Equilibrium in a Pure Exchange Economy

1. Investor endowments, objectives, and preferences

2. Arrow-Debreu equilibrium
   (a) Characterization of the sdf process
   (b) Existence and uniqueness of equilibrium
   (c) Representative agent

3. Security market equilibrium

4. Consumption CAPM

Selected Readings and References


Duffie, chapter 10.


Summary of the Continuous-Time Financial Market

- Security prices satisfy \( \frac{dS_{k,t}}{S_{k,t}} = (\mu_{k,t} - \delta_{k,t}) \, dt + \sigma_{k,t} \, dB_t \).

- Given tight tr. strat. \( \pi_t \) and consumption \( c_t \), portfolio value \( X_t \) satisfies the
  - WEE: \( dX_t = r_t X_t \, dt + \pi_t (\mu_t - r_t) \, dt + \pi_t \sigma_t \, dB_t - c_t \, dt \).

- No arbitrage \( \Rightarrow \) if \( \pi_t \sigma_t = 0 \) then \( \pi_t (\mu_t - r_t) = 0 \Rightarrow \exists \theta_t \) s.t. \( \sigma_t \theta_t = \mu_t - r_t \)
  \( \Rightarrow dX_t = r_t X_t \, dt + \pi_t \sigma_t (\theta_t \, dt + dB_t) - c_t \, dt \).

- Under emm \( P^* \) given by \( \frac{dP^*}{dP} = Z_T \) where \( Z_t = e^{-\int_0^t \theta_s \, dB_s - \frac{1}{2} \int_0^t \theta_s^2 \, ds} \).

  - \( B^*_t = B_t + \int_0^t \theta_s \, ds \) is Brownian motion.

  Let \( \beta_t = e^{-\int_0^t \theta_s \, ds} \) and sdf process \( M_t = \beta_t Z_t \). Then the WEE can also be written:
    - WEE*: \( d\beta_t X_t + \beta_t c_t \, dt = \beta_t \pi_t \sigma_t \, dB^*_t \)
    - WEE-M: \( dM_t X_t + M_t c_t \, dt = M_t [\pi_t \sigma_t - \theta_t X_t] \, dB_t \)

- So \( X_t = E_t \{ \int_t^T \frac{\beta_u}{\beta_t} c_u \, du + \frac{\beta_t}{\beta_t} X_T \} = E_t \{ \int_t^T \frac{M_u}{M_t} c_u \, du + \frac{M_T}{M_t} X_T \} \) if \( \pi \) is mgale-gen.

- If \( \sigma \) is nonsingular, every c.plan \( (c, X_T) \) can be generated by a mgale-gen. tr.strat.

Investors and Endowments

- Suppose there are \( m \) investors in the economy.

- Each investor \( j \) is endowed with an exogenous flow \( e_{j,t} \) of the single, nonstorable, consumption good. Each \( e_{j,t} \) is nonnegative and adapted on \([0, T]\).

- The aggregate endowment
  \[
  e_t = \sum_{j=1}^m e_{j,t} 
  \]
  is an Itô process described by
  \[
  \frac{de_t}{e_t} = \mu_{e,t} \, dt + \sigma_{e,t} \, dB_t ,
  \]
  where \( \mu_e \) and \( \sigma_e \) are adapted and bounded.
Investor Objectives, Preferences, and Optimal Consumption Plans

Each investor wants to maximize $E \int_0^T U_j(c_{j,t}, t) \, dt$ where each consumption plan $c_{j,t}$ is a nonnegative, adapted process satisfying $E \int_0^T c_{j,t} \, dt < \infty$ and $U_j$ is a utility function satisfying

$$U_j(c, t) = e^{-\mu_j} u_j(c),$$

where $u_j : (0, \infty) \to \mathbb{R}$ is $C^3$ and satisfies

- $u_j'(c) > 0$,
- $u_j''(c) < 0$, $\lim_{c \downarrow 0} u_j'(c) = \infty$,
- $\lim_{c \to \infty} u_j'(c) = 0$, and
- $\lim_{c \downarrow 0} \frac{u_j''(c)}{(u_j'(c))^2}$ exists and is finite.

Thus, the nonnegativity constraint on consumption is nonbinding and there exists an imuf $I_j : (0, \infty) \times [0, T] \to (0, \infty)$ s.t. $I_j(U_j'(c, t), t) = c$ for every $c \in (0, \infty)$, where $U_j'(c, t) \equiv \frac{\partial U_j(c, t)}{\partial c}$. The imuf $I_j(y, t)$ is $C^2$ and strictly decreasing in $y$.

In a complete market in which investors could trade consumption plans at prices described by a sdf process $M_t$, each investor would trade so as to

$$\max_{c_{j,t}} E \int_0^T U_j(c_{j,t}, t) \, dt \text{ s.t. } E \int_0^T M_t c_{j,t} \, dt \leq E \int_0^T M_t e_{j,t} \, dt. \quad (4)$$

From our results on optimal consumption, the solution would be

$$c^*_{j,t} = I_j(\lambda_j M_t, t), \quad (5)$$

where the Lagrange multiplier $\lambda_j$ solves

$$E \int_0^T M_t I_j(\lambda_j M_t, t) \, dt = E \int_0^T M_t e_{j,t} \, dt. \quad (6)$$
**Arrow-Debreu Equilibrium**

**Definition 1** An Arrow-Debreu equilibrium (ADE) is a sdf $M_t$ and allocations $c^*_j,t$, $j = 1, \ldots, m$, such that each $c^*_j,t$ solves investor $j$’s optimization problem (4), and markets clear, i.e.,

$$
\sum_{j=1}^m c^*_j,t = e_t, \ 0 \leq t \leq T .
$$

(7)

**Proposition 1** In an ADE, there exists $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_{++}$ satisfying

$$
E \int_0^T M_t I_j(\lambda_j M_t, t) \, dt = E \int_0^T M_t e_{j,t} \, dt, \ \forall j = 1, \ldots, m \text{ and }
$$

$$
\sum_{j=1}^m I_j(\lambda_j M_t, t) = e_t, \ 0 \leq t \leq T .
$$

(8)

(9)

Conversely, if there exist $M_t$ and $\lambda \in \mathbb{R}^m_{++}$ satisfying equations (8) and (9) then $M$ is an equilibrium sdf. In either case, $c^*_j,t = I_j(\lambda_j M_t, t)$ are the optimal consumption plans for investors $j = 1, \ldots, m$.

**Sketch of Proof** The $c^*_j$ are optimal and markets clear iff equations (8)-(9) hold.

- So the search for an Arrow-Debreu equilibrium reduces to a search for $M$ and $\lambda$ satisfying equations (8) and (9).
- First use equation (9) to define $M$ as a function of $e$ and $\lambda$ as follows. Let $\lambda \in \mathbb{R}^m_{++}$ be given. Then for each $(\omega, t)$ there exists a unique $M(\omega, t) \in (0, \infty)$ solving equation (9) (because the left-hand side is continuous and strictly decreasing in $M$, goes to $\infty$ as $M$ goes to zero, and goes to 0 as $M$ goes to $\infty$).
- Thus, define $\mathcal{M}(e_t, t; \lambda)$ as the $M_t$ that clears markets in equation (9), i.e.,

$$
\sum_{j=1}^m I_j(\lambda_j \mathcal{M}(e_t, t; \lambda), t) = e_t .
$$

(10)

- Note that $\mathcal{M}(e_t, t; \lambda/a) = a \mathcal{M}(e_t, t; \lambda) \ \forall a > 0$.

**Theorem 1 (Existence and Uniqueness of Equilibrium)**

There exists $\lambda \in \mathbb{R}^m_{++}$ that solves equations (8) and (9) with $M_t = \mathcal{M}(e_t, t; \lambda)$.

Moreover, if $-\frac{\partial u_j(e_t, \cdot)}{\partial e_t} \leq 1 \ \forall j = 1, \ldots, m$, then the equilibrium is unique in the sense that if $\lambda' \in \mathbb{R}^m_{++}$ together with $\mathcal{M}'(e_t, t; \lambda')$ are another solution, then $\mathcal{M}'(e_t, t; \lambda') = a \mathcal{M}(e_t, t; \lambda)$ and $\lambda' = \lambda/a$ for some positive constant $a$, and the allocations $c^*_j,t = I_j(\lambda_j \mathcal{M}(e_t, t; \lambda), t) = I_j(\lambda'_j \mathcal{M}'(e_t, t; \lambda'), t)$ are unique.
Construction of the “Representative Agent”

Define
\[
U(c, t; \Lambda) \equiv \max_{(c_1, \ldots, c_m) \in \mathbb{R}^m} \sum_{j=1}^{m} U_j(c_j, t) / \lambda_j \quad \text{s.t.} \quad \sum_{j=1}^{m} c_j \leq c .
\]  

(11)

Let \((\hat{c}_1(c), \ldots, \hat{c}_m(c))\) denote the solution to the maximization problem above, where

the dependence of \(\hat{c}\) on \(t\) and \(\Lambda\) is suppressed for brevity.

**Proposition 2** The function \(U(\cdot, t; \Lambda) : (0, \infty) \to \mathbb{R}\) has the following properties.

\[
U(\cdot, t; \Lambda) \in C^2 ,
\]

(12)

\[
U'(c, t; \Lambda) = U'_j(\hat{c}_j(c), t) / \lambda_j \quad \forall j = 1, \ldots, m ,
\]

(13)

\[
\lim_{c \to 0} U'(c, t; \Lambda) = \infty ,
\]

(14)

\[
\lim_{c \to \infty} U'(c, t; \Lambda) = 0 ,
\]

(15)

\[
U''(c, t; \Lambda) < 0 .
\]

(16)

**Proof** Homework

**Theorem 2**

\[
U'(c, t; \Lambda) = \mathcal{M}(c, t; \Lambda) \quad \text{and}
\]

(17)

\[
\hat{c}_j(c) = I_j(\lambda_j \mathcal{M}(c, t; \Lambda), t) .
\]

(18)

**Proof** Let \(c_j = I_j(\lambda_j \mathcal{M}(c, t; \Lambda), t)\). Note that \(\sum_{j=1}^{m} c_j = c\), by construction of \(\mathcal{M}(c, t; \Lambda)\). Now suppose \((c_1, \ldots, c_m)\) is a different feasible choice for the maximization problem in equation (11). Then \(\sum_{j=1}^{m} c_j \leq c\) and

\[
\sum_{j=1}^{m} U_j(c_j, t) / \lambda_j < \sum_{j=1}^{m} \left[ U_j(c_j^*, t) + (c_j - c_j^*)U'_j(c_j^*, t) \right] / \lambda_j
\]

(19)

\[
= \sum_{j=1}^{m} \left[ U_j(c_j^*, t) + (c_j - c_j^*)U'_j(\lambda_j \mathcal{M}(c, t; \Lambda), t) \right] / \lambda_j
\]

(20)

\[
= \sum_{j=1}^{m} U_j(c_j^*, t) / \lambda_j + \mathcal{M}(c, t; \Lambda) \sum_{j=1}^{m} (c_j - c_j^*)
\]

(21)

\[
\leq \sum_{j=1}^{m} U_j(c_j^*, t) / \lambda_j .
\]

(22)
Thus $\bar{c}_j(c) = c^*_j = I_j(\lambda_j, \mathcal{M}(c, t; \Lambda), t), j = 1, \ldots, m$, so

$$
U'(c, t; \Lambda) = U'_j(I_j(\lambda_j, \mathcal{M}(c, t; \Lambda), t), t)/\lambda_j
$$

(23)

$$
= \mathcal{M}(c, t; \Lambda). \quad \square
$$

(24)

Now fix $\Lambda$ at an equilibrium value.

Then we have $U'(e_t, t) = \mathcal{M}(e_t, t; \Lambda) = M_t$.

I.e., we can interpret the equilibrium sdf as the marginal utility of a "representative agent" consuming the aggregate endowment.

Security Market Equilibrium

The ADE can be implemented in a market where investors trade securities dynamically, rather than making one-time trades of consumption bundles.

To construct a complete, correctly priced securities market, we first deduce $r_t$ and $\theta_t$ from $M_t = \mathcal{M}(e_t, t; \Lambda) = U'(e_t, t)$, then specify a nonsingular $d \times d$-matrix-valued volatility process $\sigma_t$ and set $\mu_t = r_t 1 + \sigma_t \theta_t$.

Deduce the securities-market clearing interest rate $r_t$ and mpr $\theta_t$ from the ADE sdf by recognizing that

$$
M_t = e^{-\int_0^t r_s \, ds - \int_0^t \theta_s \, dB_s - \frac{1}{2} \int_0^t |\theta_s|^2 \, ds} = U'(e_t, t)
$$

(25)

and matching their drift and diffusion terms from Itô’s lemma:

$$
\frac{dM_t}{M_t} = -r_t \, dt - \theta'_t \, dB_t = \frac{dU'(e_t, t)}{U'(e_t, t)}
$$

(26)

$$
= \frac{1}{U'(e_t, t)} [U''(e_t, t)e_t \mu_{e.t} + \frac{\partial U'(e_t, t)}{\partial t} + \frac{1}{2} U'''(e_t, t)(e_t)^2 |\sigma_{e.t}|^2] \, dt
$$

$$
+ \frac{1}{U'(e_t, t)} U''(e_t, t)e_t \sigma_{e,t} \, dB_t.
$$

(27)
Therefore, letting

\[ R_t \equiv \frac{U''(e_t, t)e_t}{U'(e_t, t)} \tag{28} \]

denote the relative risk aversion of the representative agent,

the equilibrium interest rate is

\[ r_t = R_t \mu_{e,t} - \frac{1}{U'(e_t, t)} \left[ \frac{\partial U''(e_t, t)}{\partial e_t} + \frac{1}{2} U'''(e_t, t)(e_t)2|\sigma_{e,t}|^2 \right] \tag{29} \]

and the equilibrium mpr is

\[ \theta_t = R_t \sigma'_{e,t} . \tag{30} \]

To construct security prices, let \( \sigma_t \) be a nonsingular, \( d \times d \)-matrix-valued volatility process, let \( \mu_t = r_t 1 + \sigma_t \theta_t \), and let initial prices \( S_{k,0}, k = 0, 1, \ldots, d \), be given. Let \( S_{0,t} = S_{0,0}e^{\int_0^t r_s ds} \) and let \( S_{k,t} \) be given by

\[ \frac{dS_{k,t}}{S_{k,t}} = \mu_{k,t} dt + \sigma_{k,t} dB_t , \quad k = 1, \ldots, d . \tag{31} \]

These securities are in zero net supply ("side bets").

**Trading Strategies, Investor Wealth Evolution, and Securities Market Clearing**

Finally, suppose \( \pi \) is a trading strategy satisfying the usual integrability conditions. The wealth process generated by \( \pi \) is \( X_{j,t} \) such that

\[ dX_{j,t} = r_t X_{j,t} dt + (e_{j,t} - c_{j,t}) dt + \pi_t \sigma_t (\theta_t dt + dB_t) \tag{32} \]

The consumption-portfolio plan \((c_j, \pi^j)\) is feasible for investor \( j \) if \( X_{j,t} \) satisfies

\[ X_{j,t} + E_t\{\int_t^T \frac{M_u}{M_t} e_{j,u} du\} \geq 0 , \quad 0 \leq t \leq T . \tag{33} \]

If \( c_j \) is a consumption plan that satisfies

\[ E \int_0^T M_t c_{j,t} dt = E \int_0^T M_t e_{j,t} dt , \tag{34} \]

then there exists a trading strategy \( \pi^j \) s.t. \((c_j, \pi^j)\) is feasible for investor \( j \), and the corresponding wealth process is

\[ X_{j,t} = E_t\{\int_t^T \frac{M_u}{M_t} [c_{j,u} - e_{j,u}] du\} . \tag{35} \]
Definition 2 The securities market is in equilibrium if
1. investors’ consumption plans and trading strategies are optimal,
2. the commodities market clears, i.e., \( \sum_{j=1}^{m} c_{j,t}^* = e_t \),
3. security markets clear, i.e.,

\[
\sum_{j=1}^{m} \pi_{j,t}^* = 0 \quad \text{and} \quad \sum_{j=1}^{m} \pi_{0,t}^* \equiv \sum_{j=1}^{m} (X_{j,t}^* - \pi_{j,t}^* 1) = 0 \quad , \quad 0 \leq t \leq T .
\]  

(36)

Theorem 3 (Existence of a Securities Market Equilibrium) Let \( \Lambda \) and \( M \) be as in an Arrow-Debreu equilibrium and let \( \sigma_t \) be a given nonsingular, \( d \times d \)-matrix-valued volatility process. Define \( r_t, \theta_t, \mu_t \) and security prices as above. Then the securities market is in equilibrium.

Proof For homework, prove that security markets clear.

\[\text{Consumption CAPM}\]

Note that in equilibrium, security \( k \)'s instantaneous excess return is

\[
\mu_{k,t} - r_t = \sigma_{k,t} \theta_t = R_t \sigma_{k,t} \sigma_{e,t} = R_t \text{cov}_{k,e,t} .
\]

(37)

In other words, the excess expected return on each security is equal to the representative agent coefficient of relative risk aversion times the security’s instantaneous covariance with aggregate endowment (consumption).

Let \( \sigma_t^* = \sigma_{e,t} \) and let

\[
\mu_t^* = r_t + \sigma_t^* \theta_t = r_t + R_t \sigma_{e,t} \sigma_{e,t}^'
\]

(38)

be the volatility vector and equilibrium appreciation rate on a security that is perfectly correlated with aggregate consumption. Then we have

\[
\mu_{k,t} - r_t = \beta_{k,t} (\mu_t^* - r_t) ,
\]

(39)

where \( \beta_{k,t} = \frac{\sigma_{k,t} \sigma_{e,t}^*}{\sigma_t^* \sigma_t} \) is security \( k \)'s “consumption beta.”
Problem

Consider an economy with identical agents each deriving expected utility from consumption plan \( \{c_t\} \) equal to

\[
E \int_0^T e^{-\rho t} \frac{c_t^\gamma}{\gamma} \, dt, \quad \gamma \in (0, 1).
\]

The aggregate endowment process, \( e_t \), satisfies

\[
\frac{de_t}{e_t} = \mu_{e,t} \, dt + \sigma_{e,t} \, dB_t
\]

where \( B_t \) is \( d \)-dimensional Brownian motion and \( \mu_e \) and \( \sigma_e \) are bounded, progressively measurable processes taking values in \( \mathcal{R} \) and \( \mathcal{R}^d \), respectively.

a) Determine the interest rate process \( r_t \) in the Arrow-Debreu equilibrium for this economy.

b) Suppose \( \hat{r}_t \) is an arbitrary bounded, nonnegative, progressively measurable process. Construct an equilibrium in which the interest rate is \( \hat{r}_t \).

c) Suppose \( \hat{r}_t \) above is an Itô process with \( d\hat{r}_t = \mu_{r,t} \, dt + \sigma_{r,t} \, dB_t \). What is the drift of \( \hat{r}_t \) under the martingale measure \( \mathcal{P}^* \) in the equilibrium you constructed?