Contingent Claims Pricing

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Readings and References

Back, chapters 15 and 16.
Duffie, chapters 5 and 6.
Karatzas and Shreve, 1998, chapter 2
Summary of the Continuous-Time Financial Market

- Security prices satisfy \( \frac{dS_{0,t}}{S_{0,t}} = r_t dt \) and \( \frac{dS_{k,t}}{S_{k,t}} = (\mu_{k,t} - \delta_{k,t}) dt + \sigma_{k,t} dB_t \).
- Given tight tr. strat. \( \pi_t \) and consumption \( c_t \), portfolio value \( X_t \) satisfies the
  - WEE: \( dX_t = r_t X_t dt + \pi_t (\mu_t - r_t 1) dt + \pi_t \sigma_t dB_t - c_t dt \).
- No arbitrage \( \Rightarrow \) if \( \pi_t \sigma_t = 0 \) then \( \pi_t (\mu_t - r_t 1) = 0 \Rightarrow \exists \theta_t \text{ s.t. } \sigma_t \theta_t = \mu_t - r_t 1 \)
  \( \Rightarrow dX_t = r_t X_t dt + \pi_t \sigma_t (\theta_t dt + dB_t) - c_t dt \).
- Under emm \( \mathcal{P}^* \) given by \( \frac{dP^*}{dP} = Z_T \) where \( Z_t = e^{-\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds} \),
  - \( B_t^* = B_t + \int_0^t \theta_s ds \) is Brownian motion.
  Let \( \beta_t = e^{-\int_0^t \sigma_s ds} \) and sdf process \( M_t = \beta_t Z_t \). Then the WEE can also be written:
    - WEE*: \( d\beta_t X_t + \beta_t c_t dt = \beta_t \pi_t \sigma_t dB_t^* \)
    - WEE-M: \( dM_t X_t + M_t c_t dt = M_t [\pi_t \sigma_t - \theta_t X_t] dB_t \)

So \( X_t = E_t^* \{ \int_t^T \frac{\beta_t}{M_t} c_u du + \frac{\beta_t}{M_t} X_T \} \) if \( \pi \) is mtgale-gen.

- If \( \sigma \) is nonsingular, every c.plan \( (c, X_T) \) can be generated by a mtgale-gen. tr.strat.

European Contingent Claims

Definition 1 A European contingent claim (ecc) is a payoff \( (c, W) \in C \).
- In an incomplete market, it’s hard to nail down a unique replication cost of a ecc
  because there is so much flexibility in the choice of trading strategies that wealth
  may be only a local martingale or supermartingale.

Definition 2 The price of a ecc \( x = (c, W) \) is \( S_t^x = \min \{ \bar{\pi}_t 1 : \bar{\pi} \) is a tame, tight trading strategy that generates \( x \} \), provided the minimum exists.

Proposition 1 If the market is complete then we have the RNPE

\[
S_t^x = E^* \{ \int_t^T e^{-\int_t^u r_s ds} c_u du + e^{-\int_t^T r_s ds} W|\mathcal{F}_t} \} \forall t \in [0, T].
\] (1)

Proof In a complete market, there exists a tight, tame, martingale-generating strategy \( \pi \) that finances \( x \) and must satisfy

\[
X_t = E^* \{ e^{-\int_t^T r_s ds} c_u du + e^{-\int_t^T r_s ds} W|\mathcal{F}_t} \}.
\] (2)
For any other tame trading strategy that finances $x$, risklessly discounted cum-consumption portfolio value is a $\mathcal{P}^*$-supermartingale, so

\[ X_t \geq \mathbb{E}^* \{ e^{-\int_t^T r_s ds} c_u du + e^{-\int_t^T r_s ds} W | \mathcal{F}_t } \}. \tag{3} \]

**Definition 3**
A replicating portfolio for an ecc $x$ is a tame, martingale-generating trading strategy that tightly finances $x$.

- In what sense is $S^x$ an equilibrium price for an ecc $x$?
- If a replicating portfolio of $x$ exists, we would like to argue that the equilibrium price of $x$ must be the price of its replicating portfolio.
- But when doubling strategies are possible, the replication cost may not be unique.
- And, if we are restricted to tame trading strategies, we may not be able turn an apparent mispricing into an arbitrage, because the natural long-short position to implement might get closed out if its value gets marked below an institutionally imposed lower bound before the end of the trading horizon.
- Nevertheless, the minimum replication cost is a pretty economically natural notion of the price of $x$.

**SDF Process and Pricing Equation**

- If the market is complete, we can define the unique SDF process $M_t$ by

\[ M_t = \beta_t \mathbb{E}^* \{ \frac{d\mathbb{P}^*}{d\mathbb{P}} | \mathcal{F}_t } = \beta_t Z_t = e^{-\int_0^t r_u du - \int_0^t \theta_u dB_u - \frac{1}{2} \int_0^t \theta_u^2 du }, \tag{4} \]

- see that stochastically discounted wealth process $MX$ satisfies WEE-M:

\[ M_t X_t = x_0 + \int_0^t M_u [\pi_u \sigma_u - X_u \theta_u] dB_u - \int_0^t M_u c_u du, \tag{5} \]

and rewrite the RNPE with true expectation and stochastic discounting (SDPE):

\[ S^x_t = \mathbb{E}^* \{ \int_t^T \frac{\beta_u}{\beta_t} c_u du + \frac{\beta_T}{\beta_t} W | \mathcal{F}_t } \}
\[ = \mathbb{E}^* \{ \int_t^T \frac{\beta_u c_u}{\beta_t} du + \frac{\beta_T}{\beta_t} W | \mathcal{F}_t } \} / \mathbb{E}^* \{ \frac{d\mathbb{P}^*}{d\mathbb{P}} | \mathcal{F}_t } \}
\[ = \mathbb{E} \{ \int_t^T \frac{M_u c_u}{M_t} du + \frac{M_T}{M_t} W | \mathcal{F}_t } \} \forall t \in [0, T]. \tag{8} \]
Fundamental PDE for Path-Independent Claims

From the WEE in undiscounted terms, but with $B^*$ instead of $B$, cum-consumption or cum-dividend portfolio value always appreciates at the riskless rate $r$ under $P^*$:

$$dX_t + c_t dt = r_t X_t dt + \pi_t(\mu_t - r_t) dt + \pi_t \sigma_t dB_t = r_t X_t dt + \pi_t \sigma_t dB^*_t. \quad (9)$$

Intuitively, once $B^*$ absorbs the Sharpe ratios of all the securities, it also absorbs the Sharpe ratios of all the portfolio processes.

**Definition 4** A vector $x = (c, W) \in C^1$ is path-independent if $c_t = \varphi_1(S_{1,t}, \ldots, S_{n,t}, t)$ and $W = \varphi_2(S_{1,t}, \ldots, S_{n,t})$ for continuous functions $\varphi_1 : \mathcal{R}^n_+ \times [0, T] \to \mathcal{R}$ and $\varphi_2 : \mathcal{R}^n_+ \to \mathcal{R}$.

- In the Markovian model, if $x$ is path-independent then

$$S^x_t = E^x\{e^{-\int_t^\tau r_s ds} c_u du + e^{-\int_t^\tau r_s ds} W|\mathcal{F}_t}\} = F(S_{1,t}, \ldots, S_{n,t}, Y_T, t)$$

for some real-valued function $F$.

- If $F$ is smooth enough for an application of Itô’s lemma, then it must satisfy the following PDE, which says the drift of $F$ from the WEE must equal the drift of $F$ from Itô’s lemma:

$$rF - \varphi_1 = F'_S I_S (r_1 - \delta) + F'_Y (\mu_Y - \sigma_Y \theta) + F_t + \frac{1}{2} \tr(F_{SS} I_S \sigma \sigma'I_S) + \frac{1}{2} \tr(F_{YY} I_S \sigma \sigma'Y_S)$$

where $I_S$ is a diagonal matrix with diagonal elements $S_1, S_2, \ldots, S_n$ and the derivatives $F_S$ and $F_{SS}$ are with respect to the vector $(S_1, S_2, \ldots, S_n)$.

- This is subject to the boundary condition $F(S_{1,T}, \ldots, S_{n,T}, Y_T, T) = \varphi_2(S_{1,T}, \ldots, S_{n,T})$.

- Furthermore, by matching the diffusion coefficient from the WEE with the diffusion coefficient from Itô’s lemma, we have that the diffusion coefficient of $S^x_t$ must satisfy

$$\pi^x \sigma = F'_S I_S \sigma + F'_Y \sigma_Y,$$  \quad (10)

which gives a way to compute the replicating trading strategy $\pi^x$. 
Special Case: Complete Market with Constant Coefficients

Suppose \( r, \mu, \sigma, \) and \( \delta \) are constant.

**Definition 5** A function \( f : \mathcal{R}^d \times [0, T] \rightarrow \mathcal{R} \) satisfies a polynomial growth condition (pgc) if there exist positive constants \( k_1 \) and \( k_2 \) such that

\[
|f(x, t)| \leq k_1 (1 + |x|^{k_2}) \quad \forall (x, t) \in \mathcal{R}^d \times [0, T].
\]  

(11)

**Theorem 1 (Feynman-Kac)** Suppose the functions \( \varphi_1, \varphi_2, \) and 
\( F(S_{1, t}, \ldots, S_{n, t}, t) \equiv \mathbb{E}^* \{ \int_t^T e^{-r(u-t)} \varphi_1(S_{1, u}, \ldots, S_{n, u}, u) \, du \}
\)

\( + e^{-r(T-t)} \varphi_2(S_{1, T}, \ldots, S_{n, T}) |F_t| \)

each satisfy a pgc. Then \( F \) satisfies the PDE

\[
rF - \varphi_1 = F_s I_s (r1 - \delta) + F_t + \frac{1}{2} \text{tr}(F_{SS} \sigma \sigma' I_s)
\]

subject to \( F(S_{1, T}, \ldots, S_{n, T}, T) = \varphi_2(S_{1, T}, \ldots, S_{n, T}) \). There is no other solution to this PDE that satisfies a pgc.

In addition, the trading strategy \( \pi^x \) that generates \( x = (\varphi_1, \varphi_2) \) is given by 
\( \pi^x_k = S_k \partial F / \partial S_k \), i.e., \( N_k = \partial F / \partial S_k \), for \( k = 1, 2, \ldots, n \), and \( \pi^x_0 = F - \pi^x1 \).

Black-Scholes-Merton Call Option Model

- Assume \( n = d = 1, r \) and \( \sigma \) are constant, and for ease of exposition, begin by assuming no dividends. The price of the risky asset or “stock” follows

\[
\frac{dS_t}{S_t} = \mu_t dt + \sigma dB_t.
\]

(13)

- Consider a call on the stock with time \( T \) payoff \( \varphi_2(S_T) = (S_T - K)^+ \) for some positive constant \( K \).

- The Black-Scholes argument (assume \( \mu \) is constant):
  - The time \( t \) call price \( C_t = c(S_t, t) \) because \( S \) is Markov.
  - Consider a portfolio short one call and long \( c_S \) shares of the stock. The portfolio value is \( X = c_S S - c + \pi_0 \) where \( \pi_0 \) adjusts to make the portfolio self-financing.
  - On one hand, from the wealth evolution equation \( dX = N dS \) and Itô’s lemma,

\[
\begin{align*}
dX &= c_S dS - dc + r\pi_0 \, dt \\
&= c_S dS - (c_S dS + c_t \, dt + \frac{1}{2} c_{SS} \sigma^2 S^2 \, dt) + r\pi_0 \, dt \\
&= -c_t \, dt + \frac{1}{2} c_{SS} \sigma^2 S^2 \, dt + r\pi_0 \, dt.
\end{align*}
\]
On the other hand, by no arbitrage, since the portfolio is locally riskless, i.e., has zero diffusion, it must appreciate at the riskless rate:

$$dX = rX dt = r(cS - c + \pi_0) dt$$.

Therefore, it must be that

$$rSc + \frac{1}{2} \sigma^2 S^2 c_{SS} + c_t = ec$$.

They recognized this as the heat equation from physics, with solution

$$c(S_t, t) = \mathbb{E}^* \{ e^{-r(T-t)} (S_T - K)^+ | S_t \}$$,

which is the same as our result from equation (1).

**Alternative derivation with change of numeraire/change of measure:**

Now we can easily incorporate a constant dividend rate. Write

$$C_t = \mathbb{E}^* \{ e^{-r(T-t)} (S_T - K)_+ 1_{\{S_T > K\}} | \mathcal{F}_t \}$$

$$= S_t e^{-\delta(T-t)} \mathbb{E}^* \{ e^{\sigma (B^*_t - B_t^*) - \sigma^2 (T-t)/2} 1_{\{S_T > K\}} | \mathcal{F}_t \} - e^{-r(T-t)} K \mathbb{P}^* \{ S_T > K | \mathcal{F}_t \}$$.

Next, simplify the first term by introducing a new measure $\mathbb{P}^{(s)}$ defined by

$$d\mathbb{P}^{(s)} / d\mathbb{P}^* = e^{\sigma B^*_1 - \sigma^2 T/2}$$ (under which prices are martingales with the stock as numeraire). Then

$$C_t = S_t e^{-\delta(T-t)} \mathbb{P}^{(s)} \{ S_T > K | \mathcal{F}_t \} - e^{-r(T-t)} K \mathbb{P}^* \{ S_T > K | \mathcal{F}_t \}$$.

Under the new measure, $B^{(s)}_t = B^*_t - \sigma t$ is a standard Brownian motion, so

$$C_t = c(S_t, t) = S_t e^{-\delta(T-t)} N(d_1(S_t, t)) - e^{-r(T-t)} K N(d_2(S_t, t))$$

where $N$ is the cumulative normal distribution function,

$$d_1(S_t, t) = \frac{\ln(S_t e^{-\delta(T-t)}/(K e^{-r(T-t)})) + \sigma^2 (T - t)/2}{\sigma \sqrt{T - t}}$$,

and $d_2(S(t), t) = d_1(S(t), t) - \sigma \sqrt{T - t}$.

The option “delta,” or number of shares in the replicating portfolio, is

$$c_S = e^{-\delta(T-t)} N(d_1(S_t, t))$$. 

Forward Contracts

Consider a forward contract to buy a risky asset at time $T$ for price $F$. In this case $x = (0, S_T - F)$ and the time $t$ value of the contract is

$$V_t^F = E^*\{e^{-\int_t^t r_s ds}(S_T - F)|F_t}\} \quad (20)$$

The forward price $F_t$ at time $t$ is such that $V_t^{F_t} = 0$, i.e.,

$$F_t = \frac{E^*\{e^{-\int_t^T r_s ds}S_T|F_t}\}}{E^*\{e^{-\int_t^T r_s ds}|F_t}\}}. \quad (21)$$

If the risky asset dividend rates $\delta$ are deterministic, then

$$F_t = \frac{Se^{-\int_t^T \delta_s ds}}{P_t^T} = S_te^{y_T(T-t)+\int_t^T \delta_s ds} \quad (22)$$

where $P_t^T$ is the time $t$ price of a zero-coupon bond maturing at time $T$ and $y_T^T$ is the continuously compounded $(T-t)$-year “zero rate” or “zero yield” at time $t$.

Futures Contracts (Duffie and Stanton, 1992)

A futures contract on an underlying asset with associated futures price $G_t$, delivery date $T$, and continuous resettlement is a contract which produces a cumulative cash flow of $G_u - G_t$ between any two dates $t$ and $u$ with $0 \leq t \leq u \leq T$. In addition, at time $T$, the contract obliges the holder to buy one share of the asset at price $G_T$.

Given the futures price process $G$, the time $t$ value of a futures contract is

$$V_t^G = E^*\{\int_t^T e^{-\int_t^u r_s ds}dG_u + e^{-\int_t^T r_s ds}(S_T - G_T)|F_t}\}. \quad (23)$$

Buying and selling futures contracts is costless, so the equilibrium futures price process $G$ must be such that $V_t^G \equiv 0$ and $G_T = S_T$.

**Theorem 2** Suppose $E^*\{S_T^2\} < \infty$ and $e^{-\int_0^T r_s ds}$ is bounded above and below away from zero. Then there exists a unique $\mathcal{L}$-ô process $G$ with $E^*\langle G, G\rangle_T < \infty$ satisfying $V_t^G \equiv 0$ and $G_T = S_T$. It is given by $G_t = E^*\{S_T|F_t\}$.

**Sketch of Proof** The conditions $V_t^G \equiv 0$ and $G_T = S_T$ imply $G$ is a $\mathcal{P}^*$-martingale with last element $S_T$, so $G_t = E^*\{S_T|F_t\}$.
To compare futures and forward prices, note that
\[ F_t = G_t + \frac{\text{cov}^\ast \{S_T, e^{-\int_t^T r_s \, ds} | \mathcal{F}_t \}}{\text{E}^\ast \{e^{-\int_t^T r_s \, ds} | \mathcal{F}_t \}}. \] (23)

In particular, they are equal if and only if \( S_T \) and \( e^{-\int_t^T r_s \, ds} \) are uncorrelated.

This will be the case if interest rates are deterministic.

But if interest rates are stochastic, they may differ. For example, bond futures prices are typically lower than their forward prices.

**Forward Measure vs. Futures Measure**

When the interest rate \( r \) is stochastic, it can be convenient for contingent claims pricing to work with the so-called *forward measure*, \( \mathcal{P}^{(T)} \), under which prices normalized by the price of the zero-coupon bond maturing at time \( T \) are martingales.

The forward measure \( \mathcal{P}^{(T)} \) is defined by \( \frac{d\mathcal{P}^{(T)}}{d\mathcal{P}} = \frac{\beta_T}{P_0} \), where \( P_0 = \text{E}^\ast \{\beta_T\} \) is the time 0 price of the zero maturing at time \( T \).

Note that the forward price \( F_0 \) of a risky asset with price \( S \) satisfies
\[ 0 = \text{E}^\ast \{\beta_T(S_T - F_0)\} = P_0^T \text{E}^{(T)} \{S_T - F_0\}, \] (24)
which implies \( \text{E}^{(T)} \{S_T\} = F_0 \), which is why \( \mathcal{P}^{(T)} \) is called the forward measure.

By contrast, \( \text{E}^\ast \{S_T\} = G_0 \), so \( \mathcal{P}^\ast \) is sometimes called the *futures measure*.

Both are *risk-neutral* measures in the sense that the price of time \( T \) payoff \( W \) can be written as
\[ W_0 = \text{E}^\ast \{\beta_T W\} = P_0^T \text{E}^{(T)} \{W\} \] (25)
with discounting at a *riskless* rate. They are the same when \( r \) is nonstochastic.
The price of a call on a risky asset with strike $K$ expiring at time $T$ can be written

$$C_0 = P_0^T E^{(T)} \{(S_T - K)^+ \} = P_0^T \int_K^\infty (s - K) f^{S,T}(s) \, ds \; ,$$

(26)

assuming $S_T$ has a well-defined probability density function $f^{S,T}(s)$ under $P^{(T)}$.

If $C_0$ is a twice-differentiable function of the strike price $K$, then

$$\frac{\partial C_0}{\partial K} = P_0^T \int_K^\infty (-1) f^{S,T}(s) \, ds \quad \text{and} \quad \frac{\partial^2 C_0}{(\partial K)^2} = P_0^T f^{S,T}(K) \; ,$$

(27)

so we can recover the pdf of the future asset price with a continuum of call prices.

Then, if a claim’s payoff is a path-independent function of the future risky asset price, $W = \varphi(S_T)$, its price can be written as

$$W_0 = P_0^T \int_0^\infty \varphi(s) f^{S,T}(s) \, ds = P_0^T \int_0^\infty \varphi(s) \frac{\partial^2 C_0}{(\partial K)^2} |_{K=s} \, ds \; .$$

(28)

Finally, note that the call price above can be written

$$C_0 = P_0^T \left[ F^{P^{(S)}} \{ S_T > K \} - K P^{(T)} \{ S_T > K \} \right] \; ,$$

(29)

where $\frac{dP^{(S)}}{dP^{(T)}} = \frac{S_T}{F}$, which is a generalization of the Black-Scholes formula.
Problem Set 3

1. Derive the (Margrabe) valuation formula for a European option to exchange asset 1 for asset 2 under the assumption that each asset’s dividend rate and volatility is nonstochastic. Describe the dynamics of the replicating trading strategy.

2. Derive the (Merton) European call option valuation formula in the case that both the underlying stock and the zero-coupon bond maturing on the option expiration date have nonstochastic volatility and the stock has a nonstochastic dividend rate. Determine the replicating trading strategy.

3. Suppose the market coefficients are constant. Derive the (Black-Scholes) European call and put option valuation formulas (with dividends) and the replicating trading strategies.