Immunization

Reading

♦ Tuckman, chapter 7.
Immunizing / Hedging
Interest Rate Risk

- Suppose you have liabilities or obligations consisting of a stream of fixed cash flows you must pay in the future.
  - Bond defeasance
  - Pension liabilities?
  - Insurance liabilities?
- How can you structure an asset portfolio to fund these liabilities?

Dedication

- The only completely riskless approach is to construct an asset portfolio with cash flows that exactly match the liability cash flows.
- This funding method is called dedication.
- This approach may be infeasible or excessively costly.
- In some situations, risk managers may want more flexibility.
Immunization

Consider a more flexible but more risky approach, called *matching*.

- The liabilities have a certain market value.
- That market value changes as time passes and as interest rates change.
- Construct an asset portfolio with the same market value and the same interest rate sensitivity as the liabilities so that the asset value tracks the liability value over time.

If the assets and liabilities have
- the same market value and
- interest rate sensitivity,
the net position is said to be *hedged* or *immunized* against interest rate risk.

The approach can be extended to settings with debt instruments that do not have fixed cash flows.
Duration Matching

- The most common form of immunization: matches the duration and market value of the assets and liabilities
- This hedges the net position against small parallel shifts in the yield curve.
- Recall: Change in value ~ dollar duration x change in rates
  - Matching the dollar duration of assets and liabilities means matching their changes in value if all rates change by the same amount.
  - Matching market value means liabilities are fully funded
  - Hedging against parallel shifts is really just a first step.

Example: Duration Matching

Suppose the liabilities consist of $1,000,000 par value of a 7.5%-coupon 29-year bond.
This liability has a duration of 12.58.
Market Value of Liability
Cash Flows

Example...
Construct an asset portfolio that has the same market value and duration as the liabilities using
- a 12-year zero and
- a 15-year zero.
The following table gives information on the market value and duration of the liability and the instruments to be used in the asset portfolio.

<table>
<thead>
<tr>
<th>Coupon (%)</th>
<th>Maturity (Years)</th>
<th>Par Value ($)</th>
<th>Discount Rate (%)</th>
<th>Market Value ($)</th>
<th>Dollar Duration</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.5</td>
<td>25</td>
<td>1,000,000</td>
<td>(yield curve)</td>
<td>1,151,802</td>
<td>14,486,304</td>
<td>12.58</td>
</tr>
<tr>
<td>0</td>
<td>12</td>
<td>6.24</td>
<td>0.4784</td>
<td>5.5668</td>
<td>11.64</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>15</td>
<td>6.41</td>
<td>0.3881</td>
<td>5.6412</td>
<td>14.53</td>
<td></td>
</tr>
</tbody>
</table>

Example...

To construct the asset portfolio, solve two equations:

- Asset market value = Liability market value
- Asset dollar duration = Liability dollar duration

Notice that if the assets have
- the same market value and
- the same dollar duration

as the liability, then they have the same duration as the liability.

⇒ matching market value and dollar duration is the same as matching market value and duration
Example...

With $N_{12}$ and $N_{15}$ representing the par amounts of the 12- and 15-year zero in the portfolio, the equations become:

\[
0.4784N_{12} + 0.3881N_{15} = 1,151,802 \\
5.5668N_{12} + 5.6412N_{15} = 14,486,304
\]

Example...

The solution to the market value-matching and dollar duration-matching equations is

$N_{12} = 1,626,424$ $N_{15} = 962,969$

In other words, the immunizing asset portfolio consists of $1,626,424 face value of 12-year zeroes and $962,969 face value of 15-year zeroes. By construction it has:

- the same market value ($1,151,802) and
- the same dollar duration (14,486,304), and therefore
- the same duration (12.58), as the liability.
Market Value of Duration-Matched Portfolio Cash Flows

Performance of Duration Match for Different Parallel Yield Curve Shifts

<table>
<thead>
<tr>
<th>Market Value</th>
<th>-100 bp</th>
<th>-10 bp</th>
<th>0</th>
<th>+10 bp</th>
<th>+100 bp</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assets</td>
<td>1306889</td>
<td>1166384</td>
<td>1151802</td>
<td>1137418</td>
<td>1016080</td>
</tr>
<tr>
<td>Liabilities</td>
<td>1312293</td>
<td>1166433</td>
<td>1151802</td>
<td>1137458</td>
<td>1020267</td>
</tr>
<tr>
<td>Net Equity</td>
<td>-5604</td>
<td>-49</td>
<td>0</td>
<td>-47</td>
<td>-4188</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Dollar Duration</th>
<th>-100 bp</th>
<th>-10 bp</th>
<th>0</th>
<th>+10 bp</th>
<th>+100 bp</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assets</td>
<td>16538729</td>
<td>14678971</td>
<td>14486304</td>
<td>14296286</td>
<td>12699204</td>
</tr>
<tr>
<td>Liabilities</td>
<td>17742932</td>
<td>14777840</td>
<td>14486304</td>
<td>14201643</td>
<td>11921307</td>
</tr>
<tr>
<td>Net Equity</td>
<td>-1204203</td>
<td>-98869</td>
<td>0</td>
<td>96462</td>
<td>777898</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Duration</th>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>12.66</td>
<td>13.52</td>
</tr>
<tr>
<td></td>
<td>12.59</td>
<td>12.67</td>
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<tr>
<td></td>
<td>12.58</td>
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<td></td>
<td>12.57</td>
<td>12.49</td>
</tr>
<tr>
<td></td>
<td>12.50</td>
<td>11.68</td>
</tr>
</tbody>
</table>
Problem with this hedge?

- $\text{convexity of liabilities} = 288,068,417$, convexity = 250
- $\text{convexity of assets}$?
  - $\text{convexity of 12-yr zero}$?
  - Recall $r_{12} = 6.24\%$
  - $\text{convexity of 15-yr zero}$?
  - Recall $r_{15} = 6.41\%$
- $\text{convexity of assets} = 191,336,510$, convexity = 166

Duration and Convexity Match

- Observations:
  - The duration match performed well for small parallel shifts in the yield curve, but not for large shifts.
  - Durations and dollar durations of the assets changed with interest rates by different amounts.
- Why? The net position had zero duration but negative convexity.
- Conclusions:
  - For large interest rate changes, the duration-matched hedge has to be rebalanced.
  - A way to mitigate this problem is to match the convexity of assets and liabilities as well as duration and market value.
Consider structuring an asset portfolio that matches the convexity of the liabilities as well as their duration and market value.

Use the following instruments for the asset portfolio.

- a 2-year zero
- a 15-year zero
- a 25-year zero

Note that matching market value, duration, and convexity is the same as matching market value, dollar duration, and dollar convexity.

To construct the asset portfolio, solve three equations:

- \[ \text{Asset market value} = \text{Liability market value} \]
- \[ \text{Asset dollar duration} = \text{Liability dollar duration} \]
- \[ \text{Asset dollar convexity} = \text{Liability dollar convexity} \]
Example

The following table gives information on the market value, duration, and convexity of the liability and the instruments to be used in the asset portfolio.

<table>
<thead>
<tr>
<th>Cpn (%)</th>
<th>Maturity (Years)</th>
<th>Par ($)</th>
<th>Discount Rate (%)</th>
<th>Market Value ($)</th>
<th>Dollar Duration</th>
<th>Dollar Convexity</th>
<th>Duration</th>
<th>Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.5</td>
<td>0</td>
<td>1M</td>
<td>yield curve</td>
<td>1151802</td>
<td>14486304</td>
<td>288068417</td>
<td>12.58</td>
<td>250.10</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>1.50</td>
<td>0.8972</td>
<td>1.7463</td>
<td>4.2489</td>
<td>1.95</td>
<td>4.74</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>15</td>
<td>1.3881</td>
<td>5.6412</td>
<td>84.7226</td>
<td>14.53</td>
<td>218.38</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>25</td>
<td>0.1977</td>
<td>4.7852</td>
<td>118.1290</td>
<td>24.20</td>
<td>597.48</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example...

With N2, N15, N25 representing the par amounts of the 2-, 15-, and 25-year zeroes in the portfolio, the market-value-matching, dollar-duration-matching, and dollar-convexity-matching equations become:

\[
0.8972N_2 + 0.3881N_{15} + 0.1977N_{25} = 1,151,802 \\
1.7463N_2 + 5.6412N_{15} + 4.7852N_{25} = 14,486,304 \\
4.2489N_2 + 84.7226N_{15} + 118.129N_{25} = 288,068,417
\]

<table>
<thead>
<tr>
<th>Cpn (%)</th>
<th>Maturity (Years)</th>
<th>Par ($)</th>
<th>Discount Rate (%)</th>
<th>Market Value ($)</th>
<th>Dollar Duration</th>
<th>Dollar Convexity</th>
<th>Duration</th>
<th>Convexity</th>
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<td>0</td>
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<td>0.1977</td>
<td>4.7852</td>
<td>118.1290</td>
<td>24.20</td>
<td>597.48</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The solution to these equations is:

\[ N_2 = 497,576 \quad N_{15} = 920,680 \quad N_{25} = 1,760,379 \]

In other words, a portfolio with
$497,596$ par value of 2-year zeroes,
$920,680$ par value of 15-year zeroes, and
$1,760,379$ par value of 25-year zeroes
will have the same market value, duration and convexity as the liability.
Performance of Duration-Convexity Match for Different Parallel Yield Curve Shifts

<table>
<thead>
<tr>
<th>MARKET VALUE</th>
<th>-100 bp</th>
<th>-10 bp</th>
<th>0</th>
<th>+10 bp</th>
<th>+100 bp</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assets</td>
<td>1312210</td>
<td>1166433</td>
<td>1151802</td>
<td>1137458</td>
<td>1020328</td>
</tr>
<tr>
<td>Liabilities</td>
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<td>1166433</td>
<td>1151802</td>
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<td>1020267</td>
</tr>
<tr>
<td>Net Equity</td>
<td>-82</td>
<td>-0.07</td>
<td>0</td>
<td>0.07</td>
<td>60</td>
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<table>
<thead>
<tr>
<th>DOLLAR DURATION</th>
<th>-100 bp</th>
<th>-10 bp</th>
<th>0</th>
<th>+10 bp</th>
<th>+100 bp</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assets</td>
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<td>14777623</td>
<td>14486304</td>
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</tr>
<tr>
<td>Liabilities</td>
<td>17742932</td>
<td>14777840</td>
<td>14486304</td>
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</tr>
<tr>
<td>Net Equity</td>
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<td>0</td>
<td>-208</td>
<td>-17290</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>DURATION</th>
<th>Assets</th>
<th>13.50</th>
<th>12.67</th>
<th>12.58</th>
<th>12.49</th>
<th>11.67</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liabilities</td>
<td>13.52</td>
<td>12.67</td>
<td>12.58</td>
<td>12.49</td>
<td>11.68</td>
<td></td>
</tr>
</tbody>
</table>

Actual Yield Curve Shift
Dollar Duration of Liability Cash Flows

Dollar Duration of Asset Cash Flows in Duration-Matched Portfolio
Dollar Duration of Asset Cash Flows in Duration-Convexity-Matched Portfolio

Effect of the Actual Yield Curve Shift

- The average change in rates was +1 bp.
- If the interest rate shift had been parallel, dollar duration of 14,486,304 would have predicted a change of -14,486,304 x 0.0001 = -$1449 in the value of the liability and each asset portfolio.
- The actual change in the liability was -$2126
- The dollar duration of the liability is concentrated on year 29. The 29-year discount rate increased 2 bp.
Effect of the Actual Yield Curve Shift...

- The value of the duration-matched portfolio changed by only $-889.
  - The 12-year discount rate did not change at all.
  - The 15-year discount rate rose 2 bp.
- Net equity under this immunization would have increased to $1237.

Effect of the Actual Yield Curve Shift...

- The value of the duration-convexity-matched portfolio changed by $-3365.
  - Most of its dollar duration was on year 25.
  - The 25-year discount rate rose 3 bp.
- Net equity under this immunization would have fallen to -$1239.
Conclusion

◆ Duration or duration-convexity matching hedges against parallel shifts of the yield curve.
◆ To hedge against other shifts, the cash flows of the assets and liabilities must have similar exposure to different parts of the yield curve.
◆ The best hedges strike a balance between
  • duration matching—flexible, low transaction costs but inaccurate
  • cash flow matching—less maintenance, less risk, inflexible, very precise