The Standard Continuous-Time Financial Market

Detailed outline

1. Brownian probability space
2. Payoff space – flow rates and terminal payoffs
3. Security prices – Itô processes
4. Trading strategies – self-financing, square-integrable, tame
5. Equivalent martingale measures – characterization of Radon-Nikodym derivatives
6. Portfolio value - martingale properties, lower bounds
7. Market completeness

Readings

Domenico Cuoco’s lecture notes, parts IV and V.
Problem

1. Suppose \( \text{rank}(\sigma(t)) = n \). Let \( \nu \) be an \( d \)-dimensional process with \( \sigma(t)\nu(t) = 0 \) and let

\[
\theta(t) = \sigma(t)'(\sigma(t)\sigma(t)')^{-1}[\mu(t) - r(t)1], \tag{1}
\]

\[
\tilde{\theta}(t) = \theta(t) + \nu(t), \tag{2}
\]

\[
\tilde{Z}(t) \equiv e^{-\int_0^t \tilde{\theta}(s)'dB(s) - \frac{1}{2} \int_0^t |\tilde{\theta}(s)|^2 ds}, \tag{3}
\]

\[
\beta(t) \equiv e^{-\int_0^t r(s) ds}, \quad \text{and} \tag{4}
\]

\[
m(t) \equiv \beta(t)\tilde{Z}(t). \tag{5}
\]

Finally, let

\[
Z(t) \equiv e^{-\int_0^t \theta(s)'dB(s) - \frac{1}{2} \int_0^t |\theta(s)|^2 ds}, \tag{6}
\]

\[
m^*(t) \equiv \beta(t)Z(t), \tag{7}
\]

\[
m^* \equiv m^*(T). \tag{8}
\]

(a) Show that \( m^* \) is the only \( m \) of the form above in the payoff space, that is, it is the only such \( m \) for which there exists a trading strategy that strictly finances a consumption plan \((c, W) \in \mathcal{C}\) with \( W = m \).

(b) Show that \( m^*(t) \) is the \( m \) process with the smallest instantaneous volatility, or in other words, that its log has the smallest quadratic variation.