The Continuous-Time Financial Market

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Readings and References


Now let’s develop these results more explicitly in a rich but tractable setting with continuous trading and security price processes constructed from Brownian motion.

- There is a finite time horizon $[0, T]$.
- The filtered probability space is $(\Omega, \mathcal{F}, \mathcal{P}, \mathcal{F}^B)$ where $\mathcal{F}^B$ is the filtration (information) generated by a $d$-dim’l Brownian motion $B = (B_1, B_2, \ldots, B_d)$.
- The consumption space $C$ is the set of pairs $(c, W)$ where $c$ is an adapted consumption rate process with $\int_0^T |c_t| \, dt < \infty$ a.s. and $W$ is a random variable representing terminal (time $T$) wealth.
- There are $n + 1$ securities traded, with ex-dividend prices $S = (S_0, S_1, \ldots, S_n)$.
- Security 0 is a “bond” or locally riskless money market account earning the instantaneous riskless rate $r_t$. i.e.,
\[
\frac{dS_{0,t}}{S_{0,t}} = r_t \, dt \iff S_{0,t} = S_{0,0} \exp \int_0^t r_u \, du.
\]  
where $r$ is an adapted process with $\int_0^T |r_t| \, dt < \infty$ a.s.

- The $n$ “risky” asset prices are strictly positive Itô processes, each satisfying
\[
\frac{dS_{k,t}}{S_{k,t}} = \left[\mu_{k,t} - \delta_{k,t}\right] \, dt + \sum_{i=1}^d \sigma_{k,t,i} B_{i,t} \, dB_t .
\]  
The $n$-dimensional instantaneous expected return process $\mu = (\mu_1, \ldots, \mu_n)$ is adapted and satisfies $\int_0^T |\mu_t| \, dt < \infty$ a.s., the $n$-dimensional dividend payout rate process $\delta = (\delta_1, \ldots, \delta_n)$ is adapted and satisfies $\int_0^T |\delta_t| \, dt < \infty$ a.s., and

- the $n \times d$-matrix-valued volatility process $\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix}$ is adapted and satisfies $\int_0^T \|\sigma_{k,t}\|^2 \, dt < \infty$ a.s. for each $k$.
- Equation (2) above is shorthand for
\[
S_{k,t} = S_{k,0} \exp \int_0^t \left[\mu_{k,u} - \delta_{k,u} - \frac{1}{2} \|\sigma_{k,u}\|^2\right] \, du + \int_0^t \sigma_{k,u} \, dB_u .
\]  
W.l.o.g. assume $n \leq d$ unless you want to track redundant securities for a reason.
- Let’s briefly review Brownian motion, stochastic integrals, and Itô processes.
**Brownian Motion**

**Definition 1** A continuous, adapted process $B$ is a standard Brownian motion if $B_0 = 0$ and for any $0 \leq t \leq s \leq T$, the increment $B_s - B_t$ is independent of $\mathcal{F}_t$ and normally distributed with mean zero and variance $s - t$. A process $X$ is a Brownian motion if $X_t = X_0 + \mu t + \sigma B_t \ \forall \ \{t \in [0, T]\}$, where $\mu$ is constant.

**Proposition 1** A process $X$ is continuous with stationary independent increments if and only if $X$ is a Brownian motion.

**Proposition 2** The sample paths of a Brownian motion have infinite variation and finite quadratic variation. I.e., for all $t \in [0, T],
\[
\lim_{n \to \infty} \sum_{i=0}^{2^n-1} |B_{t+i/2^n} - B_{t+i/2^n}| = \infty \text{ a.s., } \tag{4}
\]

\[
\lim_{n \to \infty} \sum_{i=0}^{2^n-1} (B_{t+i/2^n} - B_{t+i/2^n})^2 = t \text{ a.s.} \tag{5}
\]

**Definition 2** An $d$-dimensional Brownian motion is a vector-valued process $B = (B_1, \ldots, B_d)$, where each $B_j$ is a Brownian motion, $\forall \ j = 1, \ldots, d$ and $B_i$ is independent of $B_j$ for all $i \neq j$.

**Stochastic Integrals**

Now let’s define the stochastic integral $\int_0^T \theta_s dB_s$ of a process $\theta$ w.r.t. a Brownian motion $B$. A path-by-path Riemann-Stieltjes definition won’t work when the integrand $\theta$ has infinite variation, so we build it up starting with “simple” integrands.

**Definition 3** An adapted process $\theta$ satisfying $E \int_0^T |\theta_t|^2 dt < \infty$ is simple if there exists a finite partition

$0 = t_0 < t_1 < \cdots < t_J = T$ of $[0, T]$ and random variables $\theta_j \in \mathcal{F}_{t_j}$ such that

\[
\theta_t = \begin{cases} 
\theta_0 & \text{if } t \in [t_0, t_1] \\
\theta_j & \text{if } t \in (t_j, t_{j+1}]
\end{cases} \tag{6}
\]

$\forall \ j = 0, \ldots, J - 1$.

The stochastic integral of a simple integrand $\theta$ can be defined path by path as

\[
I_t = \int_0^t \theta_s dB_s = \sum_{j=0}^{N-1} \theta_j [B_{t_{j+1}} - B_{t_j}] . \tag{7}
\]
**Proposition 3** The stochastic integral $I_t$ of a simple integrand $\theta$ is continuous, adapted to $\mathcal{F}$, linear in $\theta$, an $L^2(\mathcal{P})$-martingale, and for simple $\theta_1$ and $\theta_2$ satisfies

$$E(\int_0^t \theta_{1,s} dB_s)(\int_0^t \theta_{2,s} dB_s)) = E \int_0^t \theta_{1,s} \theta_{2,s} ds .$$

(8)

Next, it turns out that every process $\theta$ satisfying $E\{\int_0^T |\theta_t|^2 dt\} < \infty$, which we’ll call “strongly square-integrable,” has a sequence of simple processes that converge to it, and the limit of the integrals of these processes exists and is unique.

So we define the stochastic integral of a strongly square-integrable process $\theta$ as the limit of the integrals of any sequence of simple processes that converges to $\theta$.

This is the so-called Itô-integral and it satisfies the same properties as the integrals of the simple processes listed above: the Itô-integral is continuous, adapted, linear in its integrand, an $L^2(\mathcal{P})$-martingale, and the expectation of the product of stochastic integrals is the expectation of time-integral of the product of the integrands.

Finally, it is also possible to define the Itô integral of adapted processes $\theta$ that satisfy $\int_0^T |\theta_t|^2 dt < \infty$ a.s., which we’ll call “weakly square-integrable,” as follows.

**Definition 4** An $\mathcal{F}$-measurable map $\tau : \Omega \to [0,T] \cup \{\infty\}$ is a stopping time if $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}$ for all $t \in [0,T]$.

It is always possible to interpret a stopping time as the first time an event occurs.

**Definition 5** A process $X$ is a local martingale if there exists a sequence of stopping times $\tau_n \uparrow T$ a.s. s.t. each stopped process $X^{\tau_n}$ is a martingale.

Now, for any weakly square-integrable process $\theta$ there exists a sequence of stopping times $\tau_n \uparrow T$ a.s. s.t. each of the stopped processes $\theta_n = \theta \cdot 1_{t \leq \tau_n}$ is strongly square-integrable. Then we can define the Itô integral of a weakly square-integrable process as the limit of the integrals of the stopped processes that converges to it.

The Itô integral of a weakly square-integrable process will have all of the properties above, except that it may be only a local martingale, not a martingale.

An example of a local martingale that is not a martingale is a wealth process under a doubling strategy.
**Itô Processes**

**Definition 6** A process $X$ adapted to the filtration $\mathcal{F}^B$ generated by a Brownian motion $B$ is an **Itô process** if $X$ is an adapted real-valued process $\mu$ with $\int_0^T |\mu_t| \, dt < \infty$ a.s., which we'll call "absolutely integrable," and an $\mathbb{R}^n$-valued weakly square-integrable process $\sigma$ s.t.

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dB_s \quad \forall t \in [0,T] \text{ a.s.}$$

(9)

The process $\mu$ is called the **drift** of $X$ and the process $\sigma$ is called the **diffusion** of $X$.

- Equation (9) can be written in differential form as $dX_t = \mu_t \, dt + \sigma_t \, dB_t$.
- An Itô process is a local martingale iff it has zero drift.

**Definition 7** If $X_1$ and $X_2$ are Itô processes with $dX_{it} = \mu_{it} \, dt + \sigma_{it} \, dB_t$, the **quadratic variation** of $X_i$ is

$$\langle X_i, X_i \rangle_t \equiv \lim_{n \to \infty} \sum_{j=0}^{2^n-1} (X_{i,(j+1)t/2^n} - X_{i,jt/2^n})^2 = \int_0^t |\sigma_s|^2 \, ds,$$

(10)

and the **covariation** of $X_1$ and $X_2$ is

$$\langle X_1, X_2 \rangle_t \equiv \lim_{n \to \infty} \sum_{j=0}^{2^n-1} (X_{1,(i+1)t/2^n} - X_{1,it/2^n})(X_{2,(j+1)t/2^n} - X_{2,jt/2^n})$$

$$= \int_0^t \sigma_{1s} \sigma_{2s} \, ds,$$

(11)

(12)

where the convergence above is in probability.

- As a mnemonically helpful shorthand, some write $d\langle X_i, X_i \rangle_t = (dX_i)^2$ and $d\langle X_i, X_j \rangle_t = (dX_i)(dX_j)$.
- If $X$ is an Itô process, and $f$ is a smooth real-valued function, then $f(X)$ is also an Itô process, and Itô’s lemma gives its drift and diffusion:

**Itô’s Lemma** Let $X$ be an $m$-dimensional Itô process as in equation (9) and let $f : \mathbb{R}^m \times [0,T] \to \mathbb{R}$ be $C^{2,1}$. Then $f(X,t)$ is also an Itô process with $df = f_t \, dt + f_x \, dX + \frac{1}{2} \text{tr}[f_{xx} \sigma \sigma'] \, dt = \left(\frac{1}{2} \text{tr}[f_{xx} \sigma \sigma'] + f_x \mu + f_t\right) \, dt + f_x \sigma \, dB$.

- In the shorthand, the Taylor expansion underlying Itô’s lemma is more apparent, and this becomes more memorable as $df = f_t \, dt + f_x \, dX + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial X_i \partial X_j} (dX_i)(dX_j)$.
Example 1 To gain intuition for the 2nd-order term with \( f_{XX} \), let \( X_t = B_t \) and \( f(X) = X^2 \). The usual calculus \( df = f_X dX \) would yield \( f(B_t) = \int_0^t 2B_s dB_s \) and \( E\{f(B_t)\} = E\{B_t^2\} = 0 \), which is incorrect. Including the 2nd-order term gives \( f(B_t) = \int_0^t 2B_s dB_s + \frac{1}{2} \int_0^t \sigma_t^2 \, dt \) so \( E\{f(B_t)\} = E\{B_t^2\} = 1 \). The 2nd-order term captures the "Jensen's inequality" adjustment to the drift of \( f \) which is increasing in both the convexity of \( f \) and the volatility of \( X \).

Example 2 Let \( X_t = -\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \sigma_s^2 \, ds \). Let \( f(X) = e^X \). Then \( df = f dX + \frac{1}{2} f^2 \, ds \) or \( df = -\theta dB, \) so \( f = e^{-\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \sigma_s^2 \, ds} \) is a local martingale.

Example 3 Consider again the continuous-time model of security prices:

\[
\begin{align*}
\frac{dS_{0,t}}{S_{0,t}} &= r_t \, dt \text{ and } \frac{dS_{k,t}}{S_{k,t}} = [\mu_{k,t} - \delta_{k,t}] \, dt + \sigma_{k,t} \, dB_t, \quad k = 1, \ldots, n. \\
\end{align*}
\]

\[dS_{0,t} = r_t \, dt \text{ and } dS_{k,t} = [\mu_{k,t} - \delta_{k,t}] \, dt + \sigma_{k,t} \, dB_t, \quad k = 1, \ldots, n.
\]

- Use Itô's lemma to show \( S_{k,t} = S_{k,0} \Phi_t \), where \( \Phi_t = e^{\int_0^t [\mu_{k,u} - \delta_{k,u} - \frac{\sigma_{k,u}^2}{2}] \, du + \int_0^t \sigma_{k,u} \, dB_u} \).
- Note that \( r, \mu, \delta \) and \( \sigma \) can be any suitably integrable, adapted processes.
- Special Case: Markov Model It is often convenient to specialize to the case in which the coefficients \( r, \mu, \delta \) and \( \sigma \) are functions of \( (S, Y, t) \), where \( Y \) is a vector of state variables with \( Y_t = Y_0 + \int_0^t \mu_r \, (Y_u, u) \, du + \int_0^t \sigma_r (Y_u, u) \, dB_u \). Under Lipschitz and growth conditions on the coefficients, \( (S, Y) \) is Markov.

Continuous-Time Trading Strategies

- We can specify a trading strategy in the \( n+1 \) securities either in terms of the number of shares of each security held at time \( t \), \( N_t = (N_{0,t}, N_{1,t}, \ldots, N_{n,t}) \), or in terms of the value invested in each security, \( \pi_t = (\pi_{0,t}, \pi_{1,t}, \ldots, \pi_{n,t}) \equiv (\pi_{0,t}, \pi_t) \), where each \( \pi_k = N_{k,S_k} \).
- The integrability condition on the trading strategy is easier to state in terms of the (row-vector) of values invested in the \( n \) risky assets, \( \pi \).

Definition 8 A trading strategy is an \( n+1 \)-dimensional adapted process \( \pi_t = (\pi_{0,t}, \pi_{1,t}, \ldots, \pi_{n,t}) \equiv (\pi_{0,t}, \pi_t) \) with \( \int_0^T [\pi_t \sigma_t^2] \, dt < \infty \) a.s.

- We'll focus on tight trading strategies, eliminate \( \pi_0 \) and just specify \( \pi \).

Definition 9 Starting from initial wealth \( x_0 \), a tight trading strategy \( \pi \) generates consumption plan \( (c, W) \) and wealth process \( X_{t,c,x_0} = X_t \) if

\[
X_t = x_0 + \int_0^t r_u X_u \, du + \int_0^t \pi_u (\mu_u - r_u) \, du + \int_0^t \pi_u \sigma_u \, dB_u - \int_0^t c_u \, du 
\] (13)

(the continuous-time WEE) and \( X_T = W \).
The economic effect of the dividends is that if a share of security \( k \) is held in a portfolio for an instant in time, then it changes portfolio value by \( dS_k + \delta_k S_k \, dt = \mu_k S_k \, dt + \sigma_k S_k \, dB \). Thus, holding \( \mu_k \) constant, the effect on the portfolio is invariant to \( \delta_k \). Nevertheless, we keep track of the dividend rate, because it affects the ex-dividend security price, which is the basis for many derivative contracts.

Equation (13) can also be written \( N_t S_t = N_0 S_0 + \int_0^t N_u \, dS_u \) if there are no dividends and intermediate consumption.

The “tightness” of the trading strategy, i.e., the self-financing condition, is essentially the restriction that

\[
d(NS) = NdS
\]

and the additional terms from Itô’s lemma, \( S \, dN + d\langle N, S \rangle_t \) are zero. This is the continuous-time analog to more intuitive simple self-financing condition \( N_{t_j} S_{t_j} = N_{t_{j-1}} S_{t_j} \forall j = 1, \ldots, J - 1 \) that we saw for simple trading strategies before.

**Market Prices of Risk and Equivalent Martingale Measures**

**Definition 10** A market price of risk (mpr) is an adapted \( d \)-dim’l process \( \theta \) s.t.

\[
\begin{aligned}
\mu_t - r_t & = \frac{1}{n x 1} \, \sigma_t \, \theta_t \\
& \text{a.s. a.e.}
\end{aligned}
\]

\[
(15)
\]

**Proposition 4** No arbitrage \( \Rightarrow \) there exists a market price of risk \( \theta \).

The \( d \)-factor risk structure together with the wide range of available trading strategies here means that the cross-section of expected returns must respect this structure, i.e., instantaneous excess expected returns must be linear in factor loadings.

**Proof** No arbitrage \( \Rightarrow \) If \( \pi_t \sigma_t = 0 \) then \( \pi_t [\mu_t - r_t] = 0 \) a.s., a.e.. Otherwise, from WEE (13), one could construct a trading strategy that generated positive consumption from zero wealth. From linear algebra, the statement “If \( \pi_t \sigma_t = 0 \) then \( \pi_t [\mu_t - r_t] = 0 \)” is equivalent to the CSER Equation (15).

It turns out that from any well-behaved mpr \( \theta \) we can construct an equivalent martingale measure \( P^* \).
Definition 11 The riskless discount factor is $\beta_t \equiv e^{-\int_0^t r_u \, du}$, risklessly discounted security prices are $S^*_t = \beta_t S_t$, and risklessly discounted dividends are

$$D^*(t) \equiv \left( \int_0^t S^*_1 \delta_{1,u} \, du, \ldots, \int_0^t S^*_n \delta_{n,u} \, du \right).$$  \hspace{1cm} (16)

Definition 12 A probability measure $\mathcal{P}^*$ on $(\Omega, \mathcal{F}, \mathcal{P})$ is an equivalent martingale measure if $\mathcal{P}^* \sim \mathcal{P}$ and discounted cum-dividend stock prices $G^*(t) \equiv S^*_t + D^*(t)$ are local martingales under $\mathcal{P}^*$.

This is a relaxed version of the previous emm definition that’s as far as we can go without further restrictions.

Proposition 5 If there exists ampr $\theta$ s.t. $\int_0^T |\theta_t|^2 \, dt \leq \infty$ a.s. and the process

$$Z_t \equiv e^{-\int_0^t \theta_u \, dB_u - \frac{1}{2} \int_0^t |\theta_u|^2 \, du}$$  \hspace{1cm} (17)

is a martingale, then $\mathcal{P}^*$ defined by $\frac{d\mathcal{P}^*}{d\mathcal{P}} = Z_T$ is an emm.

The proof uses the Girsanov theorem. Let’s review it now.

**Girsanov Theorem** Let $B$ be an $n$-dim’l Brownian motion under $\mathcal{P}$, $\mathbb{P}^B$ its natural filtration, and $\mathcal{P}^*$ a probability measure equivalent to $\mathcal{P}$. Then $\exists$ a process $\theta$ with $\int_0^T |\theta_t|^2 \, dt \leq \infty$ a.s. s.t.

$$\frac{d\mathcal{P}^*}{d\mathcal{P}} = e^{-\int_0^T \theta_u \, dB_u - \frac{1}{2} \int_0^T |\theta_u|^2 \, du}$$  \hspace{1cm} (18)

and $B^*_t \equiv B_t + \int_0^t \theta_s \, ds$ is an $n$-dim’l Brownian motion under $\mathcal{P}^*$. Moreover, if $X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dB_s$ is an Itô process under $\mathcal{P}$, then $X$ is an Itô process under $\mathcal{P}^*$ with representation $X_t = X_0 + \int_0^t (\mu_s - \sigma_s \theta_s) \, ds + \int_0^t \sigma_s \, dB^*_s$.

Here is some intuition. Suppose $X \sim N(0, 1)$ under $\mathcal{P}$ and $X \sim N(-\theta, 1)$ under $\mathcal{P}^*$. The p.d.f. of $X$ is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ under $\mathcal{P}$ and $f^*(x) = \frac{1}{\sqrt{2\pi} e^{-\theta^2/2}}$ under $\mathcal{P}^*$. So $f^*(x) = e^{-\theta x - \theta^2/2}$. Further, for any function $g(X)$, the $\mathcal{P}^*$-mean is

$$E^*\{g(X)\} = E\{g(X) \frac{f^*(X)}{f(X)}\} = E\{g(X) e^{-\theta X - \theta^2/2}\}.$$

Girsanov says the mean shift can be done to Itô processes Brownian increment by Brownian increment, with an adapted, mean shift process $\theta$.

Note that equivalent changes of measure on a Brownian space can only change drift, not diffusion.
Proof of Proposition 5 Note \( \frac{d\mu}{d\mathcal{P}} = Z_T > 0 \) and \( E\{Z_T\} = 1 \), so \( \mathcal{P}^* \sim \mathcal{P} \). Next, it follows from the Girsanov Theorem that \( B_t^\ast \equiv B_t + \int_0^t \theta_s \, ds \) is an \( n \)-dim’l Brownian motion under \( \mathcal{P}^* \), so \( G_k^\ast(t) \) is a local martingale for all \( k = 1, \ldots, n \):

\[
\begin{align*}
    dG_{k,t}^\ast &= dS_{k,t}^\ast + dD_{k,t}^\ast = d(\beta_t S_{k,t}) + \beta_t S_{k,t} \delta_{k,t} \, dt \\
                   &= \beta_t dS_{k,t} - r_t \beta_t S_{k,t} \, dt + \beta_t S_{k,t} \{ [\mu_{k,t} - r_t] \, dt + \sigma_{k,t} \, dB_t \} \\
                   &= \beta_t S_{k,t} \sigma_{k,t} \theta_t \, dt + dB_t = \beta_t S_{k,t} \sigma_{k,t} dB_t^\ast.
\end{align*}
\]

Proposition 6 If \( \theta \) is a mpr and \( \mathcal{P}^* \) is its associated emm, then the discounted cum-consumption value of wealth \( \beta_t X_t + \int_0^t \beta_s c_s \, ds \) is also a \( \mathcal{P}^* \)-local martingale under any tight trading strategy \( \pi \).

Proof From Itô’s lemma, \( d(\beta_t X_t) = -r_t \beta_t X_t + \beta_t dX_t \), so discounting with the riskless discount factor \( \beta \) absorbs the interest term in WEE 13, and switching from \( dB \) to \( dB^\ast = dB + \theta \, dt \) absorbs excess returns, giving WEE*:

\[
\begin{align*}
    \beta_t X_t &= x_0 + \int_0^t \beta_u \pi_u (\mu_u - r_u) \, du + \int_0^t \beta_u \pi_u \sigma_u \, dB_u - \int_0^t \beta_u c_u \, du \\
                 &= x_0 + \int_0^t \beta_u \pi_u \sigma_u \, dB_u^\ast - \int_0^t \beta_u c_u \, du.
\end{align*}
\]

Tame Trading Strategies and Supermartingales

Definition 13 A trading strategy \( \pi \) starting from wealth \( x_0 \) is tame if \( \beta X_{x_0, \pi} \geq -K \) for some finite constant \( K \).

Proposition 7 A local martingale that is bounded below is a supermartingale.

The proof uses Fatou’s lemma.

Fatou’s lemma If \( \{X_n\} \) is a sequence of nonnegative random variables, then \( \liminf_{n \to \infty} E[X_n|G] \geq E[\liminf_{n \to \infty} X_n|G] \).

Apply this to the sequence \( \{X_n\} \) with \( G = F_t \) to get \( X_{t} \geq E\{X_{s}\|F_t} \).

Proposition 8 If \( \mathcal{P}^* \) is an emm and \( \pi \) is a tame, tight trading strategy generating a nonnegative consumption plan \( (c, W) \), then the discounted cum-consumption wealth \( \beta_t X_{x_0, \pi, c} + \int_0^t \beta_u c_u \, ds \equiv x_0 + \int_0^t \beta_u \pi_u \sigma_u \, dB_u^\ast \) is a \( \mathcal{P}^* \)-local martingale bounded below, and thus a \( \mathcal{P}^* \)-supermartingale. Therefore,

\[
X_{x_0, \pi, c} \geq E^\ast\{\int_t^T e^{-\int_t^r \sigma_u \, du} c_u \, du + e^{-\int_t^T r_s \, ds} W|F_t} \quad \forall t \in [0, T].
\]

Corollary 9 If there exists an emm, there are no tame arbitrage opportunities.
Complete Markets and Martingales

**Lemma 1** No arbitrage ⇒ there exists a unique mpr \( \hat{\theta} \) s.t. every mpr \( \hat{\theta} \) can be written as \( \hat{\theta} = \theta + \nu \) where \( \sigma \nu = 0 \) a.s. a.e. If \( \text{rank}(\sigma) = n \) then \( \hat{\theta} = \sigma' [\sigma \sigma']^{-1} (\mu - r1) \).

We’ll focus on this mpr \( \theta \) and its associated emm \( P^* \) and Brownian motion \( B^* \).

**Definition 14** A trading strategy \( \pi \) is martingale-generating if \( \int_0^T \beta_u \pi_u \sigma_u dB_u^* \) is a \( P^* \)-martingale, not just a \( P^* \)-local martingale.

**Proposition 10** If a tame tr. str. \( \pi \) starting from wealth \( x_0 \) generates a consumption plan \( (c, W) \) that is bounded below, then \( \pi \) is tight and martingale-generating ⇔

\[
X_t^{x_0, \pi, c} = E^* \{ \int_t^T e^{-\int_t^s r_s \, ds} c_u \, du + e^{-\int_t^T r_s \, ds} W | \mathcal{F}_t \} \quad \forall t \in [0, T].
\]  

**Definition 15** The market is complete if every cons. plan \( (c, W) \) with \( E^* \{ \int_0^T |\theta_t|^2 \, dt \} < \infty \) can be generated by a tight, martingale-generating trading strategy.

**Theorem 1** The market is complete ⇔ \( n = d \) and \( \sigma \) is nonsingular.

The proof uses the Martingale Representation Theorem.

**Martingale Representation Theorem** Let \( B \) be an \( n \)-dim'l Brownian motion, \( \mathbf{F}^B \) its natural filtration, and \( X \) a local martingale w.r.t. \( \mathbf{F}^B \). Then there exists an \( n \)-dim'l adapted process with \( \int_0^T |\theta_t|^2 \, dt < \infty \) s.t. \( X_t = X_0 + \int_0^T \theta_u \, dB_u \). Moreover, \( X \) is an \( L^p(\mathcal{P}) \)-martingale for some \( p \in [1, \infty) \) iff \( E[(\int_0^T |\theta_t|^2 \, dt)^{\frac{p}{2}}] < \infty \).

**Lemma 2 (Representation of \( P^* \)-martingales)** If \( M^* \) is a \( P^* \)-martingale then there exists an adapted, square-integrable process \( \psi \) s.t. \( M^*_t = M_0 + \int_0^t \psi_u \, dB^*_u \).

**Proof of Theorem 1** \( \Leftarrow \): Given consumption plan \( (c, W) \), let \( M^*_t = E^* \{ \int_0^T \beta_u c_u \, du + \beta_T W | \mathcal{F}_t \} \).

By the M.R.T. \( M^*_t = M_0^* + \int_0^t \psi_u \, dB^*_u \) for some adapted, square-integrable process \( \psi \). Let \( \pi = \psi' \sigma^{-1} / \beta \) and let \( x_0 = M_0^* \). Then, by the WEE*, \( \pi \) generates \( (c, W) \) starting from wealth \( x_0 \). In particular, evaluating WEE* at time \( t = T \) gives

\[
\beta_T X_T + \int_0^T \beta_u c_u \, du = x_0 + \int_0^T \beta_u \pi_u \sigma_u \, dB^*_u = M^*_T = \int_0^T \beta_u c_u \, du + \beta_T W.
\]  

**Corollary 11** In a complete market, there exists a unique mpr \( \theta = \sigma^{-1} (\mu - r1) \).
Problem Set 2

1. Suppose the price in yen $P$ of a Japanese stock and the exchange rate $X$ dollars per yen are Itô processes given by

$$dP/P = \mu_P \, dt + \sigma'_P \, dB ,$$

$$dX/X = \mu_X \, dt + \sigma'_X \, dB ,$$

(26)

(27)

where $B$ is standard 2-dimensional Brownian motion. Describe the dynamics of the price $Y$ of the stock in dollars.

2. Suppose $\text{rank}(\sigma_t) = n$. Let $\nu$ be an $d$-dimensional process with $\sigma_t \nu_t = 0$ and let

$$\theta_t = \sigma'_t (\sigma_t \sigma'_t)^{-1} [\mu_t - r_t 1] ,$$

$$\hat{\theta}_t = \theta_t + \nu_t ,$$

(28)

(29)

$$Z_t \equiv e^{\int_0^t \beta_s \, dB_s - \frac{1}{2} \int_0^t |\beta_s|^2 \, ds} ,$$

$$\beta_t \equiv e^{\int_0^t r_s \, ds}$$

and

$$m_t \equiv \beta_t Z_t .$$

(30)

(31)

(32)

Finally, let

$$Z_t \equiv e^{\int_0^t \theta_s^t \, dB_s - \frac{1}{2} \int_0^t |\theta_s^t|^2 \, ds} ,$$

$$m^*_t \equiv \beta_t Z_t ,$$

(33)

(34)

(35)

(a) Show that $m^*$ is the only $m$ of the form above in the payoff space, that is, it is the only such $m$ for which there exists a trading strategy that strictly finances a consumption plan $(c, W) \in C$ with $W = m$.

(b) Show that $m^*_t$ is the $m$ process with the smallest instantaneous volatility, or in other words, that its log has the smallest quadratic variation.

3. Verify the following: if $\theta, \theta_1, \theta_2 \in H^2_0$, then

(a) $I^\theta$ is an $L(P)^2$ martingale;

(b) $E[I^\theta_1 I^\theta_2] = E[\int_0^t \theta_{1,s} \theta_{2,s} \, ds]$.

In addition, verify that $||I^\theta_1 - I^\theta_2||_{M^2} = ||\theta_1 - \theta_2||_{H^2}$.
4. (a) Use the Martingale Representation Theorem and Itô’s Lemma to prove the following corollary:

Let \( \{B_t\} \) be an \( n \)-dimensional Brownian motion, \( \{F_t\} \) its natural filtration, and \( \{X_t\} \) a strictly positive local martingale adapted to \( \{F_t\} \). Then there exists an \( n \)-dimensional process \( \theta \in L^2 \) such that

\[
X_t = X_0 e^{\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds}.
\]

(b) Conclude that if \( Z \in L^2(\mathcal{P}) \) is a strictly positive random variable measurable with respect to \( F_t \), then \( Z \) has the representation (36).

5. Use the results in question 4 above, the lemma stated below, and Levy’s Theorem (Prop 14 of Domenico Cuoco’s lecture notes) to prove the Girsanov Theorem:

Let \( B_t \) be standard \( n \)-dimensional Brownian motion on \([0, T]\) under the probability measure \( \mathcal{P} \) with \( \{F_t\} \) its natural filtration and let \( \mathcal{P}^* \) be a probability measure equivalent to \( \mathcal{P} \). Then there exists an \( n \)-dimensional process \( \theta \in L^2 \) s.t.

\[
\frac{d\mathcal{P}^*}{d\mathcal{P}} = \exp(-\int_0^T \theta_t dB_t - \frac{1}{2} \int_0^T |\theta_t|^2 dt)
\]

and

\[
B_t^* \equiv B_t + \int_0^t \theta_s ds
\]

is standard \( n \)-dimensional Brownian motion on \([0, T]\) under \( \mathcal{P}^* \).

**Lemma** In the setting described above, define \( Z_t \equiv E \left\{ \frac{d\mathcal{P}^*}{d\mathcal{P}} \mid F_t \right\} \), a strictly positive martingale w.r.t \( \{F_t\} \). If \( Y \) is an Itô process and \( ZY \) is a \( \mathcal{P} \)-local martingale, then \( Y \) is \( \mathcal{P}^* \)-local martingale.
6. Suppose $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ and let the $\sigma$-field $\mathcal{F}$ be the set of all subsets of $\Omega$. Define the probability measure $\mathcal{P}$ by

$$
\mathcal{P}\{\omega_1\} = \mathcal{P}\{\omega_2\} = \mathcal{P}\{\omega_3\} = \mathcal{P}\{\omega_4\} = \mathcal{P}\{\omega_5\} = \frac{1}{5}.
$$

Finally, suppose $X_1, X_2$ are random variables on $(\Omega, \mathcal{F}, \mathcal{P})$ that take on the following values.

<table>
<thead>
<tr>
<th>state</th>
<th>$X_1$</th>
<th>$X_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

(a) What is the $\sigma$-field $\mathcal{F}_{X_1}$ generated by $X_1$? What is the $\sigma$-field $\mathcal{F}_{X_2}$ generated by $X_2$? (Formally, $\mathcal{F}_X$ is defined as

$$
\mathcal{F}_X = \{X^{-1}(B) | B \in \mathcal{B}\},
$$

where $\mathcal{B}$ is the Borel $\sigma$-field on $\mathbb{R}$. That is, $\mathcal{F}_X$ is the smallest $\sigma$-field with respect to which $X$ is measurable.)

(b) Specify the values of the random variables $\mathbb{E}\{X_2|\mathcal{F}_{X_1}\}$ and $\mathbb{E}\{X_1|\mathcal{F}_{X_2}\}$ for each $\omega_i$.

(c) Suppose $X_1$ and $X_2$ are the time 1 and 2 values of a stochastic process $X$ with $X_0 = 1$. Let $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$ be the filtration generated by $X$. What is $\mathcal{F}_0$? $\mathcal{F}_1$? $\mathcal{F}_2$? (In the filtration generated by a stochastic process $X$, each $\mathcal{F}_t$ is the $\sigma$-field generated by the complete history of $X$ up to and including time $t$. That is,

$$
\mathcal{F}_t = \sigma\{\mathcal{F}_{X_s} | s \leq t\},
$$

where the notation $\sigma\{\}$, or “$\sigma$-closure,” or the “$\sigma$-field generated by” is necessary because the union of $\mathcal{F}_{X_1}, \ldots, \mathcal{F}_{X_t}$ may not by itself represent a valid $\sigma$-field.)