Forward Contracts and Forward Rates

Outline and Readings

- **Outline**
  - Forward Contracts
  - Forward Prices
  - Forward Rates
  - Information in Forward Rates

- **Reading**
  - Tuckman, Chapters 2 and 16

- **Buzzwords**
  - settlement date, delivery, underlying asset
  - spot rate, spot price, spot market
  - forward purchase, forward sale, forward loan, forward lending, forward borrowing, synthetic forward
  - expectation theory, term premium
Forward Contracts

- A forward contract is an agreement to buy an asset at a future settlement date at a forward price specified today.
  - No money changes hands today.
  - The pre-specified forward price is exchanged for the asset at the settlement date.

Forward vs. Spot

- To distinguish ordinary transactions from forward transactions, we use the word “spot” (or sometimes, “cash”).
- A spot transaction is for settlement immediately.
  - Money and securities change hands today.
  - The price for such a transaction is called the “spot” price (or sometimes, “cash” price).
Motivation

- Suppose today, time 0, you now you will need to do a transaction at a future date, time t.
- One thing you can do is wait until time t and then do the transaction at prevailing market prices
  - i.e., do a spot transaction in the future.
- Alternatively, you can try to lock in the terms of the transaction today
  - i.e., arrange a forward transaction today.

What is the fair forward price?

- In some cases, the forward contract can be synthesized with transaction in the current spot market.
- In that case, no arbitrage will require that the contractual forward price must be the same as the forward price that could be synthesized.
Synthetic Forward Price

- For example, if the underlying asset doesn’t depreciate or make any payments, the synthetic forward price of the asset is Spot Price + Interest to settlement date
- How to synthesize?
  - Buy the asset now for the spot price.
  - Borrow the amount of the spot price, with repayment on the settlement date
  - You pay nothing now, and you pay the spot price plus interest at the settlement date.

Synthetic Forward Price for a Zero

Suppose the underlying asset is $1 par a zero maturing at at time $T$.
In the forward contract, you agree to buy this zero at time $t$.
The forward price you could synthesize is Spot Price + Interest to time $t$:

$$ F_T^T = d_T \times (1 + \frac{r}{2})^{2t} $$

If the contractual forward price differs, there is an arbitrage opportunity.
Example:
Suppose the underlying asset is $1 par of a zero maturing at time T=1.
In the forward contract, you agree to buy this zero at time t=0.5.
The synthetic forward price is Spot Price + Interest
\[ F_T^T = d_T \times (1 + r_t / 2)^{2t} \]
\[ F_{0.5}^1 = d_1 \times (1 + r_{0.5} / 2)^{2 \times 0.5} \]
\[ = 0.9476 \times (1+0.0554/2) \]
\[ = 0.9739 \]
What if the contractual forward price were 0.98?

Example...

1) Synthesize a long forward contract position by
   - 1a) buying $1 par of the 1-year zero and
   - 1b) borrowing the money from time 0.5 to pay for it:
2) Sell the actual forward contract with 0.98 forward price

| 1a) | -0.9476 | +1 |
| 1b) | +0.9476 | -0.9476(1+0.0554/2)=-0.9739 |
| 2) | 0 | +0.98 | -1 |
| 0 | 0.5 | 1 |
| Net arbitrage profit: | 0 | +0.0061 | 0 |
Forward Contract as a Portfolio of Zeroes

- You agree today (t=0) to pay at t the sum $F$ to get $1$ worth of par at T.
- This contract is a portfolio of cash flows:
  
  \[
  \begin{array}{ccc}
  0 & -F & +1 \\
  \hline
  t & & \\
  \end{array}
  \]

- What is the PV of this contract?
  It is a portfolio:
  Long $1$ par of $T$-year zeros
  Short $F$ par of $t$-year zeros

Zero Cost Forward Price

- At t=0 the contract “costs” zero
- The forward price is negotiated to make that true.
- What is such “F”?
  
  \[
  \begin{array}{ccc}
  0 & -F & +1 \\
  \hline
  t & & \\
  \end{array}
  \]

  \[
  V=-F \cdot d_t + 1 \cdot d_T = 0
  \]

  \[
  F= d_T / d_t
  \]

  Or

  \[
  F=d_T \times (1+r_t/2)^{2t} = F_t^T
  \]
Example

- Suppose the underlying is $1 par of the zero maturing at T=1 for settlement at t=0.5
- What is the value of the contract if the forward price is \( F=0.9739 \)?
- \( V=-0.9739 \times 0.9730 + 1 \times 0.9476 = 0 \)
- Arbitrage:
  - What if the quoted forward rate is \( F=0.98 \)?
  - \( V=-0.98 \times 0.9730 + 1 \times 0.9476 = -0.0059 \)
  - What would you do?
- Sell the actual contract for 0 (it’s overpriced). Buy the synthetic 0.98 contract for -0.0059 (collect 0.0059 when you buy this). Take 0.0059 of arbitrage profit today. (That is equivalent to taking the 0.0061 profit at time 0.5.)

Forward Contract on a Zero as a Forward Loan

- Just as we can think of the spot purchase of a zero as lending money, we can think of a forward purchase of a zero as a “forward loan.”
- The forward lender agrees today to lend \( F_t^T \) on the settlement date \( t \) and get back $1 on the date \( T \).
- The forward rate, \( f_t^T \), is the interest rate earned from lending \( F_t^T \) for \( T-t \) years and getting back $1:
  \[
  F_t^T = \frac{1}{(1 + f_t^T / 2)^{(T-t)}} \quad f_t^T = 2 \left( \frac{1}{F_t^T} \right)^{2(T-t)} - 1
  \]
- This is the same transaction, just described in terms of lending or borrowing at rate instead of buying or selling at a price.
New Riskless Lending Possibilities

- Consider the lending possibilities when a forward contract for lending from time \( t \) to time \( T \) is available. Call the forward lending rate \( f_{t}^{T} \).
- Now there are two ways to lend risklessly from time 0 to time \( T \):
  1) Lend at the current spot rate \( r_{T} \) (i.e., buy a \( T \)-year zero). A dollar invested at time 0 would grow risklessly to \( (1+r_{T}/2)^{2T} \).
  2) Lend risklessly to time \( t \) (i.e., buy a \( t \)-year zero) and roll the time \( t \) payoff into the forward contract to time \( T \). A dollar invested at time 0 would grow risklessly to \( (1+r_{t}/2)^{2t} \times (1+f_{t}^{T}/2)^{2(T-t)} \).

No Arbitrage Forward Rate

In the absence of arbitrage, the two ways of lending risklessly to time \( T \) must be equivalent:

\[
(1+r_{t}/2)^{2t} \times (1+f_{t}^{T}/2)^{2(T-t)} = (1+r_{T}/2)^{2T}
\]

Example: The forward rate from time \( t = 0.5 \) to time \( T = 1 \) must satisfy

\[
(1+0.0554/2)^{1} \times (1+f_{0.5}^{1}/2) = (1+0.0545/2)^{2}
\]

\[\Rightarrow f_{0.5}^{1} = 5.36\%\]
Connection Between Forward Prices and Forward Rates

Of course, this is the same as the no arbitrage equations we saw before:

\[
\left(1 + \frac{f_i^T}{2}\right)^{2(T-t)} = \frac{(1 + r_i/2)^{2T}}{(1 + r_f/2)^{2t}} \iff F_i^T = \frac{d_f^T}{d_i}
\]

Example

- The implied forward rate for a loan from time 0.5 to time 1 is 5.36%.
- This gives a discount factor of 0.9739, which we showed before is the synthetic forward price to pay at time 0.5 for the zero maturing at time 1.

\[
\frac{1}{\left(1 + \frac{0.0536}{2}\right)^{2}} = \frac{1 + 0.0554/2}{1 + 0.0545/2} = \frac{0.9476}{0.9222} = 0.9739
\]
Summary: One No Arbitrage Equation, Three Economic Interpretations:

1. Forward price = Spot price + Interest
   \[ F_T^t = d_r \times (1 + f_t^T / 2)^{\Delta t} \]

2. Present value of forward contract cash flows at inception = 0:
   \[ -d_r \times F_T^t + d_r \times 1 = 0 \]

3. Lending short + Rolling into forward loan = Lending long:
   \[ (1 + r_t / 2)^{\Delta t} \times (1 + f_t^T / 2)^{2(\Delta\tau - \Delta t)} = (1 + r_t / 2)^{2\tau} \]

Using the relations between prices and rates,
\[ d_r = \frac{1}{(1 + r_t / 2)^{\Delta t}} \quad \text{and} \quad F_T^t = \frac{1}{(1 + f_t^T / 2)^{2(\Delta\tau - \Delta t)}} \quad \text{or} \quad f_t^T = 2(\frac{1}{F_T^t})^{(\Delta\tau - \Delta t)} - 1 \]
we can verify that these equations are all the same. Other arrangements:
\[ F_T^t = \frac{d_r}{f_t^T} \quad \text{and} \quad (1 + f_t^T / 2)^{2(\Delta\tau - \Delta t)} = \frac{(1 + r_t / 2)^{2\tau}}{(1 + r_t / 2)^{2\tau}} \]

Spot Rates as Averages of Forward Rates

- Rolling money through a series of short-term forward contracts is a way to lock in a long term rate and therefore synthesizes an investment in a long zero.

Here are two ways to lock in a rate from time 0 to time t:
\[ (1 + r_{0.5} / 2) \times (1 + f_{0.5}^t / 2) \times \cdots \times (1 + f_{t-0.5}^t / 2) = (1 + r_t / 2)^{2t} \]

- The growth factor \((1+r/2)\) is the geometric average of the \((1+f/2)\)’s and so the interest rate \(r_t\) is approximately the average of the forward rates.
  - Recall the example
    - The spot 6-month rate is 5.54% and the forward 6-month rate is 5.36%
    - Their average is equal to the one year rate of 5.45%
Yield Curves

Forward Rate $f(t-0.5,t)$
Zero Rate $r(t)$
Par Yield $c(t)$
**Forward Rates vs. Future Spot Rates**

- The forward rate is the rate you can fix today for a loan that starts at some future date.
- By contrast, you could wait around until that future date and transact at whatever is the prevailing spot rate.
- Is the *forward rate* related to the random *future spot rate*?
- For example, *is the forward rate equal to people’s expectation of the future spot rate?*

**The Pure Expectations Hypothesis**

- The “Pure Expectations Hypothesis” says that the *forward rate is equal to the expected future spot rate*.
- That’s roughly equivalent to the hypothesis that *expected returns on all bonds over a given horizon are the same*.
- For example,
  
  \[ E(0.5 \tilde{r}_t) = f_{0.5}^1 \]
  
  \[ \Rightarrow E \{(1 + r_{0.5} / 2)(1 + 0.5 \tilde{r}_t / 2)\} = (1 + r_{0.5} / 2)(1 + f_{0.5}^1 / 2) \]
  
  \[ \Rightarrow E \{(1 + r_{0.5} / 2)(1 + 0.5 \tilde{r}_t / 2)\} = (1 + r_t / 2)^2 \]

- So the expected one-year rate of return from rolling two six-month zeroes is equal to the one-year rate of return from holding a one-year zero.
### Example in Which the Pure Expectations Hypothesis Holds

<table>
<thead>
<tr>
<th>Zero Rates</th>
<th>Rates of Return over Various Horizons</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Time 0</strong></td>
<td><strong>0.5-Year ROR</strong></td>
</tr>
<tr>
<td></td>
<td>0.5-yr zero</td>
</tr>
<tr>
<td>5.860%</td>
<td>5.540%</td>
</tr>
<tr>
<td>(w.p. 50%)</td>
<td></td>
</tr>
<tr>
<td>5.540%</td>
<td>5.700%</td>
</tr>
<tr>
<td>5.450%</td>
<td></td>
</tr>
<tr>
<td><strong>Time 0.5</strong></td>
<td><strong>1-Year ROR</strong></td>
</tr>
<tr>
<td></td>
<td>0.5-yr zero</td>
</tr>
<tr>
<td>4.860%</td>
<td>5.540%</td>
</tr>
<tr>
<td>(w.p. 50%)</td>
<td></td>
</tr>
<tr>
<td>5.200%</td>
<td>5.450%</td>
</tr>
<tr>
<td>Expected</td>
<td>5.360%</td>
</tr>
<tr>
<td></td>
<td>5.541%</td>
</tr>
<tr>
<td><strong>Forward rate:</strong></td>
<td><strong>5.360%</strong></td>
</tr>
</tbody>
</table>

If the pure expectation hypothesis holds, then the downward slope of the yield curve indicates that rates are expected to fall.
If the pure expectation hypothesis holds, then the upward slope of the yield curve indicates that rates are expected to rise.
Problem with the Pure Expectations Hypothesis: Expected Rates of Return May Differ Across Bonds

- Different bonds may have different expected rates of return because their returns have different risk properties (variance, covariance with other risks, etc.)
- In that case, the pure expectations hypothesis cannot hold.
- For example, the yield curve is typically upward sloping.
  - If the pure expectations hypothesis were true, that would mean people generally expect rates to rise.
  - An alternative explanation is that investors generally require more expected return to be willing to hold longer bonds.

Term Premiums and Information in Forward Rates

- Empirically, forward rates tend to be higher than the spot rate that ultimately prevails for that investment horizon, or equivalently, longer bonds appear to have higher average returns.
- The “term premium” is defined roughly by 
  \[
  \text{Forward rate} = \text{Expected future spot rate} + \text{Term Premium}
  \]
- A more general version of expectations hypothesis says that term premiums are roughly constant.
- If that’s true, then changes in forward rates reflect changes in expectations about future rates.
- For example, if we see forward rates fall, it may mean that people have revised their forecasts of future spot rates downward.
- On the other hand it could be because risk premiums have changed.
Example in Which Longer Bonds Have Higher Expected Returns

<table>
<thead>
<tr>
<th>Zero Rates</th>
<th>Rates of Return over Various Horizons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time 0</td>
<td>Time 0.5</td>
</tr>
<tr>
<td>5.750% (w.p. 50%)</td>
<td>0.5-Year ROR</td>
</tr>
<tr>
<td>5.000%</td>
<td>5.000% 4.751% 5.375% 5.250%</td>
</tr>
<tr>
<td>5.250%</td>
<td>4.250% (w.p. 50%) 5.000% 6.255% 4.625% 5.250%</td>
</tr>
<tr>
<td>Expected:</td>
<td>5.000% 5.000% 5.503% 5.000% 5.250%</td>
</tr>
<tr>
<td>Forward</td>
<td>rate: 5.500%</td>
</tr>
</tbody>
</table>

Here, the yield curve is upward sloping, not because rates are expected to rise, but because longer bonds have higher expected returns.