DEBT INSTRUMENTS AND MARKETS

Zeroes and Coupon Bonds
Default-Free Fixed Income Securities

- The most basic debt instrument is a "zero-coupon" or "pure discount" bond -- a security with a single cash flow equal to face value at maturity.
- Conceptually, these "zeroes" are the building blocks of all securities with fixed cash flows.
- Combining zeroes in a portfolio creates an asset with multiple fixed cash flows.
- We can structure portfolios of zeroes to replicate existing securities, such as coupon bonds.
- It is also possible to construct a portfolio of coupon bonds that replicates a zero.
- The possibility of replicating one asset from a portfolio of others means that, in the absence of arbitrage, their prices must be related.
Zero Prices or “Discount Factors”

- Let $d_t$ denote the $t$-year discount factor,
  - the price today of an asset which pays off $1$ in $t$ years,
  - …or the price of a $t$-year zero as a fraction of par value.
- Because of the "time value of money," a dollar today is worth more than a dollar to be received in the future, so the price of a zero is always less than its face value: $d_t < 1$
- The Discount Function gives the discount factor as a function of maturity.
- Because of the time value of money, longer zeroes have lower prices.
  ➔ the discount function is always downward-sloping.
**Coupon Bonds**

A Treasury bond pays semi-annual coupons every six months at an annualized fixed coupon rate $c$ and par value at maturity $T$.

Coupon bond cash flows as a percent of par value:

\[ \frac{c}{2} \quad \frac{c}{2} \quad \frac{c}{2} \quad \cdots \quad 1 + \frac{c}{2} \]

- $0.5$ years
- $1$ year
- $1.5$ years
- $\cdots$
- $T$ years

**A Coupon Bond as a Portfolio of Zeroes**

**Example:** $10,000$ par of a one and a half year, $8.5\%$ Treasury bond makes the following payments:

<table>
<thead>
<tr>
<th>6 months</th>
<th>1 year</th>
<th>1 1/2 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>$425$</td>
<td>$425$</td>
<td>$10425$</td>
</tr>
</tbody>
</table>

- This is the same as a portfolio of three different zeroes:
  - $425$ par of a 6-month zero
  - $425$ par of a 1-year zero
  - $10425$ par of a 1 1/2-year zero
The Principle of No Arbitrage or The Law of One Price

*Two assets which offer exactly the same cash flows must sell for the same price.*

Why? If not, then one could buy the cheaper asset and sell the more expensive, making a profit today with no cost in the future. This *arbitrage opportunity* cannot persist in equilibrium.

Valuing a Coupon Bond Using Zero Prices

In the absence of arbitrage, the coupon bond must have the same price as the corresponding package of zeroes.

Equivalently, in terms of the discount function,

\[ V = 425 \times d_{0.5} + 425 \times d_{1} + 10425 \times d_{1.5} \]
Valuing $10,000 par of a 1.5-year 8.5% coupon bond. The discount factors come from STRIPS prices from the WSJ.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Par of Zero</th>
<th>Discount Factor</th>
<th>Bond Cash Flow</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>$97.30</td>
<td>0.9730</td>
<td>$425</td>
<td>$414</td>
</tr>
<tr>
<td>1.0</td>
<td>$94.76</td>
<td>0.9476</td>
<td>$425</td>
<td>$403</td>
</tr>
<tr>
<td>1.5</td>
<td>$92.22</td>
<td>0.9222</td>
<td>$10425</td>
<td>$9614</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Total 10430</td>
</tr>
</tbody>
</table>

On the same day, the WSJ priced a 1.5-year 8.5%-coupon bond at 104 10/32 (=104.3125).

An Arbitrage Opportunity

- What if the 1.5-year 8.5% coupon bond were worth only 104% of par value?
- You could buy, say, $1 million par of the bond for $1,040,000 and sell the cash flows off individually as zeroes for total proceeds of $1,043,000, making $3000 of riskless profit.
- Similarly, if the bond were worth 105% of par, you could buy the portfolio of zeroes, reconstitute them, and sell the bond for riskless profit.
Summary: Zeroes and Assets with Fixed Cash Flows

Securities with fixed cash flows can be viewed as packages or portfolios of zeroes. If an asset pays cash flows $K_1, K_2, \ldots, K_n$, at times $t_1, t_2, \ldots, t_n$, then it is the same as $K_1 t_1$-year zeroes plus $K_2 t_2$-year zeroes plus... plus $K_n t_n$-year zeroes. Therefore no arbitrage requires that the asset’s value $V$ is

$$V = K_1 \times d_{t_1} + K_2 \times d_{t_2} + \ldots + K_n \times d_{t_n}$$

or

$$V = \sum_{j=1}^{n} K_j \times d_{t_j}$$

Constructing Zeroes from Coupon Bonds

• Not only can we construct a given bond from zeroes, we can also go the other way.
• Example: Constructing a 1-year zero from 6-month and 1-year coupon bonds.

• Coupon Bonds:

<table>
<thead>
<tr>
<th>Bond #</th>
<th>Maturity</th>
<th>Coupon</th>
<th>Price in 32nds</th>
<th>Price in Decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>4.25%</td>
<td>99-13</td>
<td>99.40625</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>4.375%</td>
<td>98-31</td>
<td>98.96875</td>
</tr>
</tbody>
</table>
**Constructing the One-Year Zero**

- Find portfolio of bonds 1 and 2 that replicates 1-year zero.
- Let $N_1$ be the number (par value) of bond 1 and $N_2$ be the number of bond 2 in the portfolio.
- At time 0.5, the portfolio will have a cash flow of
  $$N_1 \times (1+0.0425/2) + N_2 \times 0.04375/2$$
- At time 1, the portfolio will have a cash flow of
  $$N_1 \times 0 + N_2 \times (1+0.04375/2)$$
- We need $N_1$ and $N_2$ to solve

\[
\begin{align*}
N_1 \times (1+0.0425/2) + N_2 \times 0.04375/2 &= 0 \\
N_1 \times 0 + N_2 \times (1+0.04375/2) &= 100
\end{align*}
\]

=> $N_2 = 97.86$ and $N_1 = -2.10$

**Implied Zero Price**

- The price of the replicating portfolio,
  - long 97.86 par value of the 1-year bond
  - short 2.10 par value of the 0.5-year bond,
  is $(97.86 \times \$0.9896875) - (2.10 \times \$0.9940625)$,
  or $94.7665$.
- In the absence of arbitrage, with no market frictions, this must be the price of the 1-year zero
- Note that the price of the 1-year STRIP is $94.76
Inferring zero prices from bond prices: Short cut

- The last example showed how to construct a portfolio of bonds that synthesized (had the same cash flows as) a zero.
- We concluded that the zero price had to be the same as the price of the replicating portfolio (no arbitrage).
- If we don't need to compute the replicating portfolio, we can solve for the implied zero prices more directly:

\[
\text{Price of bond 1} = 99 \frac{13}{32} = (100 + 4.25 / 2) \times d_{0.5}
\]

\[
\text{Price of bond 2} = 98 \frac{31}{32} = (4.375 / 2) \times d_{0.5} + (100 + 4.375 / 2) \times d_1
\]

\[\Rightarrow d_{0.5} = 0.973, d_1 = 0.948\]

Implied Zero Prices

- Often people would rather work with Treasury coupon bonds than with STRIPS, because the market is more active.
- They can imply a discount function from Treasury bond prices instead of STRIPs and use this implied discount function to value more complex securities.
**Replication Possibilities**

- Since we can construct zeroes from coupon bonds, we can construct any stream of cash flows from coupon bonds.
- Uses
  - Bond portfolio dedication--creating a bond portfolio that has a desired stream of cash flows
    - funding a liability
    - defeasing an existing bond issue
  - Taking advantage of arbitrage opportunities

**Market Frictions**

- In practice, prices of Treasury STRIPS and Treasury bonds don't fit the pricing relationship exactly
  - transaction costs and search costs in stripping and reconstituting
  - bid/ask spreads
- Note: The terms "bid" and "ask" are from the viewpoint of the dealer
  - The dealer buys at the bid and sells at the ask, so the bid price is always less than the ask.
  - The customer sells at the bid and buys at the ask.
**Interest Rates**

- People try to summarize information about bond prices and cash flows by quoting interest rates.
- Buying a zero is lending money--you pay money now and get money later.
- Selling a zero is borrowing money--you get money now and pay later.
- A bond transaction can be described as buying or selling at a given price, or lending or borrowing at a given rate.
- The convention in U.S. bond markets is to use **semi-annually compounded interest rates**.

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**Annual vs. Semi-Annual Compounding**

At 10% per year, *annually* compounded, $100 grows to $110 after 1 year, and $121 after 2 years:

\[
100 \times 1.10 = 110 \\
100 \times (1.10)^2 = 121
\]

10% per year *semi-annually* compounded really means 5% every 6 months. At 10% per year, *semi-annually* compounded, $100 grows to $110.25 after 1 year, and $121.55 after 2 years:

\[
100 \times (1.05)^2 = 110.25 \\
100 \times (1.05)^4 = 121.55
\]
Annual vs. Semi-Annual Compounding...

After \( T \) years, at annually compounded rate \( r_A \), \( P \) grows to

\[
F = P(1 + r_A)^T
\]

Present value of \( F \) to be received in \( T \) years with annually compounded rate \( r_A \) is

\[
P = \frac{F}{(1 + r_A)^T}
\]

In terms of the semi-annually compounded rate \( r \), the formulas become

\[
F = P(1 + r/2)^{2T}
\]

\[
P = \frac{F}{(1 + r/2)^{2T}}
\]

The key: \((1 + r/2)^2 = 1 + r_A\)

A semi-annually compounded rate of “\( r \)” per year really means “\( r/2 \)” every six months.

Zero Rates

- If you buy a \( t \)-year zero and hold it to maturity, you lend at rate \( r_t \) where \( r_t \) is defined by

\[
d_t \times (1 + r_t/2)^{2t} = 1, \text{ or } d_t = \frac{1}{(1 + r_t/2)^{2t}},
\]

or \( r_t = 2 \times ((\frac{1}{d_t})^{1/2} - 1) \)

- Call \( r_t \) the \( t \)-year zero rate or \( t \)-year discount rate.
Example

According to convention, zero prices are quoted using rates. STRIPS rates from the WSJ:

- 6 months: 5.54%
- 1 year: 5.45%
- 5 years: 5.73%

STRIPS prices:

- 6 months: $100 \times \frac{1}{(1 + 0.0554/2)^{12}} = 97 \frac{10}{32} 
- 1 year: $100 \times \frac{1}{(1 + 0.0545/2)^{24}} = 94 \frac{10}{32} 
- 5 years: $100 \times \frac{1}{(1 + 0.0573/2)^{48}} = 75 \frac{13}{32} 

Example

The 1-year zero price implied from coupon bond prices was 94.7665.

The "implied zero rate" is

\[
r_i = 2 \times \left( \frac{1}{0.947665} \right)^\frac{1}{2} - 1 = 5.448\%
\]
Value of a Stream of Cash Flows in Terms of Zero Rates

- Recall that any asset with fixed cash flows can be viewed as a portfolio of zeroes.
- The asset price is the sum of its cash flows multiplied by the relevant zero prices.
- Equivalently, the price is the sum of the present values of the cash flows, discounted at the relevant zero rates.

\[ V = \sum_{j=1}^{n} K_j \times d_{t_j} \]

\[ V = \sum_{j=1}^{n} \frac{K_j}{(1 + r_j / 2)^{2t_j}} \]

Example

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\[ V = \frac{$425}{(1+0.0554/2)} + \frac{$425}{(1+0.0545/2)^2} + \frac{$10425}{(1+0.0547/2)^3} \]

\[ = $10430 \]