NEW YORK UNIVERSITY STERN SCHOOL OF BUSINESS
Debt Instruments and Markets
Professor Carpenter

Practice Final Exam

Assume that there are no transaction costs, default risks, or arbitrage opportunities. All interest rates are annualized with semi-annual compounding.

1) a) (2 points) Construct an interest rate tree according to the method presented in class with step size $h = 0.5$, volatility $\sigma = 0.3$, drift $m = 0.2$, and initial 0.5-year rate $r = 10\%$. The tree should give short rates out to time 0.5.

b) (2 points) What is the time 0 price of $1$ par of a zero maturing at time 0.5?

c) (3 points) What is the time 0 price of $1$ par of a zero maturing at time 1?

FOR QUESTIONS 2 THROUGH 6, PLEASE USE THE INTEREST RATE TREE ON THE LAST PAGE.

2) a) (4 points) Determine the tree of ex-coupon prices for $100$ par of a 1-year 6% semi-annual coupon noncallable bond. The tree should go out to time 0.5.

b) (4 points) Consider an American call option on the coupon bond in part (a). The strike price of the option is 100. The option can be exercised either at time 0 or at time 0.5 (immediately after the bond’s coupon is paid). What is the value of this option at time 0?

c) (4 points) A firm has issued $100$ par of a 6% coupon bond that matures at time 1. The bond is callable at par on any coupon date, immediately after the coupon is paid. The firm follows a call policy that minimizes the value of its debt. What is the value of the firm’s callable bond at time 0?

d) (6 points) Consider an American put option on the coupon bond in part (a). The strike price of the option is 100. The option can be exercised either at time 0 or at time 0.5 (immediately after the bond’s coupon is paid). What is the value of this option at time 0?

e) Consider the following putable bond. The bond has a 6% semi-annual coupon and a stated maturity of 1 year. Investors have the option to put (sell) the bond to the issuer for par value on any coupon date (immediately after the coupon is paid). Assuming that investors want to maximize the value of their assets,

(i) (2 points) is it optimal for investors to put the bond at time 0?
(ii) (4 points) what is the time 0 value of $100$ par of this putable bond?
3) a) (3 points) What is the time 0 value of a long position in a 1-year, 6%, plain vanilla semi-annual interest rate swap with $100 notional par amount?

b) (5 points) Suppose the swap is cancelable, at no cost, at the option of the party paying fixed, at either time 0 or time 0.5 (immediately after the swap payment). What is the value of this cancelable swap from the viewpoint of the party who is long the swap, i.e., from the viewpoint of the party receiving fixed?

c) (5 points) Suppose instead that the swap is cancelable, at no cost, at the option of the party receiving fixed, at either time 0 or time 0.5 (immediately after the swap payment). What then is the value of this cancelable swap from the viewpoint of the party who is long the swap?

4) Consider a 1-year semi-annual pay mortgage with a semi-annual mortgage rate of 6% and initial principal balance of $10000. The table below contains its amortization schedule.

<table>
<thead>
<tr>
<th>Period ending</th>
<th>Initial balance</th>
<th>Scheduled payment</th>
<th>Interest</th>
<th>Principal</th>
<th>Ending balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time 0.5</td>
<td>10000</td>
<td>5226</td>
<td>300</td>
<td>4926</td>
<td>5074</td>
</tr>
<tr>
<td>Time 1.0</td>
<td>5074</td>
<td>5226</td>
<td>152</td>
<td>5074</td>
<td>0</td>
</tr>
</tbody>
</table>

Suppose the mortgage is divided into two sequential tranches, A and B, each with 5000 of principal. Each period, each tranche receives interest on its beginning principal balance. Additional mortgage payments go to pay down the principal balance of tranche A, until A is paid off. Then mortgage payments in excess of interest due go to pay down tranche B’s principal balance.

a) (6 points) Suppose the mortgage has no prepayments. Determine the cash flows to each tranche by filling in the table below:

<table>
<thead>
<tr>
<th>Period ending</th>
<th>A Interest</th>
<th>A Principal</th>
<th>B Interest</th>
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<tr>
<td>Time 0.5</td>
<td></td>
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</tr>
<tr>
<td>Time 1.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Suppose the mortgagor can pay off the mortgage on any payment date by paying the remaining principal balance. Assume that the mortgagor follows a prepayment policy that minimizes the value of the mortgage.

(i) (7 points) What is the value of the whole mortgage at time 0?

(ii) (6 points) What is the time 0 value of the A tranche?

(iii) (6 points) What is the time 0 value of the B tranche?
5) a) (3 points) Consider an American call option on $100 par of the zero maturing at time 1.5. The option expires at time 0.5, and has strike price 94. The option is exercisable at time 0 or at time 0.5. What is the time 0 value of this option?

b) (5 points) Consider an American put option on $100 par of the zero maturing at time 1.5. The option expires at time 0.5, and has strike price 94. The option is exercisable at time 0 or at time 0.5. What is the time 0 value of this option?

c) (6 points) Does put-call parity hold for the values of the call and put options in parts (a) and (b)? Explain.

6) Consider a 1-year inverse floating rate note with a coupon equal to 12% minus the 0.5-year rate set 6 months prior to the coupon date, with the additional provision that the coupon cannot fall below 6%. In other words, each coupon date \( t \) the coupon payment of this note is \( \max(0.12 - t\cdot 0.5 r_t, 0.06) / 2 \) times the par amount of the note.

a) (12 points) What is the value of $100 par of this note at time 0?

b) (5 points) What par value of the 1-year 6% coupon bond of question (2a) must be held to hedge a long position in $100 par of the inverse floater in part (b) of this question?
Interest rate tree for problems 2 through 6
You may detach this page from the rest of the exam while you are working on the questions. Before you hand in your exam, you must sign this page and tuck it back inside the rest of the exam in order for your exam to count.

The first number at each node in the tree below is the 0.5-year rate at that node. The second number at each node is the price of $1 par of a 0.5-year zero at that node. The third number (if it appears) is the price of $1 par of a 1-year zero at that node. The fourth number (if it appears) is the price of $1 par of a 1.5-year zero at that node.

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
<th>Time 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>6%</td>
<td>7.11%</td>
<td>8.02%</td>
</tr>
<tr>
<td>0.9709</td>
<td>0.9657</td>
<td>0.9614</td>
</tr>
<tr>
<td></td>
<td>0.9323</td>
<td>6.31%</td>
</tr>
<tr>
<td></td>
<td>5.59%</td>
<td>0.9694</td>
</tr>
<tr>
<td></td>
<td>0.9728</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.9462</td>
<td>4.96%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.9758</td>
</tr>
</tbody>
</table>
Solutions to Practice Final Exam

1) a) (2 points) Construct an interest rate tree according to the method presented in class with step size $h = 0.5$, volatility $\sigma = 0.3$, drift $m = 0.2$, and initial 0.5-year rate $r = 10\%$. The tree should give short rates out to time 0.5.

\[
\begin{array}{c|c}
\text{Time 0} & \text{Time 0.5} \\
10\% & 0.1e^{0.2\times0.5+0.3\sqrt{0.5}} = 13.66\% \\
& 0.1e^{0.2\times0.5-0.3\sqrt{0.5}} = 8.94\%
\end{array}
\]

b) (2 points) What is the time 0 price of $1$ par of a zero maturing at time 0.5?

\[
\frac{1}{1.05} = 0.9524
\]

c) (3 points) What is the time 0 price of $1$ par of a zero maturing at time 1?

\[
\begin{array}{c|c}
\text{Time 0} & \text{Time 0.5} \\
0.5\times(0.9361+0.9572)\times0.9524 & 1/(1+0.1367/2) = 0.9361 \\
= 0.9016 & 1/(1+0.0894/2) = 0.9572
\end{array}
\]

FOR QUESTIONS 2 THROUGH 6, PLEASE USE THE INTEREST RATE TREE ON THE LAST PAGE.

2) a) (4 points) Determine the tree of ex-coupon prices for $100$ par of a 1-year 6\% semi-annual coupon noncallable bond. The tree should go out to time 0.5.

\[
\begin{array}{c|c}
\text{Time 0} & \text{Time 0.5} \\
3\times0.9709 + 103\times0.9410 = 99.836 & 103\times0.9657 = 99.467 \\
= 99.836 & 103\times0.9728 = 100.198
\end{array}
\]

b) (4 points) Consider an American call option on the coupon bond in part (a). The strike price of the option is 100. The option can be exercised either at time 0 or at time 0.5 (immediately after the bond’s coupon is paid). What is the value of this option at time 0?

\[
\begin{array}{c|c}
\text{Time 0} & \text{Time 0.5} \\
\text{ex: } 99.836-100 < 0 & \text{max}(99.467-100, 0) = 0 \\
\text{wait: } 0.5\times(0+.198)\times0.9709 = 0.096 & \text{max}(100.198-100, 0) = 0.198
\end{array}
\]

c) (4 points) A firm has issued $100$ par of a 6\% coupon bond that matures at time 1. The bond is callable at par on any coupon date, immediately after the coupon is paid.
The firm follows a call policy that minimizes the value of its debt. What is the value of the firm’s callable bond at time 0?

Callable = Noncallable from (2a) - Call Option from (2b) = 99.836 - 0.096 = 99.74

d) (6 points) Consider an American put option on the coupon bond in part (a). The strike price of the option is 100. The option can be exercised either at time 0 or at time 0.5 (immediately after the bond’s coupon is paid). What is the value of this option at time 0?

\[
\begin{align*}
\text{Time 0} & \\
\text{ex: } 100-99.836 & = 0.164 \\
\text{wait: } 0.5 \times (0.533+0) \times 0.9709 & = 0.259 \\
\text{Time 0.5} & \\
\max(100-99.467, 0) & = 0.533 \\
\max(100-100.198, 0) & = 0
\end{align*}
\]

e) Consider the following putable bond. The bond has a 6% semi-annual coupon and a stated maturity of 1 year. Investors have the option to put (sell) the bond to the issuer for par value on any coupon date (immediately after the coupon is paid). Assuming that investors want to maximize the value of their assets,

(i) (2 points) is it optimal for investors to put the bond at time 0?

No. Putable = Nonputable from (2a) + Put Option from (2d). From (2d), it is not optimal to put the bond at time 0.

(ii) (4 points) what is the time 0 value of $100 par of this putable bond?

99.836+0.259 = 100.095

3) a) (3 points) What is the time 0 value of a long position in a 1-year, 6%, plain vanilla semi-annual interest rate swap with $100 notional par amount?

Swap = Fixed rate bond from (2a) – Floater = 99.836 - 100= -0.164

b) (5 points) Suppose the swap is cancelable, at no cost, at the option of the party paying fixed, at either time 0 or time 0.5 (immediately after the swap payment). What is the value of this cancelable swap from the viewpoint of the party who is long the swap, i.e., from the viewpoint of the party who is receiving fixed?

Swap cancelable by short party = Callable swap = Swap - Receiver swaption (call on swap) with 0 strike
Call on 6% swap with 0 strike = par call on 6% bond from (2b)
Callable swap = -0.164 - 0.096 = -0.26
c) (5 points) Suppose instead that the swap is cancelable, at no cost, at the option of the party receiving fixed, at either time 0 or time 0.5 (immediately after the swap payment). What then is the value of this cancelable swap from the viewpoint of the party who is long the swap?

Swap cancelable by long party = Putable swap = Swap + Payer swaption (put on swap) with 0 strike
Put on 6% swap with 0 strike = par put on 6% bond from (2d)
\[-0.164 + 0.259 = 0.095\]

4) Consider a 1-year semi-annual pay mortgage with a semi-annual mortgage rate of 6% and initial principal balance of $10000. The table below contains its amortization schedule.

<table>
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<tr>
<th>Period ending</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Time 0.5</td>
<td>10000</td>
<td>5226</td>
<td>300</td>
<td>4926</td>
<td>5074</td>
</tr>
<tr>
<td>Time 1.0</td>
<td>5074</td>
<td>5226</td>
<td>152</td>
<td>5074</td>
<td>0</td>
</tr>
</tbody>
</table>

Suppose the mortgage is divided into two sequential tranches, A and B, each with 5000 of principal. Each period, each tranche receives interest on its beginning principal balance. Additional mortgage payments go to pay down the principal balance of tranche A, until A is paid off. Then mortgage payments in excess of interest due go to pay down tranche B’s principal balance.

a) (6 points) Suppose the mortgage has no prepayments. Determine the cash flows to each tranche by filling in the table below:

<table>
<thead>
<tr>
<th>Period ending</th>
<th>A Interest</th>
<th>A Principal</th>
<th>B Interest</th>
<th>B Principal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time 0.5</td>
<td>150</td>
<td>4926</td>
<td>150</td>
<td>0</td>
</tr>
<tr>
<td>Time 1.0</td>
<td>2</td>
<td>74</td>
<td>150</td>
<td>5000</td>
</tr>
</tbody>
</table>

b) Suppose the mortgagor can pay off the mortgage on any payment date by paying the remaining principal balance. Assume that the mortgagor follows a prepayment policy that minimizes the value of the mortgage.

(i) (7 points) What is the value of the whole mortgage at time 0?

Time 0
prepay: 10000

wait: \( (5226 + 0.5 \times (5074 + 5074)) \times 0.9709 = 9987 \)

Time 0.5
\[
\begin{align*}
\text{min}(5074, 5226 \times 0.9657) &= 5047 \\
\text{(wait)} \\
\text{wait: } (5226 + 0.5 \times (5047 + 5074)) \times 0.9709 &= 9987 \\
\end{align*}
\]

(ii) (6 points) What is the time 0 value of the A tranche?
(iii) (6 points) What is the time 0 value of the B tranche?

B tranche = whole mortgage - A tranche = 9987 - 5000 = 4987

5) a) (3 points) Consider an American call option on $100 par of the zero maturing at time 1.5. The option expires at time 0.5, and has strike price 94. The option is exercisable at time 0 or at time 0.5. What is the time 0 value of this option?

\[
\begin{align*}
\text{Time 0} \\
\text{ex: } 91.19 - 94 < 0 \\
\text{wait: } 0.5 \times (0 + 0.62) \times 0.9709 = 0.30
\end{align*}
\]

\[
\begin{align*}
\text{Time 0.5} \\
\text{max}(93.23 - 94, 0) = 0
\end{align*}
\]

b) (5 points) Consider an American put option on $100 par of the zero maturing at time 1.5. The option expires at time 0.5, and has strike price 94. The option is exercisable at time 0 or at time 0.5. What is the time 0 value of this option?

\[
\begin{align*}
\text{Time 0} \\
\text{ex: } 94 - 91.19 = 2.81 \\
\text{wait: } 0.5 \times (0.77 + 0) \times 0.9709 = 0.37
\end{align*}
\]

\[
\begin{align*}
\text{Time 0.5} \\
\text{max}(94 - 93.23, 0) = 0.77 \\
\text{max}(94 - 94.62, 0) = 0
\end{align*}
\]

c) (6 points) Does put-call parity hold for the values of the call and put options in parts (a) and (b)? Explain.

\[
C = P + V - dtK?
0.30 \neq 2.81 + 91.19 - 0.9709 \times 94 = 2.74
\]

Put-call parity need not hold for American options. It fails here because the put is exercised early, but the call is not.

6) Consider a 1-year inverse floating rate note with a coupon equal to 12% minus the 0.5-year rate set 6 months prior to the coupon date, with the additional provision that the coupon cannot fall below 6%. In other words, each coupon date \( t \) the coupon payment of this note is \( \max(0.12 - r_{t-0.5}, 0.06) / 2 \) times the par amount of the note.
a) (12 points) What is the value of $100 par of this note at time 0?

APPROACH #1
Coupon of inverse floater with floor = coupon of pure inverse floater + “coupon” of 6% cap:
\[
\max(0.12 - t - 0.5r_t, 0.06)/2 = 0.12 - t - 0.5r_t + \max(t - 0.5r_t - 0.06, 0)
\]

Par payment of inverse floater = par payment of pure inverse floater.

Therefore, inverse floater = pure inverse floater + 6% cap

pure inverse floater (12%) = 2 \times \text{Fixed rate bond (6%)} - \text{floater}
2 \times 99.836 \text{ (from (2a))} - 100 = 99.672

6% cap (= call on time 0.5 6-month rate):
\[
\begin{align*}
\text{Time 0} & \quad \text{Time 0.5} \\
0.5 \times (0.536+0) \times 0.9709 = 0.26 & \quad 0.9657 \times \max(7.11-6, 0)/2 = 0.536 \\
& \quad 0.9728 \times \max(5.59-6, 0) = 0
\end{align*}
\]

(It turns out that this call on the time 0.5 6-month rate is identical to the put in (2d); the difference is just rounding error.)

Inverse floater = 99.672 + 0.26 = 99.932

APPROACH #2
Valuing cash flows directly:
\[
\begin{align*}
\text{Time 0} & \quad \text{Time 0.5} \\
[\max(12-6, 6)/2 + 0.5 \times (99.467+100.398)] \times 0.9709 & \quad (100+\max(12-7.11, 6)/2) \times 0.9657 \\
= 99.937 & = 99.467 \\
\end{align*}
\]

(difference is rounding error)

b) (5 points) What par value of the 1-year 6% coupon bond of question (2a) must be held to hedge a long position in $100 par of the inverse floater in part (b) of this question?

\[
\text{ir}\Delta \text{ (fixed rate bond from (2a))} = -(99.467 - 100.198)/(0.0711-0.0559) = 48
\]

\[
\text{ir}\Delta \text{ (inverse floater):}
\]

With approach #1, \(\text{ir}\Delta \text{ (inverse floater)} = 2 \times \text{ir}\Delta \text{ (fixed)} + \text{ir}\Delta \text{ (cap)}\)

\[
\text{ir}\Delta \text{ (cap)} = -(0.536 - 0)/(0.0711-0.0559) = -35
\]

\[
\text{ir}\Delta \text{ (inverse floater)} = 2 \times 48 + (-35) = 61
\]
With approach #2, \( \Delta (\text{inverse floater}) = \frac{-(99.467 - 100.398)}{(0.0711 - 0.0559)} = 61 \)

Hedge ratio = 61/48 = 1.27

To hedge a \textit{long} position in the inverse floater, go \textit{short} $127 par of 6\% fixed rate bond.

Alternative approach:

Net position = \( N \) fixed rate bonds + inverse floater. Choose \( N \) to make value of net position in up state = value in down state:

\[
0.99467N + 99.467 = 1.00198N + 100.398
\]

\( \Rightarrow N = \frac{(99.467-100.398)}{(1.00198-0.99467)} = -127 \)