Options

Concepts and Buzzwords

- Contractual Features
- Payoffs
- Put-Call Parity
- Valuation
- Volatility
- Delta
- Moneyness

- Call, put, European, American, underlying asset, strike price, expiration date, option delta, delta hedging, in-the-money, out-of-the-money, at-the-money, at-the-money-forward

Readings

- Tuckman, chapter 19.
Call Option

- A *European* call option is a contract that gives the owner the *right* but not the *obligation* to buy
  - an underlying asset
  - at a pre-specified price called the *strike price*
  - at a pre-specified date called the *expiration date*.
- An *American* call option gives the owner the right to buy the asset at the strike price any time on or *before* the expiration date.

Put Option

- A European *put* option gives the owner the right to *sell* the underlying asset at the strike price on the expiration date.
- An American put option gives the owner the right to sell the asset at the strike price any time on or *before* the expiration date.
Call Payoff

- Let $V_T$ represent the value of the underlying asset on the expiration date $T$.
- Consider the payoff of a European call option with strike price $K$.
  - If the underlying is worth more than $K$ at expiration, the option holder should exercise the option and buy the asset for $K$, for a net payoff of $V_T - K$.
  - If the underlying is worth less than $K$, the option holder should leave the option unexercised, for a net payoff of 0.
- To summarize, call payoff = $\max(V_T - K, 0)$

Put Payoff

- Consider the payoff of a European put option with strike price $K$.
  - If the underlying is worth less than $K$ at expiration, the option holder should exercise the option and sell the asset for $K$, for a net payoff of $K - V_T$.
  - If the underlying is worth more than $K$, the option holder should leave the put unexercised, for a net payoff of 0.
- To summarize, put payoff = $\max(K - V_T, 0)$
Call Payoff Diagram

Call payoff =
\[ \max(V_T - K, 0) \]

Put Payoff Diagram

Put payoff =
\[ \max(K - V_T, 0) \]
Put-Call Parity: Payoffs

Consider a European call and a European put on the same underlying asset with the same strike price and the same expiration date.

Call payoff = Put payoff + $V_T - K$

$\text{Max}(V_T - K, 0) = \text{Max}(K - V_T, 0) + V_T - K.$

Check: If $V_T > K$ then the call is in the money, with a payoff of $V_T - K$, and the put is out of the money with a payoff of zero, so the payoff equation is satisfied:

$V_T - K = 0 + V_T - K.$

If $V_T < K$ then the call is out of the money, the put is in the money, and the equation still holds:

$0 = K - V_T + V_T - K.$
Put-Call Parity: Prices

- Suppose the underlying asset pays no cash flows before the option expiration date.
- Then payoff of the call is the same as the payoff of a portfolio consisting of
  - the put
  - the underlying asset
  - a short position in the riskless zero with par value K.
- Therefore, in the absence of arbitrage, the current price of the call must equal the current value of the portfolio:
  \[ C = P + V - d_fK \]

Options on a Zero

- Consider European call and put options on $100 par of a zero maturing at time 2.
- The options expire at time 1.
- The strike price of the options is $95.
Zero Prices Over the Next Year

Our interest rate model tells the possible future prices of zeroes over the next year. At each node, prices of outstanding zeroes are listed in ascending order of maturity.

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
<th>Time 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.973047</td>
<td>0.970857</td>
<td>0.966581</td>
</tr>
<tr>
<td>0.947649</td>
<td>0.941787</td>
<td>0.933802</td>
</tr>
<tr>
<td>0.922242</td>
<td>0.913180</td>
<td>0.973533</td>
</tr>
<tr>
<td>0.897166</td>
<td>0.976941</td>
<td>0.947382</td>
</tr>
<tr>
<td></td>
<td>0.953790</td>
<td>0.979071</td>
</tr>
<tr>
<td></td>
<td>0.930855</td>
<td>0.958270</td>
</tr>
</tbody>
</table>

Payoffs of Call, Put, and Underlying Zero

Possible payoffs of the options and the underlying at the expiration date, time 1:

<table>
<thead>
<tr>
<th>$100 par of zero maturing at time 2</th>
<th>Call</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>93.3802</td>
<td>0.8270</td>
</tr>
<tr>
<td>0.5</td>
<td>94.7382</td>
<td>0.8270</td>
</tr>
<tr>
<td>0.5</td>
<td>95.8270</td>
<td>0.8270</td>
</tr>
</tbody>
</table>

Note that at each possible state, up-up, up-down, or down-down, put-call parity holds for payoffs.
Valuing the Call at Time 0.5

At each state at time 0.5, the value of the call is the value of a portfolio of bonds that replicates its payoffs. This value can be computed using the risk-neutral probabilities and the riskless discount factor from time 0.5 to time 1.

<table>
<thead>
<tr>
<th>Riskless Discount Factor</th>
<th>Call Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time 0.5</td>
<td>Time 1</td>
</tr>
<tr>
<td>0.970857</td>
<td>0.5</td>
</tr>
<tr>
<td>0.5</td>
<td>0.976941</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8270</td>
</tr>
</tbody>
</table>

\[
(0.5 \times 0 + 0.5 \times 0.8270) \times 0.976941 = 0.404
\]

Valuing the Put at Time 0.5

Again, at each state at time 0.5, the value of the put is the risk-neutral expected value of its payoff at time 1, multiplied by the riskless discount factor from time 0.5 to time 1.

<table>
<thead>
<tr>
<th>Riskless Discount Factor</th>
<th>Put Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time 0.5</td>
<td>Time 1</td>
</tr>
<tr>
<td>0.970857</td>
<td>0.5</td>
</tr>
<tr>
<td>0.5</td>
<td>0.976941</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2618</td>
</tr>
</tbody>
</table>

\[
0.5(1.6198 + 0.2618) \times 0.970857 = 0.9134
\]

\[
0.5(0.2618 + 0) \times 0.976941 = 0.1279
\]
Valuing the Options at Time 0

At time 0, the value of each option is the risk-neutral expected value of its payoff at time 0.5, multiplied by the riskless discount factor from time 0 to time 0.5.

<table>
<thead>
<tr>
<th>Riskless Discount Factor</th>
<th>Time 0</th>
<th>Time 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time 0</td>
<td>0.973047</td>
<td>1</td>
</tr>
<tr>
<td>Time 0.5</td>
<td>0.5</td>
<td>1</td>
</tr>
</tbody>
</table>

Call Price

- Time 0: \(0.5(0 + 0.404) \times 0.973047 = 0.1965\)
- Time 0.5: \(0.5 \times 0.404 = 0.202\)

Put Price

- Time 0: \(0.5(0.9134 + 0.1279) \times 0.973047 = 0.5066\)
- Time 0.5: \(0.5 \times 0.1279 = 0.0639\)

Price Trees for the Call and Put with Strike Price = 0.95

At each node, the top number is the call price and the bottom number is put price.

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
<th>Time 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time 0</td>
<td>0</td>
<td>0.9134</td>
</tr>
<tr>
<td>Time 0.5</td>
<td>0.5</td>
<td>0.16198</td>
</tr>
<tr>
<td>Time 1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Time 0</td>
<td>0.1965</td>
<td>0.4040</td>
</tr>
<tr>
<td>Time 0.5</td>
<td>0.5</td>
<td>0.2618</td>
</tr>
<tr>
<td>Time 1</td>
<td>0</td>
<td>0.1279</td>
</tr>
<tr>
<td>Time 0</td>
<td>0.5066</td>
<td>0.4040</td>
</tr>
<tr>
<td>Time 0.5</td>
<td>0.5</td>
<td>0.1279</td>
</tr>
<tr>
<td>Time 1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Put-Call Parity State by State

\[ C = P + V - \Delta_d K \]
\[ C = P + 100d_2 - 95d_1 \]

Time 0:
\[ 0.1965 = 0.5066 + 89.7166 - 95 \times 0.947649 \]

Time 0.5, up:
\[ 0 = 0.9134 + 91.318 - 95 \times 0.970857 \]

Time 0.5, down:
\[ 0.404 = 0.1279 + 93.0855 - 95 \times 0.976941 \]

Option Prices and Volatility

- The higher the volatility of the underlying asset, the higher the value of both call and put options.
- Why? Here's where the mathematical device of risk-neutral probabilities can help our intuition.
  - The option payoff is a convex function of the future price of underlying asset.
  - The more disperse the (risk-neutral) distribution of the future asset price, or in other words, the more chance of extreme realizations, the higher the expected option payoff, and therefore the higher the current option price.
Graphical Illustration of the Effect of Higher Volatility on Option Value

Suppose that under distribution 1, the future asset price will be VL or VH with equal probability. Under the higher volatility distribution 2, the future asset price will be either VLL or VHH with equal probability. Then the expected option payoff will be higher under distribution 2.

Example: Increasing Volatility to $\sigma = 0.25$

Suppose we increase the interest rate volatility parameter $\sigma$ from 0.17 to 0.25. Recalibrating the model to fit the term structure sets $m_1 = -0.0955$, $m_2 = 0.0273$, and $m_3 = 0.003$. The resulting tree of prices of zeroes out to 2 years is as follows:
Price Trees for the Call and Put with Strike Price=0.95 and \( \sigma=0.25 \)

At each node, the top number is the call price and the bottom number is put price. Notice that both option prices are higher with higher volatility.

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
<th>Time 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3128</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>0.6228</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>0.5</td>
<td>1.1886</td>
<td>0.5</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6429</td>
<td>0.5</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0916</td>
<td>0.5</td>
</tr>
<tr>
<td>0.5</td>
<td>2.2649</td>
<td>0.5</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1872</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>1.3142</td>
<td>0</td>
</tr>
</tbody>
</table>

Option Delta

- The delta of an option is a measure of its sensitivity to changes in the value of the underlying asset.
  - The option's delta is not the same as its interest rate delta, although they are related.
- The option delta is a hedge ratio: the number of units of the underlying necessary to offset a position in the option.
  - To hedge a long option position, short delta units of the underlying asset.
  - To hedge a short option position, buy delta units of the underlying asset.
Option Delta in the Binomial Model

- Consider options on some underlying bond.
- How many units of the underlying bond must be held to hedge a short position in the option?
- This number of units is the option's delta.
- To be hedged, the total position must result in a riskless future payoff--the same payoff in the up state as in the down state.

Solving for Delta

Let $\Delta$ represent the number of units of the underlying bond needed to hedge a short position in a call option. $\Delta$ must make the net payoff in the up state equal to the net payoff in the down state.

$$\Delta_c B_u - C_u = \Delta_c B_d - C_d \Rightarrow \Delta_c = \frac{C_u - C_d}{B_u - B_d}$$

To hedge a written put, hold a number of units of the underlying bond equal to the put's delta:

$$\Delta_p B_u - P_u = \Delta_p B_d - P_d \Rightarrow \Delta_p = \frac{P_u - P_d}{B_u - B_d}$$
Example: Hedging the 95 Call or Put

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond</td>
<td>91.3180</td>
</tr>
<tr>
<td>Call</td>
<td>0</td>
</tr>
<tr>
<td>Put</td>
<td>0.9134</td>
</tr>
<tr>
<td></td>
<td>93.0855</td>
</tr>
</tbody>
</table>

\[ \Delta_c = \frac{C_u - C_d}{B_u - B_d} = \frac{0 - 0.4040}{91.3180 - 93.0855} = 0.229 \]

\[ \Delta_p = \frac{P_u - P_d}{B_u - B_d} = \frac{0.9134 - 0.1279}{91.3180 - 93.0855} = -0.444 \]

To hedge a short position in the call, go long 0.229 of the underlying ($22.9 par).
To hedge a short position in the put, go short 0.444 of the underlying ($44.4 par).

Change in Delta with Change in Price of Underlying

up state:

\[ \Delta_c = \frac{C_{uu} - C_{ud}}{B_{uu} - B_{ud}} = \frac{0 - 0}{93.3802 - 94.7382} = 0 \]

\[ \Delta_p = \frac{P_{uu} - P_{ud}}{B_{uu} - B_{ud}} = \frac{1.6198 - 0.2618}{93.3802 - 94.7382} = -1 \]

down state:

\[ \Delta_c = \frac{C_{dd} - C_{dd}}{B_{dd} - B_{ud}} = \frac{0 - 0.8270}{94.7382 - 95.8270} = 0.760 \]

\[ \Delta_p = \frac{P_{dd} - P_{dd}}{B_{dd} - B_{ud}} = \frac{0.2618 - 0}{94.7382 - 95.8270} = -0.240 \]

As the underlying bond price rises,
• the call delta rises from zero to one, and
• the put delta rises from minus one to zero.
Interest Rate Deltas

Remember that the interest rate delta of an instrument is like its effective dollar duration.

\[
\text{ir}\Delta_c = -\frac{C_u - C_d}{r_u - r_d} = \frac{0 - 0.4040}{0.06004 - 0.04721} = 31.49
\]

\[
\text{ir}\Delta_p = -\frac{P_u - P_d}{r_u - r_d} = \frac{0.9134 - 0.1279}{0.06004 - 0.04721} = -61.22
\]

\[
\text{ir}\Delta_{\text{underlying}} = -\frac{B_u - B_d}{r_u - r_d} = \frac{91.3180 - 93.0855}{0.06004 - 0.04721} = 137.76
\]

Connection Between Option Delta and Interest Rate Delta

The option's delta is the hedge ratio between the option and the underlying, which is the ratio of their interest rate deltas.

\[
\Delta_c = \frac{C_u - C_d}{B_u - B_d} = \frac{\text{ir}\Delta_c}{\text{ir}\Delta_{\text{underlying}}} = \frac{-\frac{C_u - C_d}{r_u - r_d}}{\frac{B_u - B_d}{r_u - r_d}} = \frac{31.49}{137.76} = 0.229
\]

\[
\Delta_p = \frac{P_u - P_d}{B_u - B_d} = \frac{\text{ir}\Delta_p}{\text{ir}\Delta_{\text{underlying}}} = \frac{-\frac{P_u - P_d}{r_u - r_d}}{\frac{B_u - B_d}{r_u - r_d}} = \frac{-61.22}{137.76} = -0.444
\]
Moneyness

- Any time at or before expiration, an option's *moneyness* describes the relation between its strike price and the price of the underlying asset.
- A call is *in the money* if the underlying asset price is greater than the strike price, and *out of the money* if the underlying is less than the strike.
- A put is *in the money* if the underlying is less than the strike, and *out of the money* if the underlying is greater than the strike.
- Both are *at the money* if the underlying equals the strike.
- Both are *at-the-money-forward* if the strike price equals the forward price of the underlying asset for settlement on the option expiration date.
- As options go out of the money, their prices and deltas fall to zero. As they get deeper in the money, their prices rise and their deltas approach 1 in absolute value.