Caps, Floors, and Collars

Concepts and Buzzwords

- Caps
- Capped Floaters
- Inverse Floaters with 0% Floor
- Floors
- Floaters with Floors
- Collars
- Floaters with Collars
- Strike rate, settlement frequency, index, notional amount, calls on yields, puts on yields, portfolio of options

Reading
Tuckman, pages 413 – 416
Debt Instruments and Markets

Interest Rate Caps

• A cap provides a guarantee to the issuer of a floating or variable rate note or adjustable rate mortgage that the coupon payment each period will be no higher than a certain amount.
• In other words, the coupon rate will be capped at a certain ceiling or cap rate or strike rate.
• Caps are either
  ➢ offered over-the-counter by dealers or
  ➢ embedded in a security.

Payoff Rule for Typical Cap

• Each payment date, the cap pays the difference, if positive, between a floating index rate and the cap rate multiplied by a prespecified notional amount of principle or par value, divided by the annual payment frequency.
• For example, a $T$-year, semi-annual cap, indexed to the 6-month rate, with $100 notional principle, and cap rate $k$ pays

$$100 \max(t_{0.5}rt-k,0)/2 \text{ at time } t, \ t = 0.5, 1.5, \ldots, T$$
Capped Floater

- Consider the net position of the issuer of $100 par a floating rate note who either buys a matching cap from a dealer or else embeds the cap in the note at issue:
- Capped Floater = Floater minus Cap
- Time $t$ Coupon Payment of Capped Floater

$$100\frac{\text{r}_t^\prime}{2} - 100\max(\frac{\text{r}_t^\prime - \text{k}}{2}, 0) / 2$$

$$= 100 \min(\frac{\text{r}_t^\prime - \text{k}}{2}) / 2$$

Diagram of Cap Payoff
Example: Cap Payments

• Consider a 2-year semi-annual cap on $100 notional amount with strike rate $k = 6\%$, indexed to the 6-month rate.
• At time 0, the 6-month rate is 5.54% so the cap is out-of-the-money, and pays 0 at time 0.5.
• The later payments of the cap depend on the path of interest rates. Suppose rates follow the path in the tree below.

Example: Payments on Capped Floater

• Now consider the coupon payments of the capped floater along the same interest rate path.
• Here, the issuer of the floater has either purchased the cap from a third party or else directly from investors by embedding the cap in the floater.
• Either way, his net coupon payment is the floating payment minus the receipt of the cap payoff.
Call on Yield

- A call on a yield is an instrument that pays off the difference between the level of some underlying yield and a given strike rate, when positive, times a given notional par or principle amount.
- Whether the cash flow is paid at the same time the yield is observed, or one period later, is a matter of convention.
- We will consider payment in arrears. For example, a call on the 6-month rate observed at time $t-0.5$ will payoff at time $t$.
- The period $t$ payoff, for $100$ notional amount and strike rate $k$, is $100\max(t-0.5r_t - k, 0) / 2$

Decomposition of Cap into Calls on Yields

- The payoffs of the cap are the same as the payoffs of a portfolio of calls on yields.
- For example, a $T$-year, semi-annual cap indexed to the 6-month rate caps $2T$ semi-annual floating coupon payments, determined by the 6-month rate at times $0$, $0.5$, $1$, ..., $T-0.5$, paying off $\frac{1}{2}$ year later.
- Each of the $2T$ cap payoffs is the payoff of a call on one of the 6-month rates.
- Therefore, the value of the cap is the sum of the values of the individual calls on the underlying 6-month rates observed at times $0$, $0.5$, $1$, ..., $T-0.5$. 
Valuing a 6% Cap on a Two-Year Floater: Call on the Six-Month Rate at Time 1.5

The value of the cap is the sum of the values of the 4 calls on the 6-month rates at times 0, 0.5, 1, and 1.5.

Let's first value the call on the 6-month rate at time 1.5

Time 0       Time 0.5       Time 1       Time 1.5
5.54%        6.004%       6.915%       7.864%
$0.133       $0.252       $0.476       $0.932
6.004%       6.915%       7.864%       $0.932/(1+0.07864/2)
$0.252       $0.476       $0.932       0.932/(1+0.07864/2)

Example: 0.043 = 0.5(0.089+0)/(1+0.05437/2)

Valuing a 6% Cap on a Two-Year Floater: Call on the Six-Month Rate at Time 1

Time 0       Time 0.5       Time 1       Time 1.5
5.54%        6.004%       6.915%       7.864%
$0.133       $0.252       $0.476       $0.932
6.004%       6.915%       7.864%       $0.932/(1+0.06915/2)
$0.252       $0.476       $0.932       0.932/(1+0.06915/2)

Example: 0.216 = 0.5(0.089+0)/(1+0.06004/2)
### Valuing a 6% Cap on a Two-Year Floater: Call on the Six-Month Rate at Time 0.5

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
<th>Time 1</th>
</tr>
</thead>
</table>
| 5.54%  | 6.004%   | $0.002=  
| $0.001= | $0.002 = 0.002/ 
| 0.5(0.002+0)/ | (1+0.06004/2)  
| (1+0.0554/2)   | 4.721%  | 0 |

### Valuing a 6% Cap on a Two-Year Floater: Summing the Values of the Calls on Yields

The value of the cap is the sum of the values of the 4 calls on the 6-month rates at times 0, 0.5, 1, and 1.5. The value of the call on the 6-month rate at time 0 is clearly zero, since the rate is "out-of-the-money". The sum of the values of the other 3 calls is the cap value, listed below for time 0 and time 0.5.

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.54%</td>
<td>6.004%</td>
</tr>
</tbody>
</table>
| $0.239 = | $0.470 =  
| 0.133 + 0.105 + 0.001 | 0.252 + 0.216 + 0.002  
| 4.721% | $0.021 =  
|        | 0.021 + 0 + 0 |
Interest Rate Sensitivity of a Cap

The cap pays off when interest rates go up. Therefore, it is a bearish position in the bond market. Indeed, its interest rate delta is negative.

\[
ir\Delta = \frac{0.470 - 0.021}{0.06004 - 0.04721} = -35
\]

Modeling a Capped Floater

• Consider an investor holding a 2-year floater with an embedded 6% cap. The floating coupon payments cannot rise above $3 per $100 par.
• The investor has therefore bought a floater, but given up the upside potential of its payments--he has sold a cap. (Capped Floater = Floater minus Cap)
• The value of the capped floater is therefore par minus the value of the cap.
• The interest rate delta of the capped floater is zero minus the delta of the cap.
Two-Year Floater with a 6% Cap

\[ \begin{align*}
\text{Time 0} & \quad \text{Time 0.5} \\
6.004\% & \quad 4.721\% \\
$99.761 = 100 - 0.239 & \quad $99.53 = 100 - 0.470 \\
6.004\% & \quad 4.721\% \\
$99.979 = 100 - 0.021 & \\
\end{align*} \]

\[ ir\Delta = -\frac{99.53 - 99.979}{0.06004 - 0.04721} = +35 \]

Caps and Inverse Floating Rate Notes

- Recall that the coupons of an inverse floater are equal to a fixed rate minus the current floating rate.
- Typically, the contract provides that the coupon of the inverse floater cannot fall below zero.
- This "floor" of 0% on the inverse floating coupon is really a cap on the floating rate, with strike rate equal to the fixed rate component of the inverse floater formula.
- For example, if the inverse floater pays \( k \) minus the 6-month rate, semi-annually, the time \( t \) coupon, per unit par, is equal to the pure inverse floating coupon plus the payoff of a cap with strike \( k \):

\[ \max(k - t^{-0.5} r_t, 0) / 2 = (k - t^{-0.5} r_t) / 2 + \max(t^{-0.5} r_t - k, 0) / 2 \]
• Consider $100 par of a 2-year inverse floater paying 6% minus the 6-month rate.
• If the coupon cannot go below zero, the value of the inverse floater is the value of the "pure" inverse floater (with no floor) plus a cap with strike rate 6%.
• Pure Inverse Floater(6%) = 2 times Fixed(3%) minus Floating.
• Real Inverse Floater(6%) = Pure Inverse Floater(6%) plus Cap (6%)

Example

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>3% fixed bond: 1.5 x (0.9730 + 0.9476 + 0.9222 + 0.8972) + 89.72</td>
<td>95.5568 / 91.1135</td>
</tr>
<tr>
<td>= 95.3268</td>
<td>91.5838</td>
</tr>
<tr>
<td>pure inv. flt.: 2 x 95.3268 - 100 = 90.6535</td>
<td>97.3779 / 94.7557</td>
</tr>
<tr>
<td>real inv. flt.: 90.6535 + 0.2391 = 90.8926</td>
<td>94.7769</td>
</tr>
</tbody>
</table>

\[\text{ir} \Delta \text{ of 3\% fixed bond} = 142\]
\[\text{ir} \Delta \text{ of pure inverse floater} = 284\]
\[\text{ir} \Delta \text{ of real inverse floater} = 249\]

Interest Rate Floors

• A floor provides a guarantee to the owner of a floating note that the coupon payment each period will be no less than a certain floor rate or strike rate.
• Floors are generally embedded in a floating rate note, but could also be purchased separately from a dealer.
Payoff Rule for Typical Floor

- Each payment date, the floor pays the difference, if positive, between the floor rate and the floating rate multiplied by the notional amount of principle or par value, divided by the annual payment frequency.
- For example, a $T$-year, semi-annual floor, indexed to the 6-month rate, with $100$ notional principle, and floor rate $k$ pays

$$\frac{100\max(k_{-0.5}r_t, 0)}{2} \; \text{at time } t, \; t = 0.5, 1.5, ..., T$$

Floater with a Floor

- Consider the net position of an investor who holds $100$ par a floating rate note with an embedded floor or else holds $100$ par of a floater and has purchased a floor separately.
- Floater with a Floor = Floater plus Floor
- Time $t$ Coupon Payment of Floater with a Floor

$$\frac{100}{2} + \max(100(k_{-0.5}r_t) / 2, 0) = \frac{100\max(r_{-0.5}k)}{2}$$
Consider a 2-year semi-annual floor on $100 notional amount with strike rate $k = 4.5\%$, indexed to the 6-month rate. At time 0, the 6-month rate is 5.54% so the floor is out-of-the-money, and pays 0 at time 0.5. The later payments of the floor depend on the path of interest rates. Suppose rates follow the path in the tree below.
Example: Payments of a Floater with a Floor

Consider the payments of a $100 par of a 2-year semi-annual floater with a 4.5% floor. Suppose rates follow the path in the tree below.

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
<th>Time 1</th>
<th>Time 1.5</th>
<th>Time 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.54%</td>
<td>6.004%</td>
<td>6.915%</td>
<td>7.864%</td>
<td>6.184%</td>
</tr>
<tr>
<td>$2.77</td>
<td>$4.721%</td>
<td>$4.275%</td>
<td>$2.25</td>
<td>$102.431</td>
</tr>
</tbody>
</table>

Put on Yield

- A put on a yield is an instrument that pays off the difference between a given strike rate and the level of some underlying yield, when positive, times a given notional amount.
- We will consider payment in arrears. For example, a put on the 6-month rate observed at time \( t-0.5 \) will payoff at time \( t \). The payoff, for $100 notional amount and strike rate \( k \), is

\[
100 \max(k_{t-0.5}r_{t-0.5},0) / 2
\]
The payoffs of the floor are the same as the payoffs of a portfolio of puts on yields.

For example, a $T$-year, semi-annual floor indexed to the 6-month rate bounds $2T$ floating coupon payments, determined by the 6-month rate at times 0, 0.5, 1, ..., $T$-0.5.

Each of the $2T$ floor payoffs is the payoff of a put on one of the 6-month rates.

Therefore, the value of the floor is the sum of the values of the individual puts on the underlying 6-month rates observed at times 0, 0.5, 1, ..., $T$-0.5.

Valuing a 4.5% Floor on a Two-Year Floater: Put on the Six-Month Rate at Time 1.5

The value of the floor is the sum of the values of the 4 puts on the 6-month rates at times 0, 0.5, 1, and 1.5.

Let's first value the put on the 6-month rate at time 1.5.

Example: $0.163 = \frac{0.163}{1+0.04275/2}$
### Valuing a 4.5% Floor on a Two-Year Floater: Put on the Six-Month Rate at Time 1

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
<th>Time 1</th>
<th>Time 1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example: 0.026 = 0.5(0.054)/(1+0.0554/2)</td>
<td>6.915%</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5.54%</td>
<td>6.004%</td>
<td>5.437%</td>
<td>0</td>
</tr>
<tr>
<td>$0.026</td>
<td>0</td>
<td>0</td>
<td>$0.112 = (4.5-4.275)/2</td>
</tr>
<tr>
<td>4.721%</td>
<td>4.275%</td>
<td>$0.112 = (4.5-4.275)/2</td>
<td></td>
</tr>
<tr>
<td>$0.054</td>
<td>$0.054</td>
<td>$0.112 = (4.5-4.275)/2</td>
<td></td>
</tr>
</tbody>
</table>

Example: $0.026 = 0.5(0.054)/(1+0.0554/2)

### Valuing a 4.5% Floor on a Two-Year Floater: Summing the Values of the Puts on Yields

The value of the floor is the sum of the values of the 4 puts on the 6-month rates at times 0, 0.5, 1, and 1.5. The values of the puts on the 6-month rate at time 0 and time 0.5 are zero because they never get in the money.

The floor pays off when rates fall, so it is a bullish position in the bond market and has a positive interest rate delta.

\[
ir\Delta = \frac{0 - 0.133}{0.06004 - 0.04721} = +10
\]
Two-Year Floater with a 4.5% Floor

Floater with a Floor = Floater plus Floor

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.54%</td>
<td>6.004%</td>
</tr>
<tr>
<td>$100.065 = 100 + 0.065</td>
<td>$100 = 100 + 0</td>
</tr>
</tbody>
</table>

\[
ir\Delta = \frac{100 - 100.133}{0.0604 - 0.04721} = +10
\]

Interest Rate Collars

- A collar is a long position in a cap and a short position in a floor.
- The issuer of a floating rate note might use this to cap the upside of his debt service, and pay for the cap with a floor.
- Collars are generally embedded in a floating rate note, but could also be purchased separately from a dealer.
### Floating Coupon

<table>
<thead>
<tr>
<th>Time</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>4%</td>
<td>4%</td>
</tr>
<tr>
<td>5%</td>
<td>5%</td>
</tr>
<tr>
<td>6%</td>
<td>6%</td>
</tr>
<tr>
<td>3%</td>
<td>3%</td>
</tr>
</tbody>
</table>

**Index**

- Cap rate: 6%
- Floor rate: 3%

### Floating Coupon with Collar

<table>
<thead>
<tr>
<th>Time</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>4%</td>
<td>4%</td>
</tr>
<tr>
<td>5%</td>
<td>5%</td>
</tr>
<tr>
<td>6%</td>
<td>6%</td>
</tr>
<tr>
<td>3%</td>
<td>3%</td>
</tr>
</tbody>
</table>

**Index, with cap & floor**

- Cap rate: 6%
- Floor rate: 3%

**Index**
### Example: Two-Year 4.5%-6% Collar

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.174 = 0.239 - 0.065$</td>
<td>$0.470 = 0.470 - 0$</td>
</tr>
<tr>
<td>$99.826 = 100 - 0.174$</td>
<td>$99.53 = 100 - 0.470$</td>
</tr>
</tbody>
</table>

- Top number: Collar = Cap minus Floor
- Bottom number: Floater with a Collar = Floater minus Collar

\[
ir\Delta_{collar} = \frac{0.470 - (-0.112)}{0.06004 - 0.04721} = -45
\]

\[
ir\Delta_{collar\text{-floater}} = -\frac{99.53 - 100.112}{0.06004 - 0.04721} = +45
\]

Both the embedded cap and embedded floor in the floater serve to increase its delta because they both make the floater more like a fixed rate note.