Models of price dynamics and order splitting

Securities Trading: Principles and Procedures
Chapters 13 and 14

Outline

- Statistical models of security prices and order impacts
- Given these statistical models, what are the best order-splitting strategies.
- The risk-return trade-off in order splitting.
Statistical models

- The basic models are constructed by starting with a simple model and adding on the features that we need.
- Random-walk model
- Random-walk + drift (“short-term alpha”)
- Impact model: Random-walk + drift + order-price impact

Random-walk model

- Let $t$ represent time (minutes, seconds, milliseconds, ticks ...)
- $p_t$ is the price at the end of interval $t$ (at the end of the minute, second, ...)
  - Usually $p_t$ is the bid-ask midpoint (BAM), but it might be the last sale price.
- $p_t = p_{t-1} + u_t$
  - where $u_t$ is some random disturbance or prediction error that reflects “new information”
    - In expectation this disturbance is zero: $E u_t = 0$.
    - The standard deviation of $u_t$ is $\sigma_u$. 
Recall the example used to analyze implementation shortfall of limit vs. market orders.

- At each step, $u_t = \pm 0.01$ with equal probability.
- $E u_t = \frac{1}{2} \times (0.01) + \frac{1}{2} \times (-0.01) = 0$
- $\sigma_u = \sqrt{\frac{1}{2} \times (0.01)^2 + \frac{1}{2} \times (-0.01)^2} = 0.01$
Random-walk with drift ("short-term alpha")

- The basic model \( p_t = p_{t-1} + u_t \) has no trend (on average).
- A trader might believe that there is a predictable trend (from momentum, over-shooting, consensus forecast error, etc.).

\[ p_t = \alpha + p_{t-1} + u_t \]

- \( \alpha \) denotes the trend (either up or down), such as, "+$0.01 per minute."
- "Alpha" is used in many finance contexts to denote superior (or, if negative, inferior) performance.
- This model is used extensively in option pricing.

Simulated random walk (with/without \( \alpha = 0.002 \) per tick)

- Non-zero alpha is difficult to detect visually and statistically.
The random-walk with drift: Limit order execution times

- Suppose that the current stock price is $S_0$ and we want to put in a limit order to sell at some price $S_{Sell} > S_0$.
- Example: $S_0 = $15 and $S_{Sell} = $17
- How long do we think it will take for the order to execute?

The situation

$S_{Sell} = $17

$S_0 = $15

Time

Execution time, $T^*$
A simple result

- If $p_t = \alpha + p_{t-1} + u_t$, then $ET^* = \frac{S_{\text{Sell}}-S_0}{\alpha}$

- Example
  - $t$ measures minutes and $\alpha = \$0.01 \text{ per minute}$.
  - Then $ET^* = \frac{17.00-15.00}{0.01} = 2.00 = 200 \text{ minutes}$

- Notes
  - The expected time to execution does not depend on volatility.
  - If $\alpha = 0$ then $ET^*$ is infinite.
    - You might get an execution, but don’t count on it.

- Embedded problem: $S_0 = 20, \alpha = -\$0.03 \text{ per minute}$. We put in a limit order to buy at $19.10. How long do we expect to wait? (Answer in online notes)

Embedded problem

- $S_0 = 20, \alpha = -\$0.03 \text{ per minute}$. We put in a limit order to buy at $19.10. How long do we expect to wait? (Answer in online notes)
  - $ET^* = \frac{20.00-19.10}{0.03} = \frac{0.90}{0.03} = 30 \text{ minutes}$
What’s useful in predicting short-term alpha?

- Advance knowledge of news announcements.
- Current/recent price changes in other stocks that are in the same industry.
- Current/recent changes in the market index.

The impact model: random-walk + drift + order/price impact

\[ p_t = p_{t-1} + \alpha + \lambda S_t + u_t \]

- \( S_t \) is the net number of shares actively purchased in interval \( t \).
  - the number of shares that lifted the ask less the number of shares that hit the bid.
  - Example: \( S_t = -100 \) → “100 shares were sold, net”
- \( \lambda > 0 \) is the impact coefficient.
  - As \( \lambda \) increases, each trade has a larger impact.
  - “Purchases drive the price up; sales drive the price down.”
Interpretation of $S_t$

- In principle $S_t$ is the net purchase computed over all trades in interval $t$.
- Including our own trades and trades of others.
- $S_t = S_{t}^{\text{own}} + S_{t}^{\text{others}}$
- For forecasting and analysis, we want to use the best available prediction of $S_{t}^{\text{others}}$.
- Often trading strategies are analyzed assuming that our expectation of others’ trades is $E S_{t}^{\text{others}} = 0$

Interpretation of $\lambda$

- The model says that order-price impact is permanent.
- Order price impact arises from the market's belief that orders might be informed.
- If we are uninformed, our trades will still move the market, but eventually the effect of our trades will vanish.
Analysis of order splitting with the impact model

- A two-period example.
- Objective: buy $S_{Total}$ shares over two periods (1 and 2) at the lowest possible expected cost.
- $S_{Total} = S_1 + S_2$
  - where $S_1 = \text{shares purchased in period 1}$ and $S_2 = \text{shares purchased in period 2}$.
- The total cost is $C = p_1S_1 + p_2S_2$
- It is now time 0. We know $p_0$, but we don’t know $p_1$ or $p_2$.
- Use the impact model to forecast $p_1$ and $p_2$.

Special case where $\alpha = 0$.

- Compute ahead:
  - $p_1 = p_0 + \lambda S_1 + u_1$
  - $p_2 = p_1 + \lambda S_2 + u_2 = p_0 + \lambda S_1 + \lambda S_2 + u_1 + u_2$
- Simplify by setting $u_1 = u_2 = 0$
  - “Looking ahead, we expect $u_1$ and $u_2$ to be zero.”
- $p_1 = p_0 + \lambda S_1$
- $p_2 = p_0 + \lambda S_1 + \lambda S_2$
- Plug these into $C$ as forecasts.
Reworking the cost

- $C = p_1 S_1 + p_2 S_2 = S_1(p_0 + \lambda S_1) + S_2(p_0 + \lambda S_1 + \lambda S_2)$
  - where $S_1$ and $S_2$ are the “unknowns”
- Remember that $S_{Total} = S_1 + S_2$, so $S_2 = S_{Total} - S_1$
- $C = \lambda S_1^2 - S_1 \lambda S_{Total} + S_{Total} (p_0 + \lambda S_{Total})$
  - This is one equation in one unknown ($S_1$)
  - Formally, this is the expected cost, “EC”

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Plot of $C$ with parameters

$S_{Total} = 10,000$ shares; $p_0 = $10; $\lambda = $0.0001/share

- The total expected cost has a minimum at $S_1 = 5,000$ shares
  (and $S_2 = 5,000$ shares)
Formally, to find the minimum, set $\frac{dc}{ds_1} = 0$

- $C = \lambda s_1^2 - s_1 \lambda S_{Total} + S_{Total} (p_0 + \lambda S_{Total})$
- $\frac{dc}{ds_1} = 2\lambda s_1 - \lambda S_{Total} = 0$ implies the optimal $s_1^* = \frac{S_{Total}}{2}$.
- In general, with $\alpha = 0$, and trading over $T$ periods,
  $$S_i^* = \frac{S_{Total}}{T}$$

What if $\alpha \neq 0$ (in the two-period problem)?

- Modified optimum: $S_1^* = \frac{\alpha + \lambda S_{Total}}{2\lambda}$
- With $\alpha > 0$, there is positive drift, so $S_1^*$ rises.
  - Future purchases will be more expensive.
- With $\alpha < 0$, there is negative drift.
  - The price is dropping: buy later.
Trading Horizon \((T)\), expected cost, and variance of cost (risk)

<table>
<thead>
<tr>
<th>(T)</th>
<th>(EC^*)</th>
<th>(Var(C^*))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(p_0S_{Total} + \lambda S^2_{Total})</td>
<td>(S^2_{Total}\sigma^2_u)</td>
</tr>
<tr>
<td>2</td>
<td>(p_0S_{Total} + \frac{3\lambda S^2_{Total}}{4})</td>
<td>(\frac{5}{4}S^2_{Total}\sigma^2_u)</td>
</tr>
<tr>
<td>3</td>
<td>(p_0S_{Total} + \frac{2\lambda S^2_{Total}}{3})</td>
<td>(\frac{14}{9}S^2_{Total}\sigma^2_u)</td>
</tr>
<tr>
<td>...</td>
<td>(\sqrt{\cdot})</td>
<td>(\uparrow)</td>
</tr>
</tbody>
</table>

\(\square\) As \(T\) increases, expected trading cost drops and risk rises.
The “efficient trading frontier”

- Expected trading cost
- Risk (volatility of trading cost)

- (By analogy with the efficient portfolio frontier.)

**Extensions and modifications**

- More complex order impacts that have both permanent and temporary (transient) effects.
- Addition of other securities (stocks in the same industry, a market-wide basket, and so on).
- Time variation in parameters.
  - \( \alpha, \lambda, \) and/or \( \sigma_u \) depend on time of day.
Order price impact and manipulation

- “Manipulation”
  - There are no universally accepted definitions in economics or law.
- Most definitions suggest something like
  - Trading to deliberately move the price, to establish an artificial price, a price that does not reflect true supply and demand.
  - Most manipulations involve deception.

- These definitions aren’t precise
  - It is almost impossible to trade without moving the price.
  - Many accepted strategies attempt to obscure the trader’s true information, intentions and plans.
- The follow illustrates certain possible manipulations based on order price impact.
- For a given impact function can an uninformed trader execute a series of profitable buys and sells based on the price movements that his orders generate?
- Note: many of the schemes discussed here are illegal. They are presented to facilitate discussion of what features make a market prone to manipulation, so that, to the greatest extent possible, these features may be avoided in actual securities markets.
Recall the impact function used to analyze order splitting.

- \( p_t = p_{t-1} + \alpha + \lambda S_t + u_t \)
  - \( S_t \) is the net number of shares actively purchased in interval \( t \).
  - \( \lambda > 0 \) is the impact coefficient.

- We will look at strategies that start “flat” (with no position) and end flat.
  - Example: buy 10 shares using 10 orders of one share, then sell using 10 orders of one share, OR two orders of five shares, OR ...

- We’ll assume that \( \alpha = 0 \).
  - If \( \alpha > 0 \), just buy. Sell after the price has gone up.
  - If \( \alpha < 0 \), ...

- We’ll also ignore risk \((u_t = 0)\)

\[ p_t = p_{t-1} + \lambda S_t \]

Attempted manipulation 1

- Starting at \( p_0 \), buy 5 shares slowly, one at a time.
  - \( p_1 = p_0 + \lambda \times 1 = p_0 + \lambda \)
  - \( p_2 = (p_0 + \lambda) + \lambda \times 1 = p_0 + 2\lambda \)
  - ...
  - \( p_5 = p_0 + 5\lambda \)
  - Average purchase price is \( \frac{p_1 + p_2 + \cdots + p_5}{5} = p_0 + 3\lambda \)

- Now sell the shares, one at a time
  - \( p_6 = p_5 - \lambda \times 1 = p_0 + 5\lambda - \lambda = p_0 + 4\lambda \)
  - \( p_7 = p_6 - \lambda \times 1 = p_0 + 4\lambda - \lambda = p_0 + 3\lambda \)
  - ...
  - \( p_{10} = p_9 - \lambda \times 1 = p_0 + \lambda - \lambda = p_0 \)
  - Average sale price is \( \frac{p_6 + p_7 + \cdots + p_{10}}{5} = p_0 + 2\lambda \)

- The average profit per share is \( -(p_0 + 3\lambda) + (p_0 + 2\lambda) = -\lambda \) (a loss)

- The receipts don’t cover the expenditure. The manipulation doesn’t work.
Attempted manipulation 2

- Buy the five shares slowly (as before)
  - Average price is $p_0 + 3\lambda$
- Sell the shares all at once:
  - $p_6 = p_5 - \lambda \times 5 = p_0 + 5\lambda - 5\lambda = p_0$
- Manipulation profits are $-(p_0 + 3\lambda) - p_0 = -3\lambda$
- This, too, leads to a loss.

“Theorem”

- If the price impact function is linear and constant over time, profitable manipulation isn’t possible.
- Are there non-linear or time-varying price impact functions that allow for manipulation.
  - Yes
Time variation in impact

- Suppose that initially $\lambda = 1$, and we know that it will drop to $\lambda = 0.1$.
- Then we can buy two units
  - $p_1 = p_0 + \lambda \times 1 = p_0 + 1$
  - $p_2 = p_1 + \lambda \times 1 = p_0 + 2$
  - Average share price is $p_0 + 1.5$
- ... and sell them when $\lambda = 0.1$
  - $p_3 = p_2 - \lambda \times 1 = p_2 - 0.1 = p_0 + 1.9$
  - $p_4 = p_3 - \lambda \times 1 = p_0 + 1.8$
  - Average share price is $p_0 + 1.85$
- Manipulation profits are $-(p_0 + 1.5) + p_0 + 1.85 = 0.35 > 0$

Asymmetry in the impact function

- Suppose that $\lambda$ for buys is $\lambda_{Buy} = 0.1$ and $\lambda$ for sells is $\lambda_{Sell} = 1$.
- We (short) sell two shares
  - $p_1 = p_0 - \lambda_{Sell} \times 1 = p_0 - 1$
  - $p_2 = p_1 - \lambda_{Sell} \times 1 = p_0 - 2$
  - Average price is $p_0 - 1.5$
- Now we cover our short sales
  - $p_3 = p_2 + \lambda_{Buy} \times 1 = p_2 - 2 + 0.1 = p_0 - 1.9$
  - $p_4 = p_3 + \lambda_{Buy} \times 1 = p_0 - 1.8$
  - Average price is $p_0 - 1.85$
- Manipulation profits are $+(p_0 - 1.5) - (p_0 - 1.85) = 0.35 > 0$
General structure of manipulations

- To establish the position, first trade to maximize the price impact.
  - This doesn’t necessarily mean “buy”; sometimes the initial position is short.
- To unwind the position, trade to minimize the price impact.
- Time variation
- Asymmetry

Nonlinearities in the impact function ...

\[ p_t \]
\[ q_t \]
The concave case, for purchases

- Suppose that \( p_t = \begin{cases} 
  p_{t-1} + \lambda \sqrt{S_t} & \text{for buy orders, } S_t > 0 \\
  p_{t-1} - \lambda \sqrt{-S_t} & \text{for sell orders, } S_t < 0 
\end{cases} \)

The average price for a purchase of \( q^* \) shares is \( p^*/q^* \).

This is lower for large traders.

A series of small trades vs. one large trade

Small trades: high impact

Large trade: lower impact
Suppose that \( p_t = \begin{cases} p_{t-1} + \lambda \sqrt{S_t} & \text{for buy orders, } S_t > 0 \\ p_{t-1} - \lambda \sqrt{-S_t} & \text{for sell orders, } S_t < 0 \end{cases} \)

- Buy 8 units with eight 1-unit trades
- Sell 8 units with two 4-unit trades

- **Purchases**
  - \( p_1 = p_0 + \lambda \times \sqrt{1} = p_0 + \lambda \)
  - ...
  - \( p_8 = p_7 + \lambda \times \sqrt{1} = p_0 + 8 \lambda \)
  - Average purchase price is \( p_0 + 4.5 \lambda \)

- **Sales**
  - \( p_9 = p_8 - \lambda \times \sqrt{4} = p_0 + 6\lambda \)
  - \( p_{10} = p_9 - \lambda \times \sqrt{4} = p_0 + 4\lambda \)
  - Average sale price is \( p_0 + 5 \lambda \)

- Profits per share are \((p_0 + 5\lambda) - (p_0 + 4.5 \lambda) = 0.5 \lambda > 0\)
The convex case, for purchases

Will the same buy-small, sell-large manipulation work?

Suppose that \[ p_t = \begin{cases} 
    p_{t-1} + \lambda S_t^2 & \text{for buy orders, } S_t > 0 \\
    p_{t-1} - \lambda S_t^2 & \text{for sell orders, } S_t < 0 
\end{cases} \]

- We’ll buy eight shares with two trades of 4 shares.
- Sell eight shares with eight sales of 1 share.
Purchases
- \( p_1 = p_0 + \lambda \times 4^2 = p_0 + 16 \lambda \)
- \( p_2 = p_1 + \lambda \times 4^2 = p_0 + 32 \lambda \)
- Average purchase price is \( p_0 + 24 \lambda \)

Sales
- \( p_3 = p_2 - \lambda \times 1 = p_0 + 31 \lambda \)
- \( p_4 = p_3 - \lambda \times 1 = p_0 + 30 \lambda \)
- ...
- \( p_{10} = p_9 - \lambda \times 1 = p_0 + 24 \lambda \)
- Average sale price is \( p_0 + 27.5 \lambda \)

Manipulation profits are 3.5 \( \lambda \) per share

Is manipulation really possible when the price impact function is non-linear, buy-sell asymmetric, or time varying?
- Other costs (like bid-ask spread, commissions) might reduce profits.
- There are risks:
  - We don’t know for sure what the price impact function looks like.
  - Prices change for reasons other than incoming orders.
- What is the empirical evidence?
The square-root “law”

- Many practitioners and academics believe that price impact goes up with the square root of order size.
  - Example: \( p_t = p_{t-1} + \lambda \sqrt{S_t} \)
- This seems to fit many samples of financial data. Typically:
  - A broker looks at the price impact of all its customer orders.
  - A hedge fund looks at the price impact of all of its own orders.

Spoofing and layering

- Spoofing: entering a bid or offer that is not intended for execution.
- Layering: entering large bids/offers not intended for execution priced away from the market.
- Priced away: a buy limit order priced below the bid or a sell limit order priced above the offer.
- Why?
BATS book in PBR (Petrobras) on April 27, 2015

- Large quantities at the best bid and offer and away from the best bid an offer.
- Conveys the sense of a liquid market.

BATS book in PRK National

- Suppose we suddenly place an order to sell 20,000 shares at 85.35.
- What inferences would the market draw?
- What action might ensue?
US v. Sarao, 2015, US District Court, Northern District of Illinois Eastern Division

- Layering case brought by the Commodities Futures Trading Commission (civil suit) and the Department of Justice (criminal prosecution).
- Read up to item 25, p. 13, ("SARAO's Responses to Queries ..."), especially items
  - 5,6 (Overview of the investigation)
  - 13-24 (Layering schemes, overview of Sarao's activity)