Convexity

Concepts and Buzzwords

- Dollar Convexity
- Convexity
- Curvature, Taylor series, Barbell, Bullet
Readings

- Tuckman, chapters 5 and 6.

Convexity

- Convexity is a measure of the curvature of the value of a security or portfolio as a function of interest rates: $V(r)$
- Duration is related to the slope, i.e., the 1st derivative: $V'(r)$
- Convexity is related to the curvature, i.e., the second derivative of the price function: $V''(r)$
- Using convexity together with duration gives a better approximation of the change in value given a change in interest rates than using duration alone.
Price of a 20-Year Zero as a Function of Its Discount Rate

\[ d_{20}(r_{20}) = \frac{1}{(1 + r_{20}/2)^{40}} \]

Correcting the Duration Error

- The price-rate function is **nonlinear**.
- Duration and dollar duration use a **linear approximation** to the price rate function to measure the change in price given a change in rates.
- The error in the approximation can be substantially reduced by making a convexity correction.
**Taylor Series**

A theorem from calculus says that the value of a function can be approximated near a given point using its "Taylor series" around that point. Using only the first two derivatives, the Taylor series approximation is:

\[ f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 \]

or:

\[ f(x) - f(x_0) \approx f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 \]

**Derivatives of the Zero Price Function**

In the case of a zero-coupon bond:

\[ d_i(r_t) = \frac{1}{(1+r_t/2)^{2t}} \text{ price} \]

\[ d'_i(r_t) = \frac{-t}{(1+r_t/2)^{2t+1}} \text{ dollar duration} \]

\[ d''_i(r_t) = \frac{t^2 + t/2}{(1+r_t/2)^{2t+2}} \text{ dollar convexity} \]
Example

For the 20-year zero at 6.5%:

\[
\begin{align*}
d_t(r_t) &= \frac{1}{(1 + r_t/2)^{2t}} = \frac{1}{(1 + 0.065/2)^{40}} = 0.278226 \\
d_t'(r_t) &= \frac{-t}{(1 + r_t/2)^{2t+1}} = \frac{-20}{(1 + 0.065/2)^{41}} = -5.389364 \\
d_t''(r_t) &= \frac{t^2 + t/2}{(1 + r_t/2)^{2t+2}} = \frac{410}{(1 + 0.065/2)^{42}} = 107.0043
\end{align*}
\]

The Convexity Correction

\[
f(x) - f(x_0) = f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2
\]

Applying the Taylor series approximation, the change in the zero price given a change in rates from \(r_{t,0}\) to \(r_{t},\) is:

\[
d_t(r_t) - d_t(r_{t,0}) \approx d_t'(r_{t,0})(r_t - r_{t,0}) + \frac{1}{2} d_t''(r_{t,0})(r_t - r_{t,0})^2
\]

Change in price = \(\Delta P = -\$D \times \Delta r + 0.5 \times \$C \times (\Delta r)^2 = \)
- dollar duration x change in rates +
(1/2) x dollar convexity x change in rates squared
Example

How does the 20-year zero price change as its discount rate changes from 6.5% to 7.5%?

The actual change is:

\[ d_{20}(0.075) - d_{20}(0.065) = \frac{1}{(1 + 0.075/2)^{40}} - \frac{1}{(1 + 0.065/2)^{40}} \]
\[ = 0.229338 - 0.278226 = -0.048888 \]

Example...

The approximate change using only dollar duration is:

change in price = -dollar duration x change in rates
\[ = -5.389364 \times 0.01 \]
\[ = -0.05389364. \]

The approximate change using both dollar duration and convexity is:

Change in price
\[ = - \text{dollar duration} \times \text{change in rates} \]
\[ + \frac{1}{2} \times \text{dollar convexity} \times \text{(change in rates squared)} \]
\[ = (-5.389364 \times 0.01) + \frac{1}{2} \times 107.0043 \times 0.0001 \]
\[ = -0.05389364 + 0.00535 = -0.048543. \]
The Convexity Correction is Always Positive

Suppose the 20-year rate fell 100 bp 5.5%. The approximate change using both dollar duration and convexity is:

Change in price =
- dollar duration x change in rates +
(1/2) x dollar convexity x change in rates squared
= (-5.389364 x (-0.01)) +((1/2) x 107.0043 x 0.0001)
= 0.05389364 +0.00535 = 0.059244.

The actual change in price is 0.059626.

---

Summary

<table>
<thead>
<tr>
<th>Rate (%)</th>
<th>20-Year Price</th>
<th>Actual Change</th>
<th>Duration Approximation</th>
<th>Duration and Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.50</td>
<td>0.337852</td>
<td>0.059626</td>
<td>0.053894</td>
<td>0.059244</td>
</tr>
<tr>
<td>6.50</td>
<td>0.278226</td>
<td>-0.048888</td>
<td>-0.053894</td>
<td>-0.048543</td>
</tr>
<tr>
<td>7.50</td>
<td>0.229338</td>
<td>-0.048888</td>
<td>-0.053894</td>
<td>-0.048543</td>
</tr>
</tbody>
</table>
Convexity

To get a scale-free measure of curvature, convexity is defined as
\[
\text{convexity} = \frac{\text{dollar convexity}}{\text{value}} = \frac{t^2 + t/2}{(1 + r/2)^{2t}}
\]

The convexity of a zero is roughly its time to maturity squared.

Example

Duration and convexity for $1 par of a 10-year, 20-year, and 30-year zero.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Rate</th>
<th>Price</th>
<th>Dollar Duration</th>
<th>Duration</th>
<th>Dollar Convexity</th>
<th>Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>6.00%</td>
<td>0.553676</td>
<td>5.375434</td>
<td>9.70874</td>
<td>54.7987</td>
<td>98.9726</td>
</tr>
<tr>
<td>20</td>
<td>6.50%</td>
<td>0.278226</td>
<td>5.389364</td>
<td>19.37048</td>
<td>107.0043</td>
<td>384.5951</td>
</tr>
<tr>
<td>30</td>
<td>6.40%</td>
<td>0.151084</td>
<td>4.391974</td>
<td>29.06977</td>
<td>128.8075</td>
<td>859.1355</td>
</tr>
</tbody>
</table>

Notice that for zeroes, duration is roughly equal to maturity, while convexity is roughly equal to maturity-squared.
Dollar Convexity of a Portfolio

Consider a portfolio with fixed cash flows at different points in time.

Just as with dollar duration, the dollar convexity of the portfolio is the sum of the dollar convexities of the component zeroes.

The dollar convexity of the portfolio gives the correction to make to the duration approximation of the change in portfolio value given a change in rates, assuming all rates shift by the same amount.

Formula for the Dollar Convexity of a Portfolio

Suppose the portfolio has cash flows $K_1, K_2, K_3, \ldots \text{ at times } t_1, t_2, t_3, \ldots$. Then the dollar convexity of the portfolio is

$$\sum_{j=1}^{n} K_j \times \frac{t_j^2 + t_j / 2}{(1 + r_{ij} / 2)^{2t_j + 2}}$$
Example

- Consider a portfolio consisting of $25,174 par value of the 10-year zero and $91,898 par value of the 30-year zero.
- The dollar convexity of the portfolio is $(25,174 \times 54.7987) + (91,898 \times 129.8015) = 13,307,997$

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Rate</th>
<th>Price</th>
<th>Dollar Duration</th>
<th>Duration</th>
<th>Dollar Convexity</th>
<th>Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>6.00</td>
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<td>20</td>
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<td>107.0043</td>
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<td>4.391974</td>
<td>29.06977</td>
<td>129.8015</td>
<td>859.1356</td>
</tr>
</tbody>
</table>

Convexity of a Portfolio

- Just as with individual zeroes, the convexity of a portfolio is its dollar convexity divided by its value:

$$\text{convexity} = \frac{\sum_{j=1}^{n} K_j \times \frac{t_j^2 + t_j / 2}{(1 + r_{t_j} / 2)^{2t_j} + 2}}{\sum_{j=1}^{n} K_j \times (1 + r_{t_j} / 2)^{2t_j}}$$
**Convexity of a Portfolio, ...**

- Rearranging terms, the convexity of the portfolio is the average of the convexity of the component zeroes weighted by market value:

  \[
  \text{convexity} = \frac{\sum_{j=1}^{n} \frac{K_j}{(1+r_j/2)^{2t_j}} \times \frac{t_j^2 + t_j/2}{(1+r_j/2)^2}}{\sum_{j=1}^{n} \frac{K_j}{(1+r_j/2)^{2t_j}}}
  \]

**Example**

Consider the portfolio of 10- and 30-year zeroes.

- The 10-year zeroes have market value $25,174 \times 0.553676 = $13,938.
- The 30-year zeroes have market value $91,898 \times 0.151084 = $13,884.
- The market value of the portfolio is $27,822.
- The convexity of the portfolio is $13,307,997/27,822 = 478.32.$
Example...

- Alternatively, the convexity of the portfolio is the average convexity of each zero weighted by market value:

\[
\frac{(13,938 \times 98.9726) + (13,884 \times 859.1356)}{13,938 + 13,884} = 478.32
\]

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Rate</th>
<th>Price</th>
<th>Dollar Duration</th>
<th>Duration</th>
<th>Dollar Convexity</th>
<th>Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.00%</td>
<td>0.553676</td>
<td>5.375493</td>
<td>9.70874</td>
<td>54.7987</td>
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<tr>
<td>20</td>
<td>6.50%</td>
<td>0.278226</td>
<td>5.389364</td>
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<tr>
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<td>4.391974</td>
<td>29.06977</td>
<td>129.8015</td>
<td>859.1358</td>
</tr>
</tbody>
</table>

Barbells and Bullets

- We can construct a portfolio of a long-term and short-term zero (a barbell) that has the same market value and duration as an intermediate-term zero (a bullet).
- The barbell will have more convexity.
Example

- **Bullet portfolio:** $100,000 par of 20-year zeroes
  - market value = $100,000 \times 0.27822 = 27,822
  - duration = 19.37
- **Barbell portfolio:** from previous example
  - $25,174 par value of the 10-year zero
  - $91,898 par value of the 30-year zero.
  - market value = 27,822
  - duration = \frac{(13,938 \times 9.70874) + (13,884 \times 29.06977)}{13,938 + 13,884} = 19.37

- The convexity of the bullet is 385.
- The convexity of the barbell is 478.

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Value of Barbell and Bullet

**Bullet:** \[ V_1(s) = \frac{100,000}{(1 + (0.065 + s)/2)^{40}} \]

**Barbell:** \[ V_2(s) = \frac{25,174}{(1 + (0.06 + s)/2)^{30}} + \frac{91,898}{(1 + (0.064 + s)/2)^{60}} \]
Does the Barbell Always Outperform the Bullet?

- If there is an immediate parallel shift in interest rates, either up or down, then the barbell will outperform the bullet.
- If the shift is not parallel, anything could happen.
- If the rates on the bonds stay exactly the same, then as time passes the bullet will actually outperform the barbell:
  - the bullet will return 6.5%
  - the barbell will return about 6.2%, the market value-weighted average of the 6% and 6.4% on the 10- and 30-year zeroes.

Value of Barbell and Bullet One Year Later

<table>
<thead>
<tr>
<th>Portfolio Value</th>
<th>Parallel Shift in Rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>45000</td>
<td>-0.015</td>
</tr>
<tr>
<td>40000</td>
<td>-0.005</td>
</tr>
<tr>
<td>35000</td>
<td>0.005</td>
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<td>30000</td>
<td>0.015</td>
</tr>
<tr>
<td>25000</td>
<td></td>
</tr>
<tr>
<td>20000</td>
<td></td>
</tr>
</tbody>
</table>

- Bullet
- Barbell