An Interest Rate Model

Concepts and Buzzwords

- Building Price Tree from Rate Tree
- Lognormal Interest Rate Model
- Nonnegativity
- Volatility and the Level Effect

Readings

- Tuckman, chapters 11 and 12.
### Review of No Arbitrage Pricing

Approach to contingent claims pricing

I. starting with the possible future payoffs of short- and long-term zeroes

II. replicate the payoffs of a derivative with a portfolio or trading strategy using two zeroes

III. use the law of one price to set the claim price equal to the price of the replicating portfolio

### Review of Risk-Neutral Probabilities

Equivalent approach

I. determine state-contingent claims prices from the original prices and payoffs of the zeroes

II. derive "risk-neutral" probabilities from the state-contingent claims prices

III. represent the no arbitrage price of a derivative as the "risk-neutral" expected value of its future payoff, discounted at the riskless rate.
### Starting with Risk-Neutral Probabilities

- Conceptually, we start with current prices and a set of future possible payoffs, and then derive the risk-neutral probabilities.
- Once we have a theory that says these risk-neutral probabilities exist, however, it is often more practical to start with them immediately.
- From a financial engineering standpoint, it is easier to set risk-neutral probabilities of the up and down states to 0.5 each, and then work out what the future payoffs must be to fit current prices.

### Interest Rate Modeling

- **GOAL:** build interest rate models that capture basic properties of interest rates while also fitting the current term structure
- Some basic properties are:
  - nonnegative interest rates
  - non-normal distribution
  - mean-reversion
  - stochastic volatility and the level effect.
- This lecture will develop a specific interest rate model and explore some of its properties.
- The next lecture will show how to calibrate the model to fit the current term structure.
Building Price Tree from Rate Tree and Risk-Neutral Probabilities

- As motivation, note that once we have a tree of one-period rates ("short" rates) and risk-neutral probabilities, we can price any term structure asset.
- For example, suppose we assume that six-month rates and risk-neutral probabilities are as follows:

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
<th>Time 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.54%</td>
<td>6.004%</td>
<td>6.915%</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Building Price Trees...

This information will determine the price trees for the 0.5-year zero, the 1-year zero, and the 1.5 year zero. Examples:

- The time 1, up-up price of the zero maturing at 1.5:
  \[ \frac{1}{1+0.06915/2} = 0.9666 \]
- The time 1, up-down price of the zero maturing at 1.5:
  \[ \frac{1}{1+0.05437/2} = 0.9735 \]
- The time 0.5 up price of the zero maturing at time 1:
  \[ \frac{1}{1+0.06004/2} = 0.9709 \]
- The time 0.5 up price of the zero maturing at 1.5:
  \[ (0.5 \times 0.9666 + 0.5 \times 0.9735) \times 0.9709 = 0.9418 \]
Eventually, we can fill out the whole tree of prices for each zero.

Each six-month zero price in the tree comes directly from the six-month rate.

The price of each long zero is the discounted, risk-neutral expected value of its future price.

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
<th>Time 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.973047</td>
<td>0.970857</td>
<td>0.966581</td>
</tr>
<tr>
<td>0.947649</td>
<td>0.941787</td>
<td>0.973533</td>
</tr>
<tr>
<td>0.922242</td>
<td>0.922242</td>
<td>0.979071</td>
</tr>
</tbody>
</table>

Once we have the tree or "model" of zero prices, we can price any interest rate derivative product.

We price derivatives at their replication cost.

We compute the replication cost by discounting risk-neutral expected payoffs.

Pricing boils down to building the interest rate model.
Lognormal Interest Rate Model

- Definition: A random variable \( Y \) has a lognormal distribution if \( \ln(Y) \) has a normal distribution (i.e., if \( Y = \exp(X) \) where \( X \) has a normal distribution).
- A lognormal model of interest rates gives both
  - non-negative interest rates
  - higher volatility at higher interest rates.
- We will work with a discrete-time binomial approximation of this lognormal model.

Log Model of Interest Rates

The short rate (the rate on \( h \)-year bonds):

\[
\begin{align*}
\text{Time 0} & \quad \text{Time } h & \quad \text{Time } 2h \\
\text{The short rate (the rate on } h \text{-year bonds):} & \quad r & \quad r_e^{m_h + m_{2h} + 2\sigma \sqrt{h}} \\
& \quad r e^{m_h + \sigma \sqrt{h}} & \quad 0.5 \\
0.5 & \quad r e^{m_{1h} - \sigma \sqrt{h}} & \quad 0.5 \\
0.5 & \quad r e^{m_{1h - m_{2h}} - 2\sigma \sqrt{h}} & \quad 0.5 \\
\end{align*}
\]

Notice that each date the short rate changes by a multiplicative term: \( e^{m_h + \sigma \sqrt{h}} \).

The exponential is always positive, which guarantees that interest rates are always positive in this model.
### Description of the Model

- $h$ is the amount of time between dates in the tree measured in years.
- For example, in a semi-annual tree, $h = 0.5$. In a monthly tree, $h = 1/12 = 0.08333$.
- Each value in the tree represents the short rate or interest rate for a zero with maturity $h$.
- Each date the (risk-neutral) probability of moving up or down is 0.5.
- The parameters of the model are
  - the drift terms $m_1, m_2, \ldots$ which are known (nonstochastic) but can change each period and
  - the *proportional* volatility $\sigma$ which is constant.

### Example: Semi-Annual Tree

- **Suppose (details later)**
  - the time steps are 6 months ($h=0.5$)
  - the current 6-month rate is 5.54%
  - the drift over the first 6 months is $m_1=-0.0797$
  - the drift over the second 6 months is $m_2=0.0422$
  - the proportional volatility $\sigma=0.17$
Example: Semi-Annual Tree

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
<th>Time 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>The short rate</td>
<td></td>
<td>6.915%</td>
</tr>
<tr>
<td>5.54%</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>0.5</td>
<td>6.004%</td>
<td>5.437%</td>
</tr>
<tr>
<td>0.5</td>
<td>4.721%</td>
<td>4.275%</td>
</tr>
<tr>
<td>0.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For example, at time 0.5, up, the 6-month zero rate is

$$0.0554e^{-0.0797 \times 0.5 + 0.17 \times \sqrt{0.5}} = 0.06004$$

$$e^x \approx 1 + x$$

<table>
<thead>
<tr>
<th>x</th>
<th>exp(x)</th>
<th>x+1</th>
<th>Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.200</td>
<td>0.819</td>
<td>0.800</td>
<td>0.019</td>
</tr>
<tr>
<td>-0.100</td>
<td>0.905</td>
<td>0.900</td>
<td>0.005</td>
</tr>
<tr>
<td>0.000</td>
<td>1.000</td>
<td>1.000</td>
<td>0.000</td>
</tr>
<tr>
<td>0.100</td>
<td>1.105</td>
<td>1.100</td>
<td>0.005</td>
</tr>
<tr>
<td>0.200</td>
<td>1.221</td>
<td>1.200</td>
<td>0.021</td>
</tr>
</tbody>
</table>
The volatility of the short rate itself is not constant, but is instead approximately proportional to the level of the short rate. To see this, note that for small x:

\[ e^x = 1 + x \]

Therefore,

\[ r^* = \begin{cases} r^{mh+\sigma\sqrt{h}} = r(1 + mh + \sigma\sqrt{h}) \\ r^{mh-\sigma\sqrt{h}} = r(1 + mh - \sigma\sqrt{h}) \end{cases} \]

\[ \text{vol(new } r) \approx \text{old } r \times \sigma\sqrt{h} \]

**Example of the Level Effect**

<table>
<thead>
<tr>
<th>Time 0.5</th>
<th>Time 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.004%</td>
<td>6.915%</td>
</tr>
<tr>
<td>5.437%</td>
<td></td>
</tr>
</tbody>
</table>

Suppose we arrive at the up state at time 0.5 so the current spot rate is 6.004%. The future spot rate is either 6.915% or 5.437%. The (risk-neutral) expected future spot rate is 0.5(6.915% + 5.437%) = 6.176%.

The volatility of the future spot rate is

\[ \sqrt{0.5(0.06915 - 0.06176)^2 + 0.5(0.05437 - 0.06176)^2} = 74 \text{ bp} \]
Example of the Level Effect...

<table>
<thead>
<tr>
<th></th>
<th>Time 0.5</th>
<th>Time 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4.721%</td>
<td>5.437%</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

In the down state at time 0.5 the current spot rate is 4.721%. The future spot rate is either 5.437% or 4.275%. The (risk-neutral) expected future spot rate is $0.5(5.437\%+4.275\%)=4.856\%$.

The volatility of the future spot rate is

$$\sqrt{0.5(0.05437 - 0.04856)^2 + 0.5(0.04275 - 0.04856)^2} = 58 \text{ bp}$$

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Basis Point Volatility

<table>
<thead>
<tr>
<th></th>
<th>Time 0.5</th>
<th>Time 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>volatility is proportional to the level of the interest rate.</td>
<td>6.004%</td>
<td>6.915%</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>vol = 74 bp</td>
<td>0.5</td>
<td>5.437%</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>vol = 58 bp</td>
<td>0.5</td>
<td>4.275%</td>
</tr>
</tbody>
</table>

The parameter $\sigma$ is called the proportional volatility. The unannualized basis point volatility is approximately $r\sigma \sqrt{h}$:

- up state: $0.06004 \times 0.17 \times \sqrt{0.5} = 72 \text{ bp}$
- down state: $0.04721 \times 0.17 \times \sqrt{0.5} = 57 \text{ bp}$

The annualized basis point volatility is approximately $r\sigma$. 

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The Log of the Short Rate

\[ \ln(r) \]

Changes in the Log of the Short Rate

The log of the rate always changes by an additive term,

\[ \frac{1}{2} mh + \sigma \sqrt{h} \]
\[ \frac{1}{2} mh - \sigma \sqrt{h} \]

The mean change is \( mh \). The standard deviation of the change is a constant, \( \sigma \sqrt{h} \).

The standard deviation of the annual change is \( \sigma \).

Why? The annual change is the sum of the changes over each period. There are \( 1/h \) changes each year. The changes or increments are independent (there is no mean reversion in this model), so the variance of the sum is the sum of the variances:

\[ \sigma^2 h \times \frac{1}{h} = \sigma^2 \]
The Limiting Distribution

- Suppose we
  - hold fixed the total calendar time spanned by the tree, but
  - divide the time into smaller intervals \( h \) goes to zero, so that the number of intervals goes to infinity.

- Then
  - the distribution of the log of the terminal short rate approaches a normal distribution
  - the distribution of the terminal short rate approaches a lognormal distribution.

Review: Using the Interest Rate Tree to Build a Bond Price Tree

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short rate</td>
<td>5.54%</td>
</tr>
<tr>
<td>Zero maturing at time 0.5</td>
<td>0.9730</td>
</tr>
<tr>
<td>Zero maturing at time 1</td>
<td>?</td>
</tr>
</tbody>
</table>

The tree implies that the price of the zero maturing at time 1 is \( 0.5 \times (0.9709 + 0.9769) \times 0.9730 = 0.9476 \).