - z\| > |X_a(z|p, \mathcal{T}) - z|$. Hence if we show that $z \in \mathcal{IR}(x|p, \mathcal{T})$ we would contradict the maximality of $X_a(z|\, p, \mathcal{T})$. Indeed,

$$\begin{aligned} V(z) - \|x - z\| = V(z) - V(y) + V(y) - \|x - z\| &\overset{(a)}{=} \|z - y\| + V(y) - \|x - z\| \\ &= \|z - y\| + V(y) - \|x - z\| - \|y - x\| + \|y - x\| \\ &\overset{(b)}{=} \|z - y\| + V(x) - \|y - x\| + \|y - x\| \\ &\overset{(c)}{\geq} V(x), \end{aligned}$$

where (a) follows from $y \in A(z|\, p, \mathcal{T})$, (b) from $y \in \mathcal{IR}(x|\, p, \mathcal{T})$ and (c) from the triangular inequality. Using the Lipchitz property of V (see Lemma 1) we conclude that $V(z) - \|x - z\| = V(x)$, that is, $z \in \mathcal{IR}(x|\, p, \mathcal{T})$.

Now we show b). Let $x \in (A^{\circ}(z|\, p, \mathcal{T}) \cup \{z\})$ and $y \in \Big(A(z|\, p, \mathcal{T})^c \cup L(z|\, p, \mathcal{T}) \setminus \{z\}\Big)$. We look into two cases: $x \neq z$ and $x = z$. In both cases we proceed by contradiction assuming that $y \in \mathcal{IR}(x|\, p, \mathcal{T})$. Let us start with $x \neq z$. Let a index the ray that contains the vector $(x - z)$. Recall that by Lemma B-1 we have that $A_a(z|\, p, \mathcal{T}) = [z, X_a(z|\, p, \mathcal{T})]$. Since, $x \in A^{\circ}(z|\, p, \mathcal{T})$ and $x \neq z$ we must have that $x \in (z, X_a(z|\, p, \mathcal{T}))$, hence

$$V(x) = V(y) - \|y - x\| \text{ and } V(X_a(z|\, p, \mathcal{T})) = V(x) - \|x - X_a(z|\, p, \mathcal{T})\|,$$

that is,

$$V(y) - V(X_a(z|\, p, \mathcal{T})) = \|y - x\| + \|x - X_a(z|\, p, \mathcal{T})\|. \tag{B-1}$$

If $y = X_a(z|\, p, \mathcal{T})$ the previous equality implies $x = X_a(z|\, p, \mathcal{T})$, but since $x \in (z, X_a(z|\, p, \mathcal{T}))$ this is not possible. If $y \neq X_a(z|\, p, \mathcal{T})$ then since $y \in \Big(A(z|\, p, \mathcal{T})^c \cup L(z|\, p, \mathcal{T}) \setminus \{z\}\Big)$ we must have that $y \notin (z, X_a(z|\, p, \mathcal{T}))$. Also, y cannot be equal to some point $x + t(z - x)$ for some $t > 1$ because that would contradict the fact that z is a sink location. Therefore, Eq. (B-1) together with the triangular inequality deliver $V(y) - V(X_a(z|\, p, \mathcal{T})) > |y - X_a(z|\, p, \mathcal{T})|$, but this contradicts the Lipschitz property of $V(\cdot)$.

To conclude, consider the case $x = z$. In this case we would have $z \in A(y|\, p, \mathcal{T})$ but this contradicts the fact that z is a sink location. □

Lemma B-3. *Let $\mathcal{L} \subset \mathcal{C} \times \mathcal{C}$. If $\mathcal{T}(\mathcal{L}) > 0$ then there exists $(x, y) \in \mathcal{L}$ such that $y \in \mathcal{IR}(x|p, \mathcal{T})$.*

Proof. Suppose $\mathcal{T}(\mathcal{L}) > 0$. We first argue that there exists a pair $(x, y) \in \mathcal{L}$ such that for all $\delta > 0$

$$\mathcal{T}(B(x, \delta) \times B(y, \delta)) > 0, \tag{B-2}$$

where $B(x, \delta)$ is an open ball of radius δ. If this is not true then for any $(x, y) \in \mathcal{L}$ we can find $\delta_{x,y} > 0$ such that Eq. (B-2) does not hold when we replace δ with $\delta_{x,y}$, that is, $\mathcal{T}(B(x, \delta_{x,y}) \times B(y, \delta_{x,y})) = 0$ for all $(x, y) \in \mathcal{L}$. The collection $\mathcal{I}$ defined by

$$\mathcal{I} = \{B(x, \delta_{x,y}) \times B(y, \delta_{x,y})\}_{(x,y) \in \mathcal{L}}$$

is an open cover of $\mathcal{L}$. Moreover the set $\mathcal{L}$ is separable because $\mathcal{C} \times \mathcal{C}$ is separable. This implies that we can find a countable sub-cover of $\mathcal{L}$ in $\mathcal{I}$, that is, there exists $\{B(x_n, \delta_{x_n,y_n}) \times B(y_n, \delta_{x_n,y_n})\}_{n\in\mathbb{N}}$ such that

$$\mathcal{L} \subset \bigcup_{n\in\mathbb{N}} B(x_n, \delta_{x_n,y_n}) \times B(y_n, \delta_{x_n,y_n}).$$

The existence of the sub-cover is guaranteed by the Lindelöf property of separable metric spaces, see e.g., Sierpiński & Krieger (1952) Theorem 69, p. 116. Since $\mathcal{T}$ is a measure we have

$$\mathcal{T}(\mathcal{L}) \leq \mathcal{T}\Big(\bigcup_{n\in\mathbb{N}} B(x_n, \delta_{x_n,y_n}) \times B(y_n, \delta_{x_n,y_n})\Big) \leq \sum_{n\in\mathbb{N}} \mathcal{T}(B(x_n, \delta_{x_n,y_n}) \times B(y_n, \delta_{x_n,y_n})) = 0,$$

a contradiction. Therefore, for some $(x, y) \in \mathcal{L}$, Eq. (B-2) holds for any $\delta > 0$.

We next show that $y \in \mathcal{IR}(x|\, p, \mathcal{T})$. First we prove that

$$\forall \epsilon > 0, \exists \delta_0 > 0 \ \text{ such that } \ \forall \delta < \delta_0 \quad \epsilon + V_{B(y,\delta)}(x) \geq V(x). \tag{B-3}$$

Let $\epsilon > 0$ and let $\delta_0 = \frac{\epsilon}{2}$. Consider $\delta < \delta_0$, from Eq. (B-2) and the equilibrium definition we have

$$\begin{aligned} 0 &< \mathcal{T}(B(x, \delta) \times B(y, \delta)) \\ &= \mathcal{T}\Big(\Big\{(x', y') \in B(x, \delta) \times B(y, \delta) : \Pi(x', y') = V(x')\Big\}\Big) \\ &\leq \mathcal{T}_2\Big(\underbrace{\Big\{y' \in B(y, \delta) : \exists x' \in B(x, \delta) \ \text{ such that } \ \Pi(x', y') = V(x')\Big\}}_{\triangleq R^{x,y,\delta}}\Big), \end{aligned}$$

since $\mathcal{T}_2 \ll \Gamma$ this implies that $\Gamma(R^{x,y,\delta}) > 0$. Now we argue that $R^{x,y,\delta} \subset \{y' \in B(y, \delta) : \Pi(x, y') \geq V(x) - \epsilon\}$. Indeed, let $y' \in R^{x,y,\delta}$ then there exists $x' \in B(x, \delta)$ for which

$$\begin{aligned} U(y') &= V(x') + \|y' - x'\| \\ &\geq V(x) - \|x' - x\| + \|y' - x'\| \\ &= V(x) - \|x' - x\| + \|y' - x'\| - \|y' - x\| + \|y' - x\| \\ &\geq V(x) - \|x' - x\| - \|x' - x\| + \|y' - x\|, \end{aligned}$$

where in the first inequality we used the Lipchitz property of V (see Lemma 1), and in the second we use triangular inequality. Since $\|x' - x\| \leq \delta_0 = \frac{\epsilon}{2}$ we have that $U(y') \geq V(x) - \epsilon + \|y' - x\|$, that is, $R^{x,y,\delta} \subset \{y' \in B(y, \delta) : \Pi(x, y') \geq V(x) - \epsilon\}$. Therefore, $\Gamma(\{y' \in B(y, \delta) : \Pi(x, y') \geq V(x) - \epsilon\}) > 0$, which implies that $V_{B(y,\delta)}(x) \geq V(x) - \epsilon$.

Next we argue that

$$V(y) - \|y - x\| + 2\delta \geq V_{B(y,\delta)}(x), \quad \forall \delta > 0. \tag{B-4}$$

Indeed, the following holds $\Gamma - a.e.\ y'$ in $B(y, \delta)$

$$\begin{aligned} V(y) - \|y - x\| + 2\delta &\geq U(y') - \|y' - y\| - \|y - x\| + 2\delta \\ &= U(y') - \|y' - x\| + \|y' - x\| - \|y' - y\| - \|y - x\| + 2\delta \\ &\geq U(y') - \|y' - x\| + \|y - x\| - \|y' - y\| - \|y' - y\| - \|y - x\| + 2\delta \\ &\geq U(y') - \|y' - x\|, \end{aligned}$$

the first inequality comes from the definition of $V(y)$, the second from the triangular inequality and the last inequality follows from $\|y' - y\| \leq \delta$.

Finally, Eq. (B-3) and Eq. (B-4) together yields that for any $\epsilon > 0$ we can find $\delta(\epsilon) > 0$ such that for all $\delta \in (0, \delta(\epsilon))$ we have $V(y) - \|y - x\| + 2\delta \geq V_{B(y,\delta)}(x) \geq V(x) - \epsilon$. We can take $\delta \downarrow 0$, and then $\epsilon \downarrow 0$ to conclude that $V(y) - \|y - x\| \geq V(x)$. But by the Lipschitz property (see Lemma 1) we have $V(y) - \|y - x\| \leq V(x)$. Therefore, $y \in \mathcal{IR}(x|\, p, \mathcal{T})$. □

B.1 Local Equilibria and Pasting

The flow separation result in Proposition B-1 will enable us to geographically decompose the platform's problem into multiple weakly coupled local problems. To that end, we introduce some additional notation that will allow us to "localize the analysis". Formally, for any measurable $\mathcal{B} \subset \mathcal{C}$ and measure $\tilde{\Theta} \in \mathcal{M}(\mathcal{B})$, we define the set of feasible flows restricted to $\mathcal{B}$ to be

$$\mathcal{F}_{\mathcal{B}}(\tilde{\Theta}) = \{\mathcal{T} \in \mathcal{M}(\mathcal{B} \times \mathcal{B}) : \mathcal{T}_1 = \tilde{\Theta}, \quad \mathcal{T}_2 \ll \Gamma|_{\mathcal{B}}\},$$

where for any measure $\mathcal{V}$ we denote its restriction to a set $\mathcal{B}$ by $\mathcal{V}|_{\mathcal{B}}$. In addition, we define local equilibria as follows.

Definition 3 (Local Equilibrium)**.** *For any $\mathcal{B} \subset \mathcal{C}$ such that $\Gamma(\mathcal{B}) > 0$ and $\tilde{\Theta} \in \mathcal{M}(\mathcal{B})$, a flow $\mathcal{T} \in \mathcal{F}_{\mathcal{B}}(\tilde{\Theta})$ is a local equilibrium in $\mathcal{B}$ if it satisfies*

$$\mathcal{T}\left(\left\{(x, y) \in \mathcal{B} \times \mathcal{B} : \Pi(x, y, p(y), s^{\mathcal{T}}(y)) = \operatorname*{ess\,sup}_{\mathcal{B}} \Pi\Big(x, \cdot, p(\cdot), s^{\mathcal{T}}(\cdot)\Big)\right\}\right) = \tilde{\Theta}(\mathcal{B}).$$

That is, a local equilibrium in $\mathcal{B}$ is a feasible flow such that no driver wishes to unilaterally change his destination when restricting attention to the set $\mathcal{B}$. With this definition in hand, we may now state our next result. Informally, this result states the following "pasting" property. Suppose we start from a price-equilibrium pair $(p, \mathcal{T})$ and a sink z and its attraction region $A(z|\, p, \mathcal{T})$. Then, we can replace the flow that occurs within $A(z|\, p, \mathcal{T})$ with any other local equilibrium within that attraction region as long as certain properties of $V(x|\, p, \mathcal{T})$ are maintained in $A(z|\, p, \mathcal{T})$.

Proposition B-2. *(Pasting) Let $(p, \mathcal{T})$ be a feasible solution of $(\mathcal{P}_2)$, and let $z \in \mathcal{C}$ be a sink. Denote $\mathcal{A} = A(z|\, p, \mathcal{T})$ and $\mathcal{L} = \bigcup_{a \in R_z} \{X_a(z|\, p, \mathcal{T})\}$. Let $\tilde{\Theta} \in \mathcal{M}(\mathcal{A})$ be the measure representing drivers that stay within $\mathcal{A}$ according to flow $\mathcal{T}$, i.e., $\tilde{\Theta}(\mathcal{B}) \triangleq \mathcal{T}(\mathcal{B} \times \mathcal{A})$ for any measurable set $\mathcal{B} \subseteq \mathcal{A}$. Suppose there exists a measurable price mapping $\tilde{p} : \mathcal{A} \to [0, \overline{V}]$ and a flow $\tilde{\mathcal{T}} \in \mathcal{F}_{\mathcal{A}}(\tilde{\Theta})$ such that $\tilde{\mathcal{T}}$ is a local equilibrium in $\mathcal{A}$ under pricing $\tilde{p}$. Furthermore, suppose $V_{\mathcal{A}}(\cdot|\, \tilde{p}, \tilde{\mathcal{T}})$ is lower or equal than $V(\cdot|\, p, \mathcal{T})$ in $\mathcal{A}$, and that $\tilde{\Theta}(\{x \in \mathcal{A} : V_{\mathcal{A}}(x|\, \tilde{p}, \tilde{\mathcal{T}}) = V(x|\, p, \mathcal{T})\}) = \tilde{\Theta}(\mathcal{A})$. Define the pasted pricing function $\hat{p} : \mathcal{C} \to [0, \overline{V}]$,*

$$\hat{p}(x) \triangleq \begin{cases} \tilde{p}(x) & \text{if } x \in \mathcal{A}; \\ p(x) & \text{if } x \in \mathcal{A}^c, \end{cases}$$

and the pasted flow $\hat{\mathcal{T}} \in \mathcal{F}(\Theta)$, where for any measurable $\mathcal{B} \subseteq \mathcal{C} \times \mathcal{C}$

$$\hat{\mathcal{T}}(\mathcal{B}) \triangleq \mathcal{T}(\mathcal{B} \cap ((\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c)) + \tilde{\mathcal{T}}(\mathcal{B} \cap (\mathcal{A} \times \mathcal{A})).$$

Then, the pasted solution $(\hat{p}, \hat{\mathcal{T}})$ is a feasible solution of problem $(\mathcal{P}_2)$ such that

$$s^{\hat{\mathcal{T}}} = \begin{cases} s^{\tilde{\mathcal{T}}}(x) & \text{if } x \in \mathcal{A}; \\ s^{\mathcal{T}}(x) & \text{if } x \in \mathcal{A}^c. \end{cases}$$

Proof of Proposition B-2. For ease of notation we use X_a to denote $X_a(z|p, \mathcal{T})$. We show that $\hat{\mathcal{T}}$ belongs to $\mathcal{F}_{\mathcal{C}}(\Theta)$ and that it is an equilibrium in $\mathcal{C}$. First we argue that $\hat{\mathcal{T}} \in \mathcal{F}_{\mathcal{C}}(\Theta)$. Since $\hat{\mathcal{T}}$ is the sum of two non-negative measures we have that $\hat{\mathcal{T}} \in \mathcal{M}(\mathcal{C} \times \mathcal{C})$. In order see why $\hat{\mathcal{T}}_1$ coincides with Θ, let B be a measurable subset of $\mathcal{C}$ then

$$\begin{aligned}\hat{\mathcal{T}}_1(B) &= \hat{\mathcal{T}}(B \times \mathcal{C})\\ &= \mathcal{T}((B \cap (\mathcal{A}^c \cup \mathcal{L})) \times \mathcal{A}^c) + \tilde{\mathcal{T}}((B \cap \mathcal{A}) \times \mathcal{A})\\ &\stackrel{(a)}{=} \mathcal{T}((B \cap (\mathcal{A}^c \cup \mathcal{L})) \times \mathcal{A}^c) + \tilde{\Theta}(B \cap \mathcal{A})\\ &= \mathcal{T}((B \cap (\mathcal{A}^c \cup \mathcal{L})) \times \mathcal{A}^c) + \mathcal{T}((B \cap \mathcal{A}) \times \mathcal{A})\\ &\stackrel{(b)}{=} \mathcal{T}((B \cap \mathcal{A}^c) \times \mathcal{A}^c) + \mathcal{T}((B \cap \mathcal{L}) \times \mathcal{A}^c) + \mathcal{T}((B \cap \mathcal{A}) \times \mathcal{A})\\ &\stackrel{(c)}{=} \mathcal{T}((B \cap \mathcal{A}^c) \times \mathcal{C}) + \mathcal{T}((B \cap \mathcal{A}) \times \mathcal{C})\\ &= \Theta(B),\end{aligned}$$

where (a) comes from the fact that $\tilde{\mathcal{T}}$ belongs to $\mathcal{F}_{\mathcal{A}}(\tilde{\Theta})$. In (b) we use the fact that $\mathcal{A}$ is a closed set. Equality (c) comes from Proposition B-1 part *(i)*. That is, $\hat{\mathcal{T}}_1$ coincides with Θ. Now, we show that $\hat{\mathcal{T}}_2 \ll \Gamma$. Let B be as before and suppose $\Gamma(B) = 0$ then

$$\hat{\mathcal{T}}_2(B) = \hat{\mathcal{T}}(\mathcal{C} \times B) = \mathcal{T}((\mathcal{A}^c \cup \mathcal{L}) \times (B \cap \mathcal{A}^c)) + \tilde{\mathcal{T}}(\mathcal{A} \times (B \cap \mathcal{A})) \leq \mathcal{T}_2(B \cap \mathcal{A}^c) + \tilde{\mathcal{T}}_2(B \cap \mathcal{A}) = 0,$$

where the last equality holds because $\mathcal{T}_2 \ll \Gamma$ and $\tilde{\mathcal{T}}_2 \ll \Gamma|_{\mathcal{A}}$. Now we show that $\hat{\mathcal{T}}$ is an equilibrium. We need to verify that $\hat{\mathcal{T}}(\hat{\mathcal{E}})$ equals $\Theta(\mathcal{C})$, where

$$\hat{\mathcal{E}} \triangleq \Big\{(x, y) \in \mathcal{C} \times \mathcal{C} : \Pi(x, y, \hat{p}(y), s^{\hat{\mathcal{T}}}(y)) = \operatorname*{ess\,sup}_{\mathcal{C}} \Pi\Big(x, \cdot, \hat{p}(\cdot), s^{\hat{\mathcal{T}}}(\cdot)\Big)\Big\}.$$

In order to verify this we compute first $s^{\hat{\mathcal{T}}}$ and $V(x|\,\hat{p}, \hat{\mathcal{T}})$. First we show that $\Gamma-$a.e we have

$$s^{\hat{\mathcal{T}}}(x) = \begin{cases} s^{\mathcal{T}}(x) & \text{if } x \in \mathcal{A}^c \\ s^{\tilde{\mathcal{T}}}(x) & \text{if } x \in \mathcal{A}. \end{cases}$$

Let B be a measurable subset of $\mathcal{A}^c$ then

$$\hat{\mathcal{T}}_2(B) = \mathcal{T}((\mathcal{C} \times B) \cap ((\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c)) = \mathcal{T}((\mathcal{A}^c \cup \mathcal{L}) \times B) \stackrel{(a)}{=} \mathcal{T}(\mathcal{C} \times B) = \mathcal{T}_2(B),$$

where (a) comes from Proposition B-1 part *(i)*. Therefore, $s^{\hat{\mathcal{T}}}(x)$ equals $s^{\mathcal{T}}(x)$ $\Gamma - a.e.\ x$ in $\mathcal{A}^c$. Similarly, for B a measurable subset of $\mathcal{A}$ we have

$$\hat{\mathcal{T}}_2(B) = \tilde{\mathcal{T}}(\mathcal{A} \times B) = \tilde{\mathcal{T}}_2(B),$$

where the second equality holds because from Proposition B-1 we have $\mathcal{T}(\mathcal{A}^c \times \mathcal{A}) = 0$, and also because $\tilde{\mathcal{T}}$ is an equilibrium in $\mathcal{A}$.

We next show that for $\Theta_m(\mathcal{B})$ defined by $\tau(\mathcal{B} \times \mathcal{A}^c)$ for $\mathcal{B} \subset \mathcal{A}^c \cup \mathcal{L}$, $V(x|\,\hat{p}, \hat{\mathcal{T}})$ satisfies

$$\Theta_m(\{x \in \mathcal{A}^c \cup \mathcal{L} : V(x|\,\hat{p}, \hat{\mathcal{T}}) = V(x|\,p, \mathcal{T})\}) = \mathcal{T}((\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c). \tag{B-5}$$

First, we argue that $V(x|\,p, \mathcal{T}) \geq V(x|\,\hat{p}, \hat{\mathcal{T}})$ for all $x \in \mathcal{A}^c \cup \mathcal{L}$. We proceed by contradiction. Let $x \in \mathcal{A}^c \cup \mathcal{L}$ and suppose that $V(x|\,p, \mathcal{T}) < V(x|\,\hat{p}, \hat{\mathcal{T}})$ then we must have that

$$\begin{aligned}0 &< \Gamma(y \in \mathcal{C} : \Pi(x, \hat{p}(y), s^{\hat{\mathcal{T}}}(y), y) > V(x|\,p, \mathcal{T}))\\ &= \Gamma(y \in \mathcal{A} : \Pi(x, \tilde{p}(y), s^{\tilde{\mathcal{T}}}(y), y) > V(x|\,p, \mathcal{T})) + \Gamma(y \in \mathcal{A}^c : \Pi(x, p(y), s^{\mathcal{T}}(y), y) > V(x|\,p, \mathcal{T}))\\ &\stackrel{(a)}{=} \Gamma(y \in \mathcal{A} : U(\tilde{p}(y), s^{\tilde{\mathcal{T}}}(y), y) - \|x - y\| > V(x|\,p, \mathcal{T}))\\ &\stackrel{(b)}{\leq} \Gamma(y \in \mathcal{A} : U(\tilde{p}(y), s^{\tilde{\mathcal{T}}}(y), y) - \|x - y\| > V_{\mathcal{A}}(x_y|\,\tilde{p}, \tilde{\mathcal{T}}) - \|x_y - x\|)\\ &\stackrel{(c)}{=} \Gamma(y \in \mathcal{A} : U(\tilde{p}(y), s^{\tilde{\mathcal{T}}}(y), y) > V_{\mathcal{A}}(x_y|\,\tilde{p}, \tilde{\mathcal{T}}) + \|x_y - y\|)\\ &\stackrel{(d)}{\leq} \Gamma(y \in \mathcal{A} : U(\tilde{p}(y), s^{\tilde{\mathcal{T}}}(y), y) > V_{\mathcal{A}}(y|\,\tilde{p}, \tilde{\mathcal{T}}))\\ &\stackrel{(e)}{=} 0,\end{aligned}$$

where (a) follows from that the definition of $V(x|\,p,\mathcal{T})$ implies that the second term in the previous line is zero. For any $y \in \mathcal{A}$ we can consider the segment $[x,y]$ that passes through the boundary of $\mathcal{A}$ (because $x \in \mathcal{A}^c \cup \mathcal{L}$), thus, in (b) we take $x_y \in [x,y] \cap \partial\mathcal{A}$ and then apply the Lipschitz property together with the assumption that $V_{\mathcal{A}}(x_y|\,\tilde{p},\tilde{\mathcal{T}}) \leq V(x_y|\,p,\mathcal{T})$. In (c) we made use of the collinearity of x, y and x_y, and in (d) we applied once again the Lipschitz property. The last line (e) follows from Lemma A-2.

Thus to prove Eq. (B-5) we need to show that $\Theta_m(\{x \in \mathcal{A}^c \cup \mathcal{L} : V(x|\,\hat{p},\hat{\mathcal{T}}) < V(x|\,p,\mathcal{T})\}) = 0$. If this is not true then since $V(x|\,\hat{p},\hat{\mathcal{T}}) \geq V_{\mathcal{A}^c}(x|\,p,\mathcal{T})$ we have

$$
\begin{aligned}
0 &< \Theta_m(\{x \in \mathcal{A}^c \cup \mathcal{L} : V_{\mathcal{A}^c}(x|\,p,\mathcal{T}) < V(x|\,p,\mathcal{T})\}) \\
&= \mathcal{T}\Big((x,y) \in (\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c : V_{\mathcal{A}^c}(x|\,p,\mathcal{T}) < V(x|\,p,\mathcal{T})\Big) \\
&\overset{(a)}{=} \mathcal{T}\Big((x,y) \in (\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c : V_{\mathcal{A}^c}(x|\,p,\mathcal{T}) < V(x|\,p,\mathcal{T}), U(p(y), s^{\mathcal{T}}(y), y) - \|x-y\| = V(x|\,p,\mathcal{T})\Big) \\
&\leq \mathcal{T}\Big((x,y) \in (\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c : U(p(y), s^{\mathcal{T}}(y), y) - \|x-y\| > V_{\mathcal{A}^c}(x|\,p,\mathcal{T})\Big) \\
&\leq \mathcal{T}_2\Big(y \in \mathcal{A}^c : \exists x \in (\mathcal{A}^c \cup \mathcal{L}), U(p(y), s^{\mathcal{T}}(y), y) - \|x-y\| > V_{\mathcal{A}^c}(x|\,p,\mathcal{T})\Big) \\
&\overset{(b)}{=} \mathcal{T}_2\Big(y \in \mathcal{A}^c : \exists x \in (\mathcal{A}^c \cup \mathcal{L}), V_{\mathcal{A}^c}(y|\,p,\mathcal{T}) - \|x-y\| > V_{\mathcal{A}^c}(x|\,p,\mathcal{T})\Big) \\
&\overset{(c)}{=} 0,
\end{aligned}
$$

where (a) is true because $\mathcal{T}$ is supported in a set where $U(p(y), s^{\mathcal{T}}(y), y) - \|x-y\| = V(x|\,p,\mathcal{T})$, (b) comes from Lemma A-2 and (c) from the Lipschitz property. This proves Eq. (B-5) .

Lastly, we verify that $\hat{\mathcal{T}}(\hat{\mathcal{E}})$ equals $\Theta(\mathcal{C})$. Define the sets

$$
\begin{aligned}
\mathcal{E}_1 &\triangleq \Big\{(x,y) \in (\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c : \Pi(x,y,\hat{p}(y), s^{\hat{\mathcal{T}}}(y)) = V(x|\,\hat{p},\hat{\mathcal{T}})\Big\} \\
\mathcal{E}_2 &\triangleq \Big\{(x,y) \in \mathcal{A} \times \mathcal{A} : \Pi(x,y,\hat{p}(y), s^{\hat{\mathcal{T}}}(y)) = V(x|\,\hat{p},\hat{\mathcal{T}})\Big\}
\end{aligned}
$$

then $\hat{\mathcal{T}}(\hat{\mathcal{E}}) = \mathcal{T}(\mathcal{E}_1) + \tilde{\mathcal{T}}(\mathcal{E}_2)$. We can replace the definition of $\hat{p}$ and what we have proved about $s^{\hat{\mathcal{T}}}$ in the expressions above to obtain

$$
\begin{aligned}
\mathcal{T}(\mathcal{E}_1) &= \mathcal{T}\Big(\Big\{(x,y) \in (\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c : \Pi(x,y,p(y), s^{\mathcal{T}}(y)) = V(x|\,\hat{p},\hat{\mathcal{T}})\Big\}\Big), \\
\tilde{\mathcal{T}}(\mathcal{E}_2) &= \tilde{\mathcal{T}}\Big(\Big\{(x,y) \in \mathcal{A} \times \mathcal{A} : \Pi(x,y,\tilde{p}(y), s^{\tilde{\mathcal{T}}}(y)) = V(x|\,\hat{p},\hat{\mathcal{T}})\Big\}\Big).
\end{aligned}
$$

From Eq. (B-5) we deduce that

$$
\mathcal{T}(\mathcal{E}_1) = \mathcal{T}\Big(\Big\{(x,y) \in (\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c : \Pi(x,y,p(y), s^{\mathcal{T}}(y)) = V(x|\,p,\mathcal{T})\Big\}\Big).
$$

Since $V_{\mathcal{A}}(x|\tilde{p},\tilde{\mathcal{T}}) \leq V(x|p,\mathcal{T})\})$ for $x \in \mathcal{A}$ we deduce that $V(x|\hat{p},\hat{\mathcal{T}}) \leq V(x|p,\mathcal{T})$ for $x \in \mathcal{A}$. Hence, because $\tilde{\Theta}(\{x \in \mathcal{A} : V_{\mathcal{A}}(x|\,\tilde{p},\tilde{\mathcal{T}}) = V(x|\,p,\mathcal{T})\}) = \tilde{\Theta}(\mathcal{A})$ and $V(x|\,\hat{p},\hat{\mathcal{T}}) \geq V_{\mathcal{A}}(x|\,\tilde{p},\tilde{\mathcal{T}})$ we must have

$$
\tilde{\Theta}(\{x \in \mathcal{A} : V_{\mathcal{A}}(x|\,\tilde{p},\tilde{\mathcal{T}}) = V(x|\,\hat{p},\hat{\mathcal{T}})\}) = \tilde{\Theta}(\mathcal{A}),
$$

in turn,

$$
\tilde{\mathcal{T}}(\mathcal{E}_2) = \tilde{\mathcal{T}}\Big(\Big\{(x,y) \in \mathcal{A} \times \mathcal{A} : \Pi(x,y,\tilde{p}(y), s^{\tilde{\mathcal{T}}}(y)) = V_{\mathcal{A}}(x|\,\tilde{p},\tilde{\mathcal{T}})\Big\}\Big) = \tilde{\Theta}(\mathcal{A}),
$$

where the second line comes from the fact that $\tilde{\mathcal{T}}$ is an equilibrium in $\mathcal{A}$. Let $\mathcal{E}$ be defined analogously to $\hat{\mathcal{E}}$ but with $(\hat{p},\hat{\mathcal{T}})$ replaced by $(p,\mathcal{T})$, then

$$
\begin{aligned}
\hat{\mathcal{T}}(\hat{\mathcal{E}}) = \mathcal{T}(\mathcal{E}_1) + \tilde{\Theta}(\mathcal{A}) \overset{(a)}{=} \mathcal{T}(\mathcal{E}_1) + \mathcal{T}(\mathcal{A} \times \mathcal{A}) \overset{(b)}{=} \mathcal{T}(\mathcal{E} \cap ((\mathcal{A}^c \cup \mathcal{L}) \times \mathcal{A}^c)) + \mathcal{T}(\mathcal{E} \cap (\mathcal{A} \times \mathcal{A})) &\overset{(c)}{=} \mathcal{T}(\mathcal{E}) \\
&\overset{(d)}{=} \Theta(\mathcal{C}),
\end{aligned}
$$

where in (a) we use the definition of $\tilde{\Theta}$. In (b) and (d) we use the fact that $\mathcal{T}$ only puts mass in $\mathcal{E}$, and in (c) we use Proposition B-1 part *(i)*.

□

C Proofs for Section 7

Proof of Theorem 1. The proof of this theorem consists of several parts. In the first part perform a disintegration step that will enable us to analyze the problem along rays stemming from z. Then we pose an optimization problem which is a relaxation of platform's optimization problem restricted to the attraction region $A(z)$. Then we introduce some notation. Given this, the relaxation has a similar structure to a continuous bounded knapsack problem, and we characterize the structure of the optimal solution as stated in the statement of the theorem. Next we construct a local price-equilibrium pair $(\hat{p}, \hat{\mathcal{T}})$ in $A(z)$ that implements the relaxation's solution. We conclude by applying the pasting result of Proposition B-2 to globally extend our price-equilibrium pair $(\hat{p}, \hat{\mathcal{T}})$ in $\mathcal{C}$ as in the statement of the theorem. In summary the parts of the proof are: Disintegration, Relaxation, Notation, Knapsack, Implementation and Conclusion. We enumerate all these parts from 0 to 5, and present them in boldface to make the presentation clearer.

Part 0: Disintegration. Recall that we use R_z to denote the set of all rays originating from z (excluding z) and index the elements of R_z by a. The advantage of this is that now we can disintegrate the city measure into a family of measures concentrated along the rays, $\{\Gamma_a\}$, which we can integrate with respect to another measure Γ^{p} in R_z to obtain Γ. That is, for any measurable set $\mathcal{B}$ we have

$$\Gamma(\mathcal{B}) = \Gamma(\{z\})\mathbf{1}_{\{z\in\mathcal{B}\}} + \int_{R_z} \Gamma_a(\mathcal{B})d\Gamma^{\mathrm{p}}(a). \tag{C-1}$$

Any measure Γ_a can be thought of as a conditional probability, and we can use it to measure quantities such as supply and demand along ray $a \in R_z$. See Ambrosio & Pratelli (2003) for a formal statement about disintegration of measures.

Part 1: Relaxation. We consider the attraction region $A(z)$. In it, the congestion bound must be satisfied. Moreover, due to our flow separation result in Proposition B-1 part *(i)* we have $\mathcal{T}_2(A(z)) = \mathcal{T}(A(z) \times A(z))$. Also, since flow is not transported across rays (see Proposition B-1 part *(ii)*), the total supply in the ray $(z, X_a]$ cannot be larger than its initial supply. Therefore, in $A(z)$ the platform's problem is bounded above by

$$\max_{s(\cdot)} \quad \int_{A(z)} V(x)\cdot s(x)\, d\Gamma(x) \tag{$\mathcal{P}_{KP}(z)$}$$

$$\text{s.t} \quad s(x) \le H_x(V(x|\, p, \mathcal{T})), \quad \Gamma - a.e.\ x \ \text{ in } A(z) \tag{CB}$$

$$\int_{A(z)} s(x)\, d\Gamma(x) = \mathcal{T}(A(z)\times A(z)) \tag{FC}$$

$$\int_{(z,X_a]} s(x)\, d\Gamma_a(x) \le \int_{(z,X_a]} s^{\mathcal{T}}(x)\, d\Gamma_a(x), \quad \Gamma^{\mathrm{p}} - a.e.\ a\in R_z, \tag{FR$_a$}$$

where $H_x(V) = \psi_x^{-1}(V)$. Observe that $s^{\mathcal{T}}$ (which defines $\mathcal{T}_2$) is a feasible solution for ($\mathcal{P}_{KP}(z)$). The supply density $s^{\hat{\mathcal{T}}}$ (as in the statement of the present theorem) will be shown to be an optimal solution for this relaxation.

Part 2: Notation.

1. Next we rename the quantities on the RHS of equations (FC) and (FR$_a$).

$$\mathcal{T}_{\text{total}} = \mathcal{T}(A(z)\times A(z)),\ \ \mathcal{T}_a = \int_{(z,X_a]} s^{\mathcal{T}}(x)\, d\Gamma_a(x),\ \ \mathcal{T}_c = \mathcal{T}(A(z)\times\{z\}).$$

2. For any measurable set $B \subseteq Az$ we define the measure

$$S^H(B) \triangleq \int_B H_x(V(x))\, d\Gamma(x),$$

   $S^H(\cdot)$ is the measure with density $H_x(V(x))$ with respect to the Γ measure. Moving forward we will use $s^H(x)$ to denote its density.

Part 3: Knapsack. We show that any optimal solution to ($\mathcal{P}_{KP}(z)$) is as $s^{\hat{\mathcal{T}}}$ in the statement of the theorem. There are two cases.

Case 1. First suppose that $0 < \mathcal{T}_{\text{total}} \le S^H(\{z\})$ (so that there is an atom at z). Then, we define $r_a = z$ for all $a \in R_z$, and let the solution to ($\mathcal{P}_{KP}(z)$) be

$$s^\star(x) = \frac{\mathcal{T}_{\text{total}}}{\mathit{\Gamma}(\{z\})} \cdot \mathbf{1}_{\{x=z\}},$$

which is feasible, and optimal because for any feasible s we have

$$\int_{A(z)} V(x) \cdot s(x) d\mathit{\Gamma}(x) \le V(z) \cdot \int_{A(z)} s(x)\, d\mathit{\Gamma}(x) = V(z) \cdot \mathcal{T}_{\text{total}},$$

which is equal to the objective function at $s^\star$. So in this case the optimal solution coincides with the description of $s^{\hat{\mathcal{T}}}$ as in the statement of the theorem.

Case 2. Now let us assume that $\mathcal{T}_{\text{total}} > S^H(\{z\})$. We start by showing that in this case we have $s^\star(z) = s^H(z)$. If z is not a point with positive $\mathit{\Gamma}$- mass then setting $s^\star(z)$ in this way is without loss of generality. If the point z has positive mass then we argue by contradiction that $s^\star(z)$ must be choose in this way. Let $s^\star$ be an optimal solution to ($\mathcal{P}_{KP}(z)$) such that $s^\star(z) < s^H(z)$. Then,

$$\mathcal{T}_{\text{total}} = \int_{A(z)\setminus\{z\}} s^\star d\mathit{\Gamma} + s^\star(z) \cdot \mathit{\Gamma}(\{z\}) < \underbrace{\int_{A(z)\setminus\{z\}} s^\star d\mathit{\Gamma}}_{K} + s^H(z) \cdot \mathit{\Gamma}(\{z\}). \tag{C-2}$$

Let $\epsilon \in (0, 1)$ be such that $(\mathcal{T}_{\text{total}} - \epsilon \cdot K)/\mathit{\Gamma}(\{z\}) = s^H(z)$, this is well defined because we are assuming $\mathcal{T}_{\text{total}} > S^H(\{z\})$. Next define a new solution $\bar{s}$ by

$$\bar{s}(x) = \begin{cases} s^H(z) & \text{if } x = z, \\ \epsilon \cdot s^\star(x) & \text{if } x \neq z. \end{cases}$$

Note that $\bar{s}$ is feasible: it satisfies (FR$_a$) for all $a \in R_z$ and (CB), and for (FC) we have

$$\int_{A(z)} \bar{s} d\mathit{\Gamma} = \epsilon \cdot K + s^H(z) \cdot \mathit{\Gamma}(\{z\}) = \mathcal{T}_{\text{total}}.$$

Furthermore, $\bar{s}$ yields an strictly larger objective than $s^\star$,

$$\begin{aligned}
\int_{A(z)} V(x) \cdot s^\star(x)\, d\mathit{\Gamma}(x) &= \int_{A(z)\setminus\{z\}} V(x) \cdot s^\star(x)\, d\mathit{\Gamma}(x) + V(z) \cdot s^\star(z) \cdot \mathit{\Gamma}(\{z\}) \\
&= \epsilon \cdot \int_{A(z)\setminus\{z\}} V(x) \cdot s^\star(x)\, d\mathit{\Gamma}(x) + (1-\epsilon) \cdot \int_{A(z)\setminus\{z\}} V(x) \cdot s^\star(x)\, d\mathit{\Gamma}(x) \\
&+ V(z) \cdot s^\star(z) \cdot \mathit{\Gamma}(\{z\}) \\
&\overset{(a)}{<} \int_{A_{\text{p}}(z)\setminus\{z\}} V(x) \cdot \bar{s}(x)\, d\mathit{\Gamma}(x) + (1-\epsilon) \cdot V(z) \cdot K \\
&+ V(z) \cdot s^\star(z) \cdot \mathit{\Gamma}(\{z\}) \\
&= \int_{A(z)\setminus\{z\}} V(x) \cdot \bar{s}(x)\, d\mathit{\Gamma}(x) + V(z) \cdot (\mathcal{T}_{\text{total}} - \epsilon \cdot K) \\
&\overset{(b)}{=} \int_{A(z)} V(x) \cdot \bar{s}(x)\, d\mathit{\Gamma}(x),
\end{aligned}$$

where (a) comes from Eq. (C-2), and (b) holds because $(\mathcal{T}_{\text{total}} - \epsilon \cdot K)/\mathit{\Gamma}(\{z\}) = s^H(z)$. Hence, whenever $\mathcal{T}_{\text{total}} > S^H(\{z\})$, we can assume that $s^\star(z) = s^H(z)$. We assume this for the reminder of the proof.

Let $s^\star(z)$ be an optimal solution. We show how to build $\hat{s}$ with the properties described in the theorem's statement. Next we construct r_a. First, note that

$$\int_{A(z)\setminus\{z\}} s^\star(x) d\Gamma = \int_{A(z)\setminus\{z\}} s^\star(x)\mathbf{1}_{\{s^\star\le s^H\}} d\Gamma = \int_{R_z} \underbrace{\int_{(z,X_a]} s^\star(x)\mathbf{1}_{\{s^\star\le s^H\}} d\Gamma_a(x)}_{q_a} d\Gamma^{\mathrm{p}}(a)$$

define r_a by

$$r_a \triangleq \inf\Big\{ r\in(z,X_a] : \int_{(z,r]} s^H(x)\mathbf{1}_{\{s^\star\le s^H\}} d\Gamma_a(x) \ge q_a \Big\}.$$

Observe that for $r = X_a$ the integral in the definition of r_a is larger or equal than q_a. Therefore, r_a is well defined. Let us define (with some abuse of notation)

$$A_r(z) \triangleq \bigcup_{a\in R_z} [z, r_a], \quad \text{and} \quad L_r(z) \triangleq \bigcup_{a\in R_z} \{r_a\}.$$

Let's define a new solution $\hat{s}$ by

$$\hat{s}(x) \triangleq \begin{cases} s^\star(z) = s^H(z) & \text{if } x = z \\ s^H(x) & \text{if } x \in A_r(z)\setminus (L_r(z)\cup\{z\}), \\ \mathbf{1}_{\{\Gamma_a(\{x\})>0\}} \frac{\left(q_a - \int_{(0,x)} s^H(y)\mathbf{1}_{\{s^\star\le s^H\}} d\Gamma_a(y)\right)}{\Gamma_a(\{x\})} & \text{if } x\in L_r(z), \end{cases}$$

and $\hat{s}(x) = 0$ otherwise. We show that $\hat{s}$ weakly revenue dominates $s^\star$ and that is feasible. Let us do first the <u>revenue dominance</u>. Note that the objective in $\{z\}$ of both solutions coincide; thus, we only need to compare the objective in the set $Q \triangleq A(z)\setminus\{z\}$. Note that $A_r(z)\setminus\{z\}\subset Q$, then

$$\begin{aligned} \int_Q V(x)\cdot s^\star(x)\,d\Gamma(x) &= \int_{A_r(z)\setminus\{z\}} V(x)\cdot s^\star(x)\,d\Gamma(x) + \int_{Q\setminus(A_r(z)\setminus\{z\})} V(x)\cdot s^\star(x)\,d\Gamma(x) \\ &= \int_{A_r(z)\setminus\{z\}} V(x)\cdot \hat{s}(x)\,d\Gamma(x) + \int_{A_r(z)\setminus\{z\}} V(x)\cdot (s^\star-\hat{s})(x)\,d\Gamma(x) \\ &+ \underbrace{\int_{Q\setminus(A_r(z)\setminus\{z\}))} V(x)\cdot s^\star(x)\mathbf{1}_{\{s^\star\le s^H\}}\,d\Gamma(x)}_{I}, \end{aligned}$$

for the last term above we have

$$\begin{aligned} I &\le \int_{R_z} V(r_a)\Big[\int_{(r_a,X_a]} s^\star(x)\mathbf{1}_{\{s^\star\le s^H\}}\,d\Gamma_a(x)\Big] d\Gamma^{\mathrm{p}}(a) \\ &= \int_{R_z} V(r_a)\Big[q_a - \int_{(z,r_a]} s^\star(x)\mathbf{1}_{\{s^\star\le s^H\}}\,d\Gamma_a(x)\Big] d\Gamma^{\mathrm{p}}(a) \\ &= \int_{R_z} V(r_a)\Big[\int_{(z,r_a)} (s^H - s^\star)(x)\mathbf{1}_{\{s^\star\le s^H\}}\,d\Gamma_a(x) + (\hat{s} - s^\star\mathbf{1}_{\{s^\star\le s^H\}})(r_a)\Gamma_a(r_a)\Big] d\Gamma^{\mathrm{p}}(a) \\ &\le \int_{R_z} \Big[\int_{(z,r_a)} V(x)(s^H - s^\star)(x)\mathbf{1}_{\{s^\star\le s^H\}}\,d\Gamma_a(x) + V(r_a)(\hat{s} - s^\star\mathbf{1}_{\{s^\star\le s^H\}})(r_a)\Gamma_a(r_a)\Big] d\Gamma^{\mathrm{p}}(a) \\ &= \int_{A_r(z)\setminus\{z\}} V(x)(\hat{s}-s^\star)(x) d\Gamma(x), \end{aligned}$$

hence

$$\int_Q V(x)\cdot s^\star(x)\,d\Gamma(x) \le \int_{A_r(z)\setminus\{z\}} V(x)\cdot\hat{s}(x)\,d\Gamma(x).$$

Since the right hand side above equals the objective under $\hat{s}$ in $A_r(z)$ we conclude that $\hat{s}$ is an optimal solution.

For the feasibility of $\hat{s}$, by construction and the definition of r_a we have that $\hat{s}$ satisfies (CB). Furthermore, because $s^\star$ satisfies (FR$_a$) and since $\hat{s}$ only redistributes the mass of $s^\star$ across rays but no between rays that originate in z, $\hat{s}$ also satisfies (FR$_a$). In order to verify (FC) note that

$$\begin{aligned}\int_{A(z)} \hat{s}(x)\, d\Gamma(x) &= \hat{s}(z)\cdot\Gamma(\{z\}) + \int_{A_r(z)\setminus\{z\}} \hat{s}(x)\, d\Gamma(x)\\
&= s^\star(z)\cdot\Gamma(\{z\}) + \int_{A_r(z)\setminus\{z\}} \hat{s}(x)\, d\Gamma(x)\\
&= s^\star(z)\cdot\Gamma(\{z\}) + \int_{R_z}\Big[\int_{(z,r_a)} s^H(z)\mathbf{1}_{\{s^\star\le s^H\}}\, d\Gamma_a(x) + \hat{s}(r_a)\Gamma_a(\{r_a\})\Big] d\Gamma^{\mathsf{p}}(a)\\
&= s^\star(z)\cdot\Gamma^{\mathsf{p}}(\{z\}) + \int_{R_z}\Big[q_a\Big] d\Gamma^{\mathsf{p}}(a)\\
&= s^\star(z)\cdot\Gamma^{\mathsf{p}}(\{z\}) + \int_{R_z}\Big[\int_{(z,X_a]} s^\star(x)\mathbf{1}_{\{s^\star\le s^H\}} d\Gamma_a(x)\Big] d\Gamma^{\mathsf{p}}(a)\\
&= s^\star(z)\cdot\Gamma^{\mathsf{p}}(\{z\}) + \int_{A(z)\setminus\{z\}} s^\star(x)\, d\Gamma(x)\\
&= \mathcal{T}_{\text{total}}.\end{aligned}$$

In conclusion, the solution $\hat{s}$ constructed is as defined in the statement of the theorem. Next, we use this solution to define prices and flows. We use $\hat{S}$ to denote the measure induced by $\hat{s}$. Observe that $\hat{S}$ has support in $A_r(z)$.

Part 4: Implementation. We construct a price-equilibrium pair $(\hat{p}, \hat{\mathcal{T}})$ in $A(z)$ with $\hat{\mathcal{T}} \in \mathcal{F}_{A(z)}(\tilde{\Theta})$ and

$$\tilde{\Theta}(\mathcal{B}) \triangleq \mathcal{T}((\mathcal{B}\cap A(z))\times A(z)), \quad \mathcal{B}\subseteq\mathcal{C} \text{ measurable}.$$

- **Prices.** Define $\hat{p} : A(z) \to [0, \overline{V}]$ by

$$\hat{p}(x) = \begin{cases} \rho_x^{loc}(\hat{s}(x)) & \text{if } x \in A_r(z)\setminus L_r(z);\\ p_a & \text{if } x = r_a,\ a\in R_z;\\ \overline{V} & \text{otherwise,}\end{cases}$$

  where p_a is such that $U(r_a, p_a, \hat{s}(r_a)) = V(r_a|\, p, \mathcal{T})$ for $a \in R_z$. By the way we constructed $\hat{s}(r_a)$, it is bounded by $H_{r_a}(V(r_a))$ and, therefore, the value p_a is always well defined (Γ-a.e).

- **Flows:** We define $\hat{\mathcal{T}}$ as a transport plan between $\tilde{\Theta}$ and $\hat{S}$. We start by defining the flow that $\hat{\mathcal{T}}$ sends to z and then the flow along rays.

  Flow to the center. Next we define the flow that $\hat{\mathcal{T}}$ sends to $\{z\}$. We define

$$\tilde{\Theta}_a(\mathcal{B}) \triangleq \int_{\mathcal{B}\cap(z,X_a]} \frac{d\tilde{\Theta}}{d\Gamma}(y) d\Gamma_a(y) \text{ and } \hat{S}_a(\mathcal{B}) \triangleq \int_{\mathcal{B}\cap(z,X_a]} \frac{d\hat{S}}{d\Gamma}(y) d\Gamma_a(y).$$

  Then,

$$\tilde{\Theta}(\mathcal{B}) = \mathcal{T}(\{z\}\times\{z\})\mathbf{1}_{\{z\in\mathcal{B}\}} + \int_{R_z}\tilde{\Theta}_a(\mathcal{B}) d\Gamma^{\mathsf{p}}(a) \text{ and } \hat{S}(\mathcal{B}) = \hat{S}(\{z\})\mathbf{1}_{\{z\in\mathcal{B}\}} + \int_{R_z}\hat{S}_a(\mathcal{B}) d\Gamma^{\mathsf{p}}(a).$$

  We define the quantities

$$\Delta_a \triangleq \tilde{\Theta}_a((z,X_a]) - \hat{S}_a((z,X_a]),$$

  note that because of (FR$_a$), $\Delta_a \ge 0$, $\Gamma^{\mathsf{p}} - a.e$ a in R_z. Further define

$$h_a \triangleq z + \inf\{\delta \ge 0 : \tilde{\Theta}_a((z, z+\delta]) \ge \Delta_a\}.$$

For any set $\mathcal{B} \subseteq A(z)$ we define the mass going to the center from ray $a \in R_z$ by the measures

$$\Theta_a^c(\mathcal{B}) \triangleq \tilde{\Theta}_a(\mathcal{B} \cap (z, h_a)) + \mathbf{1}_{\{h_a \in \mathcal{B} \cap (z, X_a]\}} \cdot (\Delta_a - \tilde{\Theta}_a(z, h_a)),$$

observe that by the definition h_a, the atoms above have non-negative mass, $\Gamma^{\mathrm{p}} - a.e\ a$ in R_z . Let $\mathcal{Q}_z \triangleq \{z\} \times \{z\}$. For any measurable set $\mathcal{R} \subseteq A(z) \times A(z)$, the measure that sends flow to the origin is defined by

$$\mathcal{T}^c(\mathcal{R}) \triangleq \mathcal{T}(\mathcal{R} \cap \mathcal{Q}_z) + \int_{R_z} \Theta_a^c(\pi_1(\mathcal{R} \cap A(z) \times \{z\}))d\Gamma^{\mathrm{p}}(a),$$

where π_1 is the mapping that to each pair (x, y) assigns the first component x. Using Lemma C-1 (which we state and prove after the present proof) we can verify that $\mathcal{T}^c \in \mathcal{M}(A(z) \times A(z))$.

Flow along rays. For any ray $a \in R_z$ define the flow $\tilde{\gamma}_a$ along that ray to be the solution to the following optimal transport problem:

$$\begin{aligned} \min \int_{(z,X_a]\times(z,X_a]} & \|x - y\| \, d\gamma_a(x, y) \\ \text{s.t } & \gamma_a \in \Pi(\tilde{\Theta}_a^{\mathbf{r}}, \hat{S}_a), \end{aligned}$$

where

$$\tilde{\Theta}_a^{\mathbf{r}}(\mathcal{B}) \triangleq \tilde{\Theta}_a(\mathcal{B} \cap (h_a, X_a]) + \mathbf{1}_{\{h_a \in \mathcal{B} \cap (z, X_a]\}} \cdot (\tilde{\Theta}_a(z, h_a] - \Delta_a),$$

where $\Pi(\tilde{\Theta}_a^{\mathbf{r}}, \hat{S}_a)$ is the set of transport plans between $\tilde{\Theta}_a^{\mathbf{r}}$ and $\hat{S}_a$. Any solution to this problem satisfies:

$$\tilde{\gamma}_a\Big(\{(x, y) \in (z, X_a] \times (z, X_a] : y > x\}\Big) = 0, \quad \Gamma^{\mathrm{p}} - a.e.\ a \in R_z. \tag{C-3}$$

We provide a proof Eq. (C-3) after **Part 5**.

We will argue that $\hat{\mathcal{T}}$ defined by

$$\hat{\mathcal{T}}(\mathcal{R}) = \mathcal{T}^c(\mathcal{R}) + \int_{R_z} \tilde{\gamma}_a(\mathcal{R})d\Gamma^{\mathrm{p}}(a)$$

yields an equilibrium, that is, for the set

$$\tilde{\mathcal{E}} \triangleq \Big\{(x, y) \in A(z) \times A(z) : U(y, \hat{p}(y), s^{\hat{\mathcal{T}}}(y)) - |y - x| = V_{A(z)}(x|\, \hat{p}, \hat{\mathcal{T}})\Big\},$$

we have that $\hat{\mathcal{T}}(\tilde{\mathcal{E}})$ equals $\tilde{\Theta}(A(z))$. Note that with this definition of $\hat{\mathcal{T}}$ there is not flow being transported across rays but only within rays. Before verifying the equilibrium condition we check that $\hat{\mathcal{T}} \in \mathcal{F}_{A(z)}(\tilde{\Theta})$. Clearly $\hat{\mathcal{T}}$ is a non-negative measure in $A(z) \times A(z)$ because is the sum of non-negative measures. Now we check that $\hat{\mathcal{T}}_1 = \tilde{\Theta}$. Consider a measurable set $\mathcal{B} \subseteq A(z)$ then

$$\begin{aligned} \tilde{\mathcal{T}}_1(\mathcal{B}) &= \tilde{\mathcal{T}}(\mathcal{B} \times A(z)) \\ &= \mathcal{T}^c(\mathcal{B} \times A(z)) + \int_{R_z} \tilde{\gamma}_a(\mathcal{B} \times A(z))d\Gamma^{\mathrm{p}}(a) \\ &= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z \in \mathcal{B}\}} + \int_{R_z} \Theta_a^c(\mathcal{B})d\Gamma^{\mathrm{p}}(a) + \int_{R_z} \tilde{\Theta}_a^{\mathbf{r}}(\mathcal{B} \cap (z, X_a])d\Gamma^{\mathrm{p}}(a) \\ &= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z \in \mathcal{B}\}} + \int_{R_z} \Big[\tilde{\Theta}_a(\mathcal{B} \cap (z, h_a)) + \mathbf{1}_{\{h_a \in \mathcal{B} \cap (z, X_a]\}} \cdot (\Delta_a - \tilde{\Theta}_a(z, h_a)) \\ &+ \tilde{\Theta}_a(\mathcal{B} \cap (h_a, X_a]) + \mathbf{1}_{\{h_a \in \mathcal{B} \cap (z, X_a]\}} \cdot (\tilde{\Theta}_a(z, h_a] - \Delta_a)\Big]d\Gamma^{\mathrm{p}}(a) \\ &= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z \in \mathcal{B}\}} + \int_{R_z} \tilde{\Theta}_a(\mathcal{B})d\Gamma^{\mathrm{p}}(a) \\ &= \tilde{\Theta}(\mathcal{B}) \end{aligned}$$

and from the definition of $\tilde{\Theta}$ we also have $\hat{\mathcal{T}}_1 \ll \Gamma$. For the second marginal of $\hat{\mathcal{T}}$ we have

$$
\begin{aligned}
\hat{\mathcal{T}}_2(\mathcal{B}) &= \tilde{\mathcal{T}}(A(z)\times \mathcal{B}) \\
&= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z\in\mathcal{B}\}} + \int_{R_z} \Theta^c_a(A(z))\mathbf{1}_{\{z\in\mathcal{B}\}} d\Gamma^{\mathrm{p}}(a) + \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a) \\
&= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z\in\mathcal{B}\}} + \mathbf{1}_{\{z\in\mathcal{B}\}} \int_{R_z} \left[\tilde{\Theta}_a((z,h_a)) + \mathbf{1}_{\{h_a\in(z,X_a]\}}\cdot(\Delta_a - \tilde{\Theta}_a(z,h_a))\right] d\Gamma^{\mathrm{p}}(a) \\
&+ \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a) \\
&= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z\in\mathcal{B}\}} + \mathbf{1}_{\{z\in\mathcal{B}\}} \int_{R_z} \Delta_a d\Gamma^{\mathrm{p}}(a) + \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a) \\
&= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z\in\mathcal{B}\}} + \mathbf{1}_{\{z\in\mathcal{B}\}} \int_{R_z} [\tilde{\Theta}_a((z,X_a]) - \hat{S}_a((z,X_a])] d\Gamma^{\mathrm{p}}(a) + \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a) \\
&= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z\in\mathcal{B}\}} + \mathbf{1}_{\{z\in\mathcal{B}\}} \int_{R_z} [\tilde{\Theta}_a(A(z)) - \hat{S}_a(A(z))] d\Gamma^{\mathrm{p}}(a) + \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a) \\
&= \mathcal{T}(\mathcal{Q}_z)\mathbf{1}_{\{z\in\mathcal{B}\}} + \mathbf{1}_{\{z\in\mathcal{B}\}} \left[\tilde{\Theta}(A(z)) - \mathcal{T}(\mathcal{Q}_z) - \hat{S}(A(z)) + \hat{S}(\{z\})\right] + \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a) \\
&= \hat{S}(\{z\})\mathbf{1}_{\{z\in\mathcal{B}\}} + \mathbf{1}_{\{z\in\mathcal{B}\}} \underbrace{\left[\tilde{\Theta}(A(z)) - \hat{S}(A(z))\right]}_{=0} + \int_{R_z} \hat{S}_a(\mathcal{B}) d\Gamma^{\mathrm{p}}(a) \\
&= \hat{S}(\mathcal{B}).
\end{aligned}
$$

Since $\hat{S}$ is such that $\hat{S} \ll \Gamma$, we conclude that $\hat{\mathcal{T}} \in \mathcal{F}_{A(z)}(\tilde{\Theta})$. Also, $s^{\hat{\mathcal{T}}}$ coincides with $\hat{s}$ Γ almost everywhere. Before we move to verify that $\hat{\mathcal{T}}$ is an equilibrium, we next compute $V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}})$ and $U(y,\hat{p}(y),s^{\hat{\mathcal{T}}}(y))$.

- **Equilibrium utilities:** From the definition of $\hat{p}$ and the value of $s^{\hat{\mathcal{T}}}$ we have that $\Gamma - a.e.$ y in $A(z)$

$$
U(y,\hat{p}(y),s^{\hat{\mathcal{T}}}(y)) = \begin{cases} V(z|\,p,\mathcal{T}) - |z-y| & \text{if } y \in A_r(z), \\ 0 & \text{if } y \in A(z)\setminus A_r(z). \end{cases}
$$

Next we verify that $V_{A(z)}(\cdot|\,\hat{p},\hat{\mathcal{T}})$ satisfies the hypothesis in Proposition B-2, the pasting result. First, for any $x \in A(z)$ we argue that $V(x|\,p,\mathcal{T}) = V(z|\,p,\mathcal{T}) - |z-x| \geq V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}})$. It is enough to show that

$$
\Gamma(y \in A(z) : U(y,\hat{p}(y),s^{\hat{\mathcal{T}}}(y)) - |y-x| > V(x|\,p,\mathcal{T})) = 0.
$$

Suppose this is not true. Lemma A-1 implies that $V(y|\,p,\mathcal{T})$ is non-negative $\Gamma - a.e.$ Also $V(y|\,p,\mathcal{T})$ equals $V(z|\,p,\mathcal{T}) - |z-y|$ for any $y \in A(z)$. Hence it must be true that $V(y|\,p,\mathcal{T})$ is larger or equal than $U(y,\hat{p}(y),s^{\hat{\mathcal{T}}}(y))$ $\Gamma - a.e$ (see the value of this expression above). Thus our current assumption implies

$$
\Gamma(y \in A(z) : V(y|\,p,\mathcal{T}) - |y-x| > V(x|\,p,\mathcal{T})) > 0,
$$

but this contradicts the Lipschitz property of $V(\cdot|\,p,\mathcal{T})$. Second, we show that

$$
\tilde{\Theta}(\{x \in A(z) : V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}}) = V(x|\,p,\mathcal{T})\}) = \tilde{\Theta}(A(z)). \tag{C-4}
$$

If this is not true then $\tilde{\Theta}(\mathcal{X}) > 0$ where $\mathcal{X} = \{x \in A(z) : V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}}) < V(x|\,p,\mathcal{T})\}$. In turn this implies that $\hat{\mathcal{T}}(\mathcal{X}\times A(z)) > 0$. Then as in the proof of Lemma B-3 (see Eq. (B-2)) we have that there exists $(x,y) \in \mathcal{X}\times A(z)$ such that $\hat{\mathcal{T}}(B(x,\delta)\times B(y,\delta)) > 0$ for all $\delta > 0$,where $B(x,\delta)$ is an open ball of radius δ. Since $x \in \mathcal{X}$ we have that $x \in A(z)$ and $V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}}) < V(x|\,p,\mathcal{T})$. Thus we can find $\delta > 0$ such that $V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}}) + 2\delta < V(x|\,p,\mathcal{T})$. Moreover since $\hat{\mathcal{T}}$ only send flows along rays originating from z we must have that $\hat{\mathcal{T}}(B(x,\delta)\times B(y,\delta)) = \hat{\mathcal{T}}(\{(\bar{x},\bar{y})\in B(x,\delta)\times B(y,\delta) : \|z-x\|+2\delta \geq \|\bar{y}-z\|+\|\bar{y}-x\|\}) > 0$. This in turn implies that

$$\begin{aligned}
0 &< \mathit{\Gamma}(\bar{y}\in B(y,\delta)\cap A_r(z) :\ 2\delta+\|z-x\| \geq \|\bar{y}-x\|+\|\bar{y}-z\|)\\
&= \mathit{\Gamma}(\bar{y}\in B(y,\delta)\cap A_r(z) : U(\bar{y},\hat{p}(\bar{y}),s^{\hat{\mathcal{T}}}(\bar{y})) +\ 2\delta+\|z-x\| \geq V(\bar{y}|\, p,\mathcal{T})+\|\bar{y}-x\|+\|\bar{y}-z\|)\\
&= \mathit{\Gamma}(\bar{y}\in B(y,\delta)\cap A_r(z) : U(\bar{y},\hat{p}(\bar{y}),s^{\hat{\mathcal{T}}}(\bar{y})) - \|\bar{y}-x\| \geq V(x|\, p,\mathcal{T}) - 2\delta)\\
&\leq \mathit{\Gamma}(\bar{y}\in B(y,\delta)\cap A_r(z) : U(\bar{y},\hat{p}(\bar{y}),s^{\hat{\mathcal{T}}}(\bar{y})) - \|\bar{y}-x\| > V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}}))\\
&\leq \mathit{\Gamma}(\bar{y}\in A(z) : U(\bar{y},\hat{p}(\bar{y}),s^{\hat{\mathcal{T}}}(\bar{y})) - \|\bar{y}-x\| > V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}})),
\end{aligned}$$

which contradicts the definition of $V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}})$. This shows that Eq. (C-4) holds.

- **Equilibrium condition:** Consider the equilibrium set

$$\tilde{\mathcal{E}} \triangleq \Big\{(x,y)\in A(z)\times A(z) : U(y,\hat{p}(y),s^{\hat{\mathcal{T}}}(y)) - \|y-x\| = V_{A(z)}(x|\,\hat{p},\hat{\mathcal{T}})\Big\},$$

we need to verify that $\hat{\mathcal{T}}(\tilde{\mathcal{E}})$ equals $\tilde{\mathit{\Theta}}(A(z))$. First, for $\hat{\mathcal{T}}(\tilde{\mathcal{E}})$ we have,

$$\hat{\mathcal{T}}(\tilde{\mathcal{E}}) \overset{(a)}{=} \hat{\mathcal{T}}\Big(\Big\{(x,y)\in A(z)\times A_r(z) :\ \|z-y\|+\|y-x\| = \|z-x\|\Big\}\Big)$$

In (a) we use Eq. (C-4), that $V(x|\,p,\mathcal{T}) = V(z|\,p,\mathcal{T}) - \|x-z\|$ and that $\hat{\mathcal{T}}_2$ only puts mass in $A_r(z)$. Consider the sets

$$\tilde{\mathcal{E}}_c \triangleq A(z)\times\{z\},\ \text{ and }\ \tilde{\mathcal{E}}_a \triangleq \Big\{(x,y)\in (z,X_a]\times(z,r_a] :\ y\leq x\Big\}.$$

Then,

$$\hat{\mathcal{T}}(\tilde{\mathcal{E}}) = \tilde{\mathcal{T}}(\tilde{\mathcal{E}}_c) + \int_{R_z} \tilde{\gamma}_a(\tilde{\mathcal{E}}_a)d\mathit{\Gamma}^{\mathrm{p}}(a) + Z.$$

For the first term we have $\hat{\mathcal{T}}(\tilde{\mathcal{E}}_c) = \hat{\mathcal{T}}_2(\{z\})$. For the second term we have that for any ray a, $\tilde{\gamma}_a(\tilde{\mathcal{E}}_a)$ equals $\hat{S}_a((z,r_a])$. This is true because the plan $\tilde{\gamma}_a$ only sends mass to $(z,r_a]$(this is the support of $\hat{S}_a$) and it does not send mass in the opposite direction of z, see Eq. (C-3). Therefore,

$$\hat{\mathcal{T}}(\tilde{\mathcal{E}}) = \hat{\mathcal{T}}_2(\{z\}) + \int_{R_z} \hat{S}_a((z,r_a])d\mathit{\Gamma}^{\mathrm{p}}(a) = \hat{S}(A_r(z))$$

Now, recall that $\tilde{\mathit{\Theta}}(A(z)) = \hat{S}(A_r(z))$ and, therefore, $\hat{\mathcal{T}}(\tilde{\mathcal{E}}) = \tilde{\mathit{\Theta}}(A(z))$, as desired.

Part 5: Conclusion. We conclude by applying Proposition B-2. The price-equilibrium pair $(\hat{p},\hat{\mathcal{T}})$ satisfies the hypothesis in Proposition B-2, so we can create a global price-equilibrium pair which we still denote by $(\hat{p},\hat{\mathcal{T}})$ in $\mathcal{C}$. This new solution has the same objective that $(p,\mathcal{T})$ in $A(z)^c$, but it dominates the platform revenue in $A(z)$. Therefore, $(\hat{p},\hat{\mathcal{T}})$ revenue dominates $(p,\mathcal{T})$.

Proof of Eq. (C-3): We show that

$$\tilde{\gamma}_a\Big(\{(x,y)\in(z,X_a]\times(z,X_a] : y > x\}\Big) = 0,\quad \mathit{\Gamma}^{\mathrm{p}} - a.e.\ a\in R_z.$$

First we show that both measures $\tilde{\mathit{\Theta}}^{\mathrm{r}}_a$ and $\hat{S}_a$ satisfy:

$$\tilde{\mathit{\Theta}}^{\mathrm{r}}_a((z,b_a]) \leq \hat{S}_a((z,c_a])\quad \forall b_a,c_a\in(z,X_a],\ b_a\leq c_a,\quad \mathit{\Gamma}^{\mathrm{p}} - a.e.\ a\in R_z, \tag{C-5}$$

where b_a and c_a lie in the ray indexed by $a\in R_z$. To see why this is true let us proceed by contradiction. Let us denote by Q de set where Eq. (C-5) is not satisfied, we have that $\mathit{\Gamma}^{\mathrm{p}}(Q) > 0$. Note that for any $a\in Q$ we can find b_a and c_a for which the inequality in Eq. (C-5) is not satisfied, so let us thus fix such collection of b_a and c_a. Moreover, from the definition of $\tilde{\Theta}^{\mathtt{r}}_a$ we deduce that for any $a \in Q$ we have $h_a \le b_a$ (otherwise $\tilde{\Theta}^{\mathtt{r}}_a((z, b_a]) = 0$ and, as a consequence, a could not belong to Q). Then,

$$
\begin{aligned}
\int_Q \tilde{\Theta}^{\mathtt{r}}_a((z, b_a]) d\Gamma^{\mathtt{p}}(a) &= \int_Q \tilde{\Theta}_a((z, b_a] \cap (h_a, X_a]) + \mathbf{1}_{\{h_a \in (z, b_a]\}} \cdot (\tilde{\Theta}_a(z, h_a] - \Delta_a) d\Gamma^{\mathtt{p}}(a) \\
&\overset{(a)}{\le} \int_Q \tilde{\Theta}_a((z, b_a] \cap (h_a, X_a]) + \mathbf{1}_{\{h_a \le b_a\}} \cdot (\tilde{\Theta}_a(z, h_a] - \Delta_a) d\Gamma^{\mathtt{p}}(a) \\
&= \int_Q (\tilde{\Theta}_a((h_a, b_a]) + \tilde{\Theta}_a((z, h_a]) - \Delta_a) d\Gamma^{\mathtt{p}}(a) \\
&= \int_Q (\tilde{\Theta}_a((z, b_a]) - \Delta_a) d\Gamma^{\mathtt{p}}(a) \\
&= \underbrace{\int_{Q \cap \{a : r_a \le b_a\}} (\tilde{\Theta}_a((z, b_a]) - \Delta_a) d\Gamma^{\mathtt{p}}(a)}_{(*)} \\
&\quad + \underbrace{\int_{Q \cap \{a : r_a > b_a\}} (\tilde{\Theta}_a((z, b_a]) - \Delta_a) d\Gamma^{\mathtt{p}}(a)}_{(**)},
\end{aligned}
$$

where (a) follows from $\tilde{\Theta}_a(z, h_a] \ge \Delta_a$. For $(*)$ we have

$$
\begin{aligned}
(*) &= \int_{Q \cap \{a : r_a \le b_a\}} (\tilde{\Theta}_a((z, b_a]) - \tilde{\Theta}((z, X_a]) + \hat{S}_a((z, X_a])) d\Gamma^{\mathtt{p}}(a) \\
&= \int_{Q \cap \{a : r_a \le b_a\}} (-\tilde{\Theta}((b_a, X_a]) + \hat{S}_a((z, X_a])) d\Gamma^{\mathtt{p}}(a) \\
&\le \int_{Q \cap \{a : r_a \le b_a\}} \hat{S}_a((z, X_a]) d\Gamma^{\mathtt{p}}(a) \\
&= \int_{Q \cap \{a : r_a \le b_a\}} \hat{S}_a((z, c_a]) d\Gamma^{\mathtt{p}}(a),
\end{aligned}
$$

the last inequality holds because

Now we analyze $(**)$. Denote by Q^r the set of rays $a \in R_z$ such that $r_a > b_a$ and $a \in Q$. Then

$$
\begin{aligned}
\int_{Q^r} \tilde{\Theta}_a((z, b_a]) d\Gamma^{\mathtt{p}}(a) &= \tilde{\Theta}\Big(\bigcup_{a \in Q^r} (z, b_a] \Big) \\
&= \mathcal{T}\Big(\bigcup_{a \in Q^r} (z, b_a] \times \bigcup_{a \in Q^r} (z, X_a] \Big) + \underbrace{\mathcal{T}\Big(\bigcup_{a \in Q^r} (z, b_a] \times \{z\} \Big)}_{\triangleq \ell_r} \\
&= \mathcal{T}\Big(\bigcup_{a \in Q^r} (z, b_a] \times \bigcup_{a \in Q^r} (z, b_a] \Big) + \mathcal{T}\Big(\bigcup_{a \in Q^r} (z, b_a] \times \bigcup_{a \in Q^r} (b_a, X_a] \Big) + \ell_r \\
&= \mathcal{T}\Big(\bigcup_{a \in Q^r} (z, b_a] \times \bigcup_{a \in Q^r} (z, b_a] \Big) + \ell_r \\
&\le \mathcal{T}_2\Big(\bigcup_{a \in Q^r} (z, b_a] \Big) + \ell_r \\
&\le \hat{S}\Big(\bigcup_{a \in Q^r} (z, b_a] \Big) + \ell_r \\
&= \int_{Q^r} \hat{S}_a((z, c_a]) d\Gamma^{\mathtt{p}}(a) + \ell_r,
\end{aligned}
$$

the first equality comes from the definition of $\tilde{\Theta}_a$ and then integrating this disintegration of measures. The second and fourth equality come from Proposition B-1 part *(ii)*. The last inequality comes from the congestion bound. For Δ_a we have

$$\begin{aligned}\int_{Q^r} \Delta_a d\Gamma^{\mathrm{p}}(a) &= \tilde{\Theta}\Big(\bigcup_{a\in Q^r}(z,X_a]\Big) - \hat{S}\Big(\bigcup_{a\in Q^r}(z,X_a]\Big)\\ &= \mathcal{T}\Big(\bigcup_{a\in Q^r}(z,X_a]\times\bigcup_{a\in Q^r}(z,X_a]\Big) + \mathcal{T}\Big(\bigcup_{a\in Q^r}(z,X_a]\times\{z\}\Big) - \hat{S}\Big(\bigcup_{a\in Q^r}(z,X_a]\Big)\\ &\geq \mathcal{T}\Big(\bigcup_{a\in Q^r}(z,X_a]\times\bigcup_{a\in Q^r}(z,X_a]\Big) + \ell_r - \hat{S}\Big(\bigcup_{a\in Q^r}(z,X_a]\Big)\\ &= \mathcal{T}_2\Big(\bigcup_{a\in Q^r}(z,X_a]\Big) + \ell_r - \hat{S}\Big(\bigcup_{a\in Q^r}(z,X_a]\Big)\\ &\geq \ell_r,\end{aligned}$$

where the last inequality comes from Eq. (FR$_a$). As a consequence we deduce that

$$(**) = \int_{Q\cap\{a:r_a>b_a\}} (\tilde{\Theta}_a((z,b_a]) - \Delta_a) d\Gamma^{\mathrm{p}}(a) \leq \int_{Q\cap\{a:r_a>b_a\}} \hat{S}_a((z,b_a])d\Gamma^{\mathrm{p}}(a)$$

Putting together the bounds for $(*)$ and $(**)$ we deduce that

$$\int_Q \tilde{\Theta}^{\mathrm{r}}_a((z,b_a])d\Gamma^{\mathrm{p}}(a) \leq \int_Q \hat{S}_a((z,c_a])d\Gamma^{\mathrm{p}}(a),$$

since $\Gamma^{\mathrm{p}}(Q)>0$ the previous inequality yields a contradiction. We conclude that Eq. (C-5) holds.

To finalize the proof of Eq. (C-3). Consider the set where Eq. (C-5) holds (the complement of this set has Γ^{p} measure equal to zero). for any ray a in this suppose that

$$\tilde{\gamma}_a\Big(\{(x,y)\in(z,X_a]\times(z,X_a]: y>x\}\Big) > 0.$$

From the proof of Lemma B-3 we deduce that there exists $(x,y)\in(z,X_a]\times(z,X_a]$ such that $y>x$ and $\tilde{\gamma}_a((z,x+\delta]\times(y-\delta,X_a))>0$, where $\delta>0$ can be taken small enough such that $x+\delta<y-\delta$. Then,

$$\begin{aligned}\hat{S}_a((z,x+\delta]) &\geq \tilde{\Theta}^{\mathrm{r}}_a((z,x+\delta])\\ &= \tilde{\gamma}_a((z,x+\delta]\times(z,X_a])\\ &= \tilde{\gamma}_a((z,x+\delta]\times(z,x+\delta]) + \tilde{\gamma}_a((z,x+\delta]\times(x+\delta,X_a])\\ &> \tilde{\gamma}_a((z,x+\delta]\times(z,x+\delta])\\ &= \hat{S}_a((z,x+\delta]) - \tilde{\gamma}_a((x+\delta,X_a]\times(z,x+\delta]),\end{aligned}$$

Thus,

$$\tilde{\gamma}_a((x+\delta,X_a]\times(z,x+\delta])>0, \text{ and we also have } \tilde{\gamma}_a((z,x+\delta]\times(y-\delta,X_a))>0,$$

but this is not possible because $\tilde{\gamma}_a$ is an optimal transport and, therefore, it is concentrated on a c-cyclically monotone set where $c(x,y)=\|x-y\|$, see Villani (2008). This concludes the proof of Eq. (C-3). □

Lemma C-1. *Let ν be a non-negative measure in $\mathcal{C}$, and π_1 a mapping be such that $\pi_1(x,y)=x$. Consider any measurable subset K of $\mathcal{C}$ and some $z\in\mathcal{C}$ then the mappings $\nu(\pi_1(\cdot\cap\mathcal{D})\cap K)$ and $\nu(\pi_1(\cdot\cap(K\times\{z\})))$, defined on the Borel sets of $\mathcal{C}\times\mathcal{C}$, belong to $\mathcal{M}(\mathcal{C}\times\mathcal{C})$.*

Proof. For any Borel set $\mathcal{L} \subset \mathcal{C} \times \mathcal{C}$ define

$$\mathcal{T}_a(\mathcal{L}) \triangleq \nu(\pi_1(\mathcal{L} \cap \mathcal{D}) \cap K) \ \text{ and } \ \mathcal{T}_b(\mathcal{L}) \triangleq \nu(\pi_1(\mathcal{L} \cap (K \times \{z\}))).$$

We show that $\mathcal{T}_a, \mathcal{T}_b \in \mathcal{M}(\mathcal{C} \times \mathcal{C})$. Note that because $\nu \in \mathcal{M}(\mathcal{C})$ for $i \in \{a, b\}$ we have that $\mathcal{T}_i(\emptyset) = 0$, and for any Borel set $\mathcal{L} \subseteq \mathcal{C} \times \mathcal{C}$ that $\mathcal{T}_i(\mathcal{L}) \in [0, \infty)$. To verify $\sigma-$ additivity consider a countable partition $\{\mathcal{L}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C} \times \mathcal{C}$, we need to show that

$$\mathcal{T}_i\Big(\bigcup_{n\in\mathbb{N}} \mathcal{L}_n\Big) = \sum_{n\in\mathbb{N}} \mathcal{T}_i(\mathcal{L}_n).$$

Note that from the definition of $\mathcal{D}$ and the fact the set $K \times \{z\}$ has second component equal to 0, both collections $\{\pi_1(\mathcal{L}_n \cap \mathcal{D})\}_{n\in\mathbb{N}}$ and $\{\pi_1(\mathcal{L}_n \cap (K \times \{z\}))\}_{n\in\mathbb{N}}$ form a partition. Given this we can verify $\sigma-$ additivity, we do it for both $\mathcal{T}_a$ and $\mathcal{T}_b$ at the same time

$$\begin{aligned}
\mathcal{T}_a\Big(\bigcup_{n\in\mathbb{N}} \mathcal{L}_n\Big) + \mathcal{T}_b\Big(\bigcup_{n\in\mathbb{N}} \mathcal{L}_n\Big) &= \nu(\pi_1(\bigcup_{n\in\mathbb{N}} \mathcal{L}_n \cap \mathcal{D}) \cap K) + \nu(\pi_1(\bigcup_{n\in\mathbb{N}} \mathcal{L}_n \cap K \times \{z\})) \\
&= \nu(\bigcup_{n\in\mathbb{N}} \pi_1(\mathcal{L}_n \cap \mathcal{D}) \cap K) + \nu(\bigcup_{n\in\mathbb{N}} \pi_1(\mathcal{L}_n \cap K \times \{z\})) \\
&= \sum_{n\in\mathbb{N}} \nu(\pi_1(\mathcal{L}_n \cap \mathcal{D}) \cap K) + \sum_{n\in\mathbb{N}} \nu(\pi_1(\mathcal{L}_n \cap K \times \{z\})) \\
&= \sum_{n\in\mathbb{N}} \mathcal{T}_a(\mathcal{L}_n) + \sum_{n\in\mathbb{N}} \mathcal{T}_b(\mathcal{L}_n),
\end{aligned}$$

where the third line comes from the $\sigma-$additivity of the ν measure. Thus $\mathcal{T} \in \mathcal{M}(\mathcal{C} \times \mathcal{C})$. □

D Proofs for Section 8

Preliminaries. We use $m \in \mathcal{M}(\mathcal{C})$ denotes the Lebesgue measure in $\mathcal{C}$. We use $\mathcal{D}$ to denote the subset of $\mathcal{C} \times \mathcal{C}$ with equal first and second components, that is, $\mathcal{D} = \{(x, y) \in \mathcal{C} \times \mathcal{C} : x = y\}$. For any measurable set $\mathcal{B} \subseteq \mathcal{C}$ and a price-equilibrium pair $(p, \mathcal{T})$ we denote the platform's revenue in $\mathcal{B}$ under $(p, \mathcal{T})$ by $\mathbf{Rev}_{\mathcal{B}}(p, \mathcal{T})$. In case that $\mathcal{B}$ is $\mathcal{C}$ we simply use $\mathbf{Rev}(p, \mathcal{T})$. For any measure $\mathcal{V}$ we denote its restriction to a set $\mathcal{B}$ by $\mathcal{V}|_{\mathcal{B}}$, and for any set $\mathcal{E}$ we denote its restriction to a set $\mathcal{B}$ by $\mathcal{E}|_{\mathcal{B}}$. Let $B(x, \delta)$ be an open ball of radius δ around x.

Proposition D-1 (Pre-demand Shock Environment)**.** *Suppose $\lambda_0 = 0$. Then, the optimal policy and corresponding supply equilibrium and flows can be characterized as follows.*

(i) (Prices) The optimal pricing policy is given by $p(x) = \rho_1$, for all x in $\mathcal{C}$.

(ii) (Flow) All supply units stay at their original locations.

Furthermore, the optimal revenue equals $\gamma \cdot \psi_1 \cdot \theta_1 \cdot 2H$.

Proof of Proposition D-1. Let $(p, \mathcal{T})$ be any feasible price-equilibrium pair by Lemma D-1 (which state and prove after this proof) we have $V(x|\, p, \mathcal{T}) \le \psi_1$, Γ almost everywhere in $\mathcal{C}_\lambda = \mathcal{C} \setminus \{0\}$. This yields the following upper bound for the platform's objective

$$\int_{\mathcal{C}_\lambda} V(x|\, p, \mathcal{T}) \cdot s^{\mathcal{T}}(x)\, dx \le \psi_1 \cdot \int_{\mathcal{C}_\lambda} s^{\mathcal{T}}(x)\, dx \le \psi_1 \cdot \theta_1 \cdot m(\mathcal{C}).$$

The maximum revenue the platform can achieve in this case is bounded above by $\gamma \cdot \psi_1 \cdot \theta_1 \cdot m(\mathcal{C})$. Next, we show that the solution given in the statement of the lemma is feasible and achieves the upper bound.

Flow feasibility. We show that $\mathcal{T} \in \mathcal{F}(\Theta)$. A complete definition of the measure $\mathcal{T}$ is $\mathcal{T}(\mathcal{L}) = \Theta(\pi_1(\mathcal{L} \cap \mathcal{D})$. From the definition of $\mathcal{T}$ it is clear that $\mathcal{T} \in \mathcal{M}(\mathcal{C})$. Furthermore, $\mathcal{T}_1$ coincides with Θ and so does $\mathcal{T}_2$. Since Θ is the Lebesgue measure times a constant and Γ is the Lebesgue measure plus an atom, we have $\mathcal{T}_1, \mathcal{T}_2 \ll \Gamma$. From this we can deduce that $m-$ a.e in $\mathcal{C}_\lambda$, $s^{\mathcal{T}}(x)$ equals θ_1.

Equilibrium utilities. We show that $V(x|\, p, \mathcal{T})$ equals ψ_1. Note that

$$U(y, p(y), s^{\mathcal{T}}(y)) = \psi_1, \quad \Gamma - a.e.\ y \text{ in } \mathcal{C}_\lambda.$$

Fix $x \in \mathcal{C}$, we have that

$$\Gamma(\{y \in \mathcal{C} : U(y, p(y), s^{\mathcal{T}}(y)) - |y - x| > \psi_1\}) = \mathbf{1}_{\{0 - |0 - x| > \psi_1\}} + \Gamma(\{y \in \mathcal{C} \setminus \{0\} : -|y - x| > 0\}) = 0.$$

Moreover, for any $\epsilon > 0$

$$\Gamma(\{y \in \mathcal{C} : U(y, p(y), s^{\mathcal{T}}(y)) - |y - x| > \psi_1 - \epsilon\}) \ge \Gamma(\{y \in \mathcal{C}_\lambda : -|y - x| > \epsilon\}) > 0,$$

where the last inequality comes from the fact that Γ corresponds to the Lebesgue measure (plus an atom). That is, $V(x|\, p, \mathcal{T})$ equals ψ_1.

Equilibrium condition. Consider the equilibrium set

$$\mathcal{E} \triangleq \Big\{(x, y) \in \mathcal{C} \times \mathcal{C} : U(y, p(y), s^{\mathcal{T}}(y)) - |y - x| = V(x|\, p, \mathcal{T})\Big\}.$$

Then,

$$\mathcal{T}(\mathcal{E}) = \mathcal{T}\Big(\Big\{(x, y) \in \mathcal{C} \times \{0\} : -|y - x| = \psi_1\Big\}\Big) + \mathcal{T}\Big(\Big\{(x, y) \in \mathcal{C} \times \mathcal{C}_\lambda : -|y - x| = 0\Big\}\Big) = \Theta(\mathcal{C}).$$

We have proven that the solution is the statement is feasible, and because of the values of $V(\cdot|\, p, \mathcal{T})$ and $s^{\mathcal{T}}(\cdot)$ we conclude that this solution achieves the upper bound. $\square$

Lemma D-1. *Let p be any price mapping and $\mathcal{T}$ a corresponding equilibrium flow. Then for any measurable set $\mathcal{B} \subseteq \mathcal{C}_\lambda$ such that $0 \notin \mathcal{B}$ and $\mathcal{T}(\mathcal{B} \times \mathcal{B}^c) = 0$ we have*

$$V(x|p, \mathcal{T}) \leq \psi_1, \quad \Gamma - a.e. \ x \ \ in \ \mathcal{B}.$$

Furthermore, in the pre-shock environment we can replace $\mathcal{B}$ with $\mathcal{C}_\lambda$ in the inequality above.

Proof. Define the set

$$\mathcal{L} \triangleq \{x \in \mathcal{B} : V(x|p, \mathcal{T}) \leq \psi_1\}.$$

We would like to show that $\Gamma(\mathcal{L}^c) = 0$ where the complement is taken with respect to $\mathcal{B}$. Suppose this is not the case, and note that

$$\theta_1 \cdot m(\mathcal{L}^c) = \Theta(\mathcal{L}^c) = \mathcal{T}(\mathcal{L}^c \times \mathcal{C}) = \mathcal{T}(\mathcal{L}^c \times \mathcal{B}) + \mathcal{T}(\mathcal{L}^c \times \mathcal{B}^c),$$

since $\mathcal{L}^c \subseteq \mathcal{B}$ and $\mathcal{T}(\mathcal{B} \times \mathcal{B}^c) = 0$, the second term in the expression above is zero. This yields,

$$\begin{aligned} \theta_1 \cdot m(\mathcal{L}^c) &= \mathcal{T}(\mathcal{L}^c \times \mathcal{B}) \\ &= \mathcal{T}(\mathcal{L}^c \times \mathcal{B} \cap \mathcal{L}^c) + \mathcal{T}(\mathcal{L}^c \times \mathcal{B} \cap \mathcal{L}) \\ &= \mathcal{T}(\mathcal{L}^c \times \mathcal{L}^c) + \mathcal{T}(\mathcal{L}^c \times \mathcal{L}) \end{aligned}$$

There are two cases. First, if $\mathcal{T}(\mathcal{L}^c \times \mathcal{L}) > 0$ then by Lemma B-3 there exists a pair $(x, y) \in \mathcal{L}^c \times \mathcal{L}$ such that $y \in \mathcal{IR}(x|p, \mathcal{T})$ thus

$$V(y|p, \mathcal{T}) = V(x|p, \mathcal{T}) + |x - y|.$$

However, since $(x, y) \in \mathcal{L}^c \times \mathcal{L}$

$$V(y|p, \mathcal{T}) \leq \psi_1 \text{ and } V(x|p, \mathcal{T}) > \psi_1.$$

Using the previous equation we can deduce that $\psi_1 > \psi_1$, which is not possible. The second case is $\mathcal{T}(\mathcal{L}^c \times \mathcal{L}) = 0$. Note that

$$\mathcal{T}_2(\mathcal{L}^c) = \mathcal{T}(\mathcal{C} \times \mathcal{L}^c) \geq \mathcal{T}(\mathcal{L}^c \times \mathcal{L}^c) = \theta_1 \cdot m(\mathcal{L}^c).$$

We also have that

$$\mathcal{T}_2(\mathcal{L}^c) = \int_{\mathcal{L}^c} s^{\mathcal{T}}(x) d\Gamma(x) \leq \int_{\mathcal{L}^c} \psi_x^{-1}(V(x|\, p, \mathcal{T})) d\Gamma(x) < \theta_1 \cdot \Gamma(\mathcal{L}^c),$$

where the first inequality comes from Proposition 2, and the second from the fact that $\psi_x(\cdot)$ is a strictly decreasing function, the definition of $\mathcal{L}^c$ and $\Gamma(\mathcal{L}^c) > 0$. Note that this inequality holds in both of the cases in the statement of the lemma. In both cases we have $0 \notin \mathcal{B}$ so $\Gamma(\mathcal{L}^c)$ equals $m(\mathcal{L}^c)$, yielding

$$\theta_1 \cdot m(\mathcal{L}^c) \leq \mathcal{T}_2(\mathcal{L}^c) < \theta_1 \cdot \Gamma(\mathcal{L}^c) = \theta_1 \cdot m(\mathcal{L}^c).$$

□

D.1 Proofs for Section 8.1

Proof of Proposition 4. The proof of this proposition consists of several steps. In the first step we establish that the origin is an attraction region, characterize some properties of it and compute the value of the equilibrium utility function outside the attraction region. After this step, the drivers utility function will be pinned down in the entire city as a function of its value in the origin, $V(0|\, p, \mathcal{T})$. The second step supplies us with a full characterization, up to $V(0|\, p, \mathcal{T})$, of the post-relocation supply $\mathcal{T}_2$ in the entire city. Finally, in step three we show how to solve for the optimal value of $V(0|\, p, \mathcal{T})$ and, therefore, we pin down both $V(\cdot|\, p, \mathcal{T})$ and $\mathcal{T}_2$. We further show how to find the optimal $p(0)$ and the corresponding optimal flow $\mathcal{T}$.

Step 1: We show that we can restrict attention to solutions $(p, \mathcal{T})$ such that $X_l < 0 < X_r$, $X_r = V(0) - \psi_1$ and $X_l = -X_r$. Furthermore, such solutions have $V(x|\, p, \mathcal{T}) = \psi_1$ for all $x \in \mathcal{C} \setminus [X_l, X_r]$.

Proof of Step 1: Let $(p, \mathcal{T})$ be a feasible solution. First, we show that at any optimal solution we must have $X_l < 0 < X_r$. By Lemma D-2 (which we state and prove after the proof of the present proposition) we have that if either of the sets $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \mathcal{T})\}$ or $\{x \in [-H, 0) : 0 \in \mathcal{IR}(x|p, \mathcal{T})\}$ is empty then the revenue the platform makes satisfies

$$\frac{1}{\gamma} \cdot \mathbf{Rev}(p, \mathcal{T}) \le \psi_1 \cdot \theta_1 \cdot 2 \cdot H.$$

Now we construct a new feasible solution $(\tilde{p}, \tilde{\mathcal{T}})$ for which both sets are non-empty and such that

$$\frac{1}{\gamma} \cdot \mathbf{Rev}(\tilde{p}, \tilde{\mathcal{T}}) > \psi_1 \cdot \theta_1 \cdot 2 \cdot H, \tag{D-1}$$

where $\tilde{p}$ equals ρ_1 in $\mathcal{C} \setminus \{0\}$ and $p(0)$ is appropriately chosen. This will imply that any optimal solution must satisfy $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \mathcal{T})\} \neq \emptyset$ and $\{x \in [-H, 0) : 0 \in \mathcal{IR}(x|p, \mathcal{T})\} \neq \emptyset$ and, therefore, $X_l < 0 < X_r$. This also implies that the optimal revenue in this case is strictly larger than the one in the pre-shock environment.

Our solution will send flow in $[-h, h]$ to the origin, where $h > 0$ is to be determined. Inside this interval, all the flow in the subinterval $[-\bar{h}(h), \bar{h}(h)]$ goes to the origin where $0 \le \bar{h}(h) \le h$. The rest of the flow in $[-h, h]$ partially stays at its original position and partially goes to the origin. We now show how to determine $\bar{h}(h)$ and h. For any given $h > 0$ we define

$$\bar{h}(h) \triangleq (\psi_1 + h - \alpha \cdot \rho_1)^+,$$

note that when ψ_1 equals $\alpha \cdot \rho_1$ we have that $\bar{h}(h)$ equals h, and we will send all the flow in $[-h, h]$ to the origin. However, when $\psi_1 < \alpha \cdot \rho_1$ not all the flow will be sent to the origin. Define

$$\theta_1(x) \triangleq \alpha \cdot \rho_1 \cdot \frac{\lambda_1 \overline{F}(\rho_1)}{\psi_1 + h - |x|},$$

then

$$\frac{\lambda_1 \overline{F}(\rho_1)}{\theta_1(x)} \le 1, \quad x \in [-h, h] \setminus [-\bar{h}(h), \bar{h}(h)].$$

The idea is that for every location $x \in K(h) \triangleq [-h, h] \setminus [-\bar{h}(h), \bar{h}(h)]$ we will leave a density $\theta_1(x)$ of flow there and send $\theta_1 - \theta_1(x)$ (note that this difference is non-negative) to the origin. In order to make this possible, we need to chose h appropriately. Observe that the total supply we will send to the origin is

$$S_T(h) = 2\bar{h}(h)\theta_1 + 2 \int_{\bar{h}(h)}^{h} (\theta_1 - \theta_1(x))\, dm(x),$$

where $\lim_{h \to 0} S_T(h) = 0$. Hence, since $\psi_1 < \overline{\alpha} \cdot \overline{V}$, we can always find $h > 0$ such that

$$\alpha \cdot \overline{V} - h \ge \alpha \cdot F^{-1}\Big(1 - \frac{S_T(h)}{\lambda_0}\Big) - h \ge \psi_1. \tag{D-2}$$

This yields

$$\overline{F}\Big(\frac{\psi_1 + h}{\alpha}\Big) \ge \frac{S_T(h)}{\lambda_0}.$$

Now we construct the solution $(\tilde{p}, \tilde{\mathcal{T}})$. Fix any h satisfying Eq. (D-2) and consider prices defined by

$$\tilde{p}(x) = \begin{cases} \frac{\psi_1 + h}{\alpha} & \text{if } x = 0 \\ \rho_1 & \text{if } x \in \mathcal{C} \setminus \{0\}, \end{cases}$$

and flows for any measurable set $\mathcal{L} \subseteq \mathcal{C} \times \mathcal{C}$ defined by

$$\begin{aligned}\tilde{\mathcal{T}}(\mathcal{L}) = \Theta(\pi_1(\mathcal{L} \cap \mathcal{D}) \cap [-h, h]^c) + \Theta(\pi_1(\mathcal{L} \cap [-\bar{h}(h), \bar{h}(h)] \times \{0\})) \\ + G_0(\pi_1(\mathcal{L} \cap K(h) \times \{0\})) + G_1(\pi_1(\mathcal{L} \cap \mathcal{D}) \cap K(h)),\end{aligned}$$

where G_0, G_1 are measures defined for any measurable set $\mathcal{B} \subseteq K(h)$ by

$$G_0(\mathcal{B}) \triangleq \int_{\mathcal{B}} (\theta_1 - \theta_1(x))\, dm(x), \quad G_1(\mathcal{B}) \triangleq \int_{\mathcal{B}} \theta_1(x)\, dm(x).$$

We argue that $(\tilde{p}, \tilde{\mathcal{T}})$ is a feasible solution that complies with Eq. (D-1). From Lemma C-1 we have that $\tilde{\mathcal{T}} \in \mathcal{M}(\mathcal{C} \times \mathcal{C})$, also note that for any measurable set $\mathcal{B} \subseteq \mathcal{C}$ the first marginal of $\tilde{\mathcal{T}}$ satisfies

$$\tilde{\mathcal{T}}_1(\mathcal{B}) = \Theta(\mathcal{B} \cap [-h, h]^c) + \Theta(\mathcal{B} \cap [-\bar{h}(h), \bar{h}(h)]) + G_0(\mathcal{B} \cap K(h)) + G_1(\mathcal{B} \cap K(h)) = \Theta(\mathcal{B}).$$

The post-relocation supply measure is

$$\tilde{\mathcal{T}}_2(\mathcal{B}) = \Theta(\mathcal{B} \cap [-h, h]^c) + S_T(h) \cdot \mathbf{1}_{\{0 \in \mathcal{B}\}} + G_1(\mathcal{B} \cap K(h)),$$

clearly $\tilde{\mathcal{T}}_2 \ll \Gamma$. Therefore, $\tilde{\mathcal{T}} \in \mathcal{F}(\Theta)$. Next, we need to show that $\tilde{\mathcal{T}}$ is a supply equilibrium. The Radon-Nikodym derivative of $\tilde{\mathcal{T}}_2$ with respect the city measure is (Γ-a.e)

$$s(x) = \begin{cases} S_T(h) & \text{if } x = 0 \\ 0 & \text{if } x \in [-\bar{h}(h), \bar{h}(h)] \setminus \{0\} \\ \theta_1(x) & \text{if } x \in K(h) \\ \theta_1 & \text{if } x \in [-h, h]^c. \end{cases}$$

Indeed,

$$\int_{\mathcal{L}} s(x)\, d\Gamma(x) = S_T(h)\mathbf{1}_{\{0 \in \mathcal{L}\}} + \int_{\mathcal{L} \cap [-h,h]^c} \theta_1\, dm(x) + \int_{\mathcal{L} \cap K(h)} \theta_1(x)\, dm(x) = \tilde{\mathcal{T}}_2(\mathcal{L}),$$

that is, $\frac{d\tilde{\mathcal{T}}_2}{d\Gamma}(\cdot)$ equals $s(\cdot)$ Γ-a.e. From this we can compute $V(\cdot|\tilde{p}, \tilde{\mathcal{T}})$. Note that ($\Gamma$-a.e)

$$\tilde{U}(y) = U\Big(y, \tilde{p}(y), \frac{d\tilde{\mathcal{T}}_2}{d\Gamma}(y)\Big) = \begin{cases} \psi_1 + h & \text{if } y = 0; \\ \alpha \cdot \rho_1 & \text{if } y \in [-\bar{h}(h), \bar{h}(h)] \setminus \{0\}; \\ \alpha \cdot \rho_1 \cdot \frac{\lambda_1 \overline{F}(\rho_1)}{\theta_1(x)} & \text{if } y \in K(h); \\ \psi_1 & \text{if } y \in [-h, h]^c. \end{cases}$$

Let $a(x)$ be defined by

$$a(x) \triangleq \begin{cases} \psi_1 + h - |x| & \text{if } x \in [-h, h], \\ \psi_1 & \text{if } x \in [-h, h]^c. \end{cases}$$

We argue that $V(\cdot|\tilde{p}, \tilde{\mathcal{T}}) \equiv a(\cdot)$. Fix $x \in \mathcal{C}$, it is not hard to verify that

$$\Gamma(y \in \mathcal{C} : \tilde{U}(y) - |y - x| > a(x)) = 0,$$

and, thus, $a(x) \geq V(x|\, \tilde{p}, \tilde{\mathcal{T}})$. Suppose that $x \in [-h, h]$ and $a(x) > V(x|\, \tilde{p}, \tilde{\mathcal{T}})$ then, because $\Gamma(\{0\}) > 0$, we have that

$$\psi_1 + h - |x| = a(x) > V(x|\, \tilde{p}, \tilde{\mathcal{T}}) \geq \Pi(x, 0) = \psi_1 + h - |0 - x|,$$

a contradiction. Thus, for $x \in [-h, h]$ we have $a(x) = V(x|\tilde{p}, \tilde{\mathcal{T}})$. For any other x we can use a similar argument to conclude that $a(x) = V(x|\tilde{p}, \tilde{\mathcal{T}})$.

Now we are ready to verify the equilibrium condition. Observe that

$$\mathcal{E} = \Big\{(x,y) \in \mathcal{C}\times\mathcal{C} : \mathit{\Pi}(x,y) = V(x|\tilde{p},\tilde{\mathcal{T}})\Big)\Big\} = ([-h,h]\times\{0\})\cup([-h,h]^c\times[-h,h]^c\cap\mathcal{D})\cup(K(h)\times K(h)\cap\mathcal{D}),$$

then

$$\begin{aligned}\tilde{\mathcal{T}}(\mathcal{E}) &= \mathit{\Theta}(\pi_1(\mathcal{E}\cap\mathcal{D})\cap[-h,h]^c) + \mathit{\Theta}(\pi_1(\mathcal{E}\cap[-\bar{h}(h),\bar{h}(h)]\times\{0\}))\\ &\quad + G_1(\pi_1(\mathcal{E}\cap\mathcal{D})\cap K(h)) + G_0(\pi_1(\mathcal{E}\cap K(h)\times\{0\}))\\ &= \mathit{\Theta}([-h,h]^c) + \mathit{\Theta}([-\bar{h}(h),\bar{h}(h)]) + G_1(K(h)) + G_0(K(h))\\ &= \mathit{\Theta}(\mathcal{C}).\end{aligned}$$

This proves that $\tilde{\mathcal{T}}$ is an equilibrium. Next we need to show $(\tilde{p},\tilde{\mathcal{T}})$ satisfies Eq. (D-1). From Proposition 1 we have

$$\begin{aligned}\gamma\mathbf{Rev}(\tilde{p},\tilde{\mathcal{T}}) &= \int_{\mathcal{C}} V(x)\cdot\frac{d\tilde{\mathcal{T}}_2}{d\mathit{\Gamma}}(x)d\mathit{\Gamma}(x)\\ &= (\psi_1+h)\cdot S_T(h) + 2\int_{\bar{h}(h)}^{h}(\psi_1+h-x)\theta_1(x)\,dm(x) + \psi_1\cdot\theta_1\cdot 2(H-h)\\ &\geq h\cdot S_T(h) + \psi_1\Big(S_T(h) + 2\int_{\bar{h}(h)}^{h}\theta_1(x)dx\Big) + \psi_1\cdot\theta_1\cdot 2(H-h)\\ &= h\cdot S_T(h) + \psi_1\Big(2\bar{h}(h)\theta_1 + 2\int_{\bar{h}(h)}^{h}(\theta_1-\theta_1(x))dx + 2\int_{\bar{h}(h)}^{h}\theta_1(x)dx\Big) + \psi_1\cdot\theta_1\cdot 2(H-h)\\ &= h\cdot S_T(h) + \psi_1\cdot\theta_1\cdot 2\cdot H.\end{aligned}$$

Since $h\cdot S_T(h) > 0$, Eq. (D-1) obtains. This proves that $X_l < 0 < X_r$ in any optimal solution.

The next step of the proof of Step 1 consists on arguing that given $V(0)$, $X_r = V(0)-\psi_1$ and $X_l = -(V(0)-\psi_1)$. Consider a feasible solution $(p,\mathcal{T})$ where $p(\cdot)$ equals ρ_1 everywhere but at the origin, and $X_l < 0 < X_r$. From Proposition B-1 and the fact that $\mathit{\Theta}(\{X_r\}) = 0$ we have that

$$\mathcal{T}([X_r,H]\times[X_r,H]^c) \leq \mathit{\Theta}(\{X_r\}) + \mathcal{T}((X_r,H]\times[X_r,H]^c) = 0.$$

Then by Lemma D-1 we have that $V(x)\leq\psi_1$, $\mathit{\Gamma}-a.e.\ x$ in $[X_r,H]$. This, together with the continuity of $V(\cdot)$ imply that $V(x)\leq\psi_1$ for all $x\in[X_r,H]$.

Suppose first that $X_r < V(0)-\psi_1$ then

$$V(X_r|\,p,\mathcal{T}) = V(0) - X_r > \psi_1,$$

but this violates the continuity of V to the right of X_r. Thus $X_r \geq V(0)-\psi_1$. On the other hand, suppose $X_r > V(0)-\psi_1$ then we must have that $\psi_1 > V(x|\,p,\mathcal{T}) = V(0)-x$ for all $x\in(V(0)-\psi_1,X_r]$. Observe that

$$\mathit{\Theta}([V(0)-\psi_1,X_r]) \geq \mathcal{T}_2([V(0)-\psi_1,X_r]) = \int_{[V(0)-\psi_1,X_r]} s^{\mathcal{T}}(x)\,d\mathit{\Gamma}(x). \tag{D-3}$$

Define the set

$$K \triangleq \{y\in[V(0)-\psi_1,X_r] : s^{\mathcal{T}}(y)\leq\theta_1\},$$

it must be that $\mathit{\Gamma}(K)=0$; otherwise, from the definition of $V(X_r|\,p,\mathcal{T})$ we have

$$\begin{aligned}V(0)-X_r = V(X_r) &\geq U(y,\rho_1,s^{\mathcal{T}}(y)) - |y-X_r|, \quad \mathit{\Gamma}-a.e.\ y \text{ in } K\\ &\geq U(y,\rho_1,\theta_1) - |y-X_r|, \quad \mathit{\Gamma}-a.e.\ y \text{ in } K\\ &= \psi_1 - (X_r - y), \quad \mathit{\Gamma}-a.e.\ y \text{ in } K,\end{aligned}$$

and $\Gamma(K) > 0$ implies that $V(0)-y \geq \psi_1$ for some $y \in (V(0)-\psi_1, X_r]$. However, we know that $\psi_1 > V(0)-y$ for $y \in (V(0) - \psi_1, X_r]$ and, therefore, we must have $\Gamma(K) = 0$. Using this in Eq. (D-3) yields

$$\Theta([V(0) - \psi_1, X_r]) > \theta_1 \cdot \Gamma([V(0) - \psi_1, X_r]) = \Theta([V(0) - \psi_1, X_r]),$$

which is not possible. Hence, $X_r = V(0) - \psi_1$ and the same arguments applies to X_l, yielding $X_l = -(V(0) - \psi_1)$.

In order to conclude the proof for Step 1 we show that we can restrict attention to solutions $(p, \mathcal{T})$ such that $V(x|\, p, \mathcal{T})$ equals ψ_1 for all $x \in [X_l, X_r]^c$. In turn, this will show that $s^{\mathcal{T}}(x)$ equals θ_1, $\Gamma - a.e.$ x in $[X_l, X_r]^c$. We base the proof of the latter statements in Lemma D-3 (which we state and prove after the proof of the present result), this lemma enables us to separate the city into two regions $[X_l, X_r]$ and $[X_l, X_r]^c$. For each region we can modify the prices and equilibria, and then paste them together to obtain a new solution that is an equilibrium for the entire city.

Consider a feasible solution $(p, \mathcal{T})$ such that $X_l < 0 < X_r$, $X_r = V(0) - \psi_1$ and $X_l = -X_r$. Since $\mathcal{T}([X_l, X_r] \times [X_l, X_r]^c) = 0$ and $0 \notin [X_l, X_r]^c$, Lemma D-1 delivers

$$\frac{1}{\gamma} \cdot \mathbf{Rev}(p, \mathcal{T}) \leq \frac{1}{\gamma} \cdot \mathbf{Rev}_{[X_l, X_r]}(p, \mathcal{T}) + 2 \cdot \theta_1 \cdot \psi_1 \cdot (H - X_r). \tag{D-4}$$

We show that we can always modify $(p, \mathcal{T})$ so that the previous upper bound is achieve. Let $\mathcal{B} = [X_l, X_r]$, since $\mathcal{T}(\mathcal{B} \times \mathcal{B}^c) = 0$ and $\mathcal{T}(\mathcal{B}^c \times \mathcal{B}) = 0$, Lemma D-3 ensures that $(p, \mathcal{T})|_{\mathcal{B}}$ is a price equilibrium pair in $\mathcal{B}$. Such equilibrium satisfies $V_{\mathcal{B}}(x) = \psi_1$ for $x \in \partial\mathcal{B}$.

Now, we choose prices $p^{\mathcal{B}^c}(x)$ equal to ρ_1 for all $x \in \mathcal{B}^c$ and a flow $\mathcal{T}^{\mathcal{B}^c}$ defines by for any measurable set $\mathcal{L}_1 \times \mathcal{L}_2 \subseteq \mathcal{B}^c \times \mathcal{B}^c$

$$\mathcal{T}^{\mathcal{B}^c}(\mathcal{L}_1 \times \mathcal{L}_2) = \Theta(\mathcal{L}_1 \cap \mathcal{L}_2).$$

Then, it is easy to verify (as we did in the pre-shock environment, see Proposition D-1) that $(p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})$ forms a price-equilibrium pair in $\mathcal{B}^c$. This solution satisfy that $V_{\mathcal{B}^c}(x) = \psi_1$ for $x \in \mathcal{B}^c$, and that $s^{\mathcal{T}^{\mathcal{B}^c}}(x)$ equals θ_1, $\Gamma - a.e.$ x in $\mathcal{B}^c$.

Lemma D-3 enables us to paste the solutions $(p, \mathcal{T})|_{\mathcal{B}}$ and $(p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})$, and generate a new solution in the entire city. Such solution preserve the prices and flows in both $\mathcal{B}$ and $\mathcal{B}^c$ and, therefore, the upper bound in Eq. (D-4) is achieved. In conclusion, we can restrict attention to solutions $(p, \mathcal{T})$ such that $V(x|\, p, \mathcal{T})$ equals ψ_1 for all $x \in [X_l, X_r]^c$, and that $s^{\mathcal{T}}(x)$ equals θ_1, $\Gamma - a.e.$ x in $[X_l, X_r]^c$.

Step 2: We characterize $s^{\mathcal{T}}(\cdot)$ (this completely characterizes $\mathcal{T}_2$). Let

$$X_r^0 = (V(0) - \alpha \cdot \rho_1))^+ \quad \text{and} \quad X_l^0 = -X_r^0,$$

and

$$\theta_1(y) \triangleq \alpha \cdot \rho_1 \cdot \frac{\lambda_1 \cdot \overline{F}(\rho_1)}{V(0) - |y|}, \quad S_T = 2 \cdot \theta_1 \cdot X_r^0 + 2 \int_{X_r^0}^{X_r} (\theta_1 - \theta_1(x)) dx.$$

In this step we show that $(\Gamma - a.e)$

$$s^{\mathcal{T}}(y) = \begin{cases} S_T & \text{if } y = 0 \\ 0 & \text{if } y \in [X_l^0, X_r^0] \setminus \{0\} \\ \theta_1(y) & \text{if } y \in [X_l, X_r] \setminus [X_l^0, X_r^0] \\ \theta_1 & \text{if } y \in [X_l, X_r]^c. \end{cases}$$

Proof of Step 2: Note that at the end of the previous step we showed the result for $y \in [X_l, X_r]^c$. So first we show

$$s^{\mathcal{T}}(y) = 0, \quad \Gamma - a.e.\ x \text{ in } [X_l^0, X_r^0] \setminus \{0\}.$$

Define the set $K_1 \triangleq \{y \in [X_l^0, X_r^0] \setminus \{0\} : s^{\mathcal{T}}(y) > 0\}$. We argue that $\Gamma(K_1) = 0$. If this is not the case then $\Gamma(K_1) > 0$ and, therefore,

$$\mathcal{T}_2(K_1) = \int_{K_1} s^{\mathcal{T}}(x)\, d\Gamma(x) > 0.$$

Then Lemma A-2 ensures that

$$U\Big(x, \rho_1, s^{\mathcal{T}}(x)\Big) \;=\; V(x|\, p, \mathcal{T}) \qquad \mathcal{T}_2 - a.e.\ x \in K_1, \tag{D-5}$$

but for $x \in K_1 \subseteq [X_l^0, X_r^0] \setminus \{0\}$ we have $V(x|\, p, \mathcal{T}) = V(0) - |x|$ and $V(0) - |x| \geq \alpha \cdot \rho_1$. Then Eq. (D-5) implies the existence of $x \in (X_l^0, X_r^0) \setminus \{0\}$ such that $\alpha \cdot \rho_1 < U\Big(x, \rho_1, s^{\mathcal{T}}(x)\Big) \leq \alpha \cdot \rho_1$, yielding a contradiction. Next we show that

$$s^{\mathcal{T}}(y) = \theta_1(y), \quad \Gamma - a.e.\ y \ \text{ in } \ [X_l, X_r] \setminus [X_l^0, X_r^0].$$

By Lemma A-2 we have that

$$U\Big(x, \rho_1, s^{\mathcal{T}}(x)\Big) = V(x) = V(0) - |x|, \quad \Gamma - a.e.\ x \ \text{ in } \ [X_l, X_r] \setminus [X_l^0, X_r^0], \tag{D-6}$$

but for any $x \in [X_l, X_r] \setminus [X_l^0, X_r^0]$ the definition of X_l^0 and X_r^0 imply that $V(0) - |x| < \alpha \cdot \rho_1$. Thus Eq. (D-6) and the definition of $U\Big(x, \rho_1, s^{\mathcal{T}}(x)\Big)$ deliver

$$\lambda_1 \cdot \overline{F}(\rho_1)/s^{\mathcal{T}}(x) < 1, \quad \Gamma - a.e.\ x \ \text{ in } \ [X_l, X_r] \setminus [X_l^0, X_r^0].$$

Using the again Eq. (D-6) and the definition of $U\Big(x, \rho_1, s^{\mathcal{T}}(x)\Big)$ we conclude that

$$s^{\mathcal{T}}(x) = \alpha \cdot \rho_1 \cdot \frac{\overline{F}(\rho_1)}{V(0) - |x|}, \quad \Gamma - a.e.\ x \ \text{ in } \ [X_l, X_r] \setminus [X_l^0, X_r^0],$$

as needed. Next we compute $s^{\mathcal{T}}(0)$,

$$\begin{aligned} s^{\mathcal{T}}(0) \cdot \Gamma(\{0\}) = \int_{\{0\}} s^{\mathcal{T}}(x)\, d\Gamma &= \mathcal{T}_2(\{0\}) \\ &= \mathcal{T}(\mathcal{C} \times \{0\}) \\ &= \mathcal{T}([X_l, X_r] \times \{0\}) \\ &= \underbrace{\mathcal{T}([X_l^0, X_r^0] \times \{0\})}_{(1)} + \underbrace{\mathcal{T}([X_l, X_r] \setminus [X_l^0, X_r^0] \times \{0\})}_{(2)}, \end{aligned}$$

for (1) we have

$$\begin{aligned} \mathcal{T}([X_l^0, X_r^0] \times \{0\}) &= \Theta([X_l^0, X_r^0]) - \mathcal{T}([X_l^0, X_r^0] \times \mathcal{C} \setminus \{0\}) \\ &= 2\theta_1 \cdot X_r^0 - \mathcal{T}([X_l^0, X_r^0] \times [X_l^0, X_r^0] \setminus \{0\}) \\ &\overset{(a)}{=} 2\theta_1 \cdot X_r^0, \end{aligned}$$

in (a) we use $s^{\mathcal{T}}(x) = 0$, $\Gamma - a.e.\ x$ in $[X_l^0, X_r^0] \setminus \{0\}$. For (2) we have

$$\begin{aligned} \mathcal{T}([X_l, X_r] \setminus [X_l^0, X_r^0] \times \{0\}) &= \Theta([X_l, X_r] \setminus [X_l^0, X_r^0]) - \mathcal{T}([X_l, X_r] \setminus [X_l^0, X_r^0] \times [X_l, X_r] \setminus \{0\}) \\ &= 2\theta_1 \cdot (X_r - X_r^0) - \mathcal{T}([X_l, X_r] \setminus [X_l^0, X_r^0] \times [X_l^0, X_r^0] \setminus \{0\}) \\ &\quad - \mathcal{T}([X_l, X_r] \setminus [X_l^0, X_r^0] \times [X_l, X_r] \setminus [X_l^0, X_r^0]) \\ &= 2\theta_1 \cdot (X_r - X_r^0) - 0 - \mathcal{T}_2([X_l, X_r] \setminus [X_l^0, X_r^0]) \\ &= 2\theta_1 \cdot (X_r - X_r^0) - \int_{[X_l, X_r] \setminus [X_l^0, X_r^0]} \theta_1(x)\, d\Gamma, \end{aligned}$$

from this we conclude that

$$s^{\mathcal{T}}(0) = 2 \cdot \theta_1 \cdot X_r^0 + 2\int_{X_r^0}^{X_r} (\theta_1 - \theta_1(x))\, dx.$$

Step 3: Now we can provide a full solution for the optimization problem. Recall that we are only optimizing over $p(0)$ or, equivalently, over $V(0)$. By our congestion bound (see Proposition 2), any solution has to satisfy $V(0|\, p, \mathcal{T}) \leq \psi_0(s^{\mathcal{T}}(0))$. Moreover, Step 2 characterizes the supply-demand ratio at every location as a function of $V(0)$. Thus, the following formulation is a natural relaxation for the platform's problem

$$\begin{aligned} \max_{V(0)} \quad & V(0) \cdot S_T + 2 \cdot \psi_1 \cdot \theta_1 \cdot (H - X_r^0) \qquad\qquad (\mathcal{P}_{loc-reac}) \\ \text{s.t} \quad & X_r^0 = (V(0) - \alpha \cdot \rho_1))^+, \quad X_r = V(0) - \psi_1 \\ & S_T = 2X_r^0\theta_1 + 2\int_{X_r^0}^{X_r} (\theta_1 - \theta_1(x))dx, \qquad \psi_1 < V(0) \leq \psi_0(S_T). \end{aligned}$$

We show that the optimal $V^\star(0)$ in $(\mathcal{P}_{loc-reac})$ is the unique solution to

$$V^\star(0) = \psi_0(S_T(V^\star(0))).$$

The optimal solution to the platform's problem set price at the origin $p^\star(0) = \rho_0^{loc}(S_T(V^\star(0)))$ such that $p^\star(0) \geq \rho_1$, and flows for any measurable set $\mathcal{B} \subset \mathcal{C} \times \mathcal{C}$ given by

$$\begin{aligned} \mathcal{T}(\mathcal{B}) &= \Theta(\pi_1(\mathcal{B} \cap \mathcal{D}) \cap [X_l, X_r]^c) + \Theta(\pi_1(\mathcal{B} \cap [X_l^0, X_r^0] \times \{0\})) \\ &\quad + G_1(\pi_1(\mathcal{B} \cap \mathcal{D}) \cap [X_l, X_r] \setminus [X_l^0, X_r^0]) + G_0(\pi_1(\mathcal{B} \cap [X_l, X_r] \setminus [X_l^0, X_r^0] \times \{0\})), \end{aligned}$$

where G_0, G_1 are measures defined for any measurable set $\mathcal{L} \subset [X_l, X_r] \setminus [X_l^0, X_r^0]$ by

$$G_0(\mathcal{L}) \triangleq \int_{\mathcal{L}} (\theta_1 - \theta_1(x))\, dm(x), \quad G_1(\mathcal{L}) \triangleq \int_{\mathcal{L}} \theta_1(x)\, dm(x).$$

Proof of Step 3: The proof consists of two parts. First, we show that $V^\star(0)$ as stated above is an optimal solution for $(\mathcal{P}_{loc-reac})$. To do this we prove that $S_T(V(0))$ is increasing for $V(0) > \psi_1$, with $S_T(\psi_1) = 0$. This implies that $\psi_0(S_T(V(0)))$ is decreasing and, therefore, it crosses with $V(0)$ at only one point. Then, we show the objective function increases with $V(0)$. These two facts imply the optimality of $V^\star(0)$. Second, we show that $(p, \mathcal{T})$ with $p(0) = p^\star(0)$ (and equal to ρ_1 for $x \neq 0$) and and $\mathcal{T}$ as stated above, are a feasible price-equilibrium pair that achieve the same revenue than the optimal solution of $(\mathcal{P}_{loc-reac})$. Since this problem is a relaxation to our original optimization problem we have optimality.

We begin with the first part. Note that

$$S_T(V(0)) = 2\theta_1 \cdot (V(0) - \psi_1) + 2\psi_1 \cdot \theta_1 \cdot \log\Big(\frac{\psi_1}{V(0) - (V(0) - \alpha\rho_1)^+}\Big).$$

From this it follows that $S_T(\psi_1) = 0$. If $V(0) \geq \alpha\rho_1$ then $S_T(V(0))$ is clearly increasing. If $V(0) \in (\psi_1, \alpha\rho_1)$ then the derivative of $S_T(V(0))$ with respect to $V(0)$ equals

$$2\theta_1 - 2\psi_1 \cdot \theta_1 \cdot \frac{V(0)}{\psi_1} \cdot \frac{\psi_1}{V(0)^2} = 2\theta_1 - 2\psi_1 \cdot \theta_1 \cdot \frac{1}{V(0)},$$

which is nonnegative if and only if $V(0) \geq \psi_1$. Since this is in our domain, we conclude that $S_T(\cdot)$ is increasing in $(\psi_1, \alpha\rho_1)$ and, therefore, is increasing for all $V(0) > \psi_1$.

Next, we show the objective is increasing in $V(0)$, the objective function is

$$V(0) \cdot S_T(V(0)) + 2 \cdot \psi_1 \cdot \theta_1 \cdot (H - (V(0) - \alpha\rho_1)^+),$$

when $V(0) \geq \alpha \cdot \rho_1$, the objective becomes

$$2\theta_1 \cdot V(0) \cdot (V(0) - \psi_1) + 2\psi_1 \cdot \theta_1 \cdot V(0) \cdot \log\left(\frac{\psi_1}{\alpha\rho_1}\right) + 2 \cdot \psi_1 \cdot \theta_1 \cdot (H - V(0) + \alpha\rho_1).$$

Its derivative is non-negative if and only if $2\frac{V(0)}{\psi_1} \geq 2 + \log\left(\frac{\alpha\rho_1}{\psi_1}\right)$, but from $V(0) \geq \alpha \cdot \rho_1$ and that the logarithm is a concave function the latter inequality is always true. Similarly, for $V(0) \in (\psi_1, \alpha \cdot \rho_1)$ the objective's derivative is non-negative if and only if $2\frac{V(0)}{\psi_1} \geq 2 + \log\left(\frac{V(0)}{\psi_1}\right)$, which, since $V(0) > \psi_1$, is always true. Observe that in both cases the inequalities for the sign of the objective's derivative is strict except when $V(0) = \psi_1$. Thus, the objective is strictly increasing in the domain.

For the second part we need to show that $(p, \mathcal{T})$ with $p(0) = p^\star(0)$ (and equal to ρ_1 for $x \neq 0$) and and $\mathcal{T}$, implement the solution of $(\mathcal{P}_{loc-reac})$. To do this we first need to argue that this solution is feasible. It can be easily seen that this flow yields the exact same flows as in Step 2, only this time we replace $V^\star(0)$ in all the quantities that depend on $V(0)$. Given the value of $s^{\mathcal{T}}$ and the fact that under $p^\star(0)$ we have $U(0, p(0), s^{\mathcal{T}}(0)) = V(0|\, p, \mathcal{T}) = V^\star(0)$, we can do the same as we did in Step 1(to show that $\tilde{\mathcal{T}}$ is an equilibrium) and show that $\mathcal{T}$ is an equilibrium. Since we have pinned the value of $V(0|\, p, \mathcal{T})$ (and thus the value of $V(|\, p, \mathcal{T})$ in the entire city) and the value of $s^{\mathcal{T}}(\cdot)$, it is easy to see (using Proposition 1) that $\frac{1}{\gamma} \cdot \mathbf{Rev}(p, \mathcal{T})$ coincides with the optimal value of $(\mathcal{P}_{loc-reac})$. Therefore, $(p, \mathcal{T})$ is the optimal solution.

To conclude we argue that $p^\star(0) \geq \rho_1$. There are two cases. If $\theta_1 \leq \lambda_1 \cdot \overline{F}(\rho_1)$ then ψ_1 equals $\alpha \cdot \rho_1$. Since $V^\star(0) > \psi$ and $V^\star(0) = \psi_0(S_T(V^\star(0)))$ we have have that

$$\alpha \cdot \rho_1 = \psi_1 < V^\star(0) = \psi_0(S_T(V^\star(0))) \leq \alpha \cdot \rho_0^{loc}(S_T(V^\star(0))) = \alpha \cdot p^\star(0),$$

that is, $\rho_1 < p^\star(0)$. The second case is $\theta_1 > \lambda_1 \cdot \overline{F}(\rho_1)$. Here ρ_1 equals ρ^u and, since $\rho_0^{loc}(S_T(V^\star(0)))$ equals $\max\{\rho_0^{bal}, \rho^u\}$, we have that $\rho_1 \leq p^\star(0)$. □

Lemma D-2. *Let $(p, \mathcal{T})$ be a feasible price-equilibrium pair for either the myopic price response environment (Section 8.1) or the global price response environment (Section 8.2). If either $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \mathcal{T})\} = \emptyset$ or $\{x \in [-H, 0) : 0 \in \mathcal{IR}(x|p, \mathcal{T})\} = \emptyset$, then the platform's objective satisfies*

$$\gamma \cdot \mathbf{Rev}(p, \mathcal{T}) \leq \psi_1 \cdot \theta_1 \cdot 2 \cdot H.$$

Proof. WLOG let us just assume that $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \mathcal{T})\} = \emptyset$. That is, for all $x \in (0, H]$ we have $0 \notin \mathcal{IR}(x|p, \mathcal{T})$. In turn, this implies that $\mathcal{T}((0, H] \times [-H, 0]) = 0$ and, therefore, by Lemma D-1 we conclude that

$$V(x|p, \mathcal{T}) \leq \psi_1 \qquad \Gamma - a.e. \text{ in } \ (0, H],$$

which, from the continuity of $V(\cdot|p, \mathcal{T})$, implies that $V(x|p, \mathcal{T}) \leq \psi_1$ for all $x \in [0, H]$. Now, we show that the same bound holds for $x \in [-H, 0)$. If $\mathcal{T}([-H, 0) \times \mathcal{B}) = 0$ for any $\mathcal{B} \subset [0, H]$, we can use Lemma D-1 to obtain the upper bound. On the other hand, if there exists $\mathcal{B} \subset [0, H]$ such that $\mathcal{T}([-H, 0) \times \mathcal{B}) > 0$ then by Lemma B-3 we know there exists a pair $(x, y) \in [-H, 0) \times \mathcal{B}$ for which $y \in \mathcal{IR}(x|\, p, \mathcal{T})$. Thus, we can define

$$\underline{x} = \inf\{z \in [-H, 0) : y \in \mathcal{IR}(z|\, p, \mathcal{T})\},$$

and by Proposition B-1, $y \in \mathcal{IR}(\underline{x}|\, p, \mathcal{T})$. Also, by definition of indifference region and Lemma B-2 we have

$$V(z|p, \mathcal{T}) = V(\underline{x}|p, \mathcal{T}) + z - \underline{x}, \quad \forall\, z \in [\underline{x}, y].$$

This implies $V(z|p, \mathcal{T}) \leq V(y|p, \mathcal{T})$ for all $z \in [\underline{x}, y]$, and because $y \in \mathcal{B} \subset [0, H]$ we have $V(y|p, \mathcal{T}) \leq \psi_1$, yielding $V(z|p, \mathcal{T}) \leq \psi_1$ for all $z \in [\underline{x}, y]$. Furthermore, from Lemma B-3 and the definition of $\underline{x}$ we can conclude that $\mathcal{T}([-H, \underline{x}] \times (\underline{x}, H]) = 0$ which together with Lemma D-1 and the continuity of V imply that $V(x|p, \mathcal{T}) \leq \psi_1$ for all $x \in [-H, \underline{x}]$. Completing the argument for the upper bound.

In order to bound the revenue, simply note that

$$\frac{1}{\gamma} \cdot \mathbf{Rev}(p, \mathcal{T}) = \int_{\mathcal{C}} V(x) s^{\mathcal{T}}(x)\, d\Gamma(x) \leq \psi_1 \cdot \int_{\mathcal{C}} s^{\mathcal{T}}(x)\, d\Gamma(x) = \psi_1 \cdot \theta_1 \cdot 2 \cdot H.$$

□

Lemma D-3. *(Equilibria Separation and Pasting) Consider a set $\mathcal{B} \subset \mathcal{C}$ such that both $\mathcal{B}$ and $\mathcal{B}^c$ are intervals or union of intervals with $\Gamma(\partial\mathcal{B}) = 0$, where $\partial\mathcal{B}$ is the boundary of $\mathcal{B}$.*

1. *(Separation) Let $(p, \mathcal{T})$ be a price-equilibrium in $\mathcal{C}$, if $\mathcal{T}(\mathcal{B} \times \mathcal{B}^c) = 0$ and $\mathcal{T}(\mathcal{B}^c \times \mathcal{B}) = 0$ then $(p|_{\mathcal{B}}, \mathcal{T}|_{\mathcal{B}\times\mathcal{B}})$ and $(p|_{\mathcal{B}^c}, \mathcal{T}|_{\mathcal{B}^c\times\mathcal{B}^c})$ are price-equilibrium pairs in $\mathcal{B}$ and $\mathcal{B}^c$, respectively. Moreover, $V(\cdot|\, p|_{\mathcal{B}}, \mathcal{T}|_{\mathcal{B}\times\mathcal{B}})$ equals $V(\cdot|\, p|_{\mathcal{B}^c}, \mathcal{T}|_{\mathcal{B}^c\times\mathcal{B}^c})$ in $\partial\mathcal{B}$, $V(\cdot|\, p|_{\mathcal{B}}, \mathcal{T}|_{\mathcal{B}\times\mathcal{B}})$ coincides with $V(\cdot|\, p, \mathcal{T})|_{\mathcal{B}}$ and the same holds for $\mathcal{B}^c$.*
2. *(Pasting) Suppose we have two price-equilibrium pairs $(p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ and $(p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})$ in $\mathcal{B}$ and $\mathcal{B}^c$ such that $\mathcal{T}^{\mathcal{B}} \in \mathcal{F}_{\mathcal{B}}(\Theta|_{\mathcal{B}})$ and $\mathcal{T}^{\mathcal{B}^c} \in \mathcal{F}_{\mathcal{B}^c}(\Theta|_{\mathcal{B}^c})$, respectively. If $V(\cdot|\, p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ equals $V(\cdot|\, p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})$ in $\partial\mathcal{B}$ then the flow $\mathcal{T}$ defined by for any measurable set $\mathcal{L} \subseteq \mathcal{C} \times \mathcal{C}$*

$$\mathcal{T}(\mathcal{L}) = \mathcal{T}^{\mathcal{B}}(\mathcal{L} \cap \mathcal{B} \times \mathcal{B}) + \mathcal{T}^{\mathcal{B}^c}(\mathcal{L} \cap \mathcal{B}^c \times \mathcal{B}^c),$$

   belongs to $\mathcal{F}(\Theta)$ and is an equilibrium in $\mathcal{C}$ for a price p equal to $p^{\mathcal{B}}$ in $\mathcal{B}$ and equal to $p^{\mathcal{B}^c}$ in $\mathcal{B}^c$. Moreover, $V(x|p, \mathcal{T}) = V(x|\, p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ in $\mathcal{B}$ and $V(x|p, \mathcal{T}) = V(x|\, p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})$ in $\mathcal{B}^c$.

Proof. **Separation.** Suppose that $\mathcal{T}(\mathcal{B} \times \mathcal{B}^c) = 0$ and $\mathcal{T}(\mathcal{B}^c \times \mathcal{B}) = 0$. Let $\mathcal{T}^{\mathcal{B}} = \mathcal{T}|_{\mathcal{B}\times\mathcal{B}}$ and $p^{\mathcal{B}} = p|_{\mathcal{B}}$, we show that $(p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ is a price-equilibrium pair. The proof for $(p|_{\mathcal{B}^c}, \mathcal{T}|_{\mathcal{B}^c\times\mathcal{B}^c})$ is analogous and, thus, omitted. We need to prove that $\mathcal{T}^{\mathcal{B}} \in \mathcal{F}_{\mathcal{B}}(\Theta^{\mathcal{B}})$, where $\Theta^{\mathcal{B}}$ coincides with $\Theta|_{\mathcal{B}}$, and that the set

$$\mathcal{E}|_{\mathcal{B}} \triangleq \left\{(x, y) \in \mathcal{B} \times \mathcal{B} : \Pi(x, y, p^{\mathcal{B}}(y), \frac{d\mathcal{T}_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(y)) = \operatorname{ess\,sup}^{\mathcal{B}} \Pi(x, \cdot, p^{\mathcal{B}}(\cdot), \frac{d\mathcal{T}_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(\cdot)\Big)\right\},$$

satisfies $\mathcal{T}_{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}) = \Theta|_{\mathcal{B}}(\mathcal{B})$.

First we verify that $\mathcal{T}^{\mathcal{B}} \in \mathcal{F}_{\mathcal{B}}(\Theta^{\mathcal{B}})$. Since $\mathcal{T}^{\mathcal{B}}$ is the restriction of $\mathcal{T}$ to $\mathcal{B} \times \mathcal{B}$ it clearly belongs to $\mathcal{M}(\mathcal{B} \times \mathcal{B})$. Also, for any $\mathcal{L}_1$ measurable subset of $\mathcal{B}$ we have that $\mathcal{T}_1^{\mathcal{B}}(\mathcal{L}_1)$ equals

$$\mathcal{T}^{\mathcal{B}}(\mathcal{L}_1 \times \mathcal{B}) = \mathcal{T}((\mathcal{L}_1 \times \mathcal{B}) \cap (\mathcal{B} \times \mathcal{B})) = \mathcal{T}(\mathcal{L}_1 \times \mathcal{B}) = \mathcal{T}(\mathcal{L}_1 \times \mathcal{C}) = \mathcal{T}_1(\mathcal{L}_1) = \Theta(\mathcal{L}_1).$$

Thus, $\mathcal{T}_1^{\mathcal{B}} = \Theta|_{\mathcal{B}}$. Now we need to prove that $\mathcal{T}_2^{\mathcal{B}} \ll \Gamma|_{\mathcal{B}}$. Observe that for any $\mathcal{L}_2$ measurable subset of $\mathcal{B}$ we have that $\mathcal{T}_2^{\mathcal{B}}(\mathcal{L}_2)$ equals

$$\mathcal{T}_{\mathcal{B}}(\mathcal{B} \times \mathcal{L}_2) = \mathcal{T}((\mathcal{B} \times \mathcal{L}_2) \cap (\mathcal{B} \times \mathcal{B})) = \mathcal{T}(\mathcal{B} \times \mathcal{L}_2) = \mathcal{T}(\mathcal{C} \times \mathcal{L}_2) = \mathcal{T}_2(\mathcal{L}_2),$$

that is, $\mathcal{T}_2^{\mathcal{B}} = \mathcal{T}_2|_{\mathcal{B}}$. Therefore, since $\mathcal{T}_2 \ll \Gamma$, we have that $\mathcal{T}_2^{\mathcal{B}} \ll \Gamma|_{\mathcal{B}}$. In turn, $\mathcal{T}^{\mathcal{B}} \in \mathcal{F}_{\mathcal{B}}$.

Now we show $\mathcal{T}^{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}) = \Theta|_{\mathcal{B}}(\mathcal{B})$. It suffices to prove that $\mathcal{T}^{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}^c) = 0$ where the complement is taken with respect to $\mathcal{B} \times \mathcal{B}$, we do this by contradiction. Assume that $\mathcal{T}^{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}^c) > 0$, this implies that $0 < \mathcal{T}^{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}^c) = \mathcal{T}(\mathcal{E}|_{\mathcal{B}}^c)$, and we must have that $\mathcal{T}_2(\mathcal{B}) > 0$, indeed

$$0 < \mathcal{T}(\mathcal{E}|_{\mathcal{B}}^c) \leq \mathcal{T}(\mathcal{C} \times \mathcal{B}) = \mathcal{T}_2(\mathcal{B}).$$

Next, observe that for any $\mathcal{L}_2$ measurable subset of $\mathcal{B}$

$$\mathcal{T}_2^{\mathcal{B}}(\mathcal{L}_2) = \mathcal{T}_2(\mathcal{L}_2) = \int_{\mathcal{L}_2} s^{\mathcal{T}}(x)\, d\Gamma(x) = \int_{\mathcal{L}_2} s^{\mathcal{T}}(x)\, d\Gamma|_{\mathcal{B}}(x),$$

therefore,

$$\frac{d\mathcal{T}_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(x) \;=\; s^{\mathcal{T}}(x), \qquad \Gamma - a.e.\ x \text{ in } \mathcal{B}. \tag{D-7}$$

This implies that

$$V(x|p^{\mathcal{B}},\mathcal{T}^{\mathcal{B}}) = \operatorname*{ess\,sup}_{\mathcal{B}} \Pi\Big(x,\cdot,p^{\mathcal{B}}(\cdot),\frac{d\mathcal{T}_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(\cdot)\Big) = \operatorname*{ess\,sup}_{\mathcal{B}} \Pi\Big(x,\cdot,p(\cdot),\frac{d\mathcal{T}_2}{d\Gamma}(\cdot)\Big) = V_{\mathcal{B}}(x|p,\mathcal{T}). \tag{D-8}$$

Consider the set $\mathcal{G} \triangleq \{y\in\mathcal{B} : \frac{d\mathcal{T}_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(y) = s^{\mathcal{T}}(y)\}$. Then, by Eq. (D-7) we have

$$\mathcal{T}(\mathcal{E}|_{\mathcal{B}}^c \cap (\mathcal{B}\times\mathcal{G}^c)) \le \mathcal{T}(\mathcal{C}\times\mathcal{G}^c) = \mathcal{T}_2(\mathcal{G}^c) = 0,$$

where the complement is take with respect to $\mathcal{B}$. Therefore, $0 < \mathcal{T}(\mathcal{E}|_{\mathcal{B}}^c) = \mathcal{T}(\mathcal{E}|_{\mathcal{B}}^c\cap(\mathcal{B}\times\mathcal{G}))$ and we can conclude that

$$\mathcal{T}\Big(\underbrace{\{(x,y)\in\mathcal{B}\times\mathcal{B} : \Pi(x,y,p(y),\frac{d\mathcal{T}_2}{d\Gamma}(y)) \ne V_{\mathcal{B}}(x|p,\mathcal{T})\}}_{\triangleq R}\Big) > 0.$$

Define the sets R^- and R^+ by

$$R^- = \{(x,y)\in\mathcal{B}\times\mathcal{B} : \Pi(x,y,p(y),\frac{d\mathcal{T}_2}{d\Gamma}(y)) > V_{\mathcal{B}}(x|p,\mathcal{T})\}$$
$$R^+ = \{(x,y)\in\mathcal{B}\times\mathcal{B} : \Pi(x,y,p(y),\frac{d\mathcal{T}_2}{d\Gamma}(y)) < V_{\mathcal{B}}(x|p,\mathcal{T})\},$$

and note that $R = R^-\cup R^+$. To obtain a contradiction we argue that $\mathcal{T}(R^-\cup R^+) = 0$. Consider first the set R^+, and note that $\mathcal{T}(R^+) = \mathcal{T}(R^+\cap\mathcal{E})$. However, any $(x,y)\in R^+\cap\mathcal{E}$ satisfies

$$\Pi(x,y,(p(y),\frac{d\mathcal{T}_2}{d\Gamma}(y)) < V_{\mathcal{B}}(x|p,\mathcal{T}) \text{ and } \Pi(x,y,(p(y),\frac{d\mathcal{T}_2}{d\Gamma}(y)) = V(x|p,\mathcal{T}),$$

but $V(x)\ge V_{\mathcal{B}}(x)$ implies that $R^+\cap\mathcal{E} = \emptyset$ and, therefore, $\mathcal{T}(R^+) = 0$.

Consider R^-. Define $A \triangleq \{y\in\mathcal{B} : U(y) = V_{\mathcal{B}}(y|p,\mathcal{T})\}$, then by Lemma A-2 we have $\mathcal{T}(R^-) = \mathcal{T}(R^-\cap(\mathcal{B}\times A))$. Take any $(x,y)\in R^-\cap(\mathcal{B}\times A)$ then $V_{\mathcal{B}}(y|p,\mathcal{T}) - |y-x| > V_{\mathcal{B}}(x|p,\mathcal{T})$, which, because of the Lipchitz property (see Lemma 1), is not possible. Thus, $R^-\cap(\mathcal{B}\times A)) = \emptyset$ and we have that $\mathcal{T}(R^-) = 0$. This proves that $\mathcal{T}^{\mathcal{B}}$ is an equilibrium in $\mathcal{B}$.

Now we show that $V(x|p^{\mathcal{B}},\mathcal{T}^{\mathcal{B}}) = V(x|p^{\mathcal{B}^c},\mathcal{T}^{\mathcal{B}^c})$ for all $x\in\partial\mathcal{B}$. From equation (D-8) we have

$$V(x|p^{\mathcal{B}},\mathcal{T}^{\mathcal{B}}) = V_{\mathcal{B}}(x|p,\mathcal{T}) \quad\text{and}\quad V(x|p^{\mathcal{B}^c},\mathcal{T}^{\mathcal{B}^c}) = V_{\mathcal{B}^c}(x|p,\mathcal{T}),$$

so we just need to show $V_{\mathcal{B}}(x|p,\mathcal{T})$ equals $V_{\mathcal{B}^c}(x|p,\mathcal{T})$ for all $x\in\partial B$. We first show that $V_{\mathcal{B}}(x|p,\mathcal{T}) = V(x|p,\mathcal{T})$ for all $x\in\mathcal{B}$. Let $x\in\mathcal{B}$, since $\mathcal{B}$ is an interval or a union of intervals we must have $\Theta(B(x,\frac{1}{n})\cap\mathcal{B}) > 0$ for all $n\in\mathbb{N}$. In turn, this implies

$$0 < \mathcal{T}(B(x,\frac{1}{n})\cap\mathcal{B}\times\mathcal{B}) = \mathcal{T}(B(x,\frac{1}{n})\cap\mathcal{B}\times(\mathcal{B}\cap A)),$$

where $A\triangleq\{y\in\mathcal{B} : V(y|p,\mathcal{T}) = V_{\mathcal{B}}(y|p,\mathcal{T})\}$, and we have used Lemma A-2. From Lemma B-3 there exists $(z_n,y_n)\in B(x,\frac{1}{n})\cap\mathcal{B}\times(\mathcal{B}\cap A)$ such that $y_n\in\mathcal{IR}(z_n|p,\mathcal{T})$, that is, $V(y_n)-\|z_n-y_n\| = V(z_n)$. Since $y_n\in A$ we have $V_{\mathcal{B}}(y_n) - \|z_n-y_n\| = V(z_n)$. From the Lipschitz property we have $V_{\mathcal{B}}(y_n) - \|z_n-y_n\| \le V_{\mathcal{B}}(z_n)$. Thus, $V(z_n)\le V_{\mathcal{B}}(z_n)$. Taking limit $n\uparrow\infty$ and noting that $z_n\to x$ we deduce that $V(x)\le V_{\mathcal{B}}(x)$ (recall that both $V(\cdot)$ and $V_{\mathcal{B}}(\cdot)$ are continuous functions). But we always have that $V(x)\ge V_{\mathcal{B}}(x)$ and, therefore, $V(x) = V_{\mathcal{B}}(x)$. The same argument shows that $V(x) = V_{\mathcal{B}^c}(x)$ for all $x\in\mathcal{B}^c$.

To conclude we need to prove that $V_{\mathcal{B}}(x|p,\mathcal{T})$ equals $V_{\mathcal{B}^c}(x|p,\mathcal{T})$ for all $x\in\partial B$. Consider $x\in\partial\mathcal{B}$. Let $\{x_n\}_{n\in\mathbb{N}}\subset\mathcal{B}$ be a sequence converging to x. Then the continuity of $V_{\mathcal{B}}$ implies $V_{\mathcal{B}}(x_n)\to V_{\mathcal{B}}(x)$. At the same time, since $x_n\in\mathcal{B}$ we have $V_{\mathcal{B}}(x_n) = V(x_n)$ and by continuity $V(x_n)\to V(x)$. Then $V_{\mathcal{B}}(x) = V(x)$ and the same is true for $\mathcal{B}^c$, which implies $V_{\mathcal{B}}(x|p,\mathcal{T}) = V_{\mathcal{B}^c}(x|p,\mathcal{T})$ for all $x\in\partial B$.

Pasting. First we check that $\mathcal{T} \in \mathcal{F}(\Theta)$. Let $\mathcal{L}_1$ be any measurable subset of $\mathcal{C}$ we have that

$$\begin{aligned}
\mathcal{T}_1(\mathcal{L}_1) &= \mathcal{T}(\mathcal{L}_1 \times \mathcal{C}) \\
&= \mathcal{T}^{\mathcal{B}}((\mathcal{L}_1 \times \mathcal{C}) \cap (\mathcal{B} \times \mathcal{B})) + \mathcal{T}^{\mathcal{B}^c}((\mathcal{L}_1 \times \mathcal{C}) \cap (\mathcal{B}^c \times \mathcal{B}^c)) \\
&= \mathcal{T}^{\mathcal{B}}((\mathcal{L}_1 \cap \mathcal{B}) \times \mathcal{B}) + \mathcal{T}^{\mathcal{B}^c}((\mathcal{L}_1 \cap \mathcal{B}^c) \times \mathcal{B}^c) \\
&= \Theta|_{\mathcal{B}}(\mathcal{L}_1 \cap \mathcal{B}) + \Theta|_{\mathcal{B}^c}(\mathcal{L}_1 \cap \mathcal{B}^c) \\
&= \Theta(\mathcal{L}_1).
\end{aligned}$$

Also, if $\Gamma(\mathcal{L}_1) = 0$ then $\Gamma|_{\mathcal{B}}(\mathcal{L}_1) = \Gamma|_{\mathcal{B}^c}(\mathcal{L}_1) = 0$. Therefore, $\mathcal{T}_2^{\mathcal{B}}(\mathcal{L}_1) = \mathcal{T}_2^{\mathcal{B}^c}(\mathcal{L}_1) = 0$, which in turn implies $\mathcal{T}_2 \ll \Gamma$. Hence $\mathcal{T} \in \mathcal{F}(\Theta)$.

Now we show the set

$$\mathcal{E} \triangleq \Big\{(x, y) \in \mathcal{C} \times \mathcal{C} : \Pi(x, y, p(y), s^{\mathcal{T}}(y)) = \operatorname*{ess\,sup}_{\mathcal{C}} \Pi(x, \cdot, p(\cdot), s^{\mathcal{T}}(\cdot)\Big)\Big\},$$

satisfies $\mathcal{T}(\mathcal{E}) = \Theta(\mathcal{C})$. Note that

$$\mathcal{E} \cap \mathcal{B} \times \mathcal{B} = \Big\{(x, y) \in \mathcal{B} \times \mathcal{B} : \Pi(x, y, p(y), s^{\mathcal{T}}(y)) = V(x|p, \mathcal{T})\Big\}.$$

It is enough to prove that $\mathcal{T}^{\mathcal{B}}(\mathcal{E} \cap \mathcal{B} \times \mathcal{B}) = \Theta(\mathcal{B})$. As we did in the first part of the proof (see Eq. (D-7)) we can show that

$$\frac{d\mathcal{T}_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(x) \;=\; s^{\mathcal{T}}(x), \qquad \Gamma - a.e.\ x \text{ in } \mathcal{B},$$

so if we prove that $V(\cdot|p, \mathcal{T})|_{\mathcal{B}} \equiv V(\cdot|p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ we will be done (the proof for $\mathcal{B}^c$ is analogous). Fix $x \in \mathcal{B}$, as in Eq. (D-8) we have

$$V(x|p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}}) = \operatorname*{ess\,sup}_{\mathcal{B}} \Pi(x, \cdot, p^{\mathcal{B}}(\cdot), \frac{d\mathcal{T}_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(\cdot)\Big) = \operatorname*{ess\,sup}_{\mathcal{B}} \Pi(x, \cdot, p(\cdot), \frac{d\mathcal{T}_2}{d\Gamma}(\cdot)\Big) = V_{\mathcal{B}}(x|p, \mathcal{T}).$$

So we just need to verify that $V(x|p, \mathcal{T}) = V_{\mathcal{B}}(x|p, \mathcal{T})$. We show that $V(x|p, \mathcal{T}) \le V_{\mathcal{B}}(x|p, \mathcal{T})$, the other inequality always holds. Let $I(x)$ be the interval in $\mathcal{B}$ to which x belongs to. Let $y_L = \inf I(x)$ and $y_U = \sup I(x)$, note that y_L and y_U do not necessarily belong to $\mathcal{B}$ but they do belong to $\partial\mathcal{B}$. By assumption $V(y|\, p^{\mathcal{B}}, \mathcal{T}_{\mathcal{B}}) = V(y|\, p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})$ for $y \in \{y_L, y_U\}$, in turn this implies that $V_{\mathcal{B}}(y|p, \mathcal{T})$ equals $V_{\mathcal{B}^c}(y|p, \mathcal{T})$ for $y \in \{y_L, y_U\}$. Consider the sets $\mathcal{B}_L^c = [H, y_L] \cap \mathcal{B}^c$ and $\mathcal{B}_U^c = [y_U, H] \cap \mathcal{B}^c$ then

$$\begin{aligned}
V_{\mathcal{B}}(x|p, \mathcal{T}) &\overset{(a)}{\ge} V_{\mathcal{B}}(y_U|p, \mathcal{T}) - |x - y_U| \\
&= V_{\mathcal{B}}(y_U|p, \mathcal{T}) - (y_U - x) \\
&\overset{(b)}{\ge} U(w, s^{\mathcal{T}}(w)) - |y_U - w| - (y_U - x), \quad \Gamma - a.e.\ w \text{ in } \mathcal{B}_U^c \\
&\overset{(c)}{\ge} U(w, s^{\mathcal{T}}(w)) - (w - y_U) - (y_U - x), \quad \Gamma - a.e.\ w \text{ in } \mathcal{B}_U^c \\
&\overset{(d)}{\ge} U(w, s^{\mathcal{T}}(w)) - |w - x|, \quad \Gamma - a.e.\ w \text{ in } \mathcal{B}_U^c,
\end{aligned}$$

where (a) follows from the Lipschitz property (see Lemma 1), and (b) from the definition of $V_{\mathcal{B}}(y_U|p, \mathcal{T})$ and $\Gamma(\mathcal{B}_U^c) > 0$; $(c), (d)$ hold because for $w \in \mathcal{B}_U^c$ we have $x \le y_U \le w$. Similarly,

$$\begin{aligned}
V_{\mathcal{B}}(x|p, \mathcal{T}) &\ge V_{\mathcal{B}}(y_L|p, \mathcal{T}) - |x - y_L| \\
&= V_{\mathcal{B}}(y_L|p, \mathcal{T}) - (x - y_L) \\
&\ge U(w, s^{\mathcal{T}}(w)) - |y_L - w| - (x - y_L), \quad \Gamma - a.e.\ w \text{ in } \mathcal{B}_L^c \\
&= U(w, s^{\mathcal{T}}(w)) - (y_L - w) - (x - y_L), \quad \Gamma - a.e.\ w \text{ in } \mathcal{B}_L^c \\
&= U(w, s^{\mathcal{T}}(w)) - |w - x|, \quad \Gamma - a.e.\ w \text{ in } \mathcal{B}_L^c.
\end{aligned}$$

Since $\mathcal{B}_L^c \cup \mathcal{B}_U^c = \mathcal{B}^c$ this implies that $V_{\mathcal{B}}(x|p, \mathcal{T}) \ge V(x|p, \mathcal{T})$. This concludes the proof. □

D.2 Proofs for Section 8.2

Proof of Lemma 2. Let $(p, \mathcal{T})$ be a feasible solution. We show that at any optimal solution we must have $X_l < 0 < X_r$, in turn this implies that 0 is a sink. By Lemma D-2 we have that if either of the sets $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \mathcal{T})\}$ or $\{x \in [-H, 0) : 0 \in \mathcal{IR}(x|p, \mathcal{T})\}$ is empty then the revenue the platform makes satisfies $\frac{1}{\gamma} \cdot \mathbf{Rev}(p, \mathcal{T}) \leq \psi_1 \cdot \theta_1 \cdot 2 \cdot H$. However, the solution $(p, \mathcal{T})$ given in Proposition 4 has both sets non-empty because $0 \in \mathcal{IR}(X_r|\, p, \mathcal{T})$ and $0 \in \mathcal{IR}(-X_r|\, p, \mathcal{T})$ with $X_r > 0$. Furthermore, $\mathbf{Rev}(p, \mathcal{T})$ is strictly large than the revenue of the pre-demand shock environment or, equivalently, strictly larger than $\psi_1 \cdot \theta_1 \cdot 2 \cdot H$. This implies that any optimal solution must satisfy $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \mathcal{T})\} \neq \emptyset$ and $\{x \in [-H, 0) : 0 \in \mathcal{IR}(x|p, \mathcal{T})\} \neq \emptyset$ and, therefore, $X_l < 0 < X_r$. □

Lemma D-4. *(Upper bound) An optimal price-equilibrium pair $(p, \mathcal{T})$ satisfies*

$$V(x) \leq \min\{V(X_r) + x - X_r, \psi_1\}, \quad \text{for all } x \in (X_r, H]. \tag{D-9}$$

Proof of Lemma D-4. If $X_r = H$ there is nothing to prove, so let's assume $X_r < H$. Fix $x \in [X_r, H]$. From the Lipschitz property (see Lemma 1) we have that $V(x|\, p, \mathcal{T}) \leq V(X_r|\, p, \mathcal{T}) + (x - X_r)$. Moreover, Proposition B-1 ensures that $\mathcal{T}([X_r, H] \times [X_r, H]^c) = 0$ and, hence, because $0 \notin [X_r, H]$ we can apply Lemma D-1 to deduce that

$$V(x|p, \mathcal{T}) \;\leq\; \psi_1, \quad \Gamma - a.e.\ x \;\text{ in }\; [X_r, H]. \tag{D-10}$$

To show that the previous inequality holds everywhere, notice that if $V(x|\, p, \mathcal{T}) > \psi_1$ the from the Lipschitz continuity property of $V(\cdot|\, p, \mathcal{T})$ we could find a subset of of $[X_r, H]$ with positive Γ measure (in this set Γ coincides with the Lebesgue measure) in which $V(\cdot|\, p, \mathcal{T})$ is strictly larger than ψ_1. This is not possible because it would contradict Eq. (D-10). Putting together both upper bounds yields the desire result. □

Proposition D-2. *(Monotonicity in the periphery) Without loss of optimality, we can focus on price-equilibrium pairs $(p, \mathcal{T})$ such that $V(\cdot)$ is non-decreasing in $(X_r, H]$. Furthermore, if $V(X_r) = \psi_1$, then $V(x) = \psi_1$ for all $x \geq X_r$.*

Proof of Proposition D-2. Let $(p, \mathcal{T})$ be optimal for problem $(\mathcal{P}_2)$ as in Lemma 2 so we have $0 < X_r$. Note that is $X_r = H$ then the result trivially holds, so let's assume $X_r < H$. Before we begin note that for any $x \geq X_r$, by Lemma D-4 and the Lipschitz continuity property of $V(\cdot|\, p, \mathcal{T})$ (see Lemma 1), we must have $V(x) \leq \psi_1$.

We first prove the second statement of the proposition. Suppose $V(X_r) = \psi_1$ and define the set $R \triangleq \{x \in [X_r, H] : V(x) = \psi_1\}$. We show by contradiction that we cannot have $\mathcal{T}_2(R^c) > 0$ (the complement is taken with respect to $[X_r, H]$). If $\mathcal{T}_2(R^c) > 0$, because ψ_1 is an upper bound from Proposition 1 we have the following

$$\begin{aligned}
\frac{1}{\gamma} \cdot \mathbf{Rev}_{[X_r,H]}(p, \mathcal{T}) &= \int_{[X_r,H]} V(x)\, d\mathcal{T}_2(x) \\
&= \int_R V(x)\, d\mathcal{T}_2(x) + \int_{R^c} V(x)\, d\mathcal{T}_2(x) \\
&< \int_R V(x)\, d\mathcal{T}_2(x) + \int_{R^c} \psi_1\, d\mathcal{T}_2(x) \\
&\leq \psi_1 \cdot \mathcal{T}_2([X_r, H]) \\
&= \psi_1 \cdot \theta_1 \cdot (H - X_r),
\end{aligned}$$

where the last line comes Proposition B-1. Thus, the quantity $\mathbf{Rev}_{[-H,X_r]}(p, \mathcal{T}) + \gamma \cdot \psi_1 \cdot \theta_1 \cdot (H - X_r)$, strictly upper bounds the platform's objective. So if we are able to construct a solution such that attains the upper bound, we will contradict the optimality of $(p, \mathcal{T})$. Observe that Lemma D-3 enables us to separate the solution $(p, \mathcal{T})$ in $[-H, X_r]$ and $(X_r, H]$. The separated solution $(p^{[-H,X_r]}, \mathcal{T}^{[-H,X_r]})$ (see Lemma D-3 for notation) in $[-H, X_r]$ has revenue equal to $\mathbf{Rev}_{[-H,X_r]}(p, \mathcal{T})$, and $V(X_r|\, p^{[-H,X_r]}, \mathcal{T}^{[-H,X_r]})$ coincides with $V(X_r|\, p, \mathcal{T})$ which equals ψ_1. For $(X_r, H]$ consider prices $\tilde{p}(x) = \rho_1$ for all $x \in (X_r, H]^c$, and flows $\tilde{\mathcal{T}}(\mathcal{L}) = \Theta(\pi_1(\mathcal{L} \cap \mathcal{D}))$ for any measurable set $\mathcal{L} \subset (X_r, H] \times (X_r, H]$. The pair $(\tilde{p}, \tilde{\mathcal{T}})$ is the same solution as in Proposition D-1 with the sole difference that we have changed the city to be $(X_r, H]$ instead of $\mathcal{C}$. Therefore, $(\tilde{p}, \tilde{\mathcal{T}})$ is a feasible price-equilibrium in $(X_r, H]$ with revenue equal to $\gamma \cdot \psi_1 \cdot \theta_1 \cdot (H - X_r)$, and such that $V(x|\, \tilde{p}, \tilde{\mathcal{T}})$ equal to ψ_1 for all $x \in (X_r, H]$. Thus we can use Lemma D-3 to paste both solution and obtain an equilibrium in the entire city. This new equilibrium achieves the upper bound.

Suppose that $\mathcal{T}_2(R^c) = 0$ and define the sets

$$L_+ \triangleq \{x : \theta_1 > s^{\mathcal{T}}(x)\}, \quad L_0 \triangleq \{x : \theta_1 = s^{\mathcal{T}}(x)\}, \quad L_- \triangleq \{x : \theta_1 < s^{\mathcal{T}}(x)\}.$$

Then by Lemma 2 it holds that $\Gamma(R \cap L_-) = 0$. Moreover, if $\Gamma(R \cap L_+) > 0$ we have

$$\Theta([X_r, H]) = \mathcal{T}_2([X_r, H]) \overset{(a)}{=} \mathcal{T}_2(R) = \int_{R\cap L_+} s^{\mathcal{T}}(x)\, d\Gamma(x) + \int_{R\cap L_0} s^{\mathcal{T}}(x)\, d\Gamma(x) < \theta_1 \Gamma(R) \le \Theta([X_r, H]),$$

not possible, where (a) comes from Proposition B-1. Thus $\Gamma(R \cap L_+) = 0$. This implies that $\Gamma(R \cap L_0) = \Gamma(R)$ and

$$\theta_1 \Gamma([X_r, H]) = \Theta([X_r, H]) = \int_{R\cap L_0} s^{\mathcal{T}}(x)\, d\Gamma(x) = \theta_1 \Gamma(R),$$

that is, $\Gamma(R) = \Gamma([X_r, H])$ or $\Gamma(R^c) = 0$. In turn, $\Gamma - a.e.\ x \in [X_r, H]$ we have that $V(x)$ equals ψ_1. Since, $V(\cdot)$ is continuous and $\Gamma|_{[X_r,H]}$ has full support in $[X_r, H]$ which has non-empty interior we conclude that $V(x) = \psi_1$ for all $x \in [X_r, H]$.

For the reminder of the proof we assume $V(X_r) < \psi_1$. We show that if $V(\cdot)$ is not non-decreasing in $[X_r, H]$ then there is an strict objective improvement. In the proof we define several critical points in the interval $[X_r, H]$ which will help us to create a flow separated region (no flow leaves this region). Then we show the objective strict improvement in this region. In Figure 11 we provide a graphical representation of the points just mentioned.

So assume that $V(x)$ is not non-decreasing in $[X_r, H]$, then there exists $\hat{x} > \hat{y} \ge X_r$ such that $V(\hat{x}) < V(\hat{y})$. Let,

$$\bar{y} \triangleq \sup\{z \in [\hat{y}, \hat{x}] : V(z) = V(\hat{y})\},$$

note that since for $z = \hat{y}$, $V(z) = V(\hat{y})$ thus the set over which we take the supremum above is both bounded and non-empty. Hence, $\bar{y}$ is well defined and it corresponds to the last point z in $[\hat{y}, \hat{x}]$ such that $V(z)$ equals $V(\hat{y})$. Moreover, because $V(\cdot)$ is continuous $\bar{y} < \hat{x}$, and for all $z \in (\bar{y}, \hat{x}]$ we have $V(z) < V(\hat{y}) = V(\bar{y})$. Let

$$y_0 \triangleq \inf\{z \in [X_r, \bar{y}] : \exists x \in (\bar{y}, H] \quad \text{such that} \quad z \in \mathcal{IR}(x)\},$$

if for all $z \in [X_r, \bar{y}]$ and for all $x \in (\bar{y}, H]$ we have $z \notin \mathcal{IR}(x)$, we let $y_0 = \bar{y}$. That is, y_0 is the smallest z in $[X_r, \bar{y}]$ to which some location in $(\bar{y}, H]$ is indifferent to travel to. Note that for all $z \in (y_0, \hat{x}]$ we have $V(z) < V(y_0)$. Also, the definition of y_0 and Lemma B-3 imply that $\mathcal{T}([-H, y_0] \times (y_0, H]) = 0$ and $\mathcal{T}((y_0, H] \times [-H, y_0]) = 0$. Let

$$y_1 \triangleq \inf\{z \in [\hat{x}, H] : V(z) > V(y_0)\},$$

that is, y_1 is the first value after $\hat{x}$ for which $V(\cdot)$ hits $V(y_0)$. Note that when well defined y_1 satisfies that $\mathcal{T}([y_1, H] \times [-H, y_1]) = 0$. If this is not the case then since points do not have measure(different from the origin) we would have $\mathcal{T}((y_1, H] \times [-H, y_1)) > 0$ and, therefore, by Lemma B-3 we can find $(x, y) \in (y_1, H] \times [-H, y_1)$ such that $y \in \mathcal{IR}(x)$. But this would contradict the definition of y_1.

There are two cases:

1. y_1 is not well defined: In this case we have that for all $z \in [\hat{x}, H]$, $V(z) \le V(y_0)$. Recall that from our previous discussion we have that $V(z) < V(y_0)$ for all $z \in (y_0, \hat{x}]$. Also, Property 1 (which we prove at

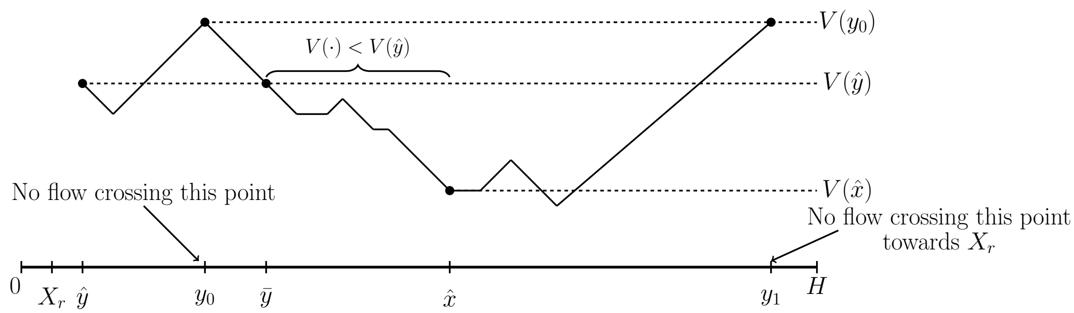

Figure 11: Graphical representation of $\hat{y}, \hat{x}, \bar{y}, y_0$ and y_1.

the end of the present proof) establishes that $\mathcal{T}_2((y_0, \hat{x}]) > 0$. Using this observations we create a new solution $(\tilde{p}, \tilde{\mathcal{T}})$ with revenue strictly larger than that of $(p, \mathcal{T})$.

Let $\mathcal{B} = [-H, y_0]$ and note that we have both $\mathcal{T}(\mathcal{B} \times \mathcal{B}^c) = 0$ and $\mathcal{T}(\mathcal{B}^c \times \mathcal{B}) = 0$, so we can use the separation result in Lemma D-3. Hence $(p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ (see Lemma D-3 for notation) is a price-equilibrium pair in $\mathcal{B}$. Its revenue equals the revenue of $(p, \mathcal{T})$ in $\mathcal{B}$, and $V(y_0| \, p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}}) = V(y_0)$.

For $\mathcal{B}^c$ we choose flows $\mathcal{T}^{\mathcal{B}^c}(\mathcal{L}) = \Theta(\pi_1(\mathcal{L} \cap \mathcal{D}))$ for all $\mathcal{L} \subset \mathcal{B}^c \times \mathcal{B}^c$. That is all drivers stay at their initial location. It is not hard to see that $s^{\mathcal{T}^{\mathcal{B}^c}}(x)$ equals θ_1, $\Gamma - a.e.\ x$ in $\mathcal{B}^c$. We choose prices $p^{\mathcal{B}^c}(x) = p_0$ for all $x \in \mathcal{B}^c$, where p_0 is such that

$$\alpha \cdot p_0 \cdot \min\{1, \frac{\lambda_1 \cdot \overline{F}(p_0)}{\theta_1}\} = V(y_0), \tag{D-11}$$

note that since $V(y_0) \leq \psi_1$, p_0 is well defined. That is, the solution $(p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})$ is the same solution as in pre-demand shock environment but in smaller city, $\mathcal{B}^c$ and with a larger price across all locations. Using Proposition 1 it is not hard to see that the revenue associated with this solution is $\gamma \cdot V(y_0) \cdot \theta_1 \cdot (H - y_0)$.

By Lemma D-3, we can paste the two previous solutions to create a new solution $(\tilde{p}, \tilde{\mathcal{T}})$ in entire city. This new solution yields a strict objective improvement. Indeed,

$$\begin{aligned}
\mathbf{Rev}_{[y_0,H]}(p, \mathcal{T}) &= \int_{[y_0,H]} V(x)\, d\mathcal{T}_2(x) \\
&= \int_{(y_0,\hat{x}]} V(x)\, d\mathcal{T}_2(x) + \int_{(\hat{x},H]} V(x)\, d\mathcal{T}_2(x) \\
&\overset{(a)}{<} V(y_0) \cdot \mathcal{T}_2((y_0, \hat{x}]) + \int_{(\hat{x},H]} V(x)\, d\mathcal{T}_2(x) \\
&\leq V(y_0) \cdot \mathcal{T}_2((y_0, \hat{x}]) + V(y_0) \cdot \mathcal{T}_2((\hat{x}, H]) \\
&\overset{(b)}{=} V(y_0) \cdot \Theta([y_0, H]) \\
&= V(y_0) \cdot \theta_1 \cdot (H - y_0) \\
&= \mathbf{Rev}_{[y_0,H]}(\tilde{p}, \tilde{\mathcal{T}}),
\end{aligned}$$

where (a) comes from $\mathcal{T}_2((y_0, \hat{x}]) > 0$, (b) comes from the fact that under $\mathcal{T}$ no flow leaves or enters $[y_0, H]$, and the last two lines from the definition of $(\tilde{p}, \tilde{\mathcal{T}})$ restricted to $[y_0, H]$.

2. y_1 is well defined: In this case there exists $z \in [\hat{x}, H]$ such that $V(z) > V(y_0)$. Also, we must have $y_1 > \hat{x}$, and we already argued that $\mathcal{T}([y_1, H] \times [-H, y_1]) = 0$. There are two more cases.

   a) $\forall y \in (y_0, y_1]$, $\forall x > y_1$, $x \notin \mathcal{IR}(y)$: This together with Lemma B-3 imply that $\mathcal{T}([y_0, y_1] \times ([-H, y_0] \cup [y_1, H])) = 0$, and we also have $\mathcal{T}(([-H, y_0] \cup [y_1, H]) \times [y_0, y_1]) = 0$. From this we can construct a new feasible solution $(\tilde{p}, \tilde{\mathcal{T}})$ with revenue strictly larger than that of $(p, \mathcal{T})$.

Let $\mathcal{B} = [-H, y_0) \cup (y_1, H]$ and note that we have both $\mathcal{T}(\mathcal{B} \times \mathcal{B}^c) = 0$ and $\mathcal{T}(\mathcal{B}^c \times \mathcal{B}) = 0$, so we can use the separation result in Lemma D-3. Thus $(p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ (see Lemma D-3 for notation) is a price-equilibrium pair in $\mathcal{B}$. Its revenue equals the revenue of $(p, \mathcal{T})$ in $\mathcal{B}$, and $V(y_0|\, p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}}) = V(y_0)$ and $V(y_1|\, p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}}) = V(y_0)$.

For $\mathcal{B}^c$ we choose flows $\mathcal{T}^{\mathcal{B}^c}(\mathcal{L}) = \Theta(\pi_1(\mathcal{L} \cap \mathcal{D}))$ for all $\mathcal{L} \subset \mathcal{B}^c \times \mathcal{B}^c$. We choose prices $p^{\mathcal{B}^c}(x) = p_0$ for all $x \in \mathcal{B}^c$, where p_0 is as in Eq. (D-11). As we argued before this solution forms an price-equilibrium pair with revenue equal to $V(y_0) \cdot \theta_1 \cdot (y_1 - y_0)$.

We can then paste both solutions (see Lemma D-3) to obtain a solution $(\tilde{p}, \tilde{\mathcal{T}})$ in the entire city. As before, it yields a strict revenue improvement.

b) $\exists\, y \in (y_0, y_1]$, $\exists\, x > y_1$ such that $x \in \mathcal{IR}(y)$: Then the following points are well defined

$$\overline{y}_1 \triangleq \sup\{x \in [y_1, H] : \exists\, y \in [y_0, y_1] \text{ such that } x \in \mathcal{IR}(y)\},$$
$$\underline{y}_1 \triangleq \inf\{y \in [y_0, y_1] : \exists\, x \in [y_1, H] \text{ such that } x \in \mathcal{IR}(y)\}.$$

That is, $\overline{y}_1$ is largest point after y_1 for which some location in $[y_0, y_1]$ has drivers indifferent to travel to it. As for $\underline{y}_1$, it corresponds to the smallest point in $[y_0, y_1]$ that has drivers willing to travel to some location in $[y_1, H]$. Note that from the definition of $\overline{y}_1$ and Lemma B-3 we can deduce that there is no flow crossing $\overline{y}_1$ in any direction, that is, $\mathcal{T}([-H, \overline{y}_1] \times [\overline{y}_1, H]) = 0$. Also, from Property 2 (which we prove at the end of the present proof) for any $z \in [\underline{y}_1, \overline{y}_1]$, $\overline{y}_1 \in \mathcal{IR}(z)$. That is, for any $z \in [\underline{y}_1, \overline{y}_1]$, $V(z|\, p, \mathcal{T}) = V(\overline{y}_1) - |\overline{y}_1 - z|$.

The idea is to again construct an strict objective improvement. First, define y^c to be such that $V(y_0) + (y^c - y_0) = V(\overline{y}_1)$, that is, $y^c = V(\overline{y}_1) - V(y_0) + y_0$. Next we argue that $y_c \in (y_0, \overline{y}_1)$. In fact, by the definition of $\overline{y}_1$ we must have $V(\overline{y}_1) > V(y_0)$ thus $y_c > y_0$. Also, if $y_c \geq \overline{y}_1$ then

$$V(y_0) + (y^c - y_0) \geq V(y_0) + (\overline{y}_1 - y_0) \Leftrightarrow V(\overline{y}_1) \geq V(y_0) + (\overline{y}_1 - y_0),$$

and since $V(\overline{y}_1) = V(y_1) + (\overline{y}_1 - y_1)$ we would have

$$V(y_1) + (\overline{y}_1 - y_1) \geq V(y_0) + (\overline{y}_1 - y_0) \Leftrightarrow V(y_1) - V(y_0) \geq y_1 - y_0,$$

which, since $y_1 > y_0$, implies that $V(y_1) > V(y_0)$, contradicting the definition of y_1. From this we can also infer that $y^c - y_0 = \overline{y}_1 - y_1$.

Second, let $h \triangleq \overline{y}_1 - y^c$ and for any set $\mathcal{L} \subseteq \mathcal{C} \times \mathcal{C}$ define the set

$$\mathcal{L}_h \triangleq \{(x + h, y + h) \in \mathcal{C} \times \mathcal{C} : (x, y) \in \mathcal{L}\}.$$

We now construct a new solution $(\tilde{p}, \tilde{\mathcal{T}})$. Let $\mathcal{B} = [-H, y_0) \cup (\overline{y}_1, H]$, so that $\mathcal{B}^c = [y_0, \overline{y}_1]$. Following our previous scheme of proof we construct two price-equilibrium pairs one in $\mathcal{B}$ and another in $\mathcal{B}^c$, and then we paste them to create $(\tilde{p}, \tilde{\mathcal{T}})$. As we did before we can use the separation result (see Lemma D-3) to obtain a solution $(p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}})$ in $\mathcal{B}$ such that $V(y_0|\, p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}}) = V(y_0)$ and $V(\overline{y}_1|\, p^{\mathcal{B}}, \mathcal{T}^{\mathcal{B}}) = V(\overline{y}_1)$.

For $\mathcal{B}^c$ define the flow $\mathcal{T}^{\mathcal{B}^c}$ for any $\mathcal{L} \subseteq \mathcal{B}^c \times \mathcal{B}^c$ by

$$\mathcal{T}^{\mathcal{B}^c}(\mathcal{L}) = \mathcal{T}\Big((\mathcal{L} \cap ([y_0, y^c] \times [y_0, \overline{y}_1]))_h\Big) + \Theta(\pi_1(\mathcal{L} \cap ([y^c, \overline{y}_1] \times [y_0, \overline{y}_1]) \cap \mathcal{D})), \qquad \text{(D-12)}$$

We next show that this flow belongs to $\mathcal{F}_{\mathcal{B}^c}(\Theta|_{\mathcal{B}^c})$ and that it is an equilibrium for some prices $p^{\mathcal{B}^c}$ yet to be defined. Indeed, for any measurable subset K of $\mathcal{B}^c$ we have

$$\begin{aligned}
\mathcal{T}_1^{\mathcal{B}^c}(K) &= \mathcal{T}\Big(((K\times\mathcal{B}^c)\cap([y_0,y^c]\times[y_0,\overline{y}_1]))_h\Big) + \Theta(\pi_1((K\times\mathcal{B}^c)\cap([y^c,\overline{y}_1]\times[y_0,\overline{y}_1])\cap\mathcal{D}))\\
&= \mathcal{T}\Big(((K\cap[y_0,y^c])\times[y_0,\overline{y}_1])_h\Big) + \Theta(K\cap[y^c,\overline{y}_1])\\
&= \mathcal{T}\Big(((K+h)\cap[y_0+h,y^c+h])\times[y_0+h,\overline{y}_1+h]\Big) + \Theta(K\cap[y^c,\overline{y}_1])\\
&= \mathcal{T}\Big(((K+h)\cap[y_1,\overline{y}_1])\times[y_1,\overline{y}_1+h]\Big) + \Theta(K\cap[y^c,\overline{y}_1])\\
&\overset{(a)}{=} \mathcal{T}\Big(((K+h)\cap[y_1,\overline{y}_1])\times\mathcal{C}\Big) + \Theta(K\cap[y^c,\overline{y}_1])\\
&= \Theta((K+h)\cap[y_1,\overline{y}_1]) + \Theta(K\cap[y^c,\overline{y}_1])\\
&= \Theta((K\cap[y_0,y^c])+h) + \Theta(K\cap[y^c,\overline{y}_1])\\
&\overset{(b)}{=} \Theta(K\cap[y_0,y_c]) + \Theta(K\cap[y^c,\overline{y}_1])\\
&= \Theta(K),
\end{aligned}$$

where (a) holds because by construction in $[y_1,\overline{y}_1]$ the flow there can be transported only inside the same set and, therefore, $\mathcal{T}([y_1,\overline{y}_1]\times[y_1,\overline{y}_1+h]^c)$ equals zero. Equality (b) comes from the fact that Θ is invariant under translation (it is a multiple of the Lebesgue measure). Therefore, $\mathcal{T}_1^{\mathcal{B}^c}$ coincides with $\Theta|_{\mathcal{B}^c}$. Also, it is clear from the definition of $\mathcal{T}^{\mathcal{B}^c}$ that $\mathcal{T}_2^{\mathcal{B}^c}\ll\Gamma$. Hence, $\mathcal{T}^{\mathcal{B}^c}$ belongs to $\mathcal{F}_{\mathcal{B}^c}(\Theta|_{\mathcal{B}^c})$. Furthermore, Property 3 (which we prove at the end of the present proof) ensures that

$$\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(x) \le \frac{d\mathcal{T}_2}{d\Gamma}(x+h)\ \ \Gamma-a.e.\ x\ \text{ in }[y_0,y^c],\ \ \text{and}\ \ \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(x)=\theta_1\ \ \Gamma-a.e.\ x\ \text{ in }[y^c,\overline{y}_1]. \tag{D-13}$$

We choose the prices $p^{\mathcal{B}^c}$ as follows. In $[y^c,\overline{y}_1]$ we set constant prices equal to p_1 such that

$$\alpha\cdot p_1\cdot\min\{1,\frac{\lambda_1\cdot\overline{F}(p_1)}{\theta_1}\} = V(\overline{y}_1),$$

this price is well defined because $V(\overline{y}_1)\le\psi_1$. For locations in $[y_0,y^c]$ consider the set

$$K \triangleq \Big\{x\in[y_0,y^c]:\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(x)\le\frac{d\mathcal{T}_2}{d\Gamma}(x+h)\Big\}, \tag{D-14}$$

note from Eq. D-13 we have $\Gamma(K^c)=0$. We set prices for $x\in K$ to be such that

$$U\Big(x,p^{\mathcal{B}^c}(x),\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(x)\Big) = U\Big(x+h,p(x+h),s^{\mathcal{T}}(x+h)\Big), \tag{D-15}$$

such prices are well defined because the new Radon-Nikodym is smaller than the old one (shifted by h) in K. For $x\in K^c$ we set the prices equal to zero. Now we need to verify that this selection of prices and flows yields an equilibrium. That is, we need show that the set

$$\mathcal{E}_{\mathcal{B}^c} = \Big\{(x,y)\in\mathcal{B}^c\times\mathcal{B}^c:\Pi(x,y,p^{\mathcal{B}^c}(y),\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(y)) = \operatorname*{ess\,sup}_{\mathcal{B}^c}\Pi\Big(x,\cdot,p^{\mathcal{B}^c}(\cdot),\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(\cdot)\Big)\Big\},$$

has $\mathcal{T}^{\mathcal{B}^c}$ measure equal to $\Theta(\mathcal{B}^c)$. First, from Property 3 we have

$$V(x|\,p^{\mathcal{B}^c},\mathcal{T}^{\mathcal{B}^c}) = \operatorname*{ess\,sup}_{\mathcal{B}^c}\Pi\Big(x,\cdot,p^{\mathcal{B}^c}(\cdot),\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(\cdot)\Big) = \begin{cases}V(y_1)+(x-y_0) & \text{if } x\in[y_0,y^c]\\ V(\overline{y}_1) & \text{if } [y^c,\overline{y}_1].\end{cases} \tag{D-16}$$

For the first term in Eq. (D-12) observe that $\mathcal{T}((\mathcal{E}_{\mathcal{B}^c}\cap[y_0,y^c]\times[y_0,\overline{y}_1])_h)$ equals

$$\mathcal{T}\Big(\Big\{(x,y)\in[y_1,\overline{y}_1]\times[y_1,\overline{y}_1]:\Pi(x-h,y-h,p^{\mathcal{B}^c}(y-h),\frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(y-h)) = V(y_1)+(x-y_1)\Big\}\Big),$$

using that $\Gamma(K^c) = 0$ and Eq. (D-24) one can verify that this expression equals

$$\mathcal{T}\Big(\Big\{(x, y) \in [y_1, \overline{y}_1] \times [y_1, \overline{y}_1] : \Pi(x, y, p(y), s^{\mathcal{T}}(y)) = V(x|\, p, \mathcal{T})\Big\}\Big).$$

In turn, from the definition of $\underline{y}_1$ and $\overline{y}_1$, and the fact that $\mathcal{T}$ is an equilibrium flow this last expression equals $\Theta([y_1, \overline{y}_1])$. For the second term in Eq. (D-12) we have

$$\mathcal{E}_{\mathcal{B}^c} \cap [y^c, \overline{y}_1] \times [y_0, \overline{y}_1] \cap \mathcal{D} = \Big\{(x, y) \in [y^c, \overline{y}_1] \times [y_0, \overline{y}_1] : \Pi(x, y, p^{\mathcal{B}^c}(y), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(y)) = V(\overline{y}_1)\Big\} \cap \mathcal{D},$$

Thus the second term in Eq. (D-12) equals

$$\Theta\Big(\Big\{x \in [y^c, \overline{y}_1] : U(x, p^{\mathcal{B}^c}(x), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(x)) = V(\overline{y}_1)\Big\}\Big) = \Theta([y^c, \overline{y}_1]) = \Theta([y_0, y_1]),$$

where the first equality comes from Eq. (D-13) and the discussion that it follows it. The second equality comes from Θ being invariant under translation and $y^c - y_0 = \overline{y}_1 - y_1$. Putting all these together yields

$$\mathcal{T}^{\mathcal{B}^c}(\mathcal{E}_{\mathcal{B}^c}) = \Theta([\overline{y}_1, y_1]) + \Theta([y_0, y_1]) = \Theta([y_0, \overline{y}_1]) = \Theta(\mathcal{B}^c).$$

In order to create the new solution $(\tilde{p}, \tilde{\mathcal{T}})$ we just use Lemma D-3 to paste the two solutions we constructed in $\mathcal{B}$ and $\mathcal{B}^c$. Note that the pasting is allowed because $V(y_0|\, p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c}) = V(y_0)$ and $V(\overline{y}_1|\, p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c}) = V(\overline{y}_1)$.

We now finally show the objective improvement. It is sufficient to prove that $\mathbf{Rev}_{[y_0, \overline{y}_1]}(\tilde{p}, \tilde{\mathcal{T}}) > \mathbf{Rev}_{[y_0, \overline{y}_1]}(p, \mathcal{T})$,

$$\begin{aligned}
\mathbf{Rev}_{[y_0, \overline{y}_1]}(p, \mathcal{T}) = \int_{[y_0, \overline{y}_1]} V(x)\, d\mathcal{T}_2(x) &\overset{(a)}{<} \int_{[y_0, \overline{y}_1]} V(y_0)\, d\mathcal{T}_2(x) \\
&\overset{(b)}{=} \int_{[y_0, \overline{y}_1]} V(y_0)\, d\mathcal{T}_2^{\mathcal{B}^c}(x) \\
&\overset{(c)}{\leq} \int_{[y_0, \overline{y}_1]} V(x|\, p^{\mathcal{B}^c}, \mathcal{T}^{\mathcal{B}^c})\, d\mathcal{T}_2^{\mathcal{B}^c}(x) \\
&= \mathbf{Rev}_{[y_0, \overline{y}_1]}(\tilde{p}, \tilde{\mathcal{T}}),
\end{aligned}$$

where in (a) use Property 1. In (b) we use that under $\mathcal{T}$ no flow leaves or enters $\mathcal{B}^c$ and, thus,

$$\mathcal{T}_2^{\mathcal{B}^c}(\mathcal{B}^c) = \mathcal{T}^{\mathcal{B}^c}(\mathcal{B}^c \times \mathcal{B}^c) = \Theta(\mathcal{B}^c) = \mathcal{T}(\mathcal{B}^c \times \mathcal{C}) = \mathcal{T}(\mathcal{B}^c \times \mathcal{B}^c) = \mathcal{T}(\mathcal{C} \times \mathcal{B}^c) = \mathcal{T}_2(\mathcal{B}^c).$$

In (c) we simply use Eq. (D-16).

In what follows we provide a complete proof of the three properties that we use to obtain the result.

Property 1. $\mathcal{T}_2((y_0, \hat{x}]) > 0$.

Proof of Property 1. First we show that $\exists\, h \in (0, \hat{x} - y_0)$ such that $\mathcal{T}((y_0, y_0 + h) \times [\hat{x}, y_1]) = 0$. Suppose this is not true then for all $n \in \mathbb{N}$ large enough we have that $\mathcal{T}((y_0, y_0 + \frac{1}{n}) \times [\hat{x}, y_1]) > 0$, which thanks to Lemma B-3 implies that for all $n \in \mathbb{N}$ large enough there exists $(x_n, y_n) \in (y_0, y_0 + \frac{1}{n}) \times [\hat{x}, y_1]$ such that $y_n \in \mathcal{IR}(x_n)$, that is, $V(x_n) = V(y_n) - |y_n - x_n|$. Since $y_n \in [\hat{x}, y_1]$ we must have $V(y_n) \leq V(y_0)$ for all $n \in \mathbb{N}$ large (when y_1 is not well defined we replaced by H and the argument still goes through). Furthermore, x_n converges to y_0 so the continuity of $V(\cdot)$ yields

$$V(y_0) = \lim_{n\to\infty} V(x_n) = \lim_{n\to\infty} V(y_n) - |y_n - x_n| \leq V(y_0) - \lim_{n\to\infty} (y_n - x_n) < V(y_0),$$

not possible. We conclude that $\exists\, h \in (0, \hat{x} - y_0)$ such that $\mathcal{T}((y_0, y_0 + h) \times [\hat{x}, y_1]) = 0$. Note that the same must be true for some $h \in (0, (\hat{x} - y_0) \wedge \frac{(y_1 - y_0)}{2})$. We fix h in this interval with the property we just proved.

Next, note we also have that $\mathcal{T}((y_0, y_0+h)\times(y_1,H])=0$; otherwise, by Lemma B-3 we can find $(x,y)\in(y_0,y_0+h)\times(y_1,H]$ such that $y\in\mathcal{IR}(x)$, which implies that $y\in\mathcal{IR}(y_1)$, that is, $V(y_1)=V(y)-|y-y_1|$ and $V(x)=V(y)-|y-x|$. Since $V(y_1)=V(y_0)$ we have $(y_1-x)=V(y_0)-V(x)$, but our choice of h implies that $y_1-x>h$ thus

$$h<(y_1-x)=V(y_0)-V(x)\le|y_0-x|\le h,$$

again a contradiction. The last inequality comes from the Lipschitz property (see Lemma 1). In summary, we have that there exists $h\in(0,(\hat{x}-y_0)\wedge\frac{(y_1-y_0)}{2})$ such that $\mathcal{T}((y_0,y_0+h)\times[\hat{x},H])=0$. To conclude the proof note the following

$$\begin{aligned}
0 &\overset{(a)}{<} \Theta((y_0,y_0+h))\\
&= \mathcal{T}((y_0,y_0+h)\times\mathcal{C})\\
&\overset{(b)}{=} \mathcal{T}((y_0,y_0+h)\times[y_0,H])\\
&= \mathcal{T}((y_0,y_0+h)\times[y_0,\hat{x}))+\mathcal{T}((y_0,y_0+h)\times[\hat{x},H])\\
&= \mathcal{T}((y_0,y_0+h)\times[y_0,\hat{x}))\\
&\le \mathcal{T}_2([y_0,\hat{x}])\\
&\overset{(c)}{=} \mathcal{T}_2((y_0,\hat{x}]),
\end{aligned}$$

where (a) comes from the fact that the measure Θ has full support in $\mathcal{C}$. The equality (b) holds because by construction no flow leaves $[y_0,H]$, and (c) is true because $\mathcal{T}_2\ll\Gamma$ and Γ does not have atoms in $[y_0,\hat{x}]$. This concludes the proof of Property 1.

Property 2. Both $\overline{y}_1$ and $\underline{y}_1$ are achieved in the set where they are defined. Furthermore, for any $z\in[\underline{y}_1,\overline{y}_1]$, $\overline{y}_1\in\mathcal{IR}(z)$.

Proof of Property 2. First we show both

$$\exists\, y_q\in[y_0,y_1] \text{ such that } \overline{y}_1\in\mathcal{IR}(y_q) \quad\text{and}\quad \exists\, x_q\in[y_1,H] \text{ such that } x_q\in\mathcal{IR}(\underline{y}_1). \tag{D-17}$$

Let us begin with the first statement. Let x^n be a sequence in A converging to $\overline{y}_1$, where

$$A=\{x\in[y_1,H]:\exists\, y\in[y_0,y_1] \text{ such that } x\in\mathcal{IR}(y)\}.$$

Then there exists a sequence $\{y^n\}\subset[y_0,y_1]$ such that $x^n\in\mathcal{IR}(y^n)$. Note that since $\{y^n\}\subset[y_0,y_1]$ and $x^n\in[y_1,H]$, Lemma B-2 implies that $x^n\in\mathcal{IR}(y_1)$. Hence, $V(x^n)-\|x^n-y_1\|=V(y_1)$, taking limit yields $V(\overline{y}_1)-\|\overline{y}_1-y_1\|=V(y_1)$, that is, $\overline{y}_1\in\mathcal{IR}(y_1)$.

Now we prove that $\underline{y}_1\in A$ where

$$A=\{y\in[y_0,y_1]:\exists\, x\in[y_1,H] \text{ such that } x\in\mathcal{IR}(y)\}.$$

By the definition of $\underline{y}_1$ we can always construct a sequence $\{y^n\}\subset A$ converging to $\underline{y}_1$. From the definition of A there exists another sequence $\{x^n\}\subset[y_1,H]$ such that $x^n\in\mathcal{IR}(y^n)$ for all n. We can extract a subsequence $\{x^{n_k}\}$ from $\{x^n\}$ that converges to some point $x_q\in[y_1,H]$. Then we have $V(x^{n_k})-\|x^{n_k}-y^{n_k}\|=V(y^{n_k})$. Taking limits and using the continuity of $V(\cdot)$ yields $V(x_q)-\|x_q-\underline{y}_1\|=V(\underline{y}_1)$, that is, $x_q\in\mathcal{IR}(\underline{y}_1)$. This concludes the proof for Eq. (D-17).

Next, we show that for all $z\in[\underline{y}_1,\overline{y}_1]$, $\overline{y}_1\in\mathcal{IR}(z)$. First, from our previos argument we know there exists y_q and x_q as in Eq. (D-17). Then Lemma B-2 implies $\overline{y}_1\in\mathcal{IR}(z)$ for all $z\in[y_q,\overline{y}_1]$. Observe that this yields $\overline{y}_1\in\mathcal{IR}(x_q)$ because $x_q\in[y_q,\overline{y}_1]$. Take $z\in[\underline{y}_1,y_q]$ then since $x_q\in\mathcal{IR}(\underline{y}_1)$ from Lemma B-2 we conclude that $x_q\in\mathcal{IR}(z)$. Hence, we have

$$V(z)=V(x_q)-(x_q-z)=V(\overline{y}_1)-(\overline{y}_1-x_q)-(x_q-z)=V(\overline{y}_1)-(\overline{y}_1-z).$$

which implies that $\overline{y}_1 \in \mathcal{IR}(z)$. This concludes the proof of Property 2.

Property 3. Both Eq. (D-13) and Eq. (D-16) hold.

Proof of Property 3. Let us star with Eq. (D-13). In order to prove the first part in Eq. (D-13) consider the following set

$$K = \Big\{x \in [y_0, y^c] : \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(x) \le \frac{d\mathcal{T}_2}{d\Gamma}(x+h)\Big\}.$$

We want to show that $\Gamma(K^c) = 0$ (the complement is taken with respect to $[y_0, y^c]$). If this is not true then $\Gamma(K^c) > 0$ and we have

$$\mathcal{T}_2^{\mathcal{B}^c}(K^c) = \int_{K^c} \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(x)\, d\Gamma(x) > \int_{K^c} \frac{d\mathcal{T}_2}{d\Gamma}(x+h)\, d\Gamma(x) = \mathcal{T}_2(K^c + h). \tag{D-18}$$

However,

$$\begin{aligned}
\mathcal{T}_2^{\mathcal{B}^c}(K^c) &= \mathcal{T}\Big(([y_0, y^c] \times K^c)_h\Big) + \Theta(\pi_1(([y^c, \overline{y}_1] \times K^c) \cap \mathcal{D})) \\
&= \mathcal{T}\Big(([y_0, y^c] \times K^c)_h\Big) \\
&= \mathcal{T}\Big([y_0 + h, y^c + h] \times (K^c + h)\Big) \\
&\le \mathcal{T}\Big(\mathcal{C} \times (K^c + h)\Big) \\
&= \mathcal{T}_2(K^c + h).
\end{aligned}$$

This together with Eq. D-18 yield a contradiction. To prove the second part of Eq. (D-13) consider any $\mathcal{R} \subset [y^c, \overline{y}_1]$, and observe that

$$\mathcal{T}_2^{\mathcal{B}^c}(\mathcal{R}) = \mathcal{T}\Big([y_1, \overline{y}_1] \times (\mathcal{R} + h)\Big) + \Theta(\mathcal{R}) = \Theta(\mathcal{R}) = \int_{\mathcal{R}} \theta_1 d\Gamma(x),$$

where the second equality comes from $\mathcal{R} + h \subset [\overline{y}_1, \overline{y}_1 + h]$ and $\mathcal{T}([y_1, \overline{y}_1] \times [\overline{y}_1, \overline{y}_1 + h]) = 0$.

Finally, we provide a proof for Eq. (D-16). Let $Z(x) \triangleq \min\{V(y_0) + (x - y_0), V(\overline{y}_1)\}$. We verify that for all $x \in \mathcal{B}^c$

$$Z(x) \ge U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w - x|, \quad \Gamma - a.e.\ w \ \text{ in } \mathcal{B}^c, \tag{D-19}$$

and that $Z(x)$ is the smallest with such property. First, fix $x \in [y^c, \overline{y}_1]$ so $Z(x) = V(\overline{y}_1)$. Note that from our choice of prices in $[y^c, \overline{y}_1]$ we have

$$Z(x) = V(\overline{y}_1) \ge V(\overline{y}_1) - |w - x| = U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w - x|, \quad \Gamma - a.e.\ w \ \text{ in } [y^c, \overline{y}_1].$$

So we only need to show the same inequality but this time for $[y_0, y^c]$. From the definition of $\underline{y}_1$ and $\overline{y}_1$ and Lemma B-2 we have that $V(\overline{y}_1) - |\overline{y}_1 - y_1|$ equals $V(y_1 | \, p, \mathcal{T})$ and, therefore,

$$\begin{aligned}
V(\overline{y}_1) &\ge U(w, p(w), s^{\mathcal{T}}(w)) - |w - y_1| + |\overline{y}_1 - y_1|, \quad \Gamma - a.e.\ w \ \text{ in } [y_1, \overline{y}_1] \\
&\ge U(w, p(w), s^{\mathcal{T}}(w)), \quad \Gamma - a.e.\ w \ \text{ in } [y_1, \overline{y}_1].
\end{aligned}$$

We can use this together with the fact that $[y_0, y^c] + h = [y_1, \overline{y}_1]$ to obtain

$$\begin{aligned}
Z(x) = V(\overline{y}_1) &\overset{(a)}{\ge} U\Big(w + h, p(w + h), s^{\mathcal{T}}(w + h)\Big), \quad \Gamma - a.e.\ w \ \text{ in } [y_0, y^c] \\
&\ge U\Big(w + h, p(w + h), s^{\mathcal{T}}(w + h)\Big) - |w - x|, \quad \Gamma - a.e.\ w \ \text{ in } [y_0, y^c] \\
&\overset{(b)}{=} U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w - x|, \quad \Gamma - a.e.\ w \ \text{ in } [y_0, y^c],
\end{aligned}$$

Inequality (a) comes from the fact that Γ in the interval under consideration is invariant under a shift; (b) comes from Eq. (D-24). That is, for $x \in [y^c, \overline{y}_1]$ Eq. (D-19) is satisfied. It is left to verify that $Z(x)$ is the smallest value satisfying Eq. (D-19). For any $\epsilon > 0$, since $x \in [y^c, \overline{y}_1]$ we have

$$\begin{aligned}
0 &< \Gamma(B(x,\epsilon) \cap [y^c, \overline{y}_1]) \\
&= \Gamma\Big(w \in [y^c, \overline{y}_1] : V(\overline{y}_1) - |w - x| > V(\overline{y}_1) - \epsilon\Big) \\
&= \Gamma\Big(w \in [y^c, \overline{y}_1] : U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w - x| > V(\overline{y}_1) - \epsilon\Big),
\end{aligned}$$

hence $V(\overline{y}_1)$ is the smallest value satisfying Eq. (D-19).

Now we show Eq. (D-19) for $x \in [y_0, y^c]$. Fix $x \in [y_0, y^c]$ so $Z(x) = V(y_0) + (x - y_0)$. Note that $V(y_0)$ equals $V(y_1)$, and from the definition of $\overline{y}_1$ and the envelope result we have that $V(y_1)$ equals $V(\overline{y}_1) - (\overline{y}_1 - y_1)$. Therefore,

$$\begin{aligned}
Z(x) &= V(\overline{y}_1) - (\overline{y}_1 - y_1) + (x - y_0) \\
&\overset{(a)}{\geq} V(\overline{y}_1) - (w - x), \quad \Gamma - a.e.\ w \ \text{in}\ [y^c, \overline{y}_1] \\
&\overset{(b)}{=} U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w - x|, \quad \Gamma - a.e.\ w \ \text{in}\ [y^c, \overline{y}_1],
\end{aligned}$$

where (a) follows from $w \geq y_c$ and $y^c - y_0 = \overline{y}_1 - y_1$. Line (b) holds from our choice of prices in $[y^c, \overline{y}_1]$. Hence, $Z(x)$ upper bounds (almost surely) the desire quantity in $[y^c, \overline{y}_1]$, so we just need to prove the same bound for $[y_0, y^c]$. Note that from the definition of $\underline{y}_1$ and $\overline{y}_1$ we have that

$$V(x + h) = V(y_1) + (x + h - y_1) = V(y_1) + (x - y_0) = Z(x),$$

and thus

$$\begin{aligned}
Z(x) &= V(x + h|\, p, \mathcal{T}) \\
&\overset{(a)}{\geq} U(w, p(w), s^{\mathcal{T}}(w)) - |w - (x + h)|, \quad \Gamma - a.e.\ w \ \text{in}\ [y_1, \overline{y}_1] \\
&\overset{(b)}{=} U(w + h, p(w + h), s^{\mathcal{T}}(w + h)) - |w + h - (x + h)|, \quad \Gamma - a.e.\ w \ \text{in}\ [y_0, y^c] \\
&\overset{(c)}{=} U(w, p^{\mathcal{B}^c}(w), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(w)) - |w - x)|, \quad \Gamma - a.e.\ w \ \text{in}\ [y_0, y^c],
\end{aligned}$$

where (a) comes from the definition of $V(x + h|\, p, \mathcal{T})$, (b) from the invariance under translation of Γ. Line (c) follows from Eq. (D-24). Therefore, $Z(x)$ satisfies Eq. (D-19). To see why $Z(x)$ is the smallest value satisfying this equation observe that

$$\begin{aligned}
0 &< \Gamma(B(y^c, \epsilon) \cap [y^c, \overline{y}_1]) \\
&\overset{(a)}{=} \Gamma\Big(w \in [y^c, \overline{y}_1] : V(\overline{y}_1) - (w - x) > V(\overline{y}_1) - (\overline{y}_1 - y_1) + (x - y_0) - \epsilon\Big) \\
&= \Gamma\Big(w \in [y^c, \overline{y}_1] : U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\mathcal{T}_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w - x| > Z(x) - \epsilon\Big),
\end{aligned}$$

where in (a) we use that $y^c - y_0 = \overline{y}_1 - y_1$. This implies that $Z(x)$ is the smallest value satisfying Eq. (D-19), completing the proof. □

Proposition D-3. *(Tight upper bound) Without loss of optimality, we can focus on price-equilibrium pairs $(p, \mathcal{T})$ such that the upper bound in Eq. (7) is tight.*

Proof of Proposition D-3. If $X_r = H$ there is nothing to prove, so assume $X_r < H$. Let $(p, \mathcal{T})$ be a feasible solution such that $V(\cdot|p, \mathcal{T})$ is non-decreasing. Due to Proposition D-2 we can always restrict attention to this type of solution. We proceed by contradiction. Assume that there exists $\tilde{x} \in (X_r, H]$ such that

$$V(\tilde{x}) < \min\{V(X_r) + (\tilde{x} - X_r), \psi_1\} \triangleq Z(\tilde{x}). \tag{D-20}$$

First, we construct an interval $\tilde{I}$ such that $\mathcal{T}_2(\tilde{I}) > 0$ and $V(x) < Z(x)$ for all $x \in \tilde{I}$. Then, we show that $Z(x)$ can be achieved in a feasible manner by appropriately creating a price-equilibrium pair $(\tilde{p}, \tilde{\mathcal{T}})$ that mimics the flow generated by $\mathcal{T}$ in $(X_r, H]$. The final step of the proof is to use the interval $\tilde{I}$ and the flow $\tilde{\mathcal{T}}$ to show an strict objective improvement.

Interval construction. From Eq. (D-20) and the continuity of $V(\cdot)$ we can deduce the existence of an interval $[\tilde{a}, \tilde{b}] \subset (X_r, H]$ such $V(x) < Z(x)$ for all $x \in [\tilde{a}, \tilde{b}]$. Furthermore, the Lipchitz property (see Lemma 1) and Lemma D-4 imply that $V(x) < Z(x)$ for all $x \in [\tilde{a}, \tilde{c}]$ where $\tilde{c}$ is the minimum between H and the value c such that $V(\tilde{a}) + (c - \tilde{a}) = \psi_1$. Also, Proposition D-2 and Lemma B-3 imply that $\mathcal{T}([\tilde{a}, \tilde{c}] \times \mathcal{C}) = \mathcal{T}([\tilde{a}, \tilde{c}] \times [\tilde{a}, \tilde{c}])$. Putting all of this together we conclude that there exists an interval $\tilde{I} = (\tilde{a}, \tilde{c})$ such that $\mathcal{T}_2(\tilde{I}) > 0$ and $V(x) < Z(x)$ for all $x \in \tilde{I}$.

Flow mimicking. Define the collection of intervals

$$\mathcal{I} \triangleq \{I \subset (X_r, H] : I = [a, b], a < b, b \in \mathcal{IR}(a), a \text{ is minimal and } b \text{ is maximal}\}.$$

There are two cases: $\mathcal{I} = \emptyset$ and $\mathcal{I} \neq \emptyset$. We only do the latter because its treatment contains the former.

Suppose $\mathcal{I} \neq \emptyset$, then there exists $X_r < a < b$ such that $b \in \mathcal{IR}(a)$, where a and b are minimal and maximal with this property, respectively. We first look at some properties of the equilibrium in each element of $\mathcal{I}$ and then we look at its complement.

Note that from the minimality of a we have that for any $x < a$, $a \notin \mathcal{IR}(x)$. Similarly, for any $x > b$ we have $x \notin \mathcal{IR}(b)$. This, together with Proposition D-2 and Lemma B-3 imply that $[a, b]$ is a flow-separated region, that is, there is no flow coming in nor flow going out of $[a, b]$, $\mathcal{T}([a, b] \times [a, b]^c) = 0$ and $\mathcal{T}([a, b]^c \times [a, b]) = 0$. Observe that our flow separation result in Lemma D-3 implies that in each interval $I \in \mathcal{I}$ we have an equilibrium. Furthermore, from the definition of $\mathcal{IR}$ we must have

$$V(x) = V(a) + (x - a), \quad \forall x \in [a, b].$$

From the previous discussion we infer that the elements in the collection $\mathcal{I}$ are disjoint intervals and, since V is non-decreasing, the collection is at most countable.

For any a, b such that $[a, b] \in \mathcal{I}$ we define

$$t(a) \triangleq V(a) - V(X_r) + X_r, \quad \text{and} \quad t(b) \triangleq V(b) - V(X_r) + X_r.$$

Note that since V is non-decreasing we have $V(a) \geq V(X_r)$ and, therefore, $t(b) > t(a) \geq X_r$. Also, for any

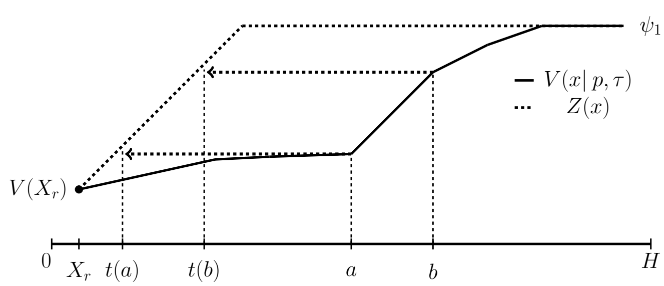

Figure 12: Graphical representation of $t(a)$ and $t(b)$.

such b we have $t(b) < Y_r$. The points $t(a), t(b)$ are the corresponding points to a, b in the interval $[X_r, Y_r]$ (see Figure 12). Furthermore, $t(\cdot)$ is a non-decreasing mapping.

We denote by $\mathcal{I}^c$ the collection of intervals whose elements are the intervals that do not belong to $\mathcal{I}$. Observe that the elements in $\mathcal{I}$ and $\mathcal{I}^c$ alternate in a consecutive manner. That is, if we have an interval

$(c, d) \in \mathcal{I}^c$ then it can only be followed by and interval $[a, b] \in \mathcal{I}$ with $a = d$. In the case that $I = (c, d) \in \mathcal{I}^c$ is not followed by an interval in $\mathcal{I}$ then I equals $(c, H]$. Define the sets

$$\mathcal{K} \triangleq \bigcup_{I \in \mathcal{I}} I \text{ and } \mathcal{K}^c \triangleq \bigcup_{I \in \mathcal{I}^c} I.$$

Note that $(X_r, H] = \mathcal{K} \cup \mathcal{K}^c$ up to a set of Γ measure zero. Also, for each interval $I \in \mathcal{I}^c$ we must have that for all measurable sets $A \subset I$, $\mathcal{T}(A \times A) = \Theta(A) = \mathcal{T}_2(A)$; otherwise, by Lemma B-3 we would get a contradiction with the definition of $\mathcal{I}$. In turn, this implies that $\frac{d\mathcal{T}_2}{d\Gamma}(x) = \theta_1$, $\Gamma - a.e.\ x$ in $\mathcal{K}^c$.

We denote by $\mathcal{I}_t$ the collection of intervals $\{[t(a), t(b)]\}_{[a,b] \in \mathcal{I}}$, and $\mathcal{I}_t^c$ is defined in analogous manner. Also, $\mathcal{K}_t$ and $\mathcal{K}_t^c$ are defined similarly to $\mathcal{K}$ and $\mathcal{K}^c$ replacing $\mathcal{I}$ with $\mathcal{I}_t$ and $\mathcal{I}^c$ with $\mathcal{I}_t^c$, respectively.

The idea now is to construct a solution $(\tilde{p}, \tilde{\mathcal{T}})$ in $(X_r, H]$ and then paste it with the old solution $(p, \mathcal{T})$ restricted to $[-H, X_r)$. To construct $(\tilde{p}, \tilde{\mathcal{T}})$ we will make use of the collections $\mathcal{I}_t$ and $\mathcal{I}_t^c$. For each element in these collections we will create a price-equilibrium. For intervals $[t(a), t(b)] \in \mathcal{I}_t$ the idea is that the solution $(\tilde{p}, \tilde{\mathcal{T}})$ has the same equilibrium than $(p, \mathcal{T})$ in $[a, b]$. For the interval in $\mathcal{I}_t^c$ we choose prices such that no drivers will have an incentive to move. Finally, using Lemma D-3 we will paste the equilibria generated in all the intervals.

First, we show how to construct prices and an equilibrium in some $[t(a), t(b)]$. Fix $[a, b] = I \in \mathcal{I}$ and denote the mimicking set $[t(a), t(b)]$ by I_t. Choose prices $p^{I_t}(x)$ equal to $p(x + (a - t(a)))$ for all $x \in I_t$. For the flows, we define $\mathcal{T}^{I_t}$ for any $\mathcal{L} \subseteq I_t \times I_t$ by

$$\mathcal{T}^{I_t}(\mathcal{L}) = \mathcal{T}\Big(\mathcal{L} + (a - t(a), a - t(a))\Big),$$

that is, $\mathcal{T}^{I_t}$ mimics $\mathcal{T}$ in $I \times I$. It can be shown that (see Property 1 at the end of this proof) $(p^{I_t}, \mathcal{T}^{I_t})$ forms a price-equilibrium pair in I_t such that $\mathcal{T}^{I_t} \in \mathcal{F}_{I_t}(\Theta|_{I_t})$. Also, $V(x|\, p^{I_t}, \mathcal{T}^{I_t})$ equals $V(x + a - t(a)|\, p, \mathcal{T})$ for all $x \in I_t$, and

$$\frac{d\mathcal{T}_2^{I_t}}{d\Gamma}(x) = \frac{d\mathcal{T}_2}{d\Gamma}(x + a - t(a)), \quad \Gamma - a.e.\ x \text{ in } I_t. \tag{D-21}$$

Furthermore, because $I \in \mathcal{I}$ we have

$$V(x|\, p^{I_t}, \mathcal{T}^{I_t}) = V(x + a - t(a)|\, p, \mathcal{T}) = V(a) + (x - t(a)) = V(X_r) + (x - X_r) = Z(x), \quad \forall x \in I_t,$$

that is, for all intervals I_t the associated solution $(p^{I_t}, \mathcal{T}^{I_t})$ achieves the upper bound $Z(x)$.

Second, we show how to set the prices and construct an equilibrium everywhere else. Consider any two consecutive sets in $\mathcal{I}$, $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$. The corresponding mimicking sets are $[t(a_1), t(b_1)]$ and $[t(a_2), t(b_2)]$. We need to set prices and define the flow in the interval $J_t = (t(b_1), t(a_2))$. We choose the prices p^{J_t} to be such that

$$U\Big(x, p^{J_t}(x), \theta_1\Big) = Z(x), \quad \forall x \in J_t.$$

Since $Z(x) \leq \psi_1$ these prices are guaranteed to exist. We define the measure $\mathcal{T}^{J_t}$ for any measurable set $\mathcal{L} \subseteq J_t \times J_t$ by

$$\mathcal{T}^{J_t}(\mathcal{L}) = \Theta(\pi_1(\mathcal{L} \cap \mathcal{D})).$$

This measure has $d\mathcal{T}_2^{J_t}/d\Gamma = \theta_1$, $\Gamma - a.e$ in J_t. It can be shown that (see Property 2 at the end of this proof) $(p^{J_t}, \mathcal{T}^{J_t})$ forms a price-equilibrium pair in J_t such that $\mathcal{T}^{J_t} \in \mathcal{F}_{J_t}(\Theta|_{J_t})$ and $V(x|\, p^{J_t}, \mathcal{T}^{J_t})$ equals $Z(x)$ for all $x \in J_t$.

Third, the solutions $\{(p^{I_t}, \mathcal{T}^{I_t})\}_{I_t \in \mathcal{I}_t}$ and $\{(p^{J_t}, \mathcal{T}^{J_t})\}_{J_t \in \mathcal{I}_t^c}$ cover the whole interval $(X_r, H]$. Moreover they are defined in disjoint interval, and are such that the respective $V(\cdot)$ functions coincide at the boundaries of the interval (these functions coincide with $Z(\cdot)$). Thus, we can apply Lemma D-3 to paste all these solutions and obtain a new solution $(\tilde{p}, \tilde{\mathcal{T}})$ in $(X_r, H]$. As mentioned before we can use the same lemma to paste this solution with the old solution restricted to $[-H, X_r]$. This would yield a solution in the entire city.

Objective improvement. Consider the revenue under $(p, \mathcal{T})$ in $(X_r, H]$, it easy to observe that

$$\begin{aligned}
\mathbf{Rev}_{(X_r,H]}(p,\mathcal{T}) &= \int_{(X_r,H]} V(x|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x) \\
&= \int_{\mathcal{K}} V(x|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x) + \int_{\mathcal{K}^c} V(x|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x) \\
&= \underbrace{\sum_{I\in\mathcal{I}} \int_I V(x|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x)}_{=(a)} + \underbrace{\int_{\mathcal{K}^c} V(x|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x)}_{=(b)}.
\end{aligned}$$

Let us develop the integral of the term (a). Let I be equal to $[a, b]$ and I_t equal to $[t(a), t(b)]$ then

$$\begin{aligned}
\int_{[a,b]} V(x|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x) &= \int_{[t(a),t(b)]} V(x+a-t(a)|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x+a-t(a))\, d\Gamma(x) \\
&= \int_{[t(a),t(b)]} V(x|p^{I_t},\mathcal{T}^{I_t}) \cdot s^{\mathcal{T}^{I_t}}(x)\, d\Gamma(x),
\end{aligned}$$

where in the first line we use the invariance under translation of Γ, and in the second line we use that $V(x|\, p^{I_t}, \mathcal{T}^{I_t})$ equals $V(x + a - t(a)|\, p, \mathcal{T})$ for all $x \in I_t$ and Eq. (D-21). Thus,

$$\begin{aligned}
\mathbf{Rev}_{(X_r,H]}(p,\mathcal{T}) &= \sum_{I_t\in\mathcal{I}_t} \int_{I_t} V(x|p^{I_t},\mathcal{T}^{I_t}) \cdot s^{\mathcal{T}^{I_t}}(x)\, d\Gamma(x) + (b) \\
&= \int_{\mathcal{K}_t} Z(x) \cdot s^{\tilde{\mathcal{T}}}(x)\, d\Gamma(x) + (b).
\end{aligned}$$

Thus, to conclude the proof we only need to show that

$$(b) = \int_{\mathcal{K}^c} V(x|p,\mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x) < \int_{\mathcal{K}_t^c} Z(x) \cdot s^{\tilde{\mathcal{T}}}(x)\, d\Gamma(x). \tag{D-22}$$

Define the following functions

$$V_e(x) = \begin{cases} V(x|p,\mathcal{T}) & \text{if } x \in \mathcal{K}^c, \\ V(a|p,\mathcal{T}) & \text{if } x \in [a,b], \text{ some } [a,b] \in \mathcal{I}, \end{cases}$$

$$Z_e(x) = \begin{cases} Z(x) & \text{if } x \in \mathcal{K}_t^c, \\ Z(t(a)) & \text{if } x \in [t(a),t(b)], \text{ some } [t(a),t(b)] \in \mathcal{I}_t. \end{cases}$$

We verify that $V_e(x) \le Z_e(x)$ for all $x \in (X_r, H]$, and the we use this inequality to prove the objective improvement. Let $x \in \mathcal{K}^c$ then there exists an interval $(c, d) \in \mathcal{I}^c$ with $x \in (c, d)$. If $x \in \mathcal{K}_t^c$ then the upper bound is trivial. If $x \notin \mathcal{K}_t^c$ then $x \in [t(a), t(b)]$ for some $[t(a), t(b)] \in \mathcal{I}_t$. We must have that $a \ge d$; otherwise, since $(c, d) \in \mathcal{I}$, it must be the case that $b \le c$. In turn, this implies that $[t(a), t(b)] \cap (c, d) = \emptyset$ which contradiction our current assumption. Therefore,

$$V_e(x) = V(x|p,\mathcal{T}) \le V(d|p,\mathcal{T}) \le V(a|p,\mathcal{T}) = Z(t(a)) = Z_e(x).$$

Let $x \in [a, b]$ for some $[a, b] \in \mathcal{I}$. If $x \in \mathcal{K}_t^c$, $t(b) < x$ otherwise we would have that $t(a) \le a \le a \le t(b)$, that is, $x \in [t(a), t(b)] \in \mathcal{I}_t$. Under our current assumption this is not possible. Then,

$$V_e(x) = V(a|p,\mathcal{T}) < V(b|p,\mathcal{T}) = Z(t(b)) \le Z(x) = Z_e(x), \tag{D-23}$$

that is, when $x \in \mathcal{K} \cap \mathcal{K}_t^c$ we have $V_e(x) < Z_e(x)$. If $x \in [t(\hat{a}), t(\hat{b})]$ for some $[t(\hat{a}), t(\hat{b})] \in \mathcal{I}_t$. Using similar arguments as before we can show that $\hat{a} \ge a$ and, therefore,

$$V_e(x) = V(a|p,\mathcal{T}) = Z(t(a)) \le Z(t(\hat{a})) = Z_e(x).$$

Now, recall that in the **Interval construction** part of the proof we defined an interval $\tilde{I} = [\tilde{a}, \tilde{c}]$ in which the function $V(\cdot|p, \mathcal{T})$ is uniformly strictly bounded by $Z(\cdot)$. Now we relate this interval to $\mathcal{K}_t^c$ by showing that there exists $\epsilon > 0$ such that $(\tilde{c} - \epsilon, \tilde{c}) \subseteq I_t^c$ with $I_t^c \in \mathcal{I}_t^c$. The idea is to use that $(\tilde{c} - \epsilon, \tilde{c}) \subset \tilde{I}$ and $(\tilde{c} - \epsilon, \tilde{c}) \subset \mathcal{K}_t^c$ together with Eq. (D-23) to show an strict objective improvement.

Note that if $\tilde{c} = H$ then

$$\begin{aligned}
\sup_{[t(a),t(b)]\in\mathcal{I}_t} t(b) &\overset{(1)}{\leq} t(\tilde{c}) \\
&= V(\tilde{c}) - V(X_r) + X_r \\
&= (V(\tilde{c}) - V(\tilde{a})) + (V(\tilde{a}) - V(X_r)) + X_r \\
&\overset{(2)}{<} (V(\tilde{c}) - V(\tilde{a})) + (Z(\tilde{a}) - Z(X_r)) + X_r \\
&\overset{(3)}{\leq} (\tilde{c} - \tilde{a}) + (\tilde{a} - X_r) + X_r \\
&= \tilde{c},
\end{aligned}$$

where (1) comes from the fact that $t(\cdot)$ is non-decreasing and $\tilde{c} = H$, line (2) follows from the $V(\tilde{a}) < Z(\tilde{a})$ and $V(X_r) = Z(X_r)$. Inequality, (3) holds because both V and Z are $1-$Lipschitz functions. In the case that $\tilde{c} < H$ we have $V(\tilde{a}) + (\tilde{c} - \tilde{a}) = \psi_1$. Also, we always have that $t(b) \leq Y_r$ where Y_r is such that $V(X_r) + (Y_r - X_r) = \psi_1$. From this we deduce that $Y_r < \tilde{c}$ and, therefore, we have that $\sup_{[t(a),t(b)]\in\mathcal{I}_t} t(b) < \tilde{c}$. Either way we can always find $\epsilon \in (0, \tilde{c} - \tilde{a})$ such that the interval $(\tilde{c} - \epsilon, \tilde{c})$ does not intersect with any interval in $\mathcal{I}_t$. Hence, since $\mathcal{I}_t^c$ are all the intervals that do not belong to $\mathcal{I}_t$ we must have that $(\tilde{c} - \epsilon, \tilde{c}) \subseteq I_t^c$ for some $I_t^c \in \mathcal{I}_t^c$.

Because $(\tilde{c} - \epsilon, \tilde{c})$ is a subset of both $\mathcal{K}_t^c$ and $(\tilde{a}, \tilde{c})$, for $x \in (\tilde{c} - \epsilon, \tilde{c}) \cap \mathcal{K}^c$ we have $V_e(x) < Z_e(x)$. Also, for $x \in (\tilde{c} - \epsilon, \tilde{c}) \cap \mathcal{K}$ from equation Eq. (D-23) we have $V_e(x) < Z_e(x)$. That is, $V_e(x) < Z_e(x)$ for all $x \in (\tilde{c} - \epsilon, \tilde{c})$ and, therefore,

$$\begin{aligned}
\int_{\mathcal{K}^c} V(x|p, \mathcal{T}) \cdot s^{\mathcal{T}}(x)\, d\Gamma(x) &= \int_{(X_r,H]} V_e(x|p, \mathcal{T}) \cdot \theta_1 d\Gamma(x) - \sum_{[a,b]\in\mathcal{I}} \int_{[a,b]} V(a|p, \mathcal{T}) \cdot \theta_1 d\Gamma(x) \\
&< \int_{(X_r,H]} Z_e(x) \cdot \theta_1 d\Gamma(x) - \sum_{[a,b]\in\mathcal{I}} \int_{[a,b]} V(a|p, \mathcal{T}) \cdot \theta_1 d\Gamma(x) \\
&= \int_{(X_r,H]} Z_e(x) \cdot \theta_1 d\Gamma(x) - \sum_{[a,b]\in\mathcal{I}} V(a|p, \mathcal{T}) \Theta([a, b]) \\
&= \int_{(X_r,H]} Z_e(x) \cdot \theta_1 d\Gamma(x) - \sum_{[t(a),t(b)]\in\mathcal{I}_t} Z(t(a)) \Theta([t(a), t(b)]) \\
&= \int_{\mathcal{K}_t^c} Z(x) \cdot \theta_1 d\Gamma(x),
\end{aligned}$$

which proves Eq. (D-22). To conclude, we provide a proof for both Property 1 and Property 2.

Property 1. $(p^{I_t}, \mathcal{T}^{I_t})$ forms a price-equilibrium pair in I_t such that $\mathcal{T}^{I_t} \in \mathcal{F}_{I_t}(\Theta|_{I_t})$. Also, $V(x|p^{I_t}, \mathcal{T}^{I_t})$ equals $V(x + a - t(a)|\, p, \mathcal{T})$ for all $x \in I_t$, and

$$\frac{d\mathcal{T}_2^{I_t}}{d\Gamma}(x) = \frac{d\mathcal{T}_2}{d\Gamma}(x + a - t(a)), \quad \Gamma - a.e.\ x \text{ in } I_t.$$

Proof of Property 1. We first show that $\mathcal{T}^{I_t} \in \mathcal{F}_{I_t}(\Theta|_{I_t})$. It is clear that $\mathcal{T}^{I_t} \in \mathcal{M}(I_t \times I_t)$, and that

$\mathcal{T}_2^{I_t} \ll \Gamma$. To see why $\mathcal{T}_1^{I_t}$ coincides with θ_{I_t} consider a set $K \subset I_t$ then $\mathcal{T}_1^{I_t}(K)$ equals

$$\begin{aligned}\mathcal{T}_1^{I_t}(K \times I_t) = \mathcal{T}((K + a - t(a)) \times (I_t + a - t(a))) &= \mathcal{T}((K + a - t(a)) \times [a, b]) \\ &= \mathcal{T}((K + a - t(a)) \times \mathcal{C}) \\ &= \Theta(K + a - t(a)) \\ &= \Theta(K),\end{aligned}$$

where the fourth line holds because the set $K + a - t(a)$ is contained in $[a, b]$, and we know there is no flow leaving this interval. Next, using a similar argument we show the property for $d\mathcal{T}_2^{I_t}/d\Gamma$, let K be a measurable subset of I_t then

$$\begin{aligned}\int_K \frac{d\mathcal{T}_2^{I_t}}{d\Gamma}(x)\, d\Gamma(x) &= \mathcal{T}^{I_t}(I_t \times K) \\ &= \mathcal{T}([a, b] \times (K + a - t(a))) \\ &= \int_{(K+a-t(a))} \frac{d\mathcal{T}_2}{d\Gamma}(x)\, d\Gamma(x) \\ &= \int_K \frac{d\mathcal{T}_2}{d\Gamma}(x + a - t(a))\, d\Gamma(x).\end{aligned}$$

Using this last property and the prices definition is easy to see that

$$\begin{aligned}V(x|\, p^{I_t}, \mathcal{T}^{I_t}) &= \inf\{u \in \mathbb{R} : \Gamma(y \in I_t : U(y, p^{I_t}(y), \frac{d\mathcal{T}_2^{I_t}}{d\Gamma}(y)) - |y - x| > u) = 0\} \\ &= \inf\{u \in \mathbb{R} : \Gamma(y \in I_t : U(y, p(y + a - t(a)), \frac{d\mathcal{T}_2}{d\Gamma}(y + a - t(a))) - |y - x| > u) = 0\} \\ &= \inf\{u \in \mathbb{R} : \Gamma(y \in I : U(y, p(y), \frac{d\mathcal{T}_2}{d\Gamma}(y)) - |y - (x + a - t(a))| > u) = 0\} \\ &= V_I(x + a - t(a)|\, p, \mathcal{T}),\end{aligned}$$

but from out flow separation result (see Lemma D-3) we have that $V_I(x+a-t(a)|p, \mathcal{T}) = V(x+a-t(a)|p, \mathcal{T})$. Using this same approach, the definition of $\mathcal{T}^{I_t}$ and the fact that $\mathcal{T}$ is an equilibrium in $[a, b]$ it is easy to verify the equilibrium condition.

Property 2. The pair $(p^{J_t}, \mathcal{T}^{J_t})$ forms a price-equilibrium pair in J_t such that $\mathcal{T}^{J_t} \in \mathcal{F}_{J_t}(\Theta|_{J_t})$ and $V(x|\, p^{J_t}, \mathcal{T}^{J_t})$ equals $Z(x)$ for all $x \in J_t$.

Proof of Property 2. From the definition of $\mathcal{T}^{J_t}$ it is clear that $\mathcal{T}^{J_t} \in \mathcal{F}_{J_t}(\Theta|_{J_t})$. Also, $d\mathcal{T}_2^{J_t}/d\Gamma = \theta_1$, $\Gamma - a.e$ in J_t. To see why $V(x|\, p^{J_t}, \mathcal{T}^{J_t})$ equals $Z(x)$ for all $x \in J_t$, note that for fixed $x \in J_t$

$$\Gamma(y \in J_t : U(y, p^{J_t}(y), \frac{d\mathcal{T}_2^{J_t}}{d\Gamma}(y)) - |y - x| > Z(x)) = \Gamma(y \in J_t : Z(y) - |x - y| > Z(x)) = 0,$$

where in the first equality we use the definition of $p^{J(t)}$ together with $d\mathcal{T}_2^{J_t}/d\Gamma = \theta_1$, $\Gamma - a.e$ in J_t. In the second equality we use the Lipschitz property of the function $Z(\cdot)$. That is, $Z(x) \geq V(x|\, p^{J_t}, \mathcal{T}^{J_t})$. This upper bound ($\Gamma-$a.e) is tight. Let $\epsilon > 0$ then

$$\begin{aligned}0 &< \Gamma(B(x, \epsilon/2) \cap J_t) \\ &\leq \Gamma(y \in B(x, \epsilon/2) \cap J_t) : \epsilon > |x - y| + (Z(x) - Z(y))) \\ &= \Gamma(y \in B(x, \epsilon/2) \cap J_t) : Z(y) - |y - x| > Z(x) - \epsilon) \\ &= \Gamma(y \in B(x, \epsilon/2) \cap J_t) : U(y, p^{J_t}(y), \frac{d\mathcal{T}_2^{J_t}}{d\Gamma}(y)) - |y - x| > Z(x) - \epsilon),\end{aligned}$$

thus $Z(x)$ is the smallest upper bound ($\Gamma-$a.e) and we have $Z(x) = V(x|\, p^{J_t}, \mathcal{T}^{J_t})$. It is not hard to verify that the equilibrium condition reduces to

$$\mathcal{T}^{J_t}((x, y) \in J_t \times J_t : Z(y) - |y - x| = Z(x)) = \Theta(J_t),$$

and by the definition of $\mathcal{T}^{J_t}$ this is immediately satisfied. $\square$

Proof of Theorem 2. The result follows directly from Proposition D-3, and the fact that $[X_l, X_r]$ is an attraction region where $V(\cdot)$ is pinned down. $\square$

Proof of Theorem 3. We separate the proof in several steps. First, we argue that there are at most three attraction regions in the any optimal solution. Then we show that any optimal solution does not have drivers moving to the interval $[W_r, X_r]$ and $[X_l, W_l]$; otherwise, the platform can incentivize the movement of a positive fraction of drivers outside of the center and make strictly larger revenue. After this we put into practice Theorem 1 which prescribes what are the optimal prices and post-relocation supply in each attraction region. In the final main step of the proof we argue that the optimal solution has to be symmetric. We present the proof of two properties that we will use during the main arguments, Property 1 and Property 2, after the main proof.

Attraction regions identification: Lemma 2 establishes that at an optimal solution the attraction region of the origin is well defined with $X_l < 0 < X_r$. So Our first attraction region is the interval $[X_l, X_r]$.

The second and third attraction regions correspond to the intervals $[Y_l, X_l]$ and $[X_r, Y_r]$ with Y_l and Y_r being sinks. WLOG consider only the right interval, if $Y_r = X_r$ we do not identify any attraction region to the right of X_r. Assume that $X_r < Y_r$, we will show that $A(Y_r) = [X_r, Y_r]$ and $Y_r \notin A(z)$ for any $z \neq Y_r$. In order to show this we first show that $Y_r \in \mathcal{IR}(X_r |\, p, \mathcal{T})$. From Theorem 2 we know that $V(x)$ equals $V(X_r) + (x - X_r)$ for all $x \in [X_r, Y_r]$. In particular, $V(X_r) + (Y_r - X_r)$. In other words, $Y_r \in \mathcal{IR}(X_r |\, p, \mathcal{T})$. Now, Y_r cannot belong to any other attraction region; otherwise, the value function would not be as in Theorem 2. Therefore, Y_r is a sink and $[X_r, Y_r] \subseteq A(Y_r)$. If there existed $x \in A(Y_r)$ but $x \notin [X_r, Y_r]$, the value function would not be as in Theorem 2. In conclusion, $A(Y_r) = [X_r, Y_r]$ and $Y_r \notin A(z)$ for any $z \neq Y_r$.

No supply in $[W_r, X_r]$**:** Next we argue that at an optimal solution $(p, \mathcal{T})$ we must have that $\mathcal{T}_2([W_r, X_r]) = 0$, the same is true for the left side. Suppose by contradiction that $\mathcal{T}_2([W_r, X_r]) > 0$ and denote this amount of supply by q_r, we construct a new solution $(\tilde{p}, \tilde{\mathcal{T}})$ that yields an strict objective improvement. Observe that,

$$0 < q_r = \mathcal{T}(\mathcal{C} \times [W_r, X_r]) = \mathcal{T}([W_r, X_r] \times [W_r, X_r]) \leq \Theta([W_r, X_r]) = \theta_1 \cdot (X_r - W_r).$$

That is, from the total amount of initial supply in $[W_r, X_r]$ we have that q_r units stay within $[W_r, X_r]$ and a total of $\theta_1 \cdot (X_r - W_r) - q_r$ units travel to $[0, W_r]$. Note that for this q_r units of mass their V is bounded by ψ_1 and, therefore, what the platform can make from them is strictly bounded by $\psi_1 \cdot q_r$ (times a scaling factor). Let $\tilde{X}_r \in [W_r, X_r)$ be such that $q_r = \theta_1 \cdot (X_r - \tilde{X}_r)$. In the new solution, we will modify the attraction region $[X_l, X_r]$ to be $[X_l, \tilde{X}_r]$. We will maintain the same prices and post-relocation supply in the origin's attraction region. However, to the right side of $\tilde{X}_r$ we will set new prices that will be consistent with a new value function and flows that upper bound those of the old solution, see Figure 13.

We begin our construction of $(\tilde{p}, \tilde{\mathcal{T}})$ with the interval $I_r^1 = [\tilde{X}_r, \tilde{Y}_r]$, where $\tilde{Y}_r$ is such that $\psi_1 = V(\tilde{X}_r) + (\tilde{Y}_r - \tilde{X}_r)$. Let $h \triangleq 2 \cdot (X_r - \tilde{X}_r)$, we define flows for any $\mathcal{L} \subseteq I_r^1 \times I_r^1$ by

$$\mathcal{T}^{I_r^1}(\mathcal{L}) = \mathcal{T}\Big(\mathcal{L} + (h, h)\Big).$$

Consider the set $K \triangleq \{x \in I_r^1 : \frac{d\mathcal{T}_2^{I_r^1}}{d\Gamma}(x) \leq \frac{d\mathcal{T}_2}{d\Gamma}(x + h)\}$. We set prices to be such that

$$U\Big(x, p^{I_r^1}(x), \frac{d\mathcal{T}_2^{I_r^1}}{d\Gamma}(x)\Big) = U\Big(x + h, p(x + h), s^{\mathcal{T}}(x + h)\Big), \quad \forall x \in K, \tag{D-24}$$

and zero otherwise. We prove, in Property 1 (see end of present proof), that $(p^{I_r^1}, \mathcal{T}^{I_r^1})$ is a price-equilibrium pair in I_r^1 such that $V(x |\, p^{I_r^1}, \mathcal{T}^{I_r^1}) = V(\tilde{X}_r) + (x - \tilde{X}_r)$ and $\Gamma(K^c) = 0$.

In the interval $I_r^2 = (\tilde{Y}_r, H]$ we can achieve the optimal solution when there is no demand shock. As in the optimal solution in the pre-demand shock environment (see Proposition D-1) we set prices equal to ρ_1 and the flows are such that $d\mathcal{T}^{I_r^2}/d\Gamma$ equals θ_1, $\Gamma - a.e$ in I_r^2.

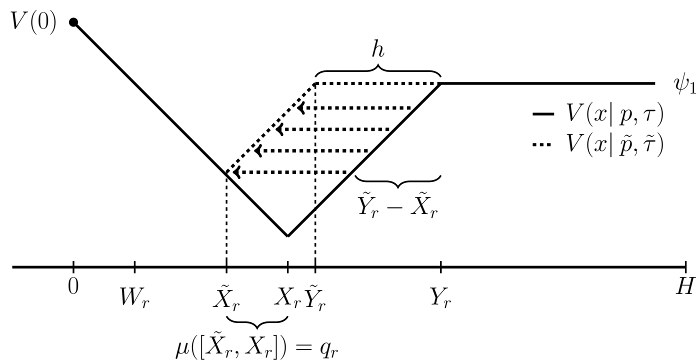

Figure 13: **No supply in** $[W_r, X_r]$. The new solution moves the right end of the attraction region from X_r to $\tilde{X}_r$, so now a mass q_r of drivers can travel towards the periphery. From this mass the platform now makes ψ_1 instead of $V(x)$ with $V(x) < \psi_1$.

The interval $I_r^0 = [X_l, \tilde{X}_r]$ is more involved. Observe that all the initial flow to the right of the origin that we have to allocate in $[0, \tilde{X}_r]$ equals $\theta_1 \cdot X_r - q_r$. This is exactly the same amount of drivers in $[0, X_r]$ that travels to $[0, W_r]$ according to $\mathcal{T}$. Our new solution will generate the same post-relocation supply than $\mathcal{T}$ in $[0, W_r]$ but this time only using drivers from $[0, \tilde{X}_r]$.

We use the same prices, that is $p^{I_r^0}(x) = p(x)$ for all $x \in [X_l, \tilde{X}_r]$. For the flows we define them through two measures: the flow that goes from $[X_l, 0]$ to $[X_l, 0]$ and the flow that goes from $[0, \tilde{X}_r]$ to $[0, \tilde{X}_r]$. For the first flow we use $\mathcal{T}^\ell = \mathcal{T}|_{[X_l,0]}$, for the second measure $\mathcal{T}^r$ we will use a monotone coupling as in the proof of Theorem 1 (see e.g, Santambrogio (2015) for details). Define the initial supply to the right measure Θ^r to be equal to $\Theta|_{[0,\tilde{X}_r]}$, and the final supply S^r to be

$$S^r(\mathcal{B}) \triangleq \mathcal{T}([0, X_r] \times \mathcal{B}), \quad \text{for any measurable set } \mathcal{B} \subseteq [0, \tilde{X}_r].$$

Note that $S^r([0, W_r])$ equals $\Theta^r([0, \tilde{X}_r])$. Given this we define $\mathcal{T}^r$ by

$$\mathcal{T}^r(\mathcal{L}) \triangleq (F_{\Theta^r}^{[-1]}, F_{S^r}^{[-1]})_{\#} m(\mathcal{L}), \quad \text{for any measurable set } \mathcal{L} \subseteq [0, \tilde{X}_r] \times [0, \tilde{X}_r],$$

where $\#$ correspond to the push-forward operator. For any measure ν defined in $[0, \tilde{X}_r]$ we define its cumulative function and pseudo-inverse by

$$F_\nu(y) \triangleq \nu([0, y]), \quad \forall\, y \geq 0 \quad \text{and} \quad F_\nu^{[-1]}(t) \triangleq \inf\{y \geq 0 : F_\nu(y) \geq t\}, \quad \forall t \in [0, \Theta^r([0, \tilde{X}_r])].$$

Effectively, $\mathcal{T}^r$ transports the initial mass in $[0, \tilde{X}_r]$ to the final supply distribution (considering only drivers that come from the right) in $[0, W_r]$ as prescribed by $\mathcal{T}$. The final flow measure $\mathcal{T}^{I_r^0}$ correspond to $\mathcal{T}^\ell + \mathcal{T}^r|_{[0,\tilde{X}_r]}$. In Property 2 below we show that $(p^{I_r^0}, \mathcal{T}^{I_r^0})$ is a price-equilibrium pair such that $\mathbf{Rev}_{[X_l,W_r]}(p^{I_r^0}, \mathcal{T}^{I_r^0}) = \mathbf{Rev}_{[X_l,W_r]}(p, \mathcal{T})$.

The solution $(\tilde{p}, \tilde{\mathcal{T}})$ is constructed by pasting (see Lemma D-3) the old solution is $[-H, X_l)$ with the new solution in I^0, I_r^1 and I_r^2. The pasting is possible because the equilibrium utility function coincide in the boundaries of these intervals. This new solution preserves the platform's revenue in $[-H, W_r] \cup [Y_r, H]$ but it strictly improves it in $[W_r, Y_r]$. Indeed, note that

$$q_r = \int_{[X_r,Y_r]} s^{\mathcal{T}}(x)\, dx - \int_{[\tilde{X}_r,\tilde{Y}_r]} s^{\tilde{\mathcal{T}}}(x)\, dx = \int_{[\tilde{X}_r,\tilde{Y}_r]} \underbrace{(s^{\mathcal{T}}(x+h) - s^{\tilde{\mathcal{T}}}(x))}_{\geq 0 \;\; \Gamma-a.e}\, dx + \int_{[X_r, X_r+(X_r-\tilde{X}_r)]} s^{\mathcal{T}}(x)\, dx, \tag{D-25}$$

thus

$$
\begin{aligned}
\frac{1}{\gamma}\cdot \mathbf{Rev}_{[W_r,Y_r]}(\tilde{p},\tilde{\mathcal{T}}) &= \int_{[W_r,\tilde{X}_r]} V(x|\tilde{p},\tilde{\mathcal{T}})\cdot s^{\tilde{\mathcal{T}}}(x)\,dx + \int_{[\tilde{X}_r,Y_r]} V(x|\tilde{p},\tilde{\mathcal{T}})\cdot s^{\tilde{\mathcal{T}}}(x)\,dx\\
&\overset{(a)}{=} \int_{[\tilde{X}_r,Y_r]} V(x|\tilde{p},\tilde{\mathcal{T}})\cdot s^{\tilde{\mathcal{T}}}(x)\,dx\\
&\overset{(b)}{=} \int_{[\tilde{X}_r,\tilde{Y}_r]} V(x|\tilde{p},\tilde{\mathcal{T}})\cdot s^{\tilde{\mathcal{T}}}(x)\,dx + \psi_1\cdot 2\cdot q_r\\
&\overset{(c)}{>} \int_{[\tilde{X}_r,\tilde{Y}_r]} V(x|\tilde{p},\tilde{\mathcal{T}})\cdot s^{\tilde{\mathcal{T}}}(x)\,dx + \psi_1\cdot q_r + \int_{[W_r,X_r]} V(x)\cdot s^{\mathcal{T}}(x)\,dx\\
&\overset{(d)}{\geq} \int_{[\tilde{X}_r,\tilde{Y}_r]} V(x|\tilde{p},\tilde{\mathcal{T}})\cdot s^{\mathcal{T}}(x+h)\,dx + \int_{[W_r,X_r+(X_r-\tilde{X}_r)]} V(x)\cdot s^{\mathcal{T}}(x)\,dx\\
&\overset{(e)}{=} \int_{[W_r,Y_r]} V(x)\cdot s^{\mathcal{T}}(x)\,dx = \frac{1}{\gamma}\cdot \mathbf{Rev}_{[W_r,Y_r]}(p,\mathcal{T}),
\end{aligned}
$$

where (a) follows because $\tilde{\mathcal{T}}$ does not put mass in $[W_r,\tilde{X}_r]$, (b) because $Y_r-\tilde{Y}_r$ equals $2\cdot(X_r-\tilde{X}_r)$. Using the fact that $\mathcal{T}_2([W_r,X_r]) = q_r$ we obtain (c), while (d) follows from Eq. (D-25) and (e) from $V(x|\,\tilde{p},\tilde{\mathcal{T}})$ being equal to $V(x+h)$ for all $x\in[\tilde{X}_r,\tilde{Y}_r]$.

In conclusion, any optimal solution must satisfy both $\mathcal{T}_2([W_r,X_r]) = 0$ and $\mathcal{T}_2([X_l,W_l]) = 0$.

Using Theorem 1: All the conditions in Theorem 1 are met. So, for any of the three attraction regions if $(p,\mathcal{T})$ is not already as in the statement of the theorem we can find at least a weak improvement. That is, we can restrict to solution as in Theorem 1. Therefore, the prices are as stated in the present theorem, and there exists $\beta_c^l\in[W_l,0]$, $\beta_c^r\in[0,W_r]$, $\beta_p^l\in[Y_l,X_l]$ and $\beta_p^r\in[X_r,Y_r]$ such that

$$
s^{\mathcal{T}}(x) = \begin{cases} 0 & \text{if } x\in(\beta_c^r,\beta_p^r)\cup(\beta_p^l,\beta_c^l),\\ \psi_x^{-1}(V(x|\,p,\mathcal{T})) & \text{otherwise,}\end{cases}
$$

with

$$
\int_{\beta_c^l}^{\beta_c^r} \psi_x^{-1}(V(x|\,p,\mathcal{T}))\,d\Gamma(x) = \theta_1\cdot(X_r-X_l)
$$

and

$$
\int_{\beta_p^r}^{Y_r} \psi_x^{-1}(V(x|\,p,\mathcal{T}))\,d\Gamma(x) = \theta_1\cdot(Y_r-X_r),\qquad \int_{Y_l}^{\beta_p^l} \psi_x^{-1}(V(x|\,p,\mathcal{T}))\,d\Gamma(x) = \theta_1\cdot(X_l-Y_l).
$$

Note that the fact that $\beta_c^l\in[W_l,0]$ and $\beta_c^r\in[0,W_r]$, does not come directly from Theorem 1 but rather is a consequence of that any optimal solution must satisfy both $\mathcal{T}_2([W_r,X_r]) = 0$ and $\mathcal{T}_2([X_l,W_l]) = 0$. Also, observe that Theorem 1 only gives us a solution in each attraction but above we have stated the solution for the entire city. The only missing interval are $[-H,Y_l]$ and $[Y_r,H]$. In this intervals, as in the pre-shock environment, the solution set prices equal to ρ_1 and the supply at every location is θ_1, in turn, the V equals ψ_1 in this region. This gives a complete solution to the platform's problem up to three values: $V(0), X_l, X_r$.

Symmetry: In the last main step of the proof we argue that the solution is symmetric. After proving this, the solution will take the exact form in the statement of the present theorem.

Note that given a value for $V(0)$ and an central attraction region characterize by X_l and X_r we can characterize the optimal solution as we did in **Using Theorem 1**. So fix these three values and the optimal solution associated to them. We now proceed to construct a new solution that yields a strict objective improvement when the solution is not symmetric. WLOG assume that $|X_l|>X_r$ and let $\delta=(|X_l|-X_r)/2$. Consider the solution $(\tilde{p},\tilde{\mathcal{T}})$ associated to the values

$$
\tilde{V}(0)=V(0),\quad \tilde{X}_l = X_l+\delta,\quad \tilde{X}_r = X_r+\delta.
$$

Note that with this values we have $|\tilde{X}_l|, \tilde{W}_r \geq W_r$ and $\tilde{Y}_i = Y_i + 2 \cdot \delta$ for $i \in \{l, r\}$. We next show that this new solution yields a weak objective improvement in the center, and a strict objective improvement in the periphery.

Note that given $\tilde{V}(0), \tilde{X}_l$ and $\tilde{X}_r$ Theorem 2 characterizes $V(\cdot|\tilde{p}, \tilde{\mathcal{T}})$. It has the same shape than $V(\cdot|p, \mathcal{T})$ except that now the dip in $[\tilde{Y}_l, W_l]$ is smaller, while the dip in $[W_r, Y_r]$ is larger. See Figure 14 for a graphical representation. Consider first the solution in the center, $[\tilde{X}_l, \tilde{X}_r]$. This interval contains the same amount

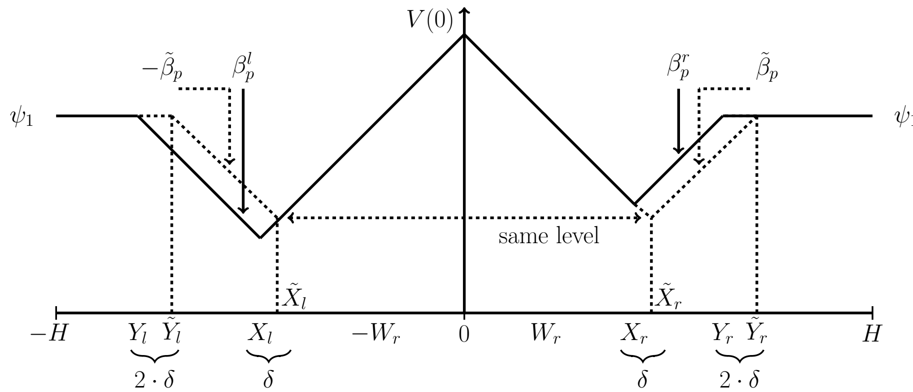

Figure 14: **Symmetry** argument.

of drivers that the old attraction region. The difference is that it lost a mass of $\theta_1 \cdot \delta$ drivers to the left and gain the same mass to the right. As in the discussion that follows Theorem 1 the optimal solution in $[\tilde{X}_l, \tilde{X}_r]$ can be obtained using a knapsack argument. This new attraction region is symmetric, $|\tilde{X}_l| = \tilde{X}_r$, with equal mass of drivers at both sides of the origin. Therefore the knapsack solution must be symmetric, with $\tilde{\beta}_c \in [0, W_r]$ such that

$$s^{\tilde{\mathcal{T}}}(x) = \psi_x^{-1}(V(x|\,\tilde{p}, \tilde{\mathcal{T}})) = \psi_x^{-1}(V(x|\,p, \mathcal{T})), \quad \forall x \in [-\tilde{\beta}_c, \tilde{\beta}_c],$$

and equals zero otherwise, and

$$\int_{-\tilde{\beta}_c}^{\tilde{\beta}_c} \psi_x^{-1}(V(x|\,\tilde{p}, \tilde{\mathcal{T}})\, d\Gamma(x) = \theta_1 \cdot (\tilde{X}_r - \tilde{X}_l) = \theta_1 \cdot (X_r - X_l).$$

Note that $\tilde{\beta}_c \in [0, W_r]$ is a consequence of the having $\beta_c^l \in [W_l, 0]$ and $\beta_c^r \in [0, W_r]$ in the old solution. Theorem 1 prescribes how to formally implement this solution through prices and flows. We omit the details of how to construct the flows, but we note that the optimal prices are given $\tilde{p}(x) = \rho_x^{loc}(s^{\tilde{\mathcal{T}}}(x))$. In the case that $\tilde{\beta} = 0$ then $s^{\tilde{\mathcal{T}}}(0) = \theta_1 \cdot (X_r - X_l)$ and $\tilde{p}(0)$ is such that $U(0, p(0), s^{\tilde{\mathcal{T}}}(0)) = V(0)$. The platform's revenue in the new center is then

$$\frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l, \tilde{X}_r]}(\tilde{p}, \tilde{\mathcal{T}}) = \int_{\tilde{X}_l}^{\tilde{X}_r} V(x|\,\tilde{p}, \tilde{\mathcal{T}}) \cdot s^{\tilde{\mathcal{T}}}(x)\, dx = \int_{-\tilde{\beta}_c}^{\tilde{\beta}_c} V(x) \cdot \psi_x^{-1}(V(x))\, dx.$$

This expression is an upper bound for the platform's revenue under $(p, \mathcal{T})$ in $[X_l, X_r]$. In fact, WLOG assume

$\beta_c^r \geq |\beta_c^l|$ which implies that $\tilde{\beta}_c \in [|\beta_c^l|, \beta_c^r]$ and we must have

$$\begin{aligned}
\frac{1}{\gamma} \cdot \mathbf{Rev}_{[X_l,X_r]}(p,\mathcal{T}) &= \int_{\beta_c^l}^{\beta_c^r} V(x) \cdot \psi_x^{-1}(V(x))\, dx \\
&= \int_{\beta_c^l}^{|\beta_c^l|} V(x) \cdot \psi_x^{-1}(V(x))\, dx + \int_{|\beta_c^l|}^{\beta_c^r} V(x) \cdot \psi_x^{-1}(V(x))\, dx \\
&= \frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l,\tilde{X}_r]}(\tilde{p},\tilde{\mathcal{T}}) - 2 \cdot \int_{|\beta_c^l|}^{\tilde{\beta}_c} V(x) \cdot \psi_x^{-1}(V(x))\, dx + \int_{|\beta_c^l|}^{\beta_c^r} V(x) \cdot \psi_x^{-1}(V(x))\, dx \\
&= \frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l,\tilde{X}_r]}(\tilde{p},\tilde{\mathcal{T}}) - \int_{|\beta_c^l|}^{\tilde{\beta}_c} V(x) \cdot \psi_x^{-1}(V(x))\, dx + \int_{\tilde{\beta}_c}^{\beta_c^r} V(x) \cdot \psi_x^{-1}(V(x))\, dx \\
&\leq \frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l,\tilde{X}_r]}(\tilde{p},\tilde{\mathcal{T}}) + V(\tilde{\beta}_c) \cdot \Big(- \int_{|\beta_c^l|}^{\tilde{\beta}_c} \psi_x^{-1}(V(x))\, dx + \int_{\tilde{\beta}_c}^{\beta_c^r} \psi_x^{-1}(V(x))\, dx \Big) \\
&= \frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l,\tilde{X}_r]}(\tilde{p},\tilde{\mathcal{T}}).
\end{aligned}$$

That is, the new solution in the center is a weakly improvement over the old solution.

Now let us consider the periphery. Since $|\tilde{X}_l| = \tilde{X}_r$ both right and left periphery are symmetric. Thus the optimal solution as given by Theorem 1 is the symmetric at both sides. The post-relocation supply is characterize by $\tilde{\beta}_p \in [\tilde{X}_r, \tilde{Y}_r]$ such that

$$s^{\tilde{\mathcal{T}}}(x) = \psi_x^{-1}(V(x|\,\tilde{p},\tilde{\mathcal{T}})) = \psi_x^{-1}(V(X_r) + (x - X_r) - 2 \cdot \delta), \quad \forall x \in [\tilde{\beta}_p, \tilde{Y}_r],$$

and equals zero otherwise, and

$$\int_{\tilde{\beta}_p}^{\tilde{Y}_r} \psi_x^{-1}(V(x|\,\tilde{p},\tilde{\mathcal{T}})\, d\Gamma(x) = \theta_1 \cdot (\tilde{Y}_r - \tilde{X}_r) = \theta_1 \cdot (Y_r - X_r) + \theta_1 \cdot \delta.$$

The optimal prices are $\tilde{p}(x) = \rho_x^{loc}(s^{\tilde{\mathcal{T}}}(x))$. As before we omit the characterization of the equilibrium flow as their existence is guaranteed by Theorem 1. The platforms revenue in the periphery is

$$\frac{1}{\gamma} \cdot \mathbf{Rev}_{[-H,\tilde{X}_l] \cup [\tilde{X}_r,H]}(\tilde{p},\tilde{\mathcal{T}}) = 2 \cdot \int_{\tilde{\beta}_p}^{\tilde{Y}_r} V(x|\,\tilde{p},\tilde{\mathcal{T}}) \cdot \psi^{-1}(V(x|\,\tilde{p},\tilde{\mathcal{T}}))\, dx + 2 \cdot \psi_1 \cdot \theta_1 \cdot (H - \tilde{Y}_r),$$

where we have dropped the subindex x from ψ_x^{-1} to stress the fact that in this part of the city this subindex does not change the congestion function. We need to compare this revenue with the revenue of the old solution in the periphery. Not that since $|X_l| > X_r$ we must have

$$Y_r - \beta_p^r < \tilde{Y}_r - \tilde{\beta}_p < \beta_p^l - Y_l.$$

Thus,

$$
\begin{aligned}
\frac{1}{\gamma}\cdot\mathbf{Rev}_{[-H,X_l]\cup[X_r,H]}(p,\mathcal{T}) &= \int_{Y_l}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx + \int_{\beta_p^r}^{Y_r} V(x)\cdot\psi^{-1}(V(x))\,dx \\
&\quad + \psi_1\cdot\theta_1\cdot(H - Y_r + Y_l + H) \\
&= \int_{Y_l+(Y_r-\beta_p^r)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx + 2\cdot\int_{\beta_p^r}^{Y_r} V(x)\cdot\psi^{-1}(V(x))\,dx \\
&\quad + \psi_1\cdot\theta_1\cdot(H - Y_r + Y_l + H) \\
&= \int_{Y_l+(Y_r-\beta_p^r)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx + 2\cdot\int_{\beta_p^r+2\cdot\delta}^{\tilde{Y}_r} V(x|\,\tilde{p},\tilde{\mathcal{T}})\cdot\psi^{-1}(V(x|\,\tilde{p},\tilde{\mathcal{T}}))\,dx \\
&\quad + 2\cdot\psi_1\cdot\theta_1\cdot(H-\tilde{Y}_r) \\
&= \underbrace{\int_{Y_l+(Y_r-\beta_p^r)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx - 2\cdot\int_{\tilde{\beta}_p}^{\beta_p^r+2\cdot\delta} V(x|\,\tilde{p},\tilde{\mathcal{T}})\cdot\psi^{-1}(V(x|\,\tilde{p},\tilde{\mathcal{T}}))\,dx}_{(a)} \\
&\quad + \frac{1}{\gamma}\cdot\mathbf{Rev}_{[-H,\tilde{X}_l]\cup[\tilde{X}_r,H]}(\tilde{p},\tilde{\mathcal{T}}),
\end{aligned}
$$

So if we show that the term (a) is strictly negative we will be done. Not that

$$
\begin{aligned}
(a) \quad &= \int_{Y_l+(Y_r-\beta_p^r)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx - 2\cdot\int_{Y_l+(Y_r-\beta_p^r)}^{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)} V(x)\cdot\psi^{-1}(V(x))\,dx \\
&= \int_{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx - \int_{Y_l+(Y_r-\beta_p^r)}^{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)} V(x)\cdot\psi^{-1}(V(x))\,dx \\
&< V(Y_l+(\tilde{Y}_r-\tilde{\beta}_p))\cdot\Big(\int_{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)}^{\beta_p^l} \psi^{-1}(V(x))\,dx - \int_{Y_l+(Y_r-\beta_p^r)}^{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)} \psi^{-1}(V(x))\,dx\Big) \\
&= 0.
\end{aligned}
$$

In conclusion, we have constructed a new symmetric solution that yields an strict revenue improvement over the old solution. Therefore, any optimal solution ought to be symmetric.

Property 1. $(p^{I_r^1},\mathcal{T}^{I_r^1})$ forms a price-equilibrium pair in I_r^1 such that $V(x|\,p^{I_r^1},\mathcal{T}^{I_r^1})$ equals $V(\tilde{X}_r)+(x-\tilde{X}_r)$ and $\Gamma(K^c)=0$.

Proof of Property 1. We first show that $\mathcal{T}^{I_r^1}\in\mathcal{F}_{I_r^1}(\Theta|_{I_r^1})$. It is clear that $\mathcal{T}^{I_r^1}\in\mathcal{M}(I_r^1\times I_r^1)$, and that $\mathcal{T}_2^{I_r^1}\ll\Gamma$. To see why $\mathcal{T}_1^{I_r^1}$ coincides with $\theta_{I_r^1}$ consider a set $I\subset I_r^1$ then $\mathcal{T}_1^{I_r^1}(K)$ equals

$$
\mathcal{T}_1^{I_r^1}(K\times I_r^1)=\mathcal{T}((I+h)\times(I_r^1+h))=\mathcal{T}((I+h)\times[\tilde{X}_r+h,Y_r])=\mathcal{T}((I+h)\times\mathcal{C})=\Theta(I+h)=\Theta(I),
$$

where the fourth line holds because the set $I+h$ is contain in $[\tilde{X}_r+h,Y_r]$, and we know there is no flow leaving this interval. Next, using a similar argument we show the property for $d\mathcal{T}_2^{I_r^1}/d\Gamma$, let I be a measurable subset of I_r^1 then

$$
\begin{aligned}
\int_I \frac{d\mathcal{T}_2^{I_r^1}}{d\Gamma}(x)\,d\Gamma(x)=\mathcal{T}^{I_r^1}(I_r^1\times I) &= \mathcal{T}([\tilde{X}_r+h,Y_r]\times(I+h)) \\
&\le \mathcal{T}([X_r,Y_r]\times(I+h)) \\
&= \int_{(I+h)}\frac{d\mathcal{T}_2}{d\Gamma}(x)\,d\Gamma(x) \\
&= \int_I \frac{d\mathcal{T}_2}{d\Gamma}(x+h)\,d\Gamma(x),
\end{aligned}
$$

that is, $\Gamma(K^c)=0$. As for the equilibrium utility function let $x\in[\tilde{X}_r,\tilde{Y}_r)$ we have

$$\begin{aligned}
V(x|\, p^{I_r^1},\mathcal{T}^{I_r^1}) &= \inf\{u\in\mathbb{R}:\Gamma(y\in I_r^1: U(y,p^{I_r^1}(y),\frac{d\mathcal{T}_2^{I_r^1}}{d\Gamma}(y))-|y-x|>u)=0\} \\
&= \inf\{u\in\mathbb{R}:\Gamma(y\in I_r^1: U(y,p(y+h),\frac{d\mathcal{T}_2}{d\Gamma}(y+h))-|y-x|>u)=0\} \\
&= \inf\{u\in\mathbb{R}:\Gamma(y\in[\tilde{X}_r+h,Y_r]: U(y,p(y),\frac{d\mathcal{T}_2}{d\Gamma}(y))-|y-(x+h)|>u)=0\} \\
&\le V(x+h|\, p,\mathcal{T}).
\end{aligned}$$

Actually this upper bound is tight. Indeed, Fix any $\epsilon>0$ and consider $\delta>0$ small enough such that $(x+h)\notin B(Y_r,\delta)$. We have $\mathcal{T}_2(\{y\in B(y,\delta)\cap[\tilde{X}_r+h,Y_r]:U(y)=V(y)\})>0$ which implies that $\Gamma(\{y\in B(Y_r,\delta)\cap[\tilde{X}_r+h,Y_r]:U(y)=V(y)\})>0$ and, therefore,

$$\begin{aligned}
0 &< \Gamma(\{y\in B(Y_r,\delta)\cap[\tilde{X}_r+h,Y_r]:U(y)=V(y),\epsilon+y-(x+h)>|y-(x+h)|\}) \\
&= \Gamma(\{y\in B(Y_r,\delta)\cap[\tilde{X}_r+h,Y_r]:U(y)=V(y),U(y)-|y-(x+h)|>V(x+h)-\epsilon\}) \\
&\le \Gamma(\{y\in[\tilde{X}_r+h,Y_r]:U(y)-|y-(x+h)|>V(x+h)-\epsilon\}) \\
&= \Gamma(\{y\in I_r^1: U(y,p^{I_r^1}(y),\frac{d\mathcal{T}_2^{I_r^1}}{d\Gamma}(y))-|y-x|>V(x+h)-\epsilon\}),
\end{aligned}$$

therefore $V(x|\,p^{I_r^1},\mathcal{T}^{I_r^1})$ equals $V(x+h)$ for all $x\in[\tilde{X}_r,\tilde{Y}_r)$, and by continuity for all $x\in I_r^1$. Since $V(x+h)$ equals $V(\tilde{X}_r)+(x-\tilde{X}_r)$ we obtain the desired result.

Now we need to verify that this selection of prices and flows yields an equilibrium. That is, we need show that the set

$$\mathcal{E}_{I_r^1}=\Big\{(x,y)\in I_r^1\times I_r^1:\Pi(x,y,p^{I_r^1}(y),\frac{d\mathcal{T}_2^{I_r^1}}{d\Gamma}(y))=V(x|\,p^{I_r^1},\mathcal{T}^{I_r^1})\Big\},$$

has $\mathcal{T}^{I_r^1}$ measure equal to $\Theta(I_r^1)$. Observe that $\mathcal{T}(\mathcal{E}_{I_r^1})$ equals

$$\mathcal{T}\Big(\Big\{(x,y)\in[\tilde{X}_r+h,Y_r]\times[\tilde{X}_r+h,Y_r]:\Pi(x-h,y-h,p^{I_r^1}(y-h),\frac{d\mathcal{T}_2^{I_r^1}}{d\Gamma}(y-h))=V(x)\Big\}\Big),$$

using that $\Gamma(K^c)=0$ and the way we chose the prices one can verify that this expression equals

$$\mathcal{T}\Big(\Big\{(x,y)\in[\tilde{X}_r+h,Y_r]\times[\tilde{X}_r+h,Y_r]:\Pi(x,y,p(y),s^{\mathcal{T}}(y))=V(x|\,p,\mathcal{T})\Big\}\Big).$$

There is no $\mathcal{T}$ flow of drivers leaving $[\tilde{X}_r+h,Y_r]$ so the fact that $\mathcal{T}$ is an equilibrium flow implies that this last expression equals $\Theta([\tilde{X}_r+h,Y_r])$, which equals $\Theta(I_r^1)$.

Property 2. $(p^{I_r^0},\mathcal{T}^{I_r^0})$ is a price-equilibrium pair such that $\mathbf{Rev}_{[X_l,W_r]}(p^{I_r^0},\mathcal{T}^{I_r^0})=\mathbf{Rev}_{[X_l,W_r]}(p,\mathcal{T})$.

Proof of Property 2. First a couple of observations, note that for any $y\in[0,\tilde{X}_r]$ and the set $[0,y]$ then

$$\begin{aligned}
\mathcal{T}_1^r([0,y])=\mathcal{T}^r([0,y]\times[0,\tilde{X}_r]) &= m\Big(t\in[0,\Theta^r([0,\tilde{X}_r])]:F_{\Theta^r}^{[-1]}(t)\in[0,y]\Big) \\
&= m\Big(t\in[0,\Theta^r([0,\tilde{X}_r])]:0\le t\le F_{\Theta^r}(y)\Big) \\
&= F_{\Theta^r}(y),
\end{aligned}$$

and the same argument holds for $\mathcal{T}_2^r$ and S^r, this characterizes the first and second marginals of $\mathcal{T}^r$. Furthermore, it's not difficult to see that for $y_1,y_2\in[0,\tilde{X}_r]$ we have

$$\mathcal{T}^r([0,y_1]\times[0,y_2])=m\Big(t\in[0,\Theta^r([0,\tilde{X}_r])]:t\le F_{\Theta^r}(y_1),\ t\le F_{S^r}(y_2)\Big)=F_{\Theta^r}(y_1)\wedge F_{S^r}(y_2). \tag{D-26}$$

Next, we show that $\mathcal{T}^{I_r^0} \in \mathcal{F}_{I_r^0}(\Theta|_{I_r^0})$ is an equilibrium in I_r^0. In order to do so we first show that$\mathcal{T}^{I_r^0} \in \mathcal{F}_{I_r^0}(\Theta|_{I_r^0})$. Second, we compute the supply density of $\mathcal{T}_2^{I_r^0}$ and corroborate they coincide with $s^{\mathcal{T}}$. Third, we compute $V_{I_r^0}(\cdot|\, p^{I_r^0}, \mathcal{T}^{I_r^0})$ and verify is coincides with $V(\cdot|\, p, \mathcal{T})$ in I_r^0. Finally, we check the equilibrium condition.

Clearly $\mathcal{T}^{I_r^0}$ is a non-negative measure in $I_r^0 \times I_r^0$ because is the sum of non-negative measures. Now we check that $\mathcal{T}_1^{I_r^0} = \Theta|_{I_r^0}$. Consider a measurable set $\mathcal{B} \subseteq I_r^0$ then

$$\begin{aligned}
\mathcal{T}_1^{I_r^0}(\mathcal{B}) &= \mathcal{T}((\mathcal{B} \cap [X_l, 0]) \times [X_l, 0]) + \mathcal{T}^r((\mathcal{B} \cap [0, \tilde{X}_r]) \times [0, \tilde{X}_r]) \\
&= \mathcal{T}((\mathcal{B} \cap [X_l, 0]) \times \mathcal{C}) + \Theta^r(\mathcal{B} \cap [0, \tilde{X}_r]) \\
&= \Theta(\mathcal{B} \cap [X_l, 0]) + \Theta(\mathcal{B} \cap [0, \tilde{X}_r]) \\
&= \Theta|_{I_r^0}(\mathcal{B})
\end{aligned}$$

and thus we also have $\mathcal{T}_1^{I_r^0} \ll \Gamma$. For the second marginal of $\mathcal{T}^{I_r^0}$ we have

$$\begin{aligned}
\mathcal{T}_2^{I_r^0}(\mathcal{B}) &= \mathcal{T}([X_l, 0] \times (\mathcal{B} \cap [X_l, 0])) + \mathcal{T}^r([0, \tilde{X}_r] \times (\mathcal{B} \cap [0, \tilde{X}_r])) \\
&= \mathcal{T}([X_l, 0] \times (\mathcal{B} \cap [X_l, 0])) + S^r(\mathcal{B} \cap [0, \tilde{X}_r]) \\
&= \mathcal{T}([X_l, 0] \times (\mathcal{B} \cap [X_l, 0])) + \mathcal{T}([0, X_r] \times (\mathcal{B} \cap [0, \tilde{X}_r])) \\
&= \mathcal{T}_2(\mathcal{B} \cap [X_l, 0)) + \mathcal{T}_2(\mathcal{B} \cap (0, \tilde{X}_r]) + \mathcal{T}_2(\mathcal{B} \cap \{0\}) \\
&= \mathcal{T}_2|_{I_r^0}(\mathcal{B}),
\end{aligned}$$

and thus $\mathcal{T}_2^{I_r^0} \ll \Gamma$. We conclude that $\mathcal{T}^{I_r^0} \in \mathcal{F}_{I_r^0}(\Theta|_{I_r^0})$. From this we can also conclude that

$$\frac{d\mathcal{T}_2^{I_r^0}}{d\Gamma}(x) = s^{\mathcal{T}}(x), \quad \Gamma - a.e.\ x \text{ in } I_r^0.$$

Next we compute the equilibrium utilities. We show that $V(x|\, p^{I_r^0}, \mathcal{T}^{I_r^0})$ equals $V(x|\, p, \mathcal{T})$ for all $x \in I_r^0$. Observe that $\Gamma - a.e.\ y$ in I_r^0 we have $U(y, p^{I_r^0}(y), s^{\mathcal{T}^{I_r^0}}(y)) = U(y, p(y), s^{\mathcal{T}}(y))$, and, therefore, $V(x|\, p, \mathcal{T}) \geq V(x|\, p^{I_r^0}, \mathcal{T}^{I_r^0})$. Using the same argument that we used for the proof of Property 1 we can argue that this upper bound is tight, that is, $V(x|\, p, \mathcal{T}) = V(x|\, p^{I_r^0}, \mathcal{T}^{I_r^0})$.

Now the equilibrium condition. Consider the equilibrium set

$$\mathcal{E}_{I_r^0} \triangleq \left\{(x, y) \in I_r^0 \times I_r^0 : U(y, p^{I_r^0}(y), s^{\mathcal{T}^{I_r^0}}(y)) - |y - x| = V(x|\, p^{I_r^0}, \mathcal{T}^{I_r^0})\right\},$$

we need to verify that $\mathcal{T}^{I_r^0}(\mathcal{E}_{I_r^0})$ equals $\Theta(I_r^0)$. First, for $\mathcal{T}^l(\mathcal{E}_{I_r^0})$ we have

$$\begin{aligned}
\mathcal{T}^l(\mathcal{E}_{I_r^0}) &= \mathcal{T}\Big(\Big\{(x, y) \in [X_l, 0] \times [X_l, 0] : U(y, p(y), s^{\mathcal{T}}(y)) - |y - x| = V(x|\, p, \mathcal{T})\Big\}\Big) \\
&= \mathcal{T}([X_l, 0] \times [X_l, 0]) \\
&= \mathcal{T}([X_l, 0] \times \mathcal{C}) \\
&= \Theta([X_l, 0])
\end{aligned}$$

where we have used our choice of prices, the relation between $d\mathcal{T}_2^{I_r^0}/d\Gamma$ and $s^{\mathcal{T}}$, and the fact that $\mathcal{T}$ is an equilibrium flow that does not setnd flow out of $[X_l, 0]$. For $\mathcal{T}^r|_{[0, \tilde{X}_r]}$, note that its second marginal is S^r and, therefore, Lemma A-2 implies that

$$\mathcal{T}^r|_{[0, \tilde{X}_r]}(\mathcal{E}_{I_r^0}) = \mathcal{T}^r\Big(\Big\{(x, y) \in [0, \tilde{X}_r] \times [0, \tilde{X}_r] : V(y|\, p, \mathcal{T}) - |y - x| = V(x|\, p, \mathcal{T})\Big\}\Big),$$

and because $V(z|\, p, \mathcal{T})$ equals $V(0) - z$ for any $z \in [0, \tilde{X}_r]$ we have

$$\begin{aligned}
\mathcal{T}^r|_{[0,\tilde{X}_r]}(\mathcal{E}_{I_r^0}) &= \mathcal{T}^r\Big(\Big\{(x,y) \in [0,\tilde{X}_r] \times [0,\tilde{X}_r] : -y - |y-x| = -x\Big\}\Big) \\
&= \mathcal{T}^r\Big(\Big\{(x,y) \in [0,\tilde{X}_r] \times [0,\tilde{X}_r] : x \geq y\Big\}\Big) \\
&= \Theta^r([0,\tilde{X}_r]) - \mathcal{T}^r\Big(\Big\{(x,y) \in [0,\tilde{X}_r] \times [0,\tilde{X}_r] : x < y\Big\}\Big),
\end{aligned}$$

but

$$\begin{aligned}
\mathcal{T}^r\Big(\Big\{(x,y) \in [0,\tilde{X}_r] \times [0,\tilde{X}_r] : x < y\Big\}\Big) &\leq \sum_{q \in \cap [0,\tilde{X}_r]} \mathcal{T}^r([0,q] \times (q,\tilde{X}_r]) \\
&= \sum_{q \in \mathbb{Q} \cap [0,\tilde{X}_r]} \mathcal{T}^r([0,q] \times [0,\tilde{X}_r]) - \mathcal{T}^r([0,q] \times [0,q]) \\
&= \sum_{q \in \mathbb{Q} \cap [0,\tilde{X}_r]} \Theta^r([0,q]) \wedge S^r([0,\tilde{X}_r]) - \Theta^r([0,q]) \wedge S^r([0,q]) \\
&= \sum_{q \in \mathbb{Q} \cap [0,\tilde{X}_r]} \Theta^r([0,q]) \wedge S^r([0,\tilde{X}_r]) - \Theta^r([0,q]) \wedge S^r([0,q]) = 0,
\end{aligned}$$

where in the last line we used that $\Theta^r([0,q]) \leq S^r([0,q])$. Adding up $\mathcal{T}^l(\mathcal{E}_{I_r^0})$ with $\mathcal{T}^r|_{[0,\tilde{X}_r]}(\mathcal{E}_{I_r^0})$, yields that $\mathcal{T}^{I_r^0}(\mathcal{E}_{I_r^0})$ equals $\Theta(I_r^0)$, and the equilibrium condition is satisfied. Finally, the revenue condition in the statement of the Property is immediately satisfied as $d\mathcal{T}_2^{I_r^0}/d\Gamma$ coincide with $s^{\mathcal{T}}$ in I_0^r, and the same is true for the equilibrium utilities. □

E Additional Numerical Results for Section 8.3

Policy structure. Figure 15 depicts the core spatial thresholds characterizing the optimal pricing policy and the myopic price response as the supply conditions θ_1 change (on the y-axis). In particular, we track the changes in X_r, β_p, β_c and Y_r for the optimal solution (cf. Theorem 3) and the changes in X_r and X_r^0 for the local price response (cf. Proposition 4).

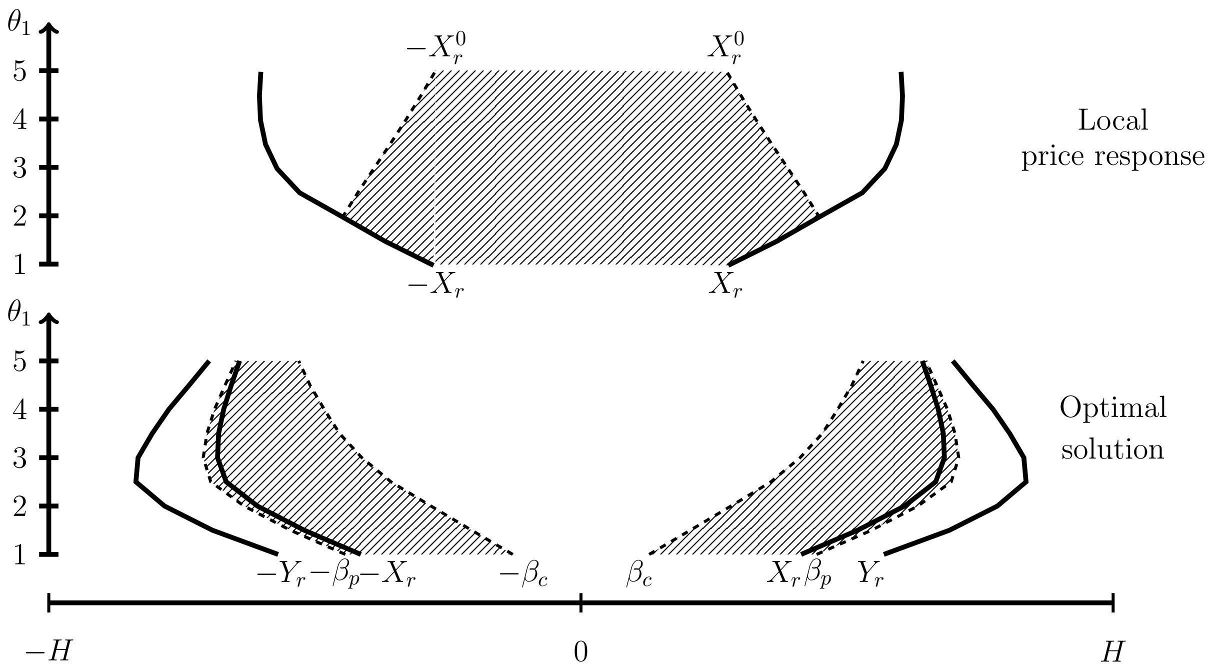

Figure 15: **Policy structure.** Spatial thresholds characterizing the optimal pricing policy and the local price response as the the supply conditions change. The shaded regions have no supply in equilibrium. The figure assumes $\lambda_0 = 9$ and $\lambda_1 = 4$.

The first thing to note is that the structure of supply in the attraction region of 0 differs significantly between the local price response and the optimal policy. In the local price response, there are no drivers who stay put around the origin; and post-relocation, drivers are either at the origin or in $[X_r^{0,\mathrm{lr}}, X_r^{\mathtt{my}}]$. In contrast for the the optimal policy, there are no drivers in a region separated from the origin $[\beta_c, X_r^{\mathtt{opt}}]$ but there are drivers in $[0, \beta_c]$. This contrast can be better understood through the reformulation of the objective in Proposition 1, in conjunction with the shape of the equilibrium utility function in the attraction region of 0 . Given the objective, the platform would ideally like to have supply as close to the origin as possible (subject to the congestion bound constraint) as it maximizes the integral of $V(x) \cdot s^{\mathcal{T}}(x)$. With a local price response, as a result of the lack of flexibility in setting prices throughout the city, the platform is unable to "optimize" the supply in the attraction region and ends up with drivers at locations with low V in $[X_r^{0,\mathrm{lr}}, X_r^{\mathtt{my}}]$ while locations with higher V's have no drivers in $(0, X_r^{\mathtt{my}}]$. Meanwhile, the optimal policy is able to set prices so as to induces the best possible distribution of supply in the attraction region.

In the periphery of the optimal solution, which is outside the origin's attraction region under the optimal pricing policy, the local price response behaves exactly as in the pre-demand shock environment. In stark contrast, the optimal solution incentivizes movement of drivers from the periphery away from the demand shock. In particular, the region $[X_r, Y_r]$, which has a non-trivial size, is artificially damaged. This region is needed for the optimal solution to steer more drivers towards the origin.

Welfare Implications. The revenue improvement of the optimal solutions relies on creating a special region in which drivers' utilities are below of what they could earn if the platform responded only locally to the demand shock. This raises the question of whether revenue-optimal pricing leads to lower or higher surpluses for drivers and consumers compared to the benchmark solution.

The social welfare (SW) equals the sum of the platform's revenue, and the driver (DS) and consumer surpluses (CS), as given by

$$DS = \int_{\mathcal{C}_{\text{diff}}} V(x)\, d\Theta(x), \quad CS = \int_{\mathcal{C}_{\text{diff}}} \mathbb{E}[(v - p(x))|v \geq p(x)] \cdot \min\left\{s^{\mathcal{T}}(x), \lambda_x \cdot \overline{F}(p(x))\right\} d\Gamma(x).$$

Driver surplus corresponds to nothing more than the integral of driver equilibrium utilities across all locations in $\mathcal{C}_{\text{diff}}$. Similarly, consumer surplus corresponds to the gains enjoyed across $\mathcal{C}_{\text{diff}}$ by all those consumers who are willing to pay and are matched to some driver.

In Table 3 we display the percentage differences of driver and consumer surpluses, as well as social welfare between the optimal and benchmark solutions. We note that there are instances where the optimal solution is a Pareto improvement over the myopic price response, in the sense that it is better for the platform, drivers and consumers. There are also instances where the platform's revenue gain is at the expense of both drivers and consumers.

θ_1		1	1.5	2	2.5	3	3.5	4	4.5	5
	$\lambda_0 = 3$	-0.67	3.09	11.3	13.64	14.6	12.44	10.00	7.53	4.92
DS	$\lambda_0 = 6$	-4.15	-3.99	-1.62	-2.01	-0.82	0.74	3.00	5.35	7.80
	$\lambda_0 = 9$	-6.22	-7.35	-7.48	-9.45	-9.72	-9.02	-8.14	-6.36	-4.32
	$\lambda_0 = 3$	-10.96	-14.1	-18.48	-7.24	-3.15	-0.44	1.01	1.57	1.58
CS	$\lambda_0 = 6$	-12.03	-10.58	-17.15	-6.32	1.18	4.18	4.24	2.85	0.69
	$\lambda_0 = 9$	-14.33	-11.94	-22.43	-12.58	-1.39	5.77	9.73	10.98	10.44
	$\lambda_0 = 3$	-1.04	0.81	4.26	8.28	9.70	8.83	7.44	5.8	3.96
SW	$\lambda_0 = 6$	-3.60	-3.56	-3.49	-1.05	1.50	3.16	4.43	5.29	5.87
	$\lambda_0 = 9$	-5.24	-5.95	-8.16	-6.84	-4.40	-2.32	-0.86	0.51	1.58

Table 3: Driver surplus, consumer surplus and social welfare difference (in %) of optimal solution over myopic price response in $\mathcal{C}_{\text{diff}}$.

For a given level of supply, the driver surplus degrades with respect to the benchmark as the demand shock becomes more intense. We also find that, independently of the size of the demand shock, the optimal solution performs better than the benchmark in terms of consumer surplus when the supply level is high. More drivers in the city imply more matches and lower prices and, thus, higher consumer surplus.