# Optimal Dynamic Mechanism Design and the Virtual Pivot Mechanism

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# Appendix

## Appendix A: Proofs for Section 3

**Lemma A.1** *For any reporting strategy $y \to z$ and initial type $x$, the partial derivative of the expected value of agent $i$ $V_i^{y\to z}(x)$ (see definition in Eq. (10)) with respect to $x$ exists and is:*

$$\frac{\partial V^{y\to z}(x)}{\partial x} = \mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t \frac{\partial}{\partial s_{i,0}} v_{i,t}(a^t, s_{i,0}, s_{i,1}, ..., s_{i,t}) \;\Big|_{s_{i,0}=x}\right]$$

*(where the expectation is under $y \to z$ and $\mathcal{T}_{-i}$). Furthermore, it is bounded by*

$$\left|\frac{\partial V_i^{y\to z}(x)}{\partial x}\right| \le \frac{\bar{V}}{1-\delta}.$$

*Proof:* From Assumption 2.2, we have that for all $i$, $t$, $a$, $x$ and $s_{i,1}, ..., s_{i,t}$,

$$\left|\frac{\partial}{\partial x} v_{i,t}(a^t, x, s_{i,1}, ..., s_{i,t})\right| \le \bar{V} < \infty.$$

Therefore, by Lebesgue's Dominated Convergence Theorem, the partial derivative $\frac{\partial \bar{V}^{y\to z}(x)}{\partial x}$ exists,

$$\frac{\partial V^{y\to z}(x)}{\partial x} = \frac{\partial}{\partial x}\mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t v_{i,t}(a^t, x, s_{i,1}, ..., s_{i,t})\right] = \mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t \frac{\partial}{\partial x} v_{i,t}(a^t, x, s_{i,1}, ..., s_{i,t})\right]$$

and $|\frac{\partial V^{y\to z}(x)}{\partial x}| \le \mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t \bar{V}\right] = \frac{\bar{V}}{1-\delta}$.

*Proof of Lemma 3.1* Any strategy available to the agents in the relaxed environment is a feasible strategy in the dynamic environment. Therefore, if all other agents are truthful, any profitable deviation from the truthful strategy in the relaxed environment implies a profitable deviation in the dynamic environment. Since no such profitable deviations exist in the dynamic environment, we obtain that the mechanism $\mathcal{M}$ is incentive compatible in the relaxed environment. Therefore, the optimal revenue in the relaxed environment provides an upper bound on the revenue in the dynamic environment.

*Proof of Lemma 3.2* For consistency with the notation used in the rest of the paper, we represent the utility of agent $i$ with initial type $s_{i,0} = z'$ and reporting his initial type as $\hat{s}_{i,0} = z$ by $U_i^{z\to z}(z')$, assuming all other agents are truthful. Respectively, $V_i^{z\to z}(z')$ and $P_i^{z\to z}(z')$ represent the expected discounted value and payment of agent $i$ under initial type $z'$ and reported initial type $z'$.

The expected utility of agent $i$ under reporting strategy $z \to z$ and initial type $x$ is

$$U_i^{z\to z}(z) = V_i^{z\to z}(z) - P_i^{z\to z}(z). \tag{27}$$

Under the same reporting strategy $z \to z$, but under initial type $z'$, the utility of agent $i$ is

$$U_i^{z\to z}(z') = V_i^{z\to z}(z') - P_i^{z\to z}(z'). \tag{28}$$

The payments are functions only of reported types, not true types, and therefore, $P_i^{z\to z}(z) = P_i^{z\to z}(z')$. Therefore, for any $z \neq z'$, combining Eqs. (27) and (28) yields

$$\frac{U_i^{z\to z}(z) - U_i^{z\to z}(z')}{z - z'} = \frac{V_i^{z\to z}(z) - V_i^{z\to z}(z')}{z - z'}.$$

At the same time, if $z > z'$, incentive compatibility yields $U_i^{z'\to z'}(z') \geq U_i^{z\to z}(z')$, hence

$$\frac{U_i^{z\to z}(z) - U_i^{z'\to z'}(z')}{z - z'} \leq \frac{U_i^{z\to z}(z) - U_i^{z\to z}(z')}{z - z'}.$$

Since the partial derivative $\frac{\partial V^{z\to z}(x)}{\partial x}$ exists for all $x$ (see Lemma A.1), we can take the limit as $z' \uparrow z$ and obtain that the left-hand side derivative of $U_i^{z\to z}(z)$ satisfies

$$\frac{d_- U_i^{z\to z}(z)}{dz} \leq \frac{\partial V_i^{z\to z}(s)}{\partial s}\Big|_{s=z}.$$

Using the same argument for $z' > z$, we obtain that the right-hand side derivative of $U_i^{z\to z}(z)$ satisfies

$$\frac{d_+ U_i^{z\to z}(z)}{dz} \geq \frac{\partial V_i^{z\to z}(s)}{\partial s}\Big|_{s=z}.$$

Since $|\frac{\partial V_i^{z\to z}(s)}{\partial s}|$ is bounded by $\frac{\bar{V}}{1-\delta}$ by Lemma A.1, we get that the absolute value of both the left-hand and right-hand side derivatives of $U_i^{z\to z}(z)$ are also bounded by $\frac{\bar{V}}{1-\delta}$. The function $U_i^{z\to z}(z)$ is, therefore, $\frac{\bar{V}}{1-\delta}$-Lipschitz-continuous and, thus, differentiable almost everywhere. At all points where the derivative exists, $\frac{dU_i^{z\to z}(z)}{dz} = \frac{\partial V_i^{z\to z}(s)}{\partial s}\Big|_{s=z}$. Therefore, the envelope condition follows:

$$U_i^{x\to x}(x) - U_i^{x'\to x'}(x') = \int_{x'}^{x} \frac{dU_i^{z\to z}(z)}{dz} dz = \int_{x'}^{x} \frac{\partial V_i^{z\to z}(s)}{\partial s}\Big|_{s=z} dz. \tag{29}$$

Plugging in the result from Lemma A.1, we obtain the desired result.

*Proof of Lemma 3.3* For notational convenience, we write:

$$\frac{\partial v_{i,t}(a^t, s_{i,0}, s_{i,1}, ..., s_{i,t})}{\partial s_{i,0}} \;\Big|_{s_{i,0}=s_{i,0}} = \frac{\partial v_{i,t}(a^t, s_i^t)}{\partial s_{i,0}}$$

where the $s_i^t$ implicitly depends on the first signal.

Consider first the utility $U_i^{\mathcal{M}}(s)$ of an agent $i$ under an initial type profile $s$, which is given by

$$U_i^{\mathcal{M}}(s_{i,0}, s_{-i,0}) - U_i^{\mathcal{M}}(0, s_{-i,0}) = \int_0^{s_{i,0}} \mathbb{E}\left[\sum_{t=1}^{\infty} \delta^t \frac{\partial v_{i,t}(a^t, s_i^t)}{\partial s_{i,0}} \middle| s_{i,0} = z, s_{-i,0}\right] dz.$$

from Lemma 3.2. Taking the expectation of this term over all possible first period signals $s_{1,0}, ..., s_{n,0}$, we obtain

$$\mathbb{E}[U_i^{\mathcal{M}}(s_{i,0}, s_{-i,0}) - U_i^{\mathcal{M}}(0, s_{-i,0})] = \int_0^1 \left(\int_0^{s_{i,0}} \mathbb{E}\left[\sum_{t=1}^{\infty} \delta^t \frac{\partial v_{i,t}(a^t, s_i^t)}{\partial s_{i,0}} \middle| s_{i,0} = z\right] dz\right) f_i(s_{i,0}) ds_{i,0}.$$

Inverting the order of integration,

$$\begin{aligned}\mathbb{E}[U_i^{\mathcal{M}}(s_{i,0},s_{-i,0})-U_i^{\mathcal{M}}(0,s_{-i,0})] &= \int_0^1\int_z^1 \mathbb{E}\left[\sum_{t=1}^{\infty}\delta^t \frac{\partial v_{i,t}(a^t,s_i^t)}{\partial s_{i,0}}\middle| s_{i,0}=z\right] f_i(s_{i,0})ds_{i,0}dz \\ &= \int_0^1 \mathbb{E}\left[\sum_{t=1}^{\infty}\delta^t \frac{\partial v_{i,t}(a^t,s_i^t)}{\partial s_{i,0}}\middle| s_{i,0}=z\right](1-F_i(z))dz.\end{aligned}$$

By multiplying and dividing the right-hand side of the equation above by the density $f_i(z)$ we obtain an unconditional expectation,

$$\mathbb{E}[U_i^{\mathcal{M}}(s_{i,0},s_{-i,0})-U_i^{\mathcal{M}}(0,s_{-i,0})]=\mathbb{E}\left[\sum_{t=1}^{\infty}\delta^t \frac{1-F_i(s_{i,0})}{f_i(s_{i,0})}\frac{\partial v_{i,t}(a^t,s_i^t)}{\partial s_{i,0}}\right].$$

Now note that the discounted sum of payments $\mathbb{E}[\sum_{t=1}^{\infty}\delta^t p_{i,t}]$ is equal to the expected discounted valuation of agent $i$ – $\mathbb{E}[\sum_{t=1}^{\infty}\delta^t v_{i,t}(a^t,s_i^t)]$ – minus her utility, which yields the claim.

*Proof of Lemma 3.4* Observe that for multiplicatively-separable value functions

$$\frac{\partial v_{i,t}(a^t,s_i^t)}{\partial s_{i,0}} = A_i'(s_{i,0})B_{i,t}(a^t,s_{i,1},...,s_{i,t})$$

and, therefore, Eqs. (7) and (6) are identical. Similarly, for additively-separable functions,

$$\frac{\partial v_{i,t}(a^t,s_i^t)}{\partial s_{i,0}} = A_i'(s_{i,0})C_i(a_t)$$

and, therefore, Eqs. (7) and (6) are again identical.

*Proof of Corollary 3.1* For an IC mechanism $\mathcal{M}$, the expected discounted sum of payments by agent $i$ is equal to

$$\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t p_{i,t}\right]=\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t\big(\alpha_i(s_{i,0})v_{i,t}(a^t,s_i^t)+\beta_{i,t}(a^t,s_{i,0})\big)\right]-\mathbb{E}\left[U_i^{\mathcal{M},\mathcal{T}}(s_{i,0}=0)\right]$$

by taking expectations over $s_{-i,0}$ (see Eq. (7)). Since the mechanism satisfies IR, $\mathbb{E}\left[U_i^{\mathcal{M},\mathcal{T}}(s_{i,0}=0)\right]\geq 0$ and, therefore,

$$\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t p_{i,t}\right]\leq\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t\big(\alpha_i(s_{i,0})v_{i,t}(a^t,s_i^t)+\beta_{i,t}(a^t,s_{i,0})\big)\right].$$

The profit of $\mathcal{M}$ is given by the sum of payments minus the cost of actions (see Eq. (4)),

$$\text{Profit}^{\mathcal{M}}\leq\mathbb{E}\left[\sum_{t=1}^{\infty}\delta^t\left(\sum_{i=1}^{n}\Big(\alpha_i(s_{i,0})v_{i,t}(a^t,s_i^t)+\beta_{i,t}(a^t,s_{i,0})\Big)-c_t(a^t)\right)\right].$$

The bound above is valid for all IC and IR mechanisms. By maximizing over the set of all possible allocation rules (payment rules do not enter the equation above), we obtain the desired result.

*Proof of Lemma 3.5* The expected utility of agent $i$ under reporting strategy $x' \to z$ and initial type $z$ is

$$U_i^{x'\to z}(z) = V_i^{x'\to z}(z) - P_i^{x'\to z}(z), \tag{30}$$

where $P_i^{x'\to z}(z)$ is the expected discounted sum of payments of agent $i$ under reporting strategy $x' \to z$ and initial type $z$ (see similar definitions of $U_i^{x'\to z}(z)$ and $V_i^{x'\to z}(z)$ in Eqs. (9) and (10)). Under the same reporting strategy $x' \to z$, but under initial type $z'$, the utility of agent $i$ is

$$U_i^{x'\to z}(z') = V_i^{x'\to z}(z') - P_i^{x'\to z}(z'). \tag{31}$$

The payments are functions only of reported types, not true types, and therefore, $P_i^{x'\to z}(z) = P_i^{x'\to z}(z')$. Therefore, for any $z \neq z'$, combining Eqs. (30) and (31) yields

$$\frac{U_i^{x'\to z}(z) - U_i^{x'\to z}(z')}{z - z'} = \frac{V_i^{x'\to z}(z) - V_i^{x'\to z}(z')}{z - z'}.$$

Periodic ex-post IC guarantees that $U_i^{x'\to z'}(z') \geq U_i^{x'\to z}(z')$. Therefore, for any $z > z'$,

$$\frac{U_i^{x'\to z}(z) - U_i^{x'\to z'}(z')}{z - z'} \leq \frac{U_i^{x'\to z}(z) - U_i^{x'\to z}(z')}{z - z'}.$$

Since the partial derivative $\frac{\partial V^{z\to z}(x)}{\partial x}$ exists for all $x$ (see Lemma A.1), we can take the limit as $z' \uparrow z$ and obtain that the left-hand side derivative of $U_i^{x'\to z}(z)$ for any constant $x'$ satisfies

$$\frac{d_- U_i^{x'\to z}(z)}{dz} \leq \frac{\partial V_i^{x'\to z}(s)}{\partial s}\Big|_{s=z}.$$

Using the same argument for $z' > z$, we obtain that the right-hand side derivative of $U_i^{x'\to z}(z)$ satisfies

$$\frac{d_+ U_i^{x'\to z}(z)}{dz} \geq \frac{\partial V_i^{x'\to z}(s)}{\partial s}\Big|_{s=z}.$$

Since $|\frac{\partial V_i^{x'\to z}(s)}{\partial s}|$ is bounded by $\frac{\bar{V}}{1-\delta}$ by Lemma A.1, we get that the absolute value of both the left-hand and right-hand side derivatives of $U_i^{x'\to z}(z)$ are also bounded by $\frac{\bar{V}}{1-\delta}$. The function $U_i^{x'\to z}(z)$ is, therefore, $\frac{\bar{V}}{1-\delta}$-Lipschitz-continuous and, thus, differentiable almost everywhere. At all points where the derivative exists, $\frac{dU_i^{x'\to z}(z)}{dz} = \frac{\partial V_i^{x'\to z}(s)}{\partial s}\Big|_{s=z}$. Therefore, the envelope condition follows:

$$U_i^{x'\to x}(x) - U_i^{x'\to x'}(x') = \int_{x'}^{x} \frac{dU_i^{x'\to z}(z)}{dz} dz = \int_{x'}^{x} \frac{\partial V_i^{x'\to z}(s)}{\partial s}\Big|_{s=z} dz.$$

*Proof of Lemma 3.6* The envelope condition from the relaxed environment (see Lemma 3.2) also applies to this setting since a deviation that is feasible in the relaxed environment (that is, using reporting strategy $z \to z$ for an initial type $z'$) is also feasible in the dynamic environment. Therefore, if the mechanism is incentive compatible, then it satisfies Eq. (29), which is identical to Eq. (15).

To see that IC implies the dynamic monotonicity condition in Eq. (16), simply note that IC is equivalent to Eq. (11) and Eqs. (15) and (13) are respectively equal to the left-hand and the right-hand side of Eq. (11). We thus obtain that IC implies Eq. (16).

We now show that if both Eqs. (15) and (16) hold, then the mechanism is IC. If both equations hold, then for all $x$ and $x'$,

$$U_i^{x\to x}(x) - U_i^{x'\to x'}(x') = \int_{x'}^{x} \frac{\partial V_i^{z\to z}(s)}{\partial s}\Big|_{s=z} dz \geq \int_{x'}^{x} \frac{\partial V_i^{x'\to z}(s)}{\partial s}\Big|_{s=z} dz = U_i^{x'\to x}(x) - U_i^{x'\to x'}(x'),$$

where the last equality follows from Lemma 3.5. The equation above is equivalent to IC (see Eq. (11)), when the mechanism is periodic ex-post IC for $t \geq 1$.

## Appendix B: Proofs for Section 4

**Lemma B.1** *Suppose Assumptions 4.1 and 4.2 hold. Then $\alpha_i$ is strictly increasing for multiplicatively separable functions and $\beta_{i,t}$ is strictly increasing for additively separable functions.*

*Proof:* For simplicity of nation, let $s = s_{i,0}$. Also, let $\eta_i(s)$ denote the hazard rate, i.e.,

$$\eta_i(s) = \frac{f_i(s)}{1 - F_i(s)}.$$

In the additive case,

$$\frac{\partial \beta_{i,t}(a^t, s)}{\partial s} = \frac{\eta_i'(s)}{\eta_i^2(s)} A_i'(s) C_{i,t}(a^t) - \frac{1}{\eta_i(s)} A_i''(s) C_{i,t}(a^t)$$

where $(\cdot)'$ denotes a partial derivative with respect to $s$. By the assumptions that $A_i$ is concave and strictly increasing, and the hazard rate is positive and strictly increasing, we have that the above has the same sign as $C_{i,t}$. In the multiplicative case, first note that $\alpha_i(s) = 1 - \frac{1}{\eta_i(s)} (\log A_i(s))'$. Therefore,

$$\alpha_i'(s) = \frac{\eta_i'(s)}{\eta_i^2} \frac{A_i'(s)}{A_i(s)} - \frac{1}{\eta_i(s)} (\log A_i(s))''$$

which is positive by the assumption.

*Proof of Lemma 4.1* If agents are truthful, by Eq. (24), the expected payment of each agent $i$ given $s_{i,0}$ is equal to $\max\{p_i^\star(s_{i,0}), 0\}$, where 0 occurs if agent $i$ is excluded from the system ($i \notin a_i$). Namely,

$$p_i^\star(\hat{s}_0) = V(s_i^t) - \int_0^{\hat{s}_{i,0}} \frac{\partial V_i^{z \to z}(s_{i,0}, \hat{s}_{-i,0})}{\partial s_{i,0}} \Big|_{s_{i,0}=z} dz \tag{32}$$

where

$$\frac{\partial V_i^{z \to z}(s_{i,0}, \hat{s}_{-i,0})}{\partial s_{i,0}} \Big|_{s_{i,0}=z} = \mathbb{E}\left[ \sum_{t=1}^{\infty} \delta^t \frac{\partial v_{i,t}(q^{\star t}, s_{i,0}, s_{i,1}, \ldots s_{i,t})}{\partial s_{i,0}} |_{s_{i,0}=z} \middle| s_{i,0} = z, s_{-i,0} = \hat{s}_{-i,0} \right]$$

For notational convenience, we write:

$$\frac{\partial v_{i,t}(a^t, s_{i,0}, s_{i,1}, ..., s_{i,t})}{\partial s_{i,0}} \ |_{s_{i,0}=s_{i,0}} = \frac{\partial v_{i,t}(a^t, s_i^t)}{\partial s_{i,0}}$$

where the $s_i^t$ implicitly depends on the first signal. The expected payment of agent $i$ is equal to:

$$\int_0^1 \max\{p_i^\star(s, s_{0,-i}), 0\} f_i(s) ds$$
$$= \int_0^1 \left( \mathbb{E}\left[ \sum_{t=1}^{\infty} \delta^t v_{i,t}(q^{\star t}, s_i^t) \middle| s_{i,0} = s, s_{-i,0} \right] - \int_0^s \mathbb{E}\left[ \sum_{t=1}^{\infty} \delta^t \frac{\partial v_{i,t}(q^{\star t}, s_i^t)}{\partial s_{i,0}} \middle| s_{i,0} = z, s_{-i,0} \right] dz \right) f_i(s) ds,$$

where we can drop the max with zero since the agent obtains value zero at all periods when she is excluded from the system. By changing the order of integration, we have

$$\int_0^1 \max\{p_i^\star(s, s_{0,-i}), 0\} f_i(s) ds$$

$$
\begin{aligned}
&= \int_0^1 \left( \mathbb{E}\left[ \sum_{t=1}^{\infty} \delta^t \left( v_{i,t}(q^{\star t}, s_i^t) - \frac{1-F_i(s)}{f_i(s)} \frac{\partial v_{i,t}(q^{\star t}, s_i^t)}{\partial s_{i,0}} \right) \middle| s_{i,0} = s, s_{-i,0} \right] \right) f_i(s) ds \\
&= \int_0^1 \left( \mathbb{E}\left[ \sum_{t=1}^{\infty} \delta^t \left( \alpha_i(s_{i,0}) v_{i,t}(q^{\star t}, s_i^t) + \beta_{i,t}(q^{\star t}, s_{i,0}) \right) \middle| s_{i,0} = s, s_{-i,0} \right] \right) f_i(s) ds \\
&= \mathbb{E}\left[ \sum_{t=1}^{\infty} \delta^t \left( \alpha_i(s_{i,0}) v_{i,t}(q^{\star t}, s_i^t) + \beta_{i,t}(q^{\star t}, s_{i,0}) \right) \middle| s_{-i,0} \right] \qquad (33)
\end{aligned}
$$

Therefore, the profit of the mechanism matches the upper-bound provided in Corollary 3.1. Hence, to prove the optimality, it suffices to show that the mechanism is individually rational. By construction, we have the utility of agent $i$ equal to 0 if $s_{i,0}=0$ for any $s_{-i,0}$. Therefore,

$$
U_i(s_0) = \int_0^{s_{i,0}} \mathbb{E}\left[ \sum_{t=1}^{\infty} \delta^t \frac{\partial v_{i,t}(q^{\star t}, s_{i,0}, s_{i,1}, \ldots s_{i,t})}{\partial s_{i,0}} |_{s_{i,0}=z} \middle| s_{i,0} = z, s_{-i,0} \right] dz.
$$

By Assumption 4.2, $\frac{\partial v_{i,t}(q^{\star t}, s_{i,0}, s_{i,1}, \ldots s_{i,t})}{\partial s_{i,0}}$ is non-negative. Hence, the mechanism is individually rational. Precisely, periodic ex-post IR at time 0.

*Proof of Lemma 4.2* Define $u_{i,t}$ to be the instantaneous utility of agent $i$ at time $t$. We get

$$
\begin{aligned}
u_{i,t} &= v_{i,t}(a^{\star t}, s_i^t) - p_{i,t} \\
&= \left( v_{i,t}(a^{\star t}, s_i^t) - v_{i,t}(a^{\star t}, \hat{s}_i^t) \right) + \frac{m_{i,t}}{\hat{\alpha}_i} \\
&= v_{i,t}(a^{\star t}, s_i^t) + \frac{\hat{\beta}_i(a^{\star t})}{\hat{\alpha}_i} \\
&\quad + \frac{1}{\hat{\alpha}_i} \left( \sum_{j \neq i} \left( \hat{\alpha}_j v_{j,t}(a^{\star t}, \hat{s}^{j,t}) \right) - c_t(a^{\star t}) - W_{-i}^{(\hat{\alpha},\hat{\beta})}(a^{\star t-1}, \hat{s}^t) + \delta \mathbb{E}\left[ W_{-i}^{(\hat{\alpha},\hat{\beta})}(a^{\star t}_{-i}, \hat{s}^{t+1}) \right] \right)
\end{aligned}
$$

The last equality follows from Eq. (21). We dropped the conditioning of $W_{-i}^{(\hat{\alpha},\hat{\beta})}(a^{\star t}, \hat{s}^{t+1})$ on $s^t = \hat{s}_t$, $a^{\star t}$, and $a^{\star}_{-i,t}$, as it is clear from the context. For ease of notation, let $s = s_0$. Because all agents except $i$ are truthful, we have

$$
\begin{aligned}
u_{it} = \frac{1}{\hat{\alpha}_i} \Bigg( & \sum_{j=1}^{n} \left( \hat{\alpha}_j v_{j,t}(a^{\star t}, s_j^t) + \hat{\beta}_j(a^{\star t}) \right) - c_t(a^{\star t}) \\
& - W_{-i}^{(\alpha(s),\beta(s))}(a^{\star t-1}, s^t) + \delta \mathbb{E}\left[ W_{-i}^{(\alpha(s),\beta(s))}(a^{\star t}_{-i}, s^{t+1}) \right] \Bigg)
\end{aligned}
$$

If agent $i$ is truthful and other agents are truthful, we have

$$
\sum_{t'=t}^{\infty} \delta^t u_{it'} = \frac{1}{\hat{\alpha}_i} \left( W^{(\alpha(s),\beta(s))}(a^{\star t-1}, s^t) - W_{-i}^{(\alpha(s),\beta(s))}(a^{\star t-1}, s^t) \right)
$$

Hence, the allocation rule is aligned with the incentive of agent $i$. She can maximize her utility by reporting truthfully.

Observe that agents with $\hat{\alpha}_i \leq 0$ would have been excluded. Hence, we have $\sum_{t'=t}^{\infty} u_{it'} \geq 0$. Therefore, the mechanism is periodic ex-post IR.

*Proof of Lemma 4.3* Observe that Eq. (15) is followed from Lemma 4.1 and Eq. (33). To establish Eq. (16), we show that the inequality holds point-wise, i.e., if $x \geq x'$, then

$$\frac{\partial V_i^{x_i \to x_i}(s)}{\partial s}\Big|_{s=x_i} \geq \frac{\partial V_i^{x'_i \to x_i}(s)}{\partial s}\Big|_{s=x_i} \tag{34}$$

By Eq. (14), this is equivalent to

$$\mathbb{E}_{x_i \to x_i}\left[\sum_{t=0}^{\infty} \delta^t \frac{\partial v_{i,t}(a^{\star t}, s_i^t)}{\partial s_{i,0}} \,\Big|_{s_{i,0}=s} \middle| s_{i,0} = x_i\right] \geq \mathbb{E}_{x'_i \to x_i}\left[\sum_{t=0}^{\infty} \delta^t \frac{\partial v_{i,t}(a'^t, s_i^t)}{\partial s_{i,0}} \,\Big|_{s_{i,0}=s} \middle| s_{i,0} = x_i\right] \tag{35}$$

where $\mathbb{E}_{x_i \to x_i}$ is the expectation under the stochastic process determined by agent $i$ reporting according to $x_i \to x_i$ (while other agents are truthful) and $a^{\star t}$ represents the allocation at time $t$ in this case. Similarly, for reporting strategy $x'_i \to x_i$, we use the notation $\mathbb{E}_{x'_i \to x_i}$ and represent the allocation at time $t$ by $a'^t$.

Recall that we have:

$$v_{i,t}(a^t, s_i^t) - \frac{1 - F_i(s_{i,0})}{f_i(s_{i,0})} \frac{\partial v_{i,t}(a^t, s_i^t)}{\partial s_{i,0}} = \alpha_i(s_{i,0}) v_{i,t}(a^t, s_i^t) + \beta_{i,t}(a^t, s_{i,0})$$

Hence, we get

$$\frac{\partial v_{i,t}(a^t, s_i^t)}{\partial s_{i,0}} = \frac{f_i(s_{i,0})}{1 - F_i(s_{i,0})}\big((1 - \alpha_i(s_{i,0})) v_{i,t}(a^t, s_i^t) - \beta_{i,t}(a^t, s_i^t)\big) \tag{36}$$

Therefore, by Eq. (36), the inequality below is equivalent to the desired equation, Eq. (34):

$$\begin{aligned} &\mathbb{E}_{x_i \to x_i}\left[\sum_{t=1}^{\infty} \delta^t \left((1 - \alpha_i(x_i)) v_{i,t}(a^t, s_i^t) - \beta_{i,t}(a^t, x_i)\right)\right] \\ &\geq \mathbb{E}_{x'_i \to x_i}\left[\sum_{t=1}^{\infty} \delta^t \Big((1 - \alpha_i(x_i)) v_{i,t}(a'^t, s_i^t) - \beta_{i,t}(a'^t, x_i)\Big)\right] \end{aligned} \tag{37}$$

In the following we prove the inequality above. For $k \neq i$, define $x_k$ and $x'_k$ to be equal $s_{k,0}$. Because $a^\star$ and $a'$ are optimal allocation rules with respect to $(\alpha(x), \beta(x))$ and $(\alpha(x'), \beta(x'))$, we have:

$$\begin{aligned} &\mathbb{E}_{x_i \to x_i}\left[\sum_{t=1}^{\infty} \delta^t \left(\sum_{j=1}^{n} \big(\alpha_j(x_j) v_{j,t}(a^{\star t}, s_j^t) + \beta_{j,t}(a^{\star t}, x_j)\big) - c_t(a^{\star t})\right)\right] \\ \geq\ &\mathbb{E}_{x'_i \to x_i}\left[\sum_{t=1}^{\infty} \delta^t \left(\sum_{j=1}^{n} \big(\alpha_j(x_j) v_{j,t}(a'^t, s_j^t) + \beta_{j,t}(a'^t, x_j)\big) - c_t(a'^t)\right)\right] \end{aligned}$$

and similarly

$$\begin{aligned} &\mathbb{E}_{x_i \to x_i}\left[\sum_{t=1}^{\infty} \delta^t \left(\sum_{j=1}^{n} \big(\alpha_j(x'_j) v_{j,t}(a^{\star t}, s_j^t) + \beta_{j,t}(a^{\star t}, x_j)\big) - c_t(a^{\star t})\right)\right] \\ \leq\ &\mathbb{E}_{x'_i \to x_i}\left[\sum_{t=1}^{\infty} \delta^t \left(\sum_{j=1}^{n} \big(\alpha_j(x'_j) v_{j,t}(a'^t, s_j^t) + \beta_{j,t}(a'^t, x'_j)\big) - c_t(a'^t)\right)\right] \end{aligned}$$

Subtracting these inequalities we get:

$$\mathbb{E}_{x_i \to x_i}\left[\sum_{t=1}^{\infty}\delta^t \sum_{j=1}^{n}\Big((\alpha_j(x_j)-\alpha_j(x'_j))v_{j,t}(a^{\star t}, s_j^t) + (\beta_{j,t}(a^{\star t}, x_j) - \beta_{j,t}(a^{\star t}, x'_j))\Big)\right]$$
$$\geq \mathbb{E}_{x'_i \to x_i}\left[\sum_{t=1}^{\infty}\delta^t \sum_{j=1}^{n}\Big((\alpha_j(x_j)-\alpha_j(x'_j))v_{j,t}(a'^t, s_j^t) + (\beta_{j,t}(a'^t, x_j) - \beta_{j,t}(a'^t, x'_j))\Big)\right]$$

Because for $k \neq i$, agents are truthful and $x'_k = x_k$, we have

$$\mathbb{E}_{x_i \to x_i}\left[\sum_{t=1}^{\infty}\delta^t \Big((\alpha_i(x_i)-\alpha_i(x'_i))v_{i,t}(a^{\star t}, s_i^t) + (\beta_{i,t}(a^{\star t}, x_i) - \beta_{i,t}(a^{\star t}, x'_i))\Big)\right] \quad (38)$$
$$\geq \mathbb{E}_{x'_i \to x_i}\left[\sum_{t=1}^{\infty}\delta^t \Big((\alpha_i(x_i)-\alpha_i(x'_i))v_{i,t}(a'^t, s_i^t) + (\beta_{i,t}(a'^t, x_i) - \beta_{i,t}(a'^t, x'_i))\Big)\right]$$

Now suppose $v_i$ is multiplicative separable (i.e., $\beta_{i,t}(\cdot,\cdot)=0$) and Assumption 4.2 holds — we consider the additive valuations later. Because $x \geq x'$, by Assumption 4.2 and Lemma B.1, we have $\alpha_i(x_i) > \alpha_i(x'_i)$; moreover $\alpha_i(x_i)$ is less than 1 for $x \in [0,1)$. Multiplying both sides of the inequality above by $\frac{1-\alpha_i(x_i)}{\alpha_i(x_i)-\alpha_i(x'_i)}$, yields the following:

$$\mathbb{E}_{x_i \to x_i}\left[\sum_{t=1}^{\infty}\delta^t (1-\alpha_i(x_i))v_{i,t}(a^{\star t}, s_i^t)\right] \geq \mathbb{E}_{x'_i \to x_i}\left[\sum_{t=1}^{\infty}\delta^t (1-\alpha_i(x_i))v_{i,t}(a'^t, s_i^t)\right]$$

which is equivalent to Eq. (37) for multiplicative-separable valuations.

Now consider the case of additive-separable value functions. We have $\alpha_i(x) = \alpha_i(x') = 1$. Plugging into Eq. (38) we get

$$\mathbb{E}_{x_i \to x_i}\left[\sum_{t=1}^{\infty}\delta^t (\beta_{i,t}(a^{\star t}, x_i) - \beta_{i,t}(a^{\star t}, x'_i))\right] \geq \mathbb{E}_{x'_i \to x_i}\left[\sum_{t=1}^{\infty}\delta^t (\beta_{i,t}(a'^t, x_i) - \beta_{i,t}(a'^t, x'_i))\right]$$

Recall that $\beta_{i,t}(a^t, x_i) = -\frac{1-F_i(x_i)}{f_i(x_i)}A'_i(x_i)C_{i,t}(a^t)$. Because $x \geq x'$, by Assumption 4.2 and Lemma B.1, we have $-\frac{1-F_i(x_i)}{f_i(x_i)}A'_i(x_i) > -\frac{1-F_i(x'_i)}{f_i(x'_i)}A'_i(x'_i)$. By multiplying both sides of the inequality above by $\frac{\frac{1-F_i(x_i)}{f_i(x_i)}A'_i(x_i)}{-\frac{1-F_i(x_i)}{f_i(x_i)}A'_i(x_i)+\frac{1-F_i(x'_i)}{f_i(x'_i)}A'_i(x'_i)}$, we get:

$$-\mathbb{E}_{x_i \to x_i}\left[\sum_{t=1}^{\infty}\delta^t \beta_{i,t}(a^{\star t}, x_i)\right] \geq -\mathbb{E}_{x'_i \to x_i}\left[\sum_{t=1}^{\infty}\delta^t \beta_{i,t}(a'^t, x'_i)\right]$$

which produces Eq. (37) and, thus, completes the proof.

## Appendix C: Proof for the Single Agent Case

*Proof of Corollary 5.1* Simply note that under the Virtual-Pivot Mechanism, if the agent is allocated the item at any time $t$, the price she pays, under the Virtual-Pivot Mechanism, is not a function of her report at time $t$ (or any report after $t=0$). Furthermore, the prices that the agent is charged at $t \geq 1$ are identical to that in the Virtual-Pivot Mechanism (see Eq. (22)). Also, the prices charged at $t=0$ is identical to that in the Virtual-Pivot Mechanism by construction.