# Corporate Bond Valuation and Hedging with Stochastic Interest Rates and Endogenous Bankruptcy

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This paper analyzes corporate bond valuation and optimal call and default rules when interest rates and firm value are stochastic. It then uses the results to explain the dynamics of hedging. Bankruptcy rules are important determinants of corporate bond sensitivity to interest rates and firm value. Although endogenous and exogenous bankruptcy models can be calibrated to produce the same prices, they can have very different hedging implications. We show that empirical results on the relation between corporate spreads and Treasury rates provide evidence on duration, and we find that the endogenous model explains the empirical patterns better than do typical exogenous models.

Corporate bonds are standard investment instruments, yet the embedded options they contain are quite complex. Most corporate bonds are callable and call provisions interact with default risk. In any case, corporate bond investors face the problem of managing interest rate and credit risk simultaneously.

This paper examines the valuation and risk management of callable defaultable bonds when both interest rates and firm value are stochastic and when the issuer follows optimal call and default rules. To our knowledge, this is the first model of coupon-bearing corporate debt that incorporates both stochastic interest rates and endogenous bankruptcy. Existing models either treat interest rates as constant or impose exogenous default rules. These assumptions can significantly impact bond pricing and hedging. Yield spreads can be sensitive to interest rate levels, volatility, and correlation with firm value. Spreads are also sensitive to assumptions about the bankruptcy process. Some exogenous bankruptcy specifications produce negative spreads. Even when they guarantee positive spreads, exogenous default models can have hedging implications that are very different from those of endogenous default models.

Working with a general Markov interest rate process, we develop analytical results about the existence and shape of optimal call and default boundaries.

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Then we numerically study the dynamics of hedging, using the results on exercise boundaries to explain patterns in bond duration and sensitivity to firm value. Finally, we link duration to the slope coefficient in a regression of changes in yield spreads on changes in interest rates and find that the endogenous bankruptcy model seems to explain empirical patterns in the spread-rate relation better than typical exogenous bankruptcy models.

To clarify the interaction between call provisions and default risk, we model the callable defaultable bond together with its pure callable and pure defaultable counterparts. We view each of the three bonds as a host bond minus a call option on that host bond. The call options differ only in their strike prices. The strike of the pure call is the provisional call price. The strike of pure default option is firm value. The strike of the option to call or default is the minimum of the two.

Treating defaultables like callables illuminates their similarities and differences. For example, spreads on all bonds, not just callables, narrow with interest rates because all embedded option values decline with the value of the underlying host bond. On the other hand, credit spreads can increase or decrease with interest rate risk, depending on how interest rates correlate with firm value.

The paper provides a number of analytical results. With regard to valuation, we prove that all three bond prices are increasing in the host bond price, but at rates less than one. The corporate bond prices are also increasing in firm value, at rates less than one. With regard to optimal call and default rules, we establish the existence and shape of optimal exercise boundaries. Like the optimal exercise policy for the pure callable, the optimal policies for corporate bonds are defined by a critical host bond price above which the bond issuer either calls or defaults and below which he continues to service the debt. In the case of the corporate bonds, this critical host bond price is a function of firm value, forming an upward-sloping boundary for noncallables and a hump-shaped boundary for callables.

We also compare the different boundaries, showing how the call and default options embedded in the callable defaultable bond interact on its optimal exercise policy. The default region of the callable defaultable bond is smaller than that of the pure defaultable and its call region is smaller than that of the pure callable. When both options are present, the value of preserving one option can make it optimal for the issuer to continue servicing the debt when it would otherwise exercise the other option.

We then numerically study the dynamics of hedging. Since duration is high when call and default are remote, the exercise boundaries explain a variety of patterns in duration. First, all bond durations are decreasing in the host bond price as increases in the host bond price bring the bonds closer to the exercise boundary. Second, as functions of firm value, bond durations inherit the shape of the boundaries, because the boundary quantifies how far away the bond is from call or default. Thus, the duration of the pure defaultable bond is increasing in firm value, while the duration of the callable defaultable bond is hump-shaped. In addition, the call and default options interact on duration. A call provision by itself reduces duration, as does default risk by itself. However, a call provision can increase the duration of a defaultable bond and default risk can increase the duration of a callable bond because the presence of one option delays the exercise of the other.

Next, we draw a link between duration and the slope coefficient in the regressions of changes in corporate yield spreads on changes in Treasury bond rates performed by Duffee (1998). The variation in this slope coefficient across bond rating gives evidence on the empirical relation between duration and firm value. In Duffee's study, these slope coefficients are increasing in bond rating for noncallable bonds and hump-shaped in bond rating for callable bonds, like the duration-firm value functions implied by our model. By contrast, in a model with exogenous default rules, as typically specified in the literature, duration is a U-shaped function of firm value near default.

Finally, we illustrate the dynamics of bond sensitivity to firm value. Sensitivity to firm value is high when default is near and low when call is near. This explains three effects. First, the pure defaultable bond's sensitivity to firm value is increasing in the host bond price, as increases in the host bond price bring the bond closer to default. Second, both the callable defaultable and the pure defaultable bond sensitivity to firm value decrease in firm value as default becomes remote. Third, the sensitivity of the callable defaultable bond is uniformly lower than that of the noncallable because default is always farther away. This last effect suggests that a call provision mitigates the underinvestment problem of levered equity described by Myers (1977).

The paper proceeds as follows. Section 1 summarizes the related literature. Section 2 describes the financial market and the bonds with embedded options and gives analytical results on valuation and numerical results on yield spreads. Section 3 contains analytical results on optimal call and default policies. Section 4 studies corporate bond risk management. Section 5 concludes.

## 1. Related Literature

Much of the existing theory of defaultable debt treats interest rates as constant in order to focus on the problems of optimal or strategic behavior of competing corporate claimants. Merton (1974) analyzes a risky zero-coupon bond and characterizes the optimal call policy for a callable coupon bond. Brennan and Schwartz (1977a) model callable convertible debt. Black and Cox (1976) and Geske (1977) value coupon-paying debt when asset sales are restricted and solve for the equity holders' optimal default policy. Fischer, Heinkel, and Zechner (1989a,b), Leland (1994), Leland and Toft (1996), Leland (1998), and Goldstein, Ju, and Leland (2000) embed the optimal default policy, and in some cases, the optimal call policy, in the problem of optimal capital structure. Models such as Anderson and Sundaresan (1996), Huang (1997), Mella-Barral and Perraudin (1997), Acharya et al. (2002), and Fan and Sundaresan (2000) introduce costly liquidations and treat bankruptcy as a bargaining game.

Other models allow for stochastic interest rates and take a different approach to the treatment of bankruptcy. Some impose exogenous bankruptcy triggers in the form of critical asset values or payout levels. These include the models of Brennan and Schwartz (1980), Kim, Ramaswamy, and Sundaresan (1993), Nielsen, Saá-Requejo, and Santa-Clara (1993), Longstaff and Schwartz (1995), Brys and de Varenne (1997), and Collin-Dufresne and Goldstein (2001). Cooper and Mello (1991) and Abken (1993) model defaultable swaps assuming that equity holders can sell assets to make swap or bond payments. Shimko, Tejima, and Van Deventer (1993) model a zero-coupon bond. Other papers model default risk with a hazard rate or stochastic credit spread. See, for example, Ramaswamy and Sundaresan (1986), Jarrow, Lando, and Turnbull (1993), Madan and Unal (1993), Jarrow and Turnbull (1995), Duffie and Huang (1996), Duffie and Singleton (1999), Das and Sundaram (2000), and Acharya, Das, and Sundaram (2002).

Another related literature analyzes callable bonds with stochastic interest rates in the absence of default risk. This includes Brennan and Schwartz (1977b) and Courtadon (1982). Related work on American options on nondefaultable bonds includes Ho, Stapleton, and Subrahmanyam (1997), Jorgensen (1997), and Peterson, Stapleton, and Subrahmanyam (1998). Amin and Jarrow (1992) provide a general analysis of American options on risky assets in the presence of stochastic interest rates.

## 2. Valuation

This section first describes the financial market and corporate setting formally and develops a framework which treats all issuer options as call options on an underlying host bond. Then we present analytical results about bond and option values and illustrate some implications for yield spreads.

### 2.1 Interest rate and firm value specifications

Suppose investors can trade continuously in a complete, frictionless, arbitrage-free financial market. There exists an equivalent martingale measure $\widetilde{\mathscr{P}}$ under which the expected rate of return on all assets at time $t$ is equal to the interest rate $r_t$. The interest rate is a nonnegative one-factor diffusion described by the equation

$$dr_t = \mu(r_t, t)\,dt + \sigma(r_t, t)\,d\widetilde{Z}_t, \tag{1}$$

where $\widetilde{Z}$ is a Brownian motion under $\widetilde{\mathcal{P}}$ and $\mu$ and $\sigma$ are continuous and satisfy Lipschitz and linear growth conditions. That is, for some constant $L$, $\mu$ and $\sigma$ satisfy

$$|\mu(x,t)-\mu(y,t)|+|\sigma(x,t)-\sigma(y,t)| \leq L|x-y|, \tag{2}$$

$$|\mu(x,t)|+|\sigma(x,t)| \leq L(1+|x|) \tag{3}$$

for all $x, y, t \in \mathcal{R}^+$.

Next, consider a firm with a single bond outstanding. The bond has a fixed continuous coupon $c$ and maturity $T$. Without loss of generality, suppose the par value of the bond is one, and all other values are in multiples of this par value. The value of the firm is equal to the value of its assets, $V$, independent of its capital structure. Firm value evolves according to the equation

$$\frac{dV_t}{V_t} = (r_t - \gamma_t)\,dt + \phi_t d\widetilde{W}_t, \tag{4}$$

where $\widetilde{W}$ is a Brownian motion under $\widetilde{\mathcal{P}}$ with $d\langle \widetilde{W}, \widetilde{Z}\rangle_t = \rho_t\,dt$ and $\gamma_t \geq 0$, $\phi_t > 0$, and $\rho_t \in (-1, 1)$ are deterministic functions of time. Protective bond covenants prevent equity holders from altering the firm's payout rate $\gamma$ or volatility $\phi$.

### 2.2 Option and bond valuation

We consider the case that the firm's bond is callable with a call price schedule $k_t$. To clarify the interaction between the call provision and default risk, we also model the pure defaultable version and the pure callable version. The pure defaultable is the noncallable bond with same coupon, maturity, and issuer. The pure callable is the nondefaultable bond with same coupon, maturity, and call provision.

The pure callable bond is equivalent to a noncallable, nondefaultable host bond with the same coupon and maturity minus a call option on that host bond with strike price equal to the provisional call price. Letting $P_t$ denote the price of the host bond, the payoff of exercising the option at time $t$ is $P_t - k_t$. We assume that $k_t$ lies below the supremum of $P_t$ for all $t \in [0, T)$, so that the option is always nontrivial.

The pure defaultable bond can also be viewed as a host bond minus a call option on that host bond, but the strike price is equal to firm value $V_t$. The firm's owners are long the firm assets, short the host bond, and long an option to default. This option to default, or buy back the bond in exchange for the firm, is a kind of call on the host bond. Its exercise value is $P_t - V_t$.

When the bond is both callable and defaultable, it is again equivalent to a host bond minus a call option on that host bond. In this case the strike price is equal to the minimum of the provisional call price and firm value, $k_t \wedge V_t$. The issuer is long the firm, short the host bond, and long the option to stop servicing the debt, i.e., buy back the host bond, either by calling and paying $k_t$ or by giving up the firm worth $V_t$. The exercise value of this option to stop servicing the debt is $P_t - k_t \wedge V_t$.

If the bond indenture includes minimum net worth or net cash flow covenants, the corporate issuer may be forced to default when firm value $V$ or asset cash flow $\gamma V$ fall too low. We suppose that no such covenants exist, so the optimal time to exercise the option to call or default is endogenous. Indeed, as Black and Cox (1976) and others have shown, when asset cash flow is insufficient to cover bond coupon payments, it may still be in equity holders' interest to meet coupon payments by raising new equity in order to retain ownership of the firm. Of course, if bond has zero coupon, it will never be optimal to default prior to maturity.

Formally, an exercise policy for the option to call or default is a stopping time of the filtration $\{\mathcal{F}_t\}$ generated by the paths of the interest rate and firm value. An optimal exercise policy maximizes the current option value. The optimal option value at an arbitrary time $t$ in the life of the option is

$$\zeta_t \equiv \sup_{t \le \tau \le T} \widetilde{E}\left[\beta_{t,\tau}(P_\tau - \kappa(V_\tau, \tau))^+ | \mathcal{F}_t\right], \tag{5}$$

where $\widetilde{E}[\cdot]$ denotes the expectation under the measure $\widetilde{\mathcal{P}}$, the strike price

$$\kappa(v, t) = k_t, v, \text{ or } k_t \wedge v, \tag{6}$$

depending on the bond in question, and the discount factor

$$\beta_{t,\tau} \equiv e^{-\int_t^\tau r_s ds}. \tag{7}$$

Under the Markov interest rate specification in Equations (1)–(3), the host bond price

$$P_t = \widetilde{E}\left[c \int_t^T \beta_{t,s} ds + 1 \cdot \beta_{t,T} \mid \mathcal{F}_t\right] \tag{8}$$

$$= p_H(r, t) \tag{9}$$

for some function $p_H$: $\mathcal{R}^+ \times [0, T] \to \mathcal{R}$. Furthermore, $p_H(\cdot, t)$ is strictly decreasing and continuous, and therefore has a continuous inverse. These properties, and the specification of the firm value process in Equation (4), allow us to invoke Theorems 3.8 and 3.10 of Krylov (1980) to conclude that, given $P_t = p$, and $V_t = v$,

$$\zeta_t = f(p, v, t) \tag{10}$$

for some continuous function $f$: $\mathcal{R}^+ \times \mathcal{R}^+ \times [0, T] \to \mathcal{R}$, satisfying

$$f(p, v, t) \ge (p - \kappa(v, t))^+. \tag{11}$$

Furthermore, the optimal stopping time is

$$\tau = \inf\{t \geq 0\colon f(P_t, V_t, t) = (P_t - \kappa(V_t, t))^+\}. \tag{12}$$

***Theorem 1.*** *The following properties hold for all three embedded options.*

1. $p^{(1)} > p^{(2)} \Rightarrow f(p^{(1)}, v, t) > f(p^{(2)}, v, t)$.
2. $v^{(1)} < v^{(2)} \Rightarrow f(p, v^{(1)}, t) \geq f(p, v^{(2)}, t)$.
3. $p^{(1)} \neq p^{(2)} \Rightarrow \frac{f(p^{(2)},v,t)-f(p^{(1)},v,t)}{p^{(2)}-p^{(1)}} \leq 1$. *(Call delta inequality)*
4. $v^{(1)} \neq v^{(2)} \Rightarrow \frac{f(p,v^{(2)},t)-f(p,v^{(1)},t)}{v^{(2)}-v^{(1)}} \geq -1$. *(Put delta inequality)*

Like ordinary calls, the option values are increasing in the underlying host bond price, but the rate of increase is bounded by one. Like ordinary puts, the defaultable bond options are decreasing in the underlying firm value, but the rate of decrease is bounded by minus one. The proofs, in Appendix A, are inspired by the analysis of Jacka (1991).

The value of the bond with an embedded option is

$$p_X(p, v, t) = p - f_X(p, v, t), \tag{13}$$

where the subscript $X$ is $C$ for the pure callable bond, $D$ for the pure defaultable, and $CD$, for the callable defaultable. Theorem 1 implies that the bond prices are increasing functions of the host bond price and firm value, but the rates of increase are bounded by one. It follows that the effective duration of the bonds, the percent increase in bond price for a decrease in the host bond yield, is nonnegative. By contrast, in models with exogenous default rules, duration can become negative, as Longstaff and Schwartz (1995) observe.

***Proposition 1.*** *The values of the different embedded options relate as follows.*

$$f_C(p, v, t) \vee f_D(p, v, t) \leq f_{CD}(p, v, t) \leq f_C(p, v, t) + f_D(p, v, t). \tag{14}$$

The combined option to call or default is worth more than either of the simple options because it has a lower strike price. However, the combined option is worth less than the sum of simple options. This is because, with the combined option, calling destroys the default option, and defaulting destroys the call option. Kim, Ramaswamy, and Sundaresan (1993) call this the "interaction effect." In terms of yields to maturity, this means that the incremental spread created by a call provision will be less for a corporate bond than for a Treasury, as Kim, Ramaswamy, and Sundaresan (1993) observe. In addition, the interaction effect implies that the "option-adjusted" credit spread between a callable defaultable bond and its callable Treasury counterpart is less than the credit spread of the noncallable issue. More generally, an option-adjusted spread computed in this fashion will vary with the nature of the call provision and may therefore be an unreliable measure of the compensation a bond offers for its credit risk.

### 2.3 Yield spreads

Practitioners typically quote corporate bond prices in terms of the spread of their yields over the yield of the comparable Treasury bond. In addition, empirical work on corporate bond pricing often focuses on yield spreads. In our model, the yield spread of a given bond over its host bond is a straightforward transformation of the bond's embedded option value, $f$. Recognizing that

$$f(p, v, t) = \widetilde{E}[\beta_{t,T}(P_T - \kappa(V_T, T))^+ | \mathcal{F}_t] \tag{15}$$

and using intuition from option theory can explain many patterns in yield spreads.

**2.3.1 Yield spreads and the level of interest rates.** Duffee (1998) finds empirically that spreads on all bonds, not just callables, narrow with interest rates. In particular, he reports significantly negative estimates for the coefficient $b_1$ in regressions of the form

$$\Delta\text{SPREAD}_t = b_0 + b_1 \Delta Y_{1/4,t} + b_2 \Delta\text{TERM}_t + \varepsilon_t, \tag{16}$$

where *SPREAD* is the mean spread of the yields of corporate bonds in a given sector over equivalent maturity Treasury bonds, $Y_{1/4}$ is the 3-month Treasury yield, and *TERM* is the difference between the 30-year constant-maturity Treasury yield and the 3-month Treasury bill yield. With *TERM* included in the regression, the coefficient $b_1$ essentially measures the change in the bond spread with respect to a parallel yield curve shift. Duffee (1996) investigates the possibility that the negative spread-rate relation stems from a positive correlation between interest rates and firm values by including S&P 500 returns in the regression. He finds that this has little effect on the estimates of $b_1$. Thus, it is reasonable to interpret estimates of $b_1$ as measures of the derivative of the spread with respect to the host bond yield, holding firm value constant.

Our model explains this pure interest rate effect on spreads by viewing all embedded options as calls on the underlying host bond: as interest rates rise, the price of the underlying host bond falls, the call option value falls, and the spread narrows. To illustrate, Figure 1 plots the three embedded option values as functions of the underlying host bond and the associated yield spreads as functions of the host yield. The exact shape of the relation varies, but in each case, the spread-rate relation is like a mirror image of the call value-underlying bond value relation. These and other examples assume that interest rates follow a Cox, Ingersoll, and Ross (1985) process. Appendix B describes how we use a two-dimensional binomial lattice to approximate the interest rate and firm values processes, extending the method of Nelson and Ramaswamy (1990).

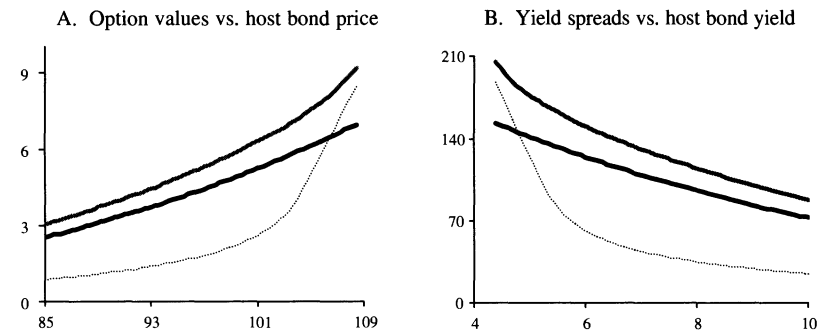


**Figure 1**
**Option values and yield spreads**
Three 5-year, 6.25%-coupon bonds: The gray line represents the callable defaultable, the black line represents the pure defaultable, and the dotted line represents the pure callable. Callable bond is currently callable at par. The default payoff to bond holders is firm value. Call and default policies minimize bond values. The instantaneous interest rate follows $dr = \kappa(\mu - r)\,dt + \sigma\sqrt{r}d\tilde{Z}$; $\kappa = 0.5$, $\mu = 6.8\%$, $\sigma = 0.10$. Firm value follows $dV/V = (r - \gamma)\,dt + \phi d\tilde{W}$; $\gamma = 0.0$, $\phi = 0.20$, $V_0 = 143$. The instantaneous correlation between the interest rate and firm value processes is zero. Numerical approximations use a two-factor binomial lattice.

Duffee (1998) documents three other empirical patterns. First, the negative spread-rate relation is usually stronger for callables. Second, among noncallables, the relation is stronger for lower grade bonds. Third, for callables, the relation is stronger for higher priced bonds. We explain these patterns by linking the spread-rate slope to the call option delta, $\frac{df_X}{dp}$. Letting $s_X \equiv y_X - y_H$ denote the spread between the yields of a given bond and its host and assuming $f$ is differentiable, we have

$$\frac{ds_X}{dy_H} = \left(1 - \frac{df_X}{dp}\right)\frac{dp/dy_H}{dp_X/dy_X} - 1. \tag{17}$$

We emphasize that the derivative $\frac{ds_X}{dy_H}$ is taken holding firm value constant. The terms $\frac{dp}{dy_H}$ and $\frac{dp_X}{dy_X}$ are derivatives of the same price-yield function, but evaluated at different points, so they differ only because of the convexity of the price-yield function. In particular, percentage changes in their ratio are small relative to percentage changes in $1 - \frac{df_X}{dp}$, so variation in the spread-rate slope is driven by variation in the call option delta. The embedded call option delta $\frac{df_X}{dp}$ tends to be larger when the option is deeper in the money. That is the case when either the strike price is lower, because of a call provision or because firm value is lower, or when the underlying host bond price is higher. This would explain the three empirical patterns listed above.

The connection between the spread-rate slope and the option delta indicates that the spread-rate slope is related to bond hedging. We develop this point in section 4.1.1, where we draw a link between the spread-rate slope and duration. We also explain why the negative spread-rate relation is not uniformly stronger for callables.

In the model of Longstaff and Schwartz (1995), spreads also narrow with interest rates, but through a different mechanism. There, default occurs when firm value falls to an exogenous boundary, and in that event, bond holders receive a fraction of the value of the host bond. As interest rates rise, the drift of firm value under the risk-neutral measure increases, decreasing the probability of default, and this causes spreads to decline. This effect of a rate increase on $V$ is at work in our model as well, but in addition, in our model, the rate increase narrows spreads by reducing $P$. By contrast, in the Longstaff and Schwartz model, the reduction in $P$ by itself may actually serve to widen spreads because it reduces the expected default payoff to bond holders.

**2.3.2 Yield spreads and interest rate volatility and correlation.** Existing models of corporate debt with endogenous default policies treat interest rates as constant. In the example of Brennan and Schwartz (1980), the assumption of constant interest rates has only a small effect on bond value. Similarly, Kim, Ramaswamy, and Sundaresan (1993) report that in their examples, spreads are fairly insensitive to the level of interest rate risk or correlation with firm value. However, such results do not generalize. Introducing stochastic interest rates can materially affect pricing.

To understand the impact of an increase in interest rate volatility on corporate spreads, it is again useful to view the spread as a transformation of the value of the issuer's option. From Equation (15), the corporate issuer's option value should increase in the volatility of $P-V$. The effect of an increase in interest rate volatility on the volatility of $P-V$ depends on the correlation $\rho$ between $r$ and $V$. When $\rho \geq 0$, interest rate volatility increases the volatility of $P-V$ and widens spreads. However, when $\rho < 0$, $P$ hedges changes in $V$, and, if the volatility of $P$ is low, an increase in the volatility of $P$ can improve this hedge, decreasing the volatility of $P-V$. Consequently, for negative values of $\rho$, option values and corporate yield spreads can decline as interest rate volatility rises. Figure 2 illustrates both cases.

Figure 3 illustrates the relation between yield spreads and the correlation between interest rates and firm value. The relation is positive. The higher the correlation between interest rates and firm value, the lower the covariance between $P$ and $V$, the higher the volatility of $P-V$, the higher the option value, and thus, the higher the corporate yield spread.

Shimko, Tejima, and Van Deventer (1993) and Longstaff and Schwartz (1995) also find that spreads widen with the correlation between firm value and interest rates. Shimko, Tejima, and Van Deventer (1993) analyze a zero-coupon bond. Longstaff and Schwartz (1995) model coupon-bearing bonds, as we do, but the mechanism for their correlation effect is different. When innovations in firm value correlate positively with interest rates, then they correlate positively with the drift of firm value under the risk-neutral measure, which increases the total variance of firm value. This increases the probability of default and widens spreads. A comparison of their examples

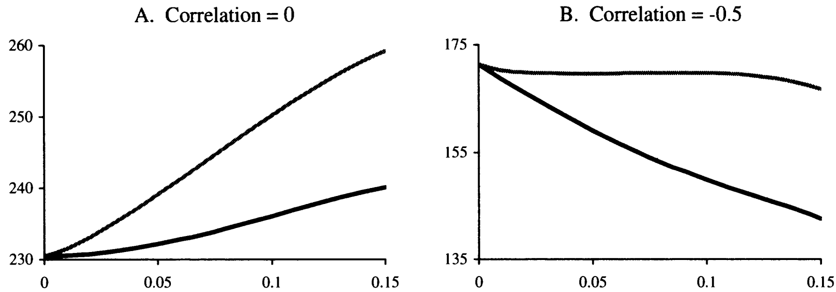


**Figure 2**
**Yield spreads vs. interest rate volatility**
Two 10-year, 9%-coupon corporate bonds—one noncallable, represented by the black line, and one callable at par, represented by the gray line. Call and default policies minimize bond values. The instantaneous interest rate follows $dr = \kappa(\mu - r)\,dt + \sigma\sqrt{r}\,d\tilde{Z}$; $\kappa = 0.5$, $\mu = 9\%$, $r_0 = 9\%$. Firm value follows $dV/V = (r-\gamma)\,dt + \phi\, d\tilde{W}$; $\gamma = 0.05$, $\phi = 0.15$, $V_0 = 93$. $\rho$ is the instantaneous correlation between the interest rate and firm value processes. Numerical approximations use a two-factor binomial lattice.

with ours suggests that the magnitude of the correlation effect is smaller in their model. One reason for this may be the difference in assumptions about default payoffs. In the event of default, bond holders in the Longstaff and Schwartz model get a fraction of host bond value, not the value of the firm. This means that the higher the correlation between interest rates and firm value, the higher the expected default payoff, which should by itself narrow spreads.

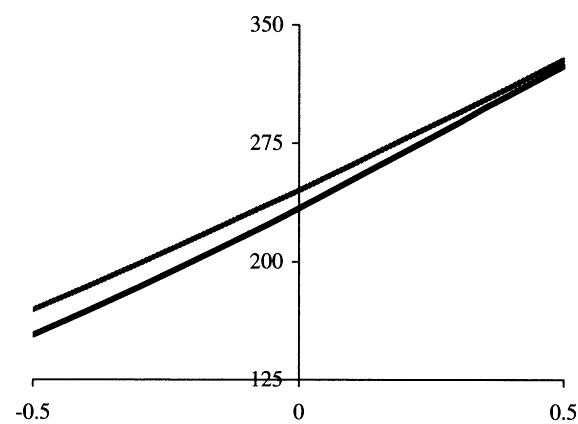


**Figure 3**
**Yield spreads vs. interest rate correlation with firm value**
Two 10-year, 9%-coupon corporate bonds—one noncallable, represented by the black line, and one callable at par, represented by the gray line. Call and default policies minimize bond values. The instantaneous interest rate follows $dr = \kappa(\mu - r)\,dt + \sigma\sqrt{r}\,d\tilde{Z}$; $\kappa = 0.5$, $\mu = 9\%$, $r_0 = 9\%$. Firm value follows $dV/V = (r-\gamma)\,dt + \phi\, d\tilde{W}$; $\gamma = 0.05$, $\phi = 0.15$, $V_0 = 93$. $\rho$ is the instantaneous correlation between the interest rate and firm value processes. Numerical approximations use a two-factor binomial lattice.

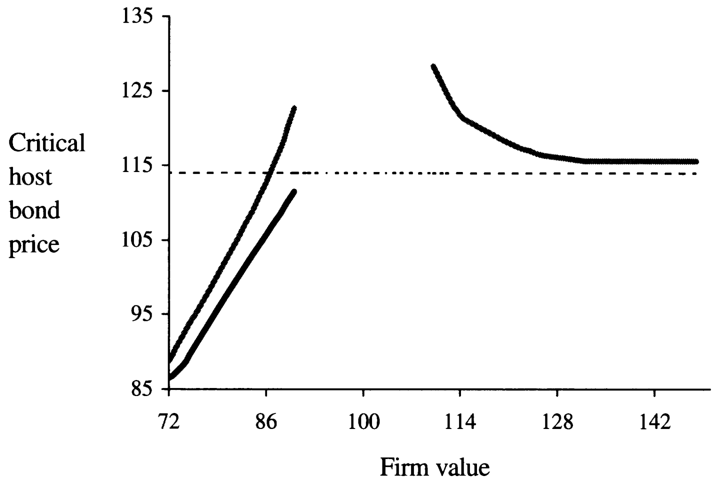


**Figure 4**
**Optimal call and default boundaries**
Critical host bond prices $b(v,t)$ for three 5-year, 10.25%-coupon bonds. The gray line corresponds to the callable defaultable, the black line corresponds to the pure defaultable, and the dotted line corresponds to the pure callable. For host bond prices below $b(v,t)$, it is optimal to continue, and for host bond prices above $b(v,t)$, it is optimal to default or call. Callable bonds are currently callable at par. The default payoff to bond holders is firm value. Call and default policies minimize bond values. The instantaneous interest rate follows $dr = \kappa(\mu - r)\,dt + \sigma\sqrt{r}d\tilde{Z}$; $\kappa = 0.5$, $\mu = 6.8\%$, $\sigma = 0.10$. Firm value follows $dV/V = (r-\gamma)\,dt + \phi d\tilde{W}$; $\gamma = 0.0$, $\phi = 0.20$. The instantaneous correlation between the interest rate and firm value processes is zero. Numerical approximations use a two-factor binomial lattice.

## 3. Optimal Call and Default Policies

This section proves that for each of the three bonds, pure callable, pure defaultable, and callable defaultable, a simple boundary separates the region of host bond and firm values where it is optimal for the bond issuer to continue servicing the debt from the region where it is optimal to call or default. The first theorem establishes the existence of a boundary of critical host bond prices. The second theorem describes the boundary in terms of critical firm values. The third theorem characterizes the shape and relation of the boundaries for the different types of bonds. Figure 4 illustrates the results.

***Theorem 2.*** *Let $t \in [0, T)$ and $v > 0$. If there is any bond price $p$ such that it is optimal to exercise the embedded option at time $t$ given $P_t = p$ and $V_t = v$, then there exists a critical bond price $b(v,t) > \kappa(v,t)$ such that it is optimal to exercise the option if and only if $p \geq b(v,t)$.*

We use the notation $b_C$, $b_D$, and $b_{CD}$ to distinguish the boundaries for the three bonds. In this orientation, an increase in the host bond price, or a decline in interest rates, triggers the option exercise. While it is natural to think of interest rate declines triggering bond calls, Theorem 2 implies that interest rate declines can also trigger defaults.

Models with constant interest rates describe the optimal call and default rules in terms of critical firm values, a lower critical firm value below which default is optimal and an upper critical firm value above which call is optimal [see, for example, Merton (1974), Black and Cox (1976), and Leland (1994, 1998), and Goldstein, Ju, and Leland (2000)]. Our next result states that this characterization is also valid when interest rates are stochastic, only now the critical firm values are functions of interest rates.

***Theorem 3.*** *Let $t \in [0, T)$ and $p > 0$.*

1. *For the pure defaultable bond, there exists a critical firm value $v_D(p, t) < p$ such that, at time $t$, given $P_t = p$ and $V_t = v$, it is optimal to default if and only if $v \leq v_D(p, t)$.*
2. *For the callable defaultable bond, there exists a critical firm value $v_{CD}(p, t)$, satisfying $v_{CD}(p, t) \leq k_t$ and $v_{CD}(p, t) < p$, such that, at time $t$, given $P_t = p$ and $V_t = v$, it is optimal to default if and only if $v \leq v_{CD}(p, t)$. In addition, if there exists any firm value $v$ at which it is optimal to call, then there exists a critical firm value $\bar{v}_{CD}(p, t) \geq k_t$ such that it is optimal to call if and only if $v \geq \bar{v}_{CD}(p, t)$.*

The next theorem describes the shape and relation of the different boundaries.

***Theorem 4.*** *For each $t \in [0, T)$,*

1. $v_1 < v_2 \Rightarrow b_D(v_1, t) \leq b_D(v_2, t)$.
2. $p_1 < p_2 \Rightarrow v_D(p_1, t) \leq v_D(p_2, t)$.
3. $v_1 < v_2 \leq k_t \Rightarrow b_{CD}(v_1, t) \leq b_{CD}(v_2, t)$.
4. $k_t < v_1 < v_2 \Rightarrow b_{CD}(v_1, t) \geq b_{CD}(v_2, t)$.
5. $v \leq k_t \Rightarrow b_{CD}(v, t) \geq b_D(v, t)$.
6. $v > k_t \Rightarrow b_{CD}(v, t) \geq b_C(v, t)$.

First, consider the default option embedded in the pure defaultable bond. Part 1 of Theorem 4 states that the critical bond price above which the firm should default, $b_D(v, t)$, is increasing in the firm value $v$. That is, the higher the firm value, the lower the interest rates must be to trigger a default. Conversely, Part 2 indicates that the critical firm value below which the equity holders should default is increasing in the host bond price $p$. In other words, in high interest rate environments, it takes lower firm values to make equity holders stop servicing the debt and give up the firm.

Next, consider the option to call or default embedded in a callable defaultable bond. For firm value below the call price, $v \leq k_t$, exercising the option means defaulting. For firm value greater than the call price, $v > k_t$, exercising means calling the bond. Part 3 of Theorem 4 indicates that the critical host bond price, $b_{CD}(v, t)$, above which it is optimal to default, is increasing in $v$, like $b_D(v, t)$. Part 4 indicates that the critical host bond price, $b_{CD}(v, t)$, above which it is optimal to call, is decreasing in $v$. At lower firm values, it takes lower interest rates to trigger a bond call.

Parts 5 and 6 of Theorem 4 describe the interaction of the call and default options on the optimal exercise policy. Part 5 states that the callable defaultable has a smaller default region than the pure defaultable. Part 6 states that the callable defaultable has a smaller call region than the pure callable. When both options are present, the value of preserving one option can make it optimal for the issuer to continue servicing the debt in states in which it would otherwise exercise the other option. These results will be useful for understanding the patterns in the risk measures presented below.

## 4. Hedging Interest Rate Risk and Credit Risk

A corporate bond is subject to both bond market risk and firm risk. In principle, a portfolio containing Treasuries and shares of the issuer's equity could serve to hedge both risks. The number of units of these instruments in the hedge portfolio, the hedge ratios, explicitly spell out the trading strategy for hedging and quantify the exposure to risks that the corporate bond imparts.

The market for the issuer's equity is generally much more active than the market for the firm's assets, so the hedge ratios in a hedge using host bonds and equity have more practical application than the hedge ratios in a hedge using host bonds and firm assets. However, the two pairs of hedge ratios are related through a simple transformation, and we find that their dynamics are qualitatively very similar. For ease of exposition, we work with the host bond-firm value hedge, because its dynamics can be understood through a more direct application of our model.

We use a bond's hedge ratio with respect to firm value, $\frac{dp_X}{dv}$, to measure its firm risk. However, instead of using a bond's hedge ratio with respect to the host bond, $\frac{dp_X}{dp}$, to measure its bond market risk, we use a similar but more widely recognized risk measure,

$$\text{duration} \equiv -\frac{dp_X/p_X}{dy_H}. \tag{18}$$

The measures $-\frac{dp_X/p_X}{dy_H}$ and $\frac{dp_X}{dp}$ behave similarly because their dynamics are both driven by the dynamics of the option delta $\frac{df_X}{dp}$. More precisely, $-\frac{dp_X/p_X}{dy_H} = \frac{dp_X}{dp}(-\frac{dp}{dy_H}\frac{1}{p_X})$ and the percentage changes in the factor $(-\frac{dp}{dy_H}\frac{1}{p_X})$ associated with changes in firm value and interest rates are small relative to the percentage changes in $\frac{dp_X}{dp} = 1 - \frac{df_X}{dp}$. The duration defined in Equation (18), sometimes called "effective duration," essentially measures price sensitivity to Treasury yields. By contrast, so-called "modified duration," $-\frac{dp_X/p_X}{dy_X}$, measures a bond's price sensitivity to its own yield. Both measures are used by practitioners, but effective duration is generally a more appropriate measure to use for risk management.

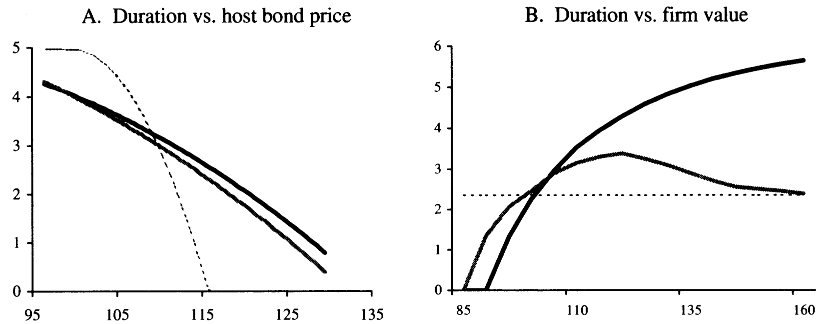


**Figure 5**
**Dynamics of duration**
Three 10-year, 11%-coupon bonds: the gray line represents the callable defaultable, the black line represents the pure defaultable, and the dotted line represents the pure callable. Duration is $-\frac{dp_X/p_X}{dy_H}$ where $p_X$ is the price of the bond in question and $y_H$ is the yield of its host bond. Callable bonds are currently callable at par. The default payoff to bond holders is firm value. Call and default policies minimize bond values. The instantaneous riskless rate follows $dr = \kappa(\mu - r)\,dt + \sigma\sqrt{r}\,d\tilde{Z}$; $\kappa = 0.5$, $\mu = 9\%$, $\sigma = 0.078$. Firm value follows $dV/V = (r - \gamma)\,dt + \phi\,d\tilde{W}$; $\gamma = 0.05$, $\phi = 0.15$. The instantaneous correlation between the interest rate and firm value processes is $\rho = -0.2$. Numerical approximations use a two-factor binomial lattice.

## 4.1 Duration

The duration of a bond tends to decline as call or default becomes imminent. This observation, together with our results on optimal exercise boundaries, explains how bond duration changes as interest rates and firm value change. Figure 5 illustrates the dynamics of duration for the pure callable, pure defaultable, and callable defaultable bonds. A number of properties are apparent. Figure 5A shows, for example, that all durations are decreasing in the host bond price. An increase in the host bond price is a move upward in the plot of the corresponding exercise boundaries shown in Figure 4, taking the bond closer to either call or default, and reducing its duration.

The relation between duration and firm value, shown in Figure 5B, has a similar explanation. A change in firm value moves the current state to the left or the right in Figure 4. A move to the left brings the bond closer to default, while a move to the right may bring the bond closer to call. But the vertical distance from the current state to the boundary is a measure of how close the bond is to stopping for all three kinds of bonds. Thus, to the extent that duration is high when the boundary is far away, the duration-firm value function should inherit the shape of the boundary. Figure 5B shows that the properties of the duration-firm value relation do indeed correspond closely to the properties of the boundary $b(v, t)$ listed in Theorem 4:

- The duration of the pure defaultable bond is increasing in firm value.
- The duration of the callable defaultable bond is increasing in firm value for low firm values and decreasing in firm value for high firm values.
- The duration of the callable defaultable bond exceeds that of the pure defaultable for low firm values.

- The duration of the callable defaultable bond exceeds that of the pure callable for high firm values.

These last two points describe an interaction effect on duration: a call provision by itself reduces duration, as does default risk by itself, but a call provision can increase the duration of a defaultable bond and default risk can increase the duration of a callable bond because the presence of one option delays the exercise of the other.

**4.1.1 Duration and the slope of the spread-rate relation.** The slope of the spread-rate relation studied empirically by Duffee (1998) and described in Section 2.3.1 is related to duration through the following equation.

$$\frac{ds_X}{dy_H} = \frac{\frac{dp_X/p_X}{dy_H}}{\frac{dp_X/p_X}{dy_X}} - 1 = \frac{\text{duration}_X}{\text{modified duration}_X} - 1. \tag{19}$$

Figure 6A shows that, as a function of firm value, the shape of the spread-rate slope is the same as that of duration as a function of firm value in Figure 5B.

To the extent that bond rating proxies for asset-debt ratio, the variation in Duffee's estimates for the spread-rate slope $b_1$ in Equation (16) across rating classes gives evidence on the empirical relation between duration and firm value. Now the duration-firm value relation shown in Figure 5B holds the

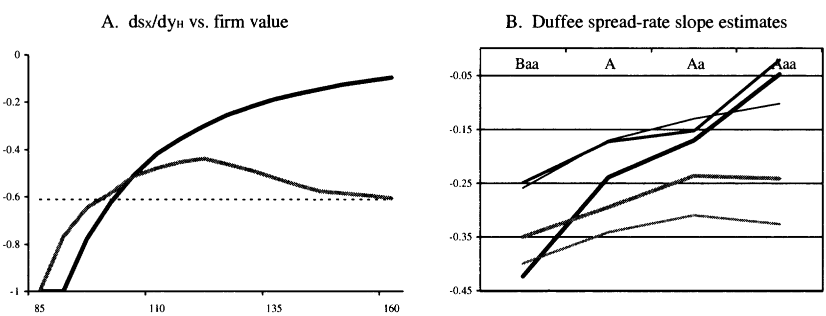


**Figure 6**
**Theoretical and empirical slopes of the spread-rate relation**
The theoretical slope in 6A is $ds_X/dy_H$, where $s_X$ is the yield spread of the bond in question and $y_H$ is the yield of its host bond. Callable bonds are currently callable at par. The default payoff to bond holders is firm value. Call and default policies minimize bond values. The instantaneous riskless rate follows $dr = \kappa(\mu - r)\,dt + \sigma\sqrt{r}d\tilde{Z}$; $\kappa = 0.5$, $\mu = 9\%$, $\sigma = 0.078$, $r_0 = 9\%$. Firm value follows $dV/V = (r-\gamma)\,dt + \phi d\tilde{W}$; $\gamma = 0.05$, $\phi = 0.15$. The instantaneous correlation between the interest rate and firm value processes is $\rho = -0.2$. Numerical approximations use a two-factor binomial lattice. The empirical slope in 6B is the estimate of $b_1$ in a regression of the form $\Delta\text{SPREAD}_t = b_0 + b_1\Delta Y_{1/4,t} + b_2\Delta\text{TERM}_t + \varepsilon_t$, where *SPREAD* is the mean spread of the yields of corporate bonds in a given sector over equivalent maturity Treasury bonds, $Y_{1/4}$ is the 3-month Treasury yield, and *TERM* is the difference between the 30-year constant-maturity Treasury yield and the 3-month Treasury bill yield, from Duffee (1998). Panel A: solid line, noncallable defaultable; light line, callable defaultable; broken line, callable. Panel B: thick solid line, noncallable–long; medium solid line, noncallable–medium; thin solid line, noncallable–short; thick light line, callable–long; medium light line, callable–intermediate.

bond coupon rate constant, while in the data, coupon rates decline as rating increases. However, examples suggest that if coupon rate varies to keep all bonds priced at par, then the noncallable bond duration remains upward-sloping in firm value, although the callable bond duration becomes flat at zero because the bonds are callable at par. The data most likely reflect a mixture of these cases. Coupon rates decline with rating, but not by so much that all bonds are at par. Lower rating classes contain more discount bonds and higher rating classes contain more premium bonds.

Figure 6B plots Duffee's (1998) estimates for $b_1$ for various bond rating classes within a given maturity sector. Like the duration-firm value graphs in Figure 5B, the curves for noncallable bonds are upward-sloping, while the curves for callable bonds are hump-shaped. Again, our model's explanation for these shapes lies in the shape of the endogenous default and call boundaries analyzed in Theorem 4 and illustrated in Figure 4. As bonds move away from default, the sensitivity of spreads to rates moves toward zero, but as callable bonds approach call, this sensitivity becomes large again.

**4.1.2 Duration in models with exogenous default boundaries.** In corporate bond models with exogenous default boundaries, default occurs when firm value falls to a pre-specified critical level [see, for example, Brennan and Schwartz (1980), Kim, Ramaswamy, and Sundaresan (1993), Longstaff and Schwartz (1995), and Brys and de Varenne (1997)]. If the model specifies a fixed payoff to bond holders in the event of default, then, in states when host bond prices are lower than this level, default can be a windfall to bond holders and bond spreads can become negative. Instead of specifying a fixed default payoff to bond holders, most exogenous default models set the bond default payoff equal to a fraction $\delta$ of the host bond price, which guarantees that default is not a benefit to bond holders [see, for example, Kim, Ramaswamy, and Sundaresan (1993) and Longstaff and Schwartz (1995)]. This, however, makes duration a sort of U-shaped function of firm value near default. Duration increases not only as firm value rises and the bond becomes like a nondefaultable, but also as firm value falls to the default level, and the bond tracks the host bond.

Figure 7 illustrates this effect with plots of duration vs. firm value for the exogenous model with $\delta = 0.8$ and $\delta = 0.4$. The range of firm values for these plots gives the noncallable bond in the case $\delta = 0.8$ approximately the same range of prices that it has in the plot of duration vs. firm value for the endogenous default model in Figure 5B. A comparison of these plots shows that although the two models can be calibrated to imply the same bond values, their implications for both the level and dynamics of duration can be very different.

Setting the bond payoff in default equal to a fraction of the host bond price also does not guarantee nonnegative duration. In particular, when the default payoff is set very low, bond duration can become negative near default. This

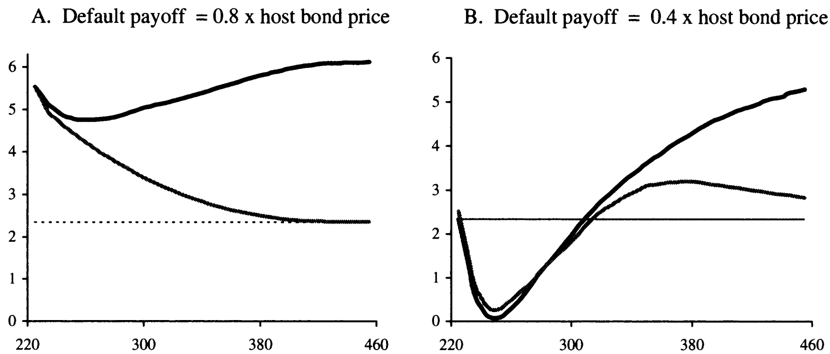


**Figure 7**
**Duration vs. firm value in exogenous default model**
Three 10-year, 11%-coupon bonds: the gray line represents the callable defaultable, the black line represents the pure defaultable, and the dotted line represents the pure callable. Duration is $-\frac{dp_X/p_X}{dy_H}$ where $p_X$ is the price of the bond in question and $y_H$ is the yield of its host bond. Callable bonds are currently callable at par. Default occurs when firm value hits 220. The default payoff to bond holders is a fraction of the host bond price. Call policies minimize bond values. The instantaneous riskless rate follows $dr = \kappa(\mu - r)\,dt + \sigma\sqrt{r}d\tilde{Z}$; $\kappa = 0.5$, $\mu = 9\%$, $\sigma = 0.078$, $r_0 = 9\%$. Firm value follows $dV/V = (r-\gamma)\,dt + \phi d\tilde{W}$; $\gamma = 0.05$, $\phi = 0.15$. The instantaneous correlation between the interest rate and firm value processes is $\rho = -0.2$. Numerical approximations use a two-factor binomial lattice.

is because increases in interest rates increase the drift of firm value, reducing the risk of default. Near default, the benefit of reducing the risk of default, which is catastrophic when the default payoff is very low, offsets the cost of the loss in the value of the bond's future promised payments.

By contrast, part 3 of Theorem 1 implies that duration is always nonnegative in our model. Table 1 compares the durations of bonds under the two models. The two bonds have the same coupon, maturity, and price, but bond duration is positive under the endogenous bankruptcy model and negative under the exogenous bankruptcy model. Again, the two models can imply the same bond price and yet have very different implications about hedging. More generally, in the presence of stochastic interest rates, it seems difficult to devise an exogenous default specification that has both the same pricing and same hedging implications as the endogenous model.

**Table 1**
**Duration under alternative bankruptcy assumptions**

| Default boundary | Default payoff | Maturity | Coupon rate | Yield spread | Duration | $V_0$ | $E\{V\vert\text{at default}\}$ |
|---|---|---|---|---|---|---|---|
| Endogenous | $V$ | 10 years | 9% | 720 bp | 0.8 | 65 | 60 |
| Exogenous | $0.2 \times P$ | 10 years | 9% | 720 bp | −0.6 | 118 | 75 |

Both bonds are noncallable. $P$ is the price of the noncallable, nondefaultable host bond with the same coupon and maturity. Duration is $\frac{dp_X/p_X}{dy_H}$ where $p_X$ is the price of the bond in question and $y_H$ is the yield of its host bond. The instantaneous riskless rate follows $dr = \kappa(\mu - r)\,dt + \sigma\sqrt{r}d\tilde{Z}$; $\kappa = 0.5$, $\mu = 9\%$, $\sigma = 0.078$, $r_0 = 9\%$. Firm value follows $dV/V = (r-\gamma)\,dt + \phi d\tilde{W}$; $\gamma = 0.12$, $\phi = 0.15$. The instantaneous correlation between the interest rate and firm value processes is $\rho = -0.2$. Numerical approximations use a two-factor binomial lattice.

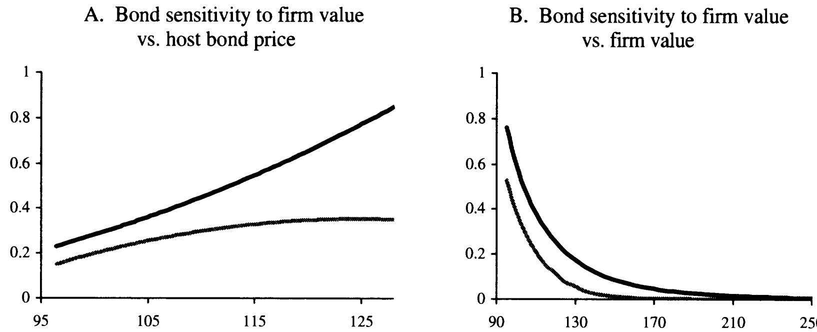


**Figure 8**
**Dynamics of bond sensitivity to firm value**
Two 10-year, 11%-coupon corporate bonds. One bond is noncallable, represented by the black line, and one bond is callable at par, represented by the gray line. Bond sensitivity to firm value is $\frac{dp_X}{dv}$ where $p_X$ is the price of the bond in question and $v$ is firm value. The default payoff to bond holders is firm value. Call and default policies minimize bond values. The instantaneous riskless rate follows $dr = \kappa(\mu - r)\,dt + \sigma\sqrt{r}d\tilde{Z}$; $\kappa = 0.5$, $\mu = 9\%$, $\sigma = 0.078$. Firm value follows $dV/V = (r - \gamma)\,dt + \phi d\tilde{W}$; $\gamma = 0.05$, $\phi = 0.15$. The instantaneous correlation between the interest rate and firm value processes is $\rho = -0.2$. Numerical approximations use a two-factor binomial lattice.

## 4.2 Bond price sensitivity to firm value

A bond's sensitivity to firm value is high when default is imminent and low when call is imminent. Therefore, we can again use the results on optimal call and default boundaries in Theorem 4 to explain the dynamics of hedging credit risk. Figure 8 plots corporate bond sensitivity to firm value as a function of the host bond price and as a function of firm value. Three effects are clear.

- The sensitivity of the noncallable bond increases in the host bond price, because increases in the host bond price bring the bond closer to default.
- The sensitivities of both the noncallable and callable bonds decrease in firm value, because as firm value rises, both bonds move away from the default boundary.
- The callable bond's sensitivity to firm value is uniformly lower than that of the noncallable,

because the callable bond is always farther from the default boundary. This last effect suggests that the presence of a call provision mitigates the underinvestment problem of levered equity. As Myers (1977) shows, since levered equity holders share increases in firm value with bond holders, they may pass up positive net present value projects that require all-equity financing. By increasing equity sensitivity to firm value, the call provision makes the underinvestment problem less severe.

## 5. Conclusion

This paper analyzes corporate bonds in a model in which the interest rate is a one-factor diffusion process and the issuer follows optimal call and default rules. By incorporating both stochastic interest rates and endogenous bankruptcy, the model bridges a gap in the corporate bond literature. The combination of these elements is of particular interest because bankruptcy assumptions significantly impact interest rate hedging.

A single framework encompasses the callable defaultable bond and its pure callable and pure defaultable counterparts, viewing each bond as a riskless, noncallable host bond minus a call on that host bond. This perspective provides intuition for the sensitivity of spreads to interest rate levels, volatility, and correlation with firm value. It also leads us to extend results for callable bonds to defaultable bonds.

The paper develops analytical results on corporate bond valuation and optimal call and default boundaries. Previous corporate bond models that provide analytical results with stochastic interest rates either work with a zero-coupon bond or else treat bankruptcy as exogenous, and thus avoid the issue of optimal default rules. By characterizing the solution to the two-dimensional optimal stopping time problem analytically, this paper makes an incremental theoretical contribution.

The optimal exercise boundaries explain the dynamics of hedging. For example, the critical host bond price above which it is optimal to exercise the embedded option is an increasing function of firm value for noncallable bonds. For callable bonds however, it is an increasing function at low firm values and a decreasing function at high firm values. This explains why noncallable bond duration is an increasing function of firm value while callable bond duration is a hump-shaped function of firm value. By contrast, under typical exogenous bankruptcy specifications, duration is a U-shaped function of firm value near default.

We interpret recent evidence on the relation between corporate bond yield spreads and Treasury bond yields as information about hedging and find that the empirical patterns in the spread-rate slope mirror the duration patterns implied by endogenous bankruptcy. In particular, our results on boundaries and durations explain why the slope of the empirical spread-rate relation is increasing in bond rating for noncallables, but hump-shaped for callables. A formal empirical test of the model's hedging implications would be an interesting subject for future research.

Much of the recent work in the corporate bond literature has focused on optimal or strategic behavior in the presence of frictions such as taxes or bankruptcy costs. In order to focus on optimal issuer behavior with stochastic interest rates, our paper uses a relatively simple contingent claims approach that abstracts from such frictions. One extension would be straightforward, however. For expositional purposes, the model presented here assumes that in the event of default, bond holders get the full value of the firm $V$ and equity holders get nothing. Yet we could easily incorporate deviations from absolute priority of the following form: in the event of default, bond holders get $\alpha V$ and equity holders get $(1-\alpha)V$, where $\alpha$ is a constant between zero and one. In that case, corporate bond values would become

$$p_X^{\alpha}(p, v, t) = p - f_X(p, \alpha v, t) = p_X(p, \alpha v, t). \tag{20}$$

Bond prices, yields, and durations would be invariant to $\alpha$ holding $\alpha v$ constant and sensitivity to firm value would adjust in a straightforward fashion. All of our analytical results would continue to hold, as would the qualitative nature of our numerical results.

This paper focuses on how changes in market conditions affect prices, spreads, durations, hedge ratios, and call and default decisions in the absence of frictions. However, many other issues surround the subject of corporate debt. One area of interest is term structure. Another is the dynamics of optimal capital structure with taxes, bankruptcy costs, and refinancing costs. The framework developed here could provide the foundation for research in a variety of different directions.

## Appendix A: Proofs

The proof of Theorem 1 makes use of a number of so-called *no-crossing properties*. The first follows from Proposition 2.18 of Karatzas and Shreve (1987):

***Proposition 2.*** *Consider two values of interest rates at time 0, $r_0^{(1)}$ and $r_0^{(2)}$ such that $r_0^{(1)} \leq r_0^{(2)}$, and denote the corresponding interest rate processes as $r_t^{(1)}$ and $r_t^{(2)}$, respectively. Then*

$$\widetilde{\mathcal{P}}\left[r_t^{(1)} \leq r_t^{(2)}, 0 \leq t < \infty\right] = 1. \tag{21}$$

This no-crossing property of $r$ implies no-crossing properties for $\beta$, $P$, $\beta P$, and $V$. For ease of exposition, let

$$\beta_t \equiv \beta_{0,t}. \tag{22}$$

***Corollary 1.*** *Let $\beta_t^{(1)}$ and $\beta_t^{(2)}$ be the discount factor processes corresponding to initial interest rates $r_0^{(1)}$ and $r_0^{(2)}$, respectively. Then*

$$r_0^{(1)} < r_0^{(2)} \Rightarrow \beta_t^{(1)} > \beta_t^{(2)}, \widetilde{\mathcal{P}} - a.s. \ \forall\ 0 < t < \infty. \tag{23}$$

*Proof.* From Proposition 2, we have $r_s^{(1)} \leq r_s^{(2)}$, $\forall\ 0 \leq s \leq t$. The paths of $r^{(1)}$ and $r^{(2)}$ are continuous, so there exists a neighborhood around $t = 0$ on which $r^{(1)} < r^{(2)}$. Consequently, $e^{-\int_0^t r_s^{(1)} ds} > e^{-\int_0^t r_s^{(2)} ds}$. ■

The monotonicity of the host bond price in level of the interest rate implies:

***Corollary 2.*** $r_0^{(1)} \leq r_0^{(2)} \Rightarrow P_t^{(1)} \geq P_t^{(2)}, \widetilde{\mathcal{P}} - a.s. \ \forall\ 0 \leq t \leq T.$

Combining Corollaries 1 and 2 yields:

***Corollary 3.*** $r_0^{(1)} < r_0^{(2)} \Rightarrow \beta_t^{(1)} P_t^{(1)} > \beta_t^{(2)} P_t^{(2)}, \widetilde{\mathcal{P}} - a.s. \ \forall\ 0 \leq t \leq T.$

Under the firm value specification (4),

$$V_t = V_0 \cdot e^{\int_0^t r_u du - \int_0^t \gamma_u du - \frac{1}{2}\int_0^t \phi_u^2 du + \int_0^t \phi_u d\widetilde{W}_u}. \tag{24}$$

It follows that:

***Corollary 4.*** $r_0^{(1)} < r_0^{(2)} \Rightarrow V_t^{(1)} < V_t^{(2)}, \widetilde{\mathcal{P}} - a.s. \; \forall \; 0 < t \le T.$

The following lemma also serves in the proof of Theorem 1.

***Lemma 1.*** $r_0^{(1)} \le r_0^{(2)} \Rightarrow \widetilde{E}[\beta_t^{(2)} P_t^{(2)} - \beta_t^{(1)} P_t^{(1)}] \ge P_0^{(2)} - P_0^{(1)}, \; \forall \; 0 \le t \le T.$

*Proof.* Define the $\widetilde{\mathcal{P}}$-martingale $\beta P^*$ by

$$\beta_t P_t^* \equiv \widetilde{E}\left[c \int_0^T \beta_s \, ds + 1 \cdot \beta_T | \mathcal{F}_t\right], \; \forall \; 0 \le t \le T. \tag{25}$$

Note that

$$\beta_t P_t = \widetilde{E}\left[c \int_t^T \beta_s \, ds + 1 \cdot \beta_T | \mathcal{F}_t\right], \tag{26}$$

so

$$\beta_t P_t^* = \beta_t P_t + c \int_0^t \beta_s \, ds. \tag{27}$$

Rearranging,

$$\beta_t P_t - P_0 = \beta_t P_t^* - c \int_0^t \beta_t \, dt - P_0 \tag{28}$$

$$\Rightarrow \widetilde{E}[\beta_t P_t] - P_0 = -\widetilde{E}\left[c \int_0^t \beta_s ds\right]. \tag{29}$$

Corollary 1 implies that

$$\widetilde{E}\left[c \int_0^t \beta_s^{(1)} ds\right] \ge \widetilde{E}\left[c \int_0^t \beta_s^{(2)} ds\right], \tag{30}$$

and the result follows. ■

*Proof of Theorem 1.* 1. Consider the stopping problem at time $t < T$. Let $p^{(1)} > p^{(2)}$ be two possible values of the time $t$ bond price. Note that, from the strict monotonicity of $p_H(\cdot, t)$, there are corresponding values of the time $t$ interest rate process, $r^{(1)}$ and $r^{(2)}$, satisfying $r^{(1)} < r^{(2)}$. Let $\tau$ be the optimal stopping time given the state at time $t$ is $P_t = p^{(2)}$ and $V_t = v$. Then its feasibility as a stopping time for the state $P_t = p^{(1)}$ and $V_t = v$ implies that

$$\begin{aligned} f(p^{(1)}, v, t) - f(p^{(2)}, v, t) &\ge \widetilde{E}[\beta_{t,\tau}^{(1)}(P_\tau^{(1)} - \kappa(V_\tau^{(1)}, \tau))^+ \\ &\quad - \beta_{t,\tau}^{(2)}(P_\tau^{(2)} - \kappa(V_\tau^{(2)}, \tau))^+ | \mathcal{F}_t] \qquad (31) \\ &> 0. \qquad (32) \end{aligned}$$

To establish the last inequality, note that if $\tau = t$, the expectation above is $p^{(1)} - p^{(2)} > 0$. If $\tau > t$, $r^{(1)} < r^{(2)} \Rightarrow \beta_{t,\tau}^{(1)} > \beta_{t,\tau}^{(2)}$, and $P_\tau^{(1)} \ge P_\tau^{(2)}$. Furthermore, Corollary (4) implies that $\kappa(V_\tau^{(1)}, \tau) \le \kappa(V_\tau^{(2)}, \tau)$. It follows that $(P_\tau^{(1)} - \kappa(V_\tau^{(1)}, \tau))^+ \ge (P_\tau^{(2)} - \kappa(V_\tau^{(2)}, \tau))^+$. Now $P_\tau^{(2)} - \kappa(V_\tau^{(2)}, \tau) > 0$ with positive probability, so $\beta_{t,\tau}^{(1)}(P_\tau^{(1)} - \kappa(V_\tau^{(1)}, \tau))^+ \ge \beta_{t,\tau}^{(2)}(P_\tau^{(2)} - \kappa(V_\tau^{(2)}, \tau))^+$ a.s. and $\beta_{t,\tau}^{(1)}(P_\tau^{(1)} - \kappa(V_\tau^{(1)}, \tau))^+ > \beta_{t,\tau}^{(2)}(P_\tau^{(2)} - \kappa(V_\tau^{(2)}, \tau))^+$ with positive probability.

2. Consider the cases $\kappa(V_t, t) = V_t$ and $\kappa(V_t, t) = k_t \wedge V_t$, and let $t < T$. Let $v^{(1)} < v^{(2)}$ be two possible values of the time $t$ firm value, $V_t$. From Equation (24), $V_s^{(1)} < V_s^{(2)}$, $\forall s \in [t, T]$. It follows that $\kappa(V_\tau^{(1)}, \tau) \leq \kappa(V_\tau^{(2)}, \tau)$, where $\tau$ is the optimal stopping time given that the state at time $t$ is $P_t = p$ and $V_t = v^{(2)}$. The feasibility of $\tau$ as a stopping time for the state $P_t = p$ and $V_t = v^{(1)}$ implies that

$$f(p, v^{(1)}, t) - f(p, v^{(2)}, t) \geq \widetilde{E}\big[\beta_{t,\tau}\big(P_\tau - \kappa\big(V_\tau^{(1)}, \tau\big)\big)^+ - \beta_{t,\tau}\big(P_\tau - \kappa\big(V_\tau^{(2)}, \tau\big)\big)^+ |\mathcal{F}_t\big] \tag{33}$$

$$\geq 0. \tag{34}$$

In the case of the pure default option, $\kappa(V_t, t) = V_t$, the last inequality is strict.

3. We let $p^{(1)} > p^{(2)}$ and prove that $f(p^{(2)}, v, t) - f(p^{(1)}, v, t) \geq p^{(2)} - p^{(1)}$. Let $r^{(1)} < r^{(2)}$ denote the time $t$ interest rates corresponding to the two possible values for the time $t$ bond price, $p^{(1)}$ and $p^{(2)}$, respectively. Let $\tau$ be the optimal stopping time for $p^{(1)}$. Then $\tau$ is a feasible stopping time for $p^{(2)}$ as well.

$$f(p^{(2)}, v, t) - f(p^{(1)}, v, t)$$

$$\geq \widetilde{E}\big[\beta_{t,\tau}^{(2)}\big(P_\tau^{(2)} - \kappa\big(V_\tau^{(2)}, \tau\big)\big)^+ - \beta_{t,\tau}^{(1)}\big(P_\tau^{(1)} - \kappa\big(V_\tau^{(1)}, \tau\big)\big)^+ |\mathcal{F}_t\big] \tag{35}$$

$$= \widetilde{E}\Big\{\big[\beta_{t,\tau}^{(2)}\big(P_\tau^{(2)} - \kappa\big(V_\tau^{(2)}, \tau\big)\big)^+ - \beta_{t,\tau}^{(1)}\big(P_\tau^{(1)} - \kappa\big(V_\tau^{(1)}, \tau\big)\big)\big] \cdot 1_{\left(P_\tau^{(1)} > \kappa\left(V_\tau^{(1)}, \tau\right)\right)} |\mathcal{F}_t\Big\} \tag{36}$$

$$\geq \widetilde{E}\Big\{\big[\beta_{t,\tau}^{(2)}\big(P_\tau^{(2)} - \kappa\big(V_\tau^{(2)}, \tau\big)\big) - \beta_{t,\tau}^{(1)}\big(P_\tau^{(1)} - \kappa\big(V_\tau^{(1)}, \tau\big)\big)\big] \cdot 1_{\left(P_\tau^{(1)} > \kappa\left(V_\tau^{(1)}, \tau\right)\right)} |\mathcal{F}_t\Big\} \tag{37}$$

$$= \widetilde{E}\Big\{\big[\beta_{t,\tau}^{(2)} P_\tau^{(2)} - \beta_{t,\tau}^{(1)} P_\tau^{(1)}\big] \cdot 1_{\left(P_\tau^{(1)} > \kappa\left(V_\tau^{(1)}, \tau\right)\right)} |\mathcal{F}_t\Big\} + \widetilde{E}\Big\{\big[\beta_{t,\tau}^{(1)}\kappa\big(V_\tau^{(1)}, \tau\big) - \beta_{t,\tau}^{(2)}\kappa\big(V_\tau^{(2)}, \tau\big)\big] \cdot 1_{\left(P_\tau^{(1)} > \kappa\left(V_\tau^{(1)}, \tau\right)\right)} |\mathcal{F}_t\Big\} \tag{38}$$

$$\geq \widetilde{E}\big\{\big[\beta_{t,\tau}^{(2)} P_\tau^{(2)} - \beta_{t,\tau}^{(1)} P_\tau^{(1)}\big] \cdot 1_{\left(P_\tau^{(1)} > \kappa\left(V_\tau^{(1)}, \tau\right)\right)} |\mathcal{F}_t\big\} \tag{39}$$

$$\geq \widetilde{E}\big[\beta_{t,\tau}^{(2)} P_\tau^{(2)} - \beta_{t,\tau}^{(1)} P_\tau^{(1)} |\mathcal{F}_t\big] \tag{40}$$

$$\geq p^{(2)} - p^{(1)}. \tag{41}$$

Equation (36) follows from the fact that $r^{(1)} < r^{(2)} \Rightarrow P_\tau^{(1)} \geq P_\tau^{(2)}$ (Corollary 2), and $\kappa(V_\tau^{(1)}, \tau) \leq \kappa(V_\tau^{(2)}, \tau)$ (Corollary 4) which in turn imply that $P_\tau^{(1)} \leq \kappa(V_\tau^{(1)}, \tau) \Rightarrow P_\tau^{(2)} \leq \kappa(V_\tau^{(2)}, \tau)$. Inequality (39) follows from the fact that $r^{(1)} < r^{(2)} \Rightarrow \beta_{t,\tau}^{(1)}\kappa(V_\tau^{(1)}, \tau) \geq \beta_{t,\tau}^{(2)}\kappa(V_\tau^{(2)}, \tau)$ (Corollary 1 and Equation 24). Inequality (40) follows from the fact that $r^{(1)} < r^{(2)} \Rightarrow \beta_{t,\tau}^{(1)} P_\tau^{(1)} \geq \beta_{t,\tau}^{(2)} P_\tau^{(2)}$ (Corollary 3). Finally, Inequality (41) follows from Lemma 1.

4. We let $v^{(2)} > v^{(1)}$ and prove that $f(p, v^{(2)}, t) - f(p, v^{(1)}, t) \geq v^{(1)} - v^{(2)}$. Let $\tau$ be the optimal stopping time for $v^{(1)}$. Then $\tau$ is a feasible stopping time for $v^{(2)}$.

$$f(p, v^{(2)}, t) - f(p, v^{(1)}, t) \geq \widetilde{E}\big[\beta_{t,\tau}\big(P_\tau - \kappa\big(V_\tau^{(2)}, \tau\big)\big)^+ - \beta_{t,\tau}\big(P_\tau - \kappa\big(V_\tau^{(1)}, \tau\big)\big)^+ |\mathcal{F}_t\big] \tag{42}$$

$$= \widetilde{E}\Big\{\big[\beta_{t,\tau}\big(P_\tau - \kappa\big(V_\tau^{(2)}, \tau\big)\big)^+ - \beta_{t,\tau}\big(P_\tau - \kappa\big(V_\tau^{(1)}, \tau\big)\big)\big] \cdot 1_{\left(P_\tau > \kappa\left(V_\tau^{(1)}, \tau\right)\right)} |\mathcal{F}_t\Big\} \tag{43}$$

$$\geq \widetilde{E}\Big\{\big[\beta_{t,\tau}(P_\tau - \kappa(V_\tau^{(2)}, \tau)) - \beta_{t,\tau}(P_\tau - \kappa(V_\tau^{(1)}, \tau))\big] \cdot 1_{\left(P_\tau > \kappa\left(V_\tau^{(1)}, \tau\right)\right)} | \mathcal{F}_t\Big\} \tag{44}$$

$$= \widetilde{E}\Big\{\big[\beta_{t,\tau}(\kappa(V_\tau^{(1)}, \tau) - \kappa(V_\tau^{(2)}, \tau))\big] \cdot 1_{\left(P_\tau > \kappa\left(V_\tau^{(1)}, \tau\right)\right)} | \mathcal{F}_t\Big\} \tag{45}$$

$$\geq \widetilde{E}\big[\beta_{t,\tau}(\kappa(V_\tau^{(1)}, \tau) - \kappa(V_\tau^{(2)}, \tau)) | \mathcal{F}_t\big] \tag{46}$$

$$\geq \widetilde{E}\big[\beta_{t,\tau}(V_\tau^{(1)} - V_\tau^{(2)}) | \mathcal{F}_t\big] \tag{47}$$

$$= e^{-\int_t^\tau \gamma_u \, du}\big(v^{(1)} - v^{(2)}\big) \tag{48}$$

$$\geq v^{(1)} - v^{(2)}. \tag{49}$$

Inequalities (43) and (46) follow from the fact that $v^{(2)} > v^{(1)} \Rightarrow \kappa(V_\tau^{(2)}, \tau) \geq \kappa(V_\tau^{(1)}, \tau)$. ■

*Proof of Proposition 1.* The first inequality is obvious. We establish the second inequality as follows.

$$f_{CD}(p, v, t) = \sup_{t \leq \tau \leq T} \widetilde{E}\big[\beta_{t,\tau}(P_\tau - k_\tau \wedge V_\tau)^+ | \mathcal{F}_t\big] \tag{50}$$

$$= \sup_{t \leq \tau \leq T} \widetilde{E}\big[\beta_{t,\tau}((P_\tau - k_\tau)^+ \vee (P_\tau - V_\tau)^+) | \mathcal{F}_t\big] \tag{51}$$

$$\leq \sup_{t \leq \tau \leq T} \widetilde{E}\big[\beta_{t,\tau}((P_\tau - k_\tau)^+ + (P_\tau - V_\tau)^+) | \mathcal{F}_t\big] \tag{52}$$

$$\leq \sup_{t \leq \tau \leq T} \widetilde{E}\big[\beta_{t,\tau}(P_\tau - k_\tau)^+ | \mathcal{F}_t\big] + \sup_{t \leq \tau \leq T} \widetilde{E}\big[\beta_t^\tau (P_\tau - V_\tau)^+ | \mathcal{F}_t\big] \tag{53}$$

$$= f_C(p, v, t) + f_D(p, v, t). \tag{54}$$

■

For the proofs of Theorems 2–4, note that the continuation region for each option is the open set

$$U \equiv \big\{(p, v, t) \in \mathcal{R}^+ \times \mathcal{R}^+ \times [0, T]\colon f(p, v, t) > (p - \kappa(v, t))^+\big\}. \tag{55}$$

In addition, note that for all $t \in [0, T)$, $f(p, v, t) > 0$.

*Proof of Theorem 2.* Suppose it is optimal to continue at $p_1$ and $p_1 > p_2$. We show that it is then optimal to continue at $p_2$. Using the call delta inequality, we have

$$f(p_2, v, t) \geq f(p_1, v, t) + p_2 - p_1 > (p_1 - \kappa(v, t))^+ + p_2 - p_1 \geq p_2 - \kappa(v, t). \tag{56}$$

In addition, $f(p_2, v, t) > 0$, so

$$f(p_2, v, t) > (p_2 - \kappa(v, t))^+. \tag{57}$$

Let $b(v, t)$ be the supremum of $p$ such that $(p, v, t) \in U$. The point $(b(v, t), v, t)$ cannot lie in $U$ because $U$ is open, so $f(b(v, t), v, t) = b(v, t) - \kappa(v, t) > 0$, which implies $b(v, t) > \kappa(v, t)$. ■

*Proof of Theorem 3.*
1. Note that it must be optimal to default at $v = 0$. Suppose it is optimal to continue at $v_1$ and $v_1 < v_2$. We show that it is then optimal to continue at $v_2$. Using the put delta inequality,

$$f(p, v_2, t) \geq f(p, v_1, t) + v_1 - v_2 > (p - v_1)^+ + v_1 - v_2 \geq p - v_2, \tag{58}$$

and thus $f(p, v_2, t) > (p - v_2)^+$. Let $v_D(p, t)$ be the infimum of $v$ such that $(p, v, t) \in U$. Since $f(p, v_D(p, t), t) > 0$, $v_D(p, t) < p$.

2. First, suppose it is optimal not to default at $v_1$ and $v_1 < v_2$. We show that it is then also optimal not to default at $v_2$. From the put delta inequality,

$$f(p, v_2, t) \geq f(p, v_1, t) + v_1 - v_2 > (p - v_1 \wedge k_t)^+ + v_1 - v_2$$
$$\geq p - v_2, \tag{59}$$

and thus $f(p, v_2, t) > (p - v_2)^+$.

Note that it must be optimal to default at $v = 0$. Therefore, there exists a critical value $v_{CD}(p, t)$ such that it is optimal to default $\forall v, v \leq v_{CD}(p, t)$. Further, $v_{CD}(p, t) < p$ must hold. Otherwise $f(p, v_{CD}(p, t), t) = 0$, a contradiction. In addition, $v_{CD}(p, t) \leq k_t$ must hold. Otherwise, there would exist a firm value greater than $k_t$ at which it is optimal to default, which is impossible.

Next, suppose it is optimal to call at $v_1$, and $v_1 < v_2$. We show that then it is then optimal to call at $v_2$. Note that $k_t \leq v_1$ must hold. Now, on one hand, $f(p, v_2, t) \geq p - k_t \wedge v_2 = p - k_t$. On the other hand, from part 2 of Theorem 1, $f(p, v_2, t) \leq f(p, v_1, t) = p - k_t$. Let $\bar{v}_{CD}(p, t) \geq k_t$ be the minumum of $v$ such that it is optimal to call at $(p, v, t)$. ∎

*Proof of Theorem 4.* 1. Suppose $0 < p < b_D(v_1, t)$. Then $p < b_D(v_2, t)$ as well:

$$f(p, v_2, t) \geq f(p, v_1, t) + v_1 - v_2 > p - v_1 + v_1 - v_2 = p - v_2. \tag{60}$$

2. Suppose $v > v_D(p_2, t)$. Then $v > v_D(p_1, t)$ as well:

$$f(p_1, v, t) \geq f(p_2, v, t) + p_1 - p_2 > p_2 - v + p_1 - p_2 \geq p_1 - v. \tag{61}$$

3. The proof is essentially the same as that in part 1.
4. Suppose $0 < p < b_{CD}(v_2, t)$. Then $p < b_{CD}(v_1, t)$ as well:

$$f(p, v_1, t) \geq f(p, v_2, t) > g(p, v_2, t) = (p - k_t)^+ = g(p, v_1, t). \tag{62}$$

5. If $p < b_D(v, t)$, then $f_{CD}(p, v, t) \geq f_D(p, v, t) > p - v = p - v \wedge k_t$, so $p < b_{CD}(v, t)$.
6. If $p < b_C(v, t)$, then $f_{CD}(p, v, t) \geq f_C(p, t) > p - k_t = p - v \wedge k_t$, so $p < b_{CD}(v, t)$. ∎

## Appendix B: Numerical Implementation

Nelson and Ramaswamy (1990) show how to use binomial processes to approximate a general class of single-factor diffusions. To extend their analysis to multi-factor diffusion models, we first transform the state variables into new diffusion processes that are uncorrelated and have constant volatility. Then we construct a recombining, two-dimensional binomial lattice for the resulting orthogonalized diffusions. Finally, we transform the lattice for the orthogonalized state variables into a lattice for the original variables and price the callable and defaultable bonds using backward induction. This appendix describes the construction of the two-dimensional binomial lattice. Other papers illustrating the implementation of bivariate diffusions are Boyle, Evnine, and Gibbs (1989), Hilliard, Schwartz, and Tucker (1996), who consider lognormal processes, and Hull and White (1994a,b, 1996), Ho, Stapleton, and Subrahmanyam (1995), and Peterson, Stapleton, and Subrahmanyam (1998), who consider two-factor term structure models.

As a special case of our model, we consider the Cox, Ingersoll, and Ross (1985) interest rate process $r_t$, where

$$dr_t = \kappa(\mu - r_t)\, dt + \sigma\sqrt{r_t}\, d\widetilde{Z}_t. \tag{63}$$

Firm value $V_t$ follows the log-normal process

$$\frac{dV_t}{V_t} = (r_t - \gamma_t)\,dt + \phi_t d\widetilde{W}_t. \tag{64}$$

The instantaneous correlation between $\widetilde{Z}_t$ and $\widetilde{W}_t$ is denoted as $\rho_t$. Note that $\gamma_t \geq 0$, $\phi_t > 0$, and $\rho_t \in (-1, 1)$ are deterministic functions of time.

To orthogonalize these interest rate and firm value processes, let $G_t = \frac{ln(V_t)}{\phi_t}$ and $H_t = \frac{2\sqrt{r_t}}{\sigma}$. Then, by Ito's Lemma,

$$\begin{aligned} dG_t &= \mu_t dt + d\widetilde{W}_t, \quad \mu_t = \frac{r_t - \gamma_t - \frac{\phi_t^2}{2}}{\phi_t}, \text{ and} \\ dH_t &= \nu_t dt + d\widetilde{Z}_t, \quad \nu_t = \frac{\kappa\mu - \frac{\sigma^2}{4} - \kappa r_t}{\sigma\sqrt{r_t}}. \end{aligned} \tag{65}$$

Second, let $X_t = G_t$ and $Y_t = \frac{1}{\sqrt{1-\rho_t^2}}(-\rho_t G_t + H_t)$. Then $X$ and $Y$ are diffusions with unit instantaneous variance and zero cross-variation. The drift of $X$ is $\mu_t^+ = \mu_t$ and the drift of $Y$ is $\mu_t^- = \frac{1}{\sqrt{1-\rho_t^2}}(-\rho_t \mu_t + \nu_t)$.

The inverse transformation to obtain $r_t$ and $V_t$ from $X_t$ and $Y_t$ are

$$V_t = e^{\phi_t X_t},\ r_t = \left[\frac{\sigma}{2} \cdot (\sqrt{1-\rho_t^2}Y_t + \rho_t X_t)\right]^2. \tag{66}$$

To get a lattice for $r$ and $V$, we apply this inverse transformation at each node of the lattice for $X$ and $Y$.

To construct a recombining, two-dimensional binomial lattice for the variables $X$ and $Y$, we divide the time-interval $[0, T]$ into $N$ equal intervals of length $\Delta t$. From a node $(X_t, Y_t)$ at time $t$, the lattice evolves to four nodes, $(X_t^+, Y_t^+)$, $(X_t^+, Y_t^-)$, $(X_t^-, Y_t^+)$, and $(X_t^-, Y_t^-)$, where

$$\begin{aligned} X_t^+ &= X_t + (2k_1+1)\sqrt{\Delta t},\ X_t^- = X_t + (2k_1 - 1)\sqrt{\Delta t}. \\ Y_t^+ &= Y_t + (2k_2+1)\sqrt{\Delta t},\ Y_t^- = Y_t + (2k_2 - 1)\sqrt{\Delta t}, \end{aligned} \tag{67}$$

and $k_1$ and $k_2$ are integers such that

$$(2k_1 - 1)\sqrt{\Delta t} \leq \mu_t^+ \Delta t \leq (2k_1 + 1)\sqrt{\Delta t}, \tag{68}$$

$$(2k_2 - 1)\sqrt{\Delta t} \leq \mu_t^- \Delta t \leq (2k_2 + 1)\sqrt{\Delta t}. \tag{69}$$

The four nodes have associated risk-neutral probabilities $pq, p(1-q), (1-p)q$, and $(1-p)(1-q)$, respectively. The probabilities, $p$, of an up-jump in $X_t$ process, and $q$, of an up-jump in $Y_t$ process, are picked to ensure the right first moments at the node $(X_t, Y_t)$:

$$p = \frac{1}{2} + \frac{\mu_t^+ \sqrt{\Delta t}}{2} - k_1,\ q = \frac{1}{2} + \frac{\mu_t^- \sqrt{\Delta t}}{2} - k_2. \tag{70}$$

Equations (68) and (69) ensure that the probabilities are between 0 and 1. While the first moment of the process $(X, Y)$ is matched exactly by the scheme above, the second moment is approximated with an error that is $O(\Delta t)$. The two-factor binomial process converges in distribution to the original continuous-time process as $\Delta t \to 0$.

To make the lattice for each state variable recombine, the variable can only move an integral number of increments $\sqrt{\Delta t}$, as Equation (67) indicates. When the drift terms $\mu_t^+$ and $\mu_t^-$ are large in magnitude, for instance, at low interest rates when the speed of mean reversion is high, multiple jumps, that is, nonzero $k_1$ or $k_2$, occur. However, the lattice for each variable has only

$n+1$ nodes at each time $n\Delta t$, so an up or down move from any node at time $(n-1)\Delta t$ must lead to one of the $n+1$ nodes at time $n\Delta t$. Therefore, the moves described in Equations (68) and (69) require that $\Delta t$ be sufficiently small. The numerical examples employ 35 to 40 time steps per year. We check the convergence by matching the price of a zero-coupon bond maturing at $T$, which can be calculated analytically, and by matching the price of a European default option on the zero-coupon bond with an expiration at $T$, under the two-factor specification, which can be calculated using Monte Carlo simulation.

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