Problem Set 2: Duration, Convexity, Immunization

1) The 0.5-year zero rate is 10% and the 1-year zero rate is 12%.
   a) What is the price of
      (i) $1 par of a 0.5-year zero?
      (ii) $1 par of a 1-year zero?
      (iii) $100 par of a 1-year 12%-coupon bond, in the absence of arbitrage?

   b) What is the dollar duration of
      (i) $1 par of a 0.5-year zero?
      (ii) $1 par of a 1-year zero?
      (iii) $100 par of a 1-year 12%-coupon bond?

   c) What is the duration of
      (i) $1 par of a 0.5-year zero?
      (ii) $1 par of a 1-year zero?
      (iii) $100 par of a 1-year 12%-coupon bond?

2) The current price of $1 par of a zero maturing at time 2 is $0.90
   a) What is the 2-year spot rate?
   b) What is the dollar duration of $1 par of the 2-year zero?

   The current price of $1 par of a zero maturing at time 3 is $0.84
   c) What is the 3-year spot rate?
   d) What is the dollar duration of $1 par of the 3-year zero?
   e) What is the dollar convexity of $1 par of the 3-year zero?
   f) Using dollar duration alone, approximate the change in the value of $1,000,000 par of the
      3-year zero given an immediate decline in all discount rates of 50 basis points.
   g) Using dollar duration and dollar convexity, approximate the change in the value of
      $1,000,000 par of the 3-year zero given an immediate decline in all discount rates of 50
      basis points.

   You can enter into a forward contract today to buy, at time 2, $1 par of a zero maturing at
   time 3. The price you would pay at time 2 is the forward price. The cost today of entering
   into this contract is zero.
   h) Construct a portfolio of 2- and 3-year zeroes that synthesizes this forward contract.
   i) What is the no arbitrage forward price?
   j) What is the forward rate from time 2 to time 3?
   k) What is the dollar duration of the forward contract?
3) Suppose you have a short position in a 30-year 6%-coupon bond and a long position in a zero-coupon bond with exactly the same market value and duration. If all zero rates fall by 20 basis points, will your net position rise or fall in value? Explain.

4) Your liabilities have market value 1,000,000, duration of 10, and convexity of 125. Write down but do not solve equations that determine a portfolio of the 3 assets listed below that immunizes your net position by matching the market value, duration, and convexity of your liabilities. Let \( N_1, N_2, \) and \( N_3 \) represent the number of units of assets 1, 2, and 3 in the portfolio.

<table>
<thead>
<tr>
<th></th>
<th>Market Value</th>
<th>Dollar Duration</th>
<th>Dollar Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset #1</td>
<td>744</td>
<td>3612</td>
<td>19288</td>
</tr>
<tr>
<td>Asset #2</td>
<td>554</td>
<td>5375</td>
<td>54799</td>
</tr>
<tr>
<td>Asset #3</td>
<td>412</td>
<td>6000</td>
<td>90288</td>
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</tbody>
</table>

5) a) Your liabilities have a market value of $1,000,000, a duration of 6, and a convexity of 50.

   (i) Using duration alone (without convexity), approximate the change in the value of your liabilities if all zero rates decline 40 basis points.
   (ii) Using duration and convexity, approximate the change in the value of your liabilities if all zero rates decline 40 basis points.

b) You want to immunize your position by constructing a portfolio of the three assets below that has the same market value, duration, and convexity as your liabilities. Write down but do not solve equations that determine the number of units of each asset in the portfolio. Use notation \( N_1, N_2, \) and \( N_3 \) to represent the number of units of assets 1, 2, and 3, respectively.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Market Value</th>
<th>Duration</th>
<th>Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
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<td>2</td>
<td>4</td>
</tr>
<tr>
<td>#2</td>
<td>200</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>#3</td>
<td>300</td>
<td>10</td>
<td>200</td>
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