

# Mathematics and Statistics Background

## Detailed outline

1. Basic analysis and probability theory
  - (a) Vector spaces
  - (b) Topologies
  - (c) Topological vector spaces (Separating Hyperplane Theorem)
  - (d) Normed vector spaces
  - (e) Measure spaces and probability spaces
  - (f) Random variables
  - (g) Expectation and the Lebesgue integral
  - (h) Conditional expectation
  - (i) Radon-Nikodym derivative
  - (j)  $L^p$  Spaces (Riesz Representation Theorem)
2. Stochastic processes
  - (a) Filtrations
  - (b) Stochastic processes
  - (c) Stopping times
  - (d) Martingales
  - (e) Brownian motion
  - (f) Stochastic integration (Martingale Representation Theorem)
  - (g) Itô processes (Itô's Lemma)
  - (h) Quadratic Variation and Covariation
  - (i) Girsanov's Theorem

## Readings

This material is taken from the first two chapters of Domenico Cuoco's lecture notes.

For additional references, consult the recommended mathematics and statistics books or Steve Shreve's lecture notes at <http://www.math.cmu.edu/users/shreve/LectureNotes.pdf>.

## Problems

1. Suppose  $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$  and let the  $\sigma$ -field  $\mathcal{F}$  be the set of all subsets of  $\Omega$ . Define the probability measure  $\mathcal{P}$  by  $\mathcal{P}\{\omega_1\} = \mathcal{P}\{\omega_2\} = \mathcal{P}\{\omega_3\} = \mathcal{P}\{\omega_4\} = \mathcal{P}\{\omega_5\} = 1/5$ . Finally, suppose  $X_1, X_2$  are random variables on  $(\Omega, \mathcal{F}, \mathcal{P})$  that take on the following values.

state	$X_1$	$X_2$
$\omega_1$	1	2
$\omega_2$	1	3
$\omega_3$	2	3
$\omega_4$	3	4
$\omega_5$	3	4

- (a) What is the  $\sigma$ -field  $\mathcal{F}_{X_1}$  generated by  $X_1$ ? What is the  $\sigma$ -field  $\mathcal{F}_{X_2}$  generated by  $X_2$ ?

(Formally,  $\mathcal{F}_X$  is defined as

$$\mathcal{F}_X = \{X^{-1}(B) \mid B \in \mathcal{B}\}, \quad (1)$$

where  $\mathcal{B}$  is the Borel  $\sigma$ -field on  $\mathcal{R}$ . That is,  $\mathcal{F}_X$  is the smallest  $\sigma$ -field with respect to which  $X$  is measurable.)

- (b) Specify the values of the random variables  $E\{X_2 \mid \mathcal{F}_{X_1}\}$  and  $E\{X_1 \mid \mathcal{F}_{X_2}\}$ .
- (c) Suppose  $X_1$  and  $X_2$  are the time 1 and 2 values of a stochastic process  $X$  with  $X_0 = 1$ . Let  $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$  be the filtration generated by  $X$ . What is  $\mathcal{F}_0$ ?  $\mathcal{F}_1$ ?  $\mathcal{F}_2$ ? (In the filtration generated by a stochastic process  $X$ , each  $\mathcal{F}_t$  is the  $\sigma$ -field generated by the complete history of  $X$  up to and including time  $t$ . That is,

$$\mathcal{F}_t = \sigma\{\mathcal{F}_{X_s} \mid s \leq t\}, \quad (2)$$

where the notation  $\sigma\{\}$ , or “ $\sigma$ -closure,” or the “ $\sigma$ -field generated by” is necessary because the union of  $\mathcal{F}_{X_1}, \dots, \mathcal{F}_{X_t}$  may not by itself represent a valid  $\sigma$ -field.)

2. Suppose the price in yen  $P$  of a Japanese stock and the exchange rate  $X$  dollars per yen are Itô processes given by

$$dP/P = \mu_P dt + \sigma'_P dB, \quad (3)$$

$$dX/X = \mu_X dt + \sigma'_X dB, \quad (4)$$

where  $B$  is standard 2-dimensional Brownian motion. Describe the dynamics of the price  $Y$  of the stock in dollars.

3. (a) Use the Martingale Representation Theorem and Itô's Lemma to prove Corollary 1:

Let  $\{B_t\}$  be an  $n$ -dimensional Brownian motion,  $\{\mathcal{F}_t\}$  its natural filtration, and  $\{X_t\}$  a strictly positive local martingale adapted to  $\{\mathcal{F}_t\}$ . Then there exists an  $n$ -dimensional process  $\theta \in \mathcal{L}^2$  such that

$$X_t = X_0 e^{\int_0^t \theta'_s dB_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds} . \quad (5)$$

- (b) Conclude that if  $Z \in L^2(\mathcal{P})$  is a strictly positive random variable measurable with respect to  $\mathcal{F}_t$ , then  $Z$  has the representation (5).
4. Use the results in question 4 above, the lemma stated below, and Levy's Theorem (Proposition 14 of Domenico Cuoco's lecture notes) to prove the Girsanov Theorem:

Let  $B_t$  be standard  $n$ -dimensional Brownian motion on  $[0, T]$  under the probability measure  $\mathcal{P}$  with  $\{\mathcal{F}_t\}$  its natural filtration and let  $\mathcal{P}^*$  be a probability measure equivalent to  $\mathcal{P}$ . Then there exists an  $n$ -dimensional process  $\theta \in \mathcal{L}^2$  s.t.

$$\frac{d\mathcal{P}^*}{d\mathcal{P}} = \exp\left(-\int_0^T \theta'_t dB_t - \frac{1}{2} \int_0^T |\theta_t|^2 dt\right) \quad (6)$$

and

$$B_t^* \equiv B_t + \int_0^t \theta_s ds \quad (7)$$

is standard  $n$ -dimensional Brownian motion on  $[0, T]$  under  $\mathcal{P}^*$ .

**Lemma** In the setting described above, define  $Z_t \equiv E\left\{\frac{d\mathcal{P}^*}{d\mathcal{P}} \mid \mathcal{F}_t\right\}$ , a strictly positive martingale w.r.t  $\{\mathcal{F}_t\}$ . If  $Y$  is an Itô process and  $ZY$  is a  $\mathcal{P}$ -local martingale, then  $Y$  is  $\mathcal{P}^*$ -local martingale.