Portfolio Constraints

Detailed outline

1. Portfolio constraint sets
   Support function, effective domain

2. Budget constraint characterization of feasible consumption plans
   Adjusted coefficients, discount factor, state-price density

3. Dual problem

4. Existence of constrained optimal consumption/investment policy

5. Characterization of constrained optimal consumption/investment policy

Readings


Problem
Consider a complete, standard continuous-time financial market with a single risky asset. Suppose an investor derives utility from consumption plan \( \{c_t\} \) equal to

\[
E \int_0^T \log c_t \, dt
\]

The investor has initial wealth \( X_0 \) and no subsequent income.

(a) Determine the investor’s optimal consumption and investment policies in the absence of any portfolio constraint.

(b) Now suppose the investor’s portfolio holdings of the riskless and risky asset, \((\pi_0(t), \pi(t))\), are constrained to satisfy \( \pi_0(t) + \pi(t) \geq 0 \) and

\[
\frac{\pi(t)}{\pi_0(t) + \pi(t)} \in [w_1, w_2],
\]

where \( w_1 \leq 0 \) and \( w_2 \geq 0 \). Determine the investor’s constrained optimal consumption and investment policies. Show that at each date and state, the investor’s constrained optimal portfolio is either equal to his unconstrained optimal portfolio or else is at the boundary of the constraint set.

Hints:

(i) The constraint set \( \mathcal{A} \) containing the feasible portfolio holdings is a closed convex cone in \( \mathbb{R}^2 \), and thus, the support function \( \delta \) of \( -\mathcal{A} \) has value zero on its effective domain \( \hat{\mathcal{A}} \). Apply Proposition 1 of Cuoco (1997) to characterize the investor’s constrained optimal consumption plan by solving the dual shadow state-price problem. Find that the minimization over processes \( \nu \in \mathcal{N}^* \) amounts to a pointwise minimization of linear and quadratic terms on \( \hat{\mathcal{A}} \). The set \( \hat{\mathcal{A}} \) is a closed, convex cone and \( \{\nu_0 : \nu \equiv (\nu_0, \nu_-) \in \hat{\mathcal{A}}\} \) is bounded below by zero. So \( \{\nu_0 : \nu \in \mathcal{A}\} \) is a ray and for each \( \nu_0 \), \( \{\nu_- : (\nu_0, \nu_-) \in \mathcal{A}\} \) is an interval or ray.

(ii) Derive and expression for the constrained optimal wealth process for the investor from the equation in the proof of Proposition 1 at the bottom of p.63. From the expression for the optimal wealth process, derive an expression for the constrained optimal trading strategy. Then show that the minimization over processes \( \nu \in \mathcal{N}^* \) in the dual problem implies that the investor’s constrained optimal portfolio is either equal to his unconstrained optimal portfolio or else is at the boundary of the constraint set.