# Distribution-valued heavy-traffic limits for the $G/GI/\infty$ queue

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**Abstract**

We study the $G/GI/\infty$ queue from two different perspectives in the same heavy-traffic regime. First, we represent the dynamics of the system using a measure-valued process that keeps track of the age of each customer in the system. Using the continuous-mapping approach together with the martingale functional central limit theorem, we obtain fluid and diffusion limits for this process in a space of distribution-valued processes. Next, we study a measure-valued process that keeps track of the residual service time of each customer in the system. In this case, using the functional central limit theorem and the random time change theorem together with the continuous-mapping approach, we again obtain fluid and diffusion limits in our space of distribution-valued processes. In both cases, we find that our diffusion limits may be characterized as distribution-valued Ornstein-Uhlenbeck processes. Further, these diffusion limits can be analyzed using standard results from the theory of Markov processes.

## 1 Introduction

Limit theorems for infinite-server queues in heavy-traffic have a rich history starting with the seminal paper by Iglehart [20] on the $M/M/\infty$ queue. This work initiated a line of research aiming to extend Iglehart's results to additional classes of service time distributions. Whitt [33] studies the $GI/PH/\infty$ queue, having phase-type service-time distributions, and Glynn and Whitt [14] consider the $GI/GI/\infty$ queue with service times taking values in a finite set. Furthermore, in [6], [24] and [31], the $G/GI/\infty$ queue is studied with general service time distributions. [28] gives a survey of these results.

In this paper, we study the $G/GI/\infty$ queue as a Markov process. This is accomplished using two different methods. In the first method we construct a process that tracks the age of each customer in the system and in the second method we construct a process that tracks the residual service time of each customer in the system. Although analyzing these processes might at first appear to be a complicated task, one of the themes that runs throughout the paper is that techniques originally developed for establishing heavy-traffic limits for finite-dimensional state descriptors may successfully be applied to the somewhat more abstract infinite-dimensional setting. In our first approach we establish fluid and diffusion limits for a measure-valued process tracking the age of each customer in the system using the continuous-mapping approach together with the martingale functional central limit theorem. In our second approach we establish fluid and diffusion limits

for a measure-valued process tracking the residual service time of each customer in the system. The representation we use in the second approach was also used by Decreusefond and Moyal [8] to analyze the $M/G/\infty$ queue. Indeed, many of the results and techniques found in this paper have been inspired by them. However, our proofs are quite different. In particular, in the second approach we establish the fluid and diffusion limits using the functional central limit theorem and the random time change theorem together with the continuous mapping approach. We find that for both the age and residual service time representations the diffusion limit is a distribution-valued Ornstein-Uhlenbeck process. We then utilize the highly developed theory of Markov processes in order to study our limits.

Another paper related to ours is Kaspi and Ramanan [23]. Although this work analyzes many-server queues with general service time distributions, the measure-valued representation of the system is similar to our representation. Fluid limits are established for the system in a space of Radon-measure-valued processes. However, when establishing diffusion limits for such processes, the limit process evidently falls out of the space of Radon-measure-valued process. Indeed, a significant challenge in our study was choosing a reasonable infinite-dimensional space to work in. In the work of [8], the space of test functions used is the Schwartz space, or the space of rapidly decreasing infinitely differentiable functions. This space has the disadvantage of not containing test functions that would allow one to obtain corresponding heavy-traffic limits for useful functionals such as number-in-system and workload. In the present paper, we find that the Sobolev space of infinite order (see [10], [1]) with respect to $L^2(\mu_e)$, where $\mu_e$ is the excess distribution of the service-time distribution, is the tightest space that has all the properties we need to prove limit theorems and also enables one to use our results to obtain corresponding limit theorems for useful functionals.

Besides identifying an appropriate infinite-dimensional space to work in, another major contribution of our work is making a connection between the literature on infinite-dimensional heavy-traffic limits for queueing systems ([16], [15], [7], [23], [8], [17], [9]) and the vast literature on infinite-dimensional Ornstein-Uhlenbeck processes motivated by applications to interacting particle systems ([19], [26], [3], [5], [18], [27], [21], [22], [4]). Our work especially relies on [21] and [22] to prove continuity of our regulator map.

In the forthcoming paper [29], the authors build on the work of [14] and [24] to prove heavy-traffic limits for the $G/GI/\infty$ queue in a two-parameter function space. They analyze both age and residual processes as we do. The main difference between our work and their work is that our framework, which uses distribution-valued processes, allows one to apply the continuous mapping approach and other standard techniques to obtain the heavy-traffic limits.

The remainder of the paper is organized as follows. In §2 we derive system equations for both the ages of customers in the system and the residual service times of customers in the system. These equations will be the starting point for the main results of the paper. In §3 we present a regulator map result to be used with the continuous mapping theorem. In §4 we give martingale results that will be used with the regulator map of §3, to obtain our fluid and diffusion limits. In §5 and §6 we prove our fluid and diffusion limits, respectively. In §7 we analyze the diffusion limit for our age process as a Markov process. A corresponding analysis for the limit of residual process could be conducted similarly.

### 1.1 Notation

The set of reals, nonnegative reals and nonpositive reals are denoted by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. We denote by $C^\infty$ the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ and by $\|\cdot\|_{L^2}$ the standard

norm on the space $L^2(\mu)$, where $\mu$ is some measure on $\mathbb{R}$, i.e. for $f \in L^2(\mu)$, $\|f\|_{L^2} = \left(\int_{\mathbb{R}} |f|^2 \, d\mu\right)^{1/2}$. Letting $\varphi^{(i)}$ denote the $i$th derivative of $\varphi \in C^\infty$ for $i \geq 0$, we denote the Sobolev space of order $m$ for $m \geq 0$ by

$$\Phi^m \equiv \left\{ \varphi \in C^\infty, \ \|\varphi\|_m \equiv \sum_{i=0}^{m} \|\varphi^{(i)}\|_{L^2} < \infty \right\}. \tag{1}$$

For each $m \geq 0$, this space is known to be a Hilbert space (see §5.2 of [12]). Furthermore, we denote the projective limit of the spaces $(\Phi^m)_{m\geq 0}$ by

$$\Phi \equiv \bigcap_{m=0}^{\infty} \Phi^m,$$

and call $\Phi$ a *Sobolev space of infinite order* (see [10]). It is shown in Lemma 5 of [1] that $\Phi$ is a *nuclear Fréchet space* with the topology induced by the sequence of seminorms $(\|\cdot\|_m)_{m\geq 0}$. Furthermore, since for each $m \geq 0$, $\Phi_m$ is a Hilbert space, $\Phi$ is a *countably Hilbertian nuclear space*. $\Phi$ is also a Polish space: It is a complete metric space since it is Fréchet and it is separable by Assertion 11 of [1].

Our primary objects of study are processes that takes values in the topological dual of $\Phi$, denoted by $\Phi'$. To be precise, $\Phi'$ is the space of all continuous linear functionals on $\Phi$ and we refer to elements of this space as *distributions*. For $\mu \in \Phi'$ and $\varphi \in \Phi$ we denote the *duality product* of $\mu$ and $\varphi$ by $\langle \mu, \varphi \rangle \equiv \mu(\varphi)$. For $\mu \in \Phi'$, its *distributional derivative*, denoted by $\mu'$, is the unique element of $\Phi'$ such that

$$\langle \mu', \varphi \rangle = -\langle \mu, \varphi' \rangle, \quad \varphi \in \Phi.$$

It is clear that $\mu'$ is well-defined by the definition of $\Phi$. For $\mu \in \Phi'$ and $t \in \mathbb{R}$, we can define $\tau_t \mu$ as the unique element of $\Phi'$ (when it exists) so that

$$\langle \tau_t \mu, \varphi \rangle = \langle \mu, \tau_t \varphi \rangle, \quad \varphi \in \Phi,$$

where $\tau_t \varphi$ is the function defined by $\tau_t \varphi(\cdot) \equiv \varphi(\cdot - t)$ (when it exists).

For $0 < T \leq \infty$ and Polish space $E$, we denote by $D([0,T], E)$ the space of functions from $[0,T]$ to $E$ that are right-continuous with left limits everywhere on $(0,T]$. We equip this space with the Skohorod $J_1$ topology (see [2] or [34]). In the sequel we will be concerned with the cases $E = \mathbb{R}$, $E = D \equiv D([0,\infty), \mathbb{R})$ and $E = \Phi'$. The quadratic covariation of two martingales $M$ and $N$ in $D([0,\infty), \mathbb{R})$ is denoted by $(< M, N >_t)_{t\geq 0}$ and the quadratic variation of a martingale $M \in D([0,\infty), \mathbb{R})$ is denoted by $(< M >_t)_{t\geq 0} \equiv (< M, M >_t)_{t\geq 0}$

In general, nuclear Fréchet spaces are infinite dimensional spaces that possess many desirable properties of finite dimensional spaces. For instance, $(\mu_t)_{t\geq 0} \in D([0,\infty), \Phi')$ if and only if $(\langle \mu_t, \varphi \rangle)_{t\geq 0} \in D([0,\infty), \mathbb{R})$ for each $\varphi \in \Phi$. In proving our main results we make use of the following theorem of Mitoma [25].

**Theorem 1.1** (Mitoma [25])**.** *Let $\mathcal{S}$ be a nuclear Fréchet space and let $(\mu^n)_{n\geq 1}$ be a sequence of elements of $D([0,T], \mathcal{S}')$, where $\mathcal{S}'$ denotes the topological dual of $\mathcal{S}$. Then $\mu^n \Rightarrow \mu$ in $D([0,T], \mathcal{S}')$ if and only if for each $\varphi \in \mathcal{S}$, $(\langle \mu^n_t, \varphi \rangle)_{t\geq 0} \Rightarrow (\langle \mu_t, \varphi \rangle)_{t\geq 0}$ in $D([0,T], \mathbb{R})$.*

If $(\mu_t)_{t\geq 0} \in D([0,\infty), \Phi')$ and $t \in [0,T]$, we can define the distribution $\int_0^t \mu_s \, ds$ as the element of $\Phi'$ such that for all $\varphi \in \Phi$,

$$\left\langle \int_0^t \mu_s \, ds, \varphi \right\rangle = \int_0^t \langle \mu_s, \varphi \rangle \, ds.$$

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A process $M \in D([0,\infty), \Phi^{'})$ is a $\Phi^{'}$*-valued* $\mathcal{F}_t$*-martingale* if for all $\varphi \in \Phi$, $(\langle M_t, \varphi\rangle)_{t\geq 0}$ is a $\mathbb{R}$-valued $\mathcal{F}_t$-martingale. For two $\Phi^{'}$-valued martingales, $M$ and $N$ in $D([0,\infty), \Phi^{'})$, their *tensor quadratic covariation* $(< M, N >_t)_{t\geq 0}$ is given for all $t \geq 0$ and all $\varphi, \psi \in \Phi$ by

$$< M, N >_t (\varphi, \psi) \equiv < \langle M., \varphi\rangle, \langle N., \psi\rangle >_t,$$

and the *tensor quadratic variation* $(<< M >>_t)_{t\geq 0}$ of a $\Phi^{'}$-valued martingale $M \in D([0,\infty), \Phi^{'})$ is given by $(<< M >>_t)_{t\geq 0} \equiv (< M, M >_t)_{t\geq 0}$. Two $\Phi^{'}$-valued martingales, $M$ and $N$, are said to be *orthogonal* if $< M, N >= 0$ identically. Corresponding notions for the optional quadratic variation process $[M]$ are defined analogously.

# 2 System Equations

In this section, we obtain semi-martingale decompositions of the distribution-valued process $\mathcal{A} \equiv (\mathcal{A}_t)_{t\geq 0}$, which keeps track of the age of each customer in the system, and the distribution-valued process $\mathcal{R} \equiv (\mathcal{R}_t)_{t\geq 0}$, which keeps track of the residual service time of each customer in the system. We begin by treating the age process $\mathcal{A}$ in §2.1 and then move on to the residual service time process $\mathcal{R}$ in §2.2.

## 2.1 Ages

Consider a $G/GI/\infty$ queue with general arrival process $(E_t)_{t\geq 0} \in D([0,\infty), \mathbb{R})$. We denote by $\tau_i$ and $\eta_i$ the arrival time and service time, respectively, of the $i$th customer to enter the system after time $0-$, for $i \geq 1$. These service times are independent and identically distributed (iid) with cumulative distribution function (cdf) $F$ with mean 1, complementary cumulative distribution function $\bar{F} \equiv 1 - F$, probability density function (pdf) $f$ and hazard rate function $h \in C_b^\infty$.

Define $(A_t)_{t\geq 0} \in D([0,\infty), D)$ so that $A_t(y)$ denotes the number of customers in the system at time $t \geq 0$ that have been in the system for less than or equal to $y \geq 0$ units of time at time $t$. At time $0-$, we assume that there are $A_0(y)$ customers present who have been in the system for less than $y \geq 0$ units of time and we denote by

$$-\tilde{\tau}_i \equiv \inf\{y \geq 0 | A_0(y) \geq i\}$$

the "arrival" time of the $i$th initial customer to the system for $i \geq 1$. $A_0 \equiv A_0(\infty)$ denotes the total number of customers in the system at time $0-$. We also denote by $\tilde{\eta}_i$ the remaining service time at time 0 of the $i$th initial customer in the system. The distribution of $\tilde{\eta}_i$, conditional on the arrival time $\tilde{\tau}_i$, is given for $x \geq 0$ by

$$\mathbb{P}\left[\tilde{\eta}_i > x | \tilde{\tau}_i\right] = \frac{1 - F(-\tilde{\tau}_i + x)}{1 - F(-\tilde{\tau}_i)}. \tag{2}$$

We denote by $f_{\tilde{\tau}_i}$ and $h_{\tilde{\tau}_i}$ the conditional pdf and hazard rate function associated with this distribution, respectively.

We now derive system equations for a measure-valued process that tracks the age of each customer in service. First note that by first principles we have for $y \geq 0$,

$$A_t(y) = \sum_{i=1}^{A_0} \mathbf{1}_{\{t-\tilde{\tau}_i \leq y\}} \mathbf{1}_{\{t<\tilde{\eta}_i\}} + \sum_{i=1}^{E_t} \mathbf{1}_{\{t-\tau_i \leq y\}} \mathbf{1}_{\{t-\tau_i<\eta_i\}}. \tag{3}$$

Our first result provides an alternative way to write (3).

**Proposition 2.1.** *For each $t \geq 0$,*

$$A_t(y) = A_0(y) - \sum_{i=1}^{A_0} \mathbf{1}_{\{\tilde{\eta}_i \leq t \wedge (y+\tilde{\tau}_i)\}} - \sum_{i=A_0(y-t)+1}^{A_0(y)} \mathbf{1}_{\{\tilde{\eta}_i > y+\tilde{\tau}_i\}} + E_t - \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i \leq (t-\tau_i)\wedge y\}} - \sum_{i=1}^{E_{t-y}} \mathbf{1}_{\{\eta_i > y\}}. \tag{4}$$

*Proof.* By (3), we have that

$$\begin{aligned} A_t(y) &= \sum_{i=1}^{A_0} \mathbf{1}_{\{t-\tilde{\tau}_i \leq y\}} \mathbf{1}_{\{t<\tilde{\eta}_i\}} + \sum_{i=1}^{E_t} \mathbf{1}_{\{t-\tau_i \leq y\}} \mathbf{1}_{\{t-\tau_i<\eta_i\}} \\ &= \sum_{i=1}^{A_0(y)} \mathbf{1}_{\{t-\tilde{\tau}_i \leq y\}} \mathbf{1}_{\{t<\tilde{\eta}_i\}} + \sum_{i=1}^{E_t} \mathbf{1}_{\{t-\tau_i \leq y\}} \mathbf{1}_{\{t-\tau_i<\eta_i\}} \\ &= A_0(y) + \sum_{i=1}^{A_0(y)} (\mathbf{1}_{\{t-\tilde{\tau}_i \leq y\}} \mathbf{1}_{\{t<\tilde{\eta}_i\}} - 1) + E_t + \sum_{i=1}^{E_t} (\mathbf{1}_{\{t-\tau_i \leq y\}} \mathbf{1}_{\{t-\tau_i<\eta_i\}} - 1). \end{aligned} \tag{5}$$

However,

$$\begin{aligned} \mathbf{1}_{\{t-\tilde{\tau}_i \leq y\}} \mathbf{1}_{\{t<\tilde{\eta}_i\}} - 1 &= \mathbf{1}_{\{t-\tilde{\tau}_i \leq y\}} \mathbf{1}_{\{\tilde{\eta}_i \leq t\}} + \mathbf{1}_{\{t-\tilde{\tau}_i > y\}} \\ &= \left(\mathbf{1}_{\{t-\tilde{\tau}_i \leq y\}} \mathbf{1}_{\{\tilde{\eta}_i \leq t\}} + \mathbf{1}_{\{t-\tilde{\tau}_i > y\}} \mathbf{1}_{\{-\tilde{\tau}_i+\tilde{\eta}_i \leq y\}}\right) + \mathbf{1}_{\{t-\tilde{\tau}_i > y\}} \mathbf{1}_{\{-\tilde{\tau}_i+\tilde{\eta}_i > y\}}. \end{aligned} \tag{6}$$

and, similarly,

$$\begin{aligned} \mathbf{1}_{\{t-\tau_i \leq y\}} \mathbf{1}_{\{t-\tau_i<\eta_i\}} - 1 &= \mathbf{1}_{\{t-\tau_i \leq y\}} \mathbf{1}_{\{\eta_i \leq t-\tau_i\}} + \mathbf{1}_{\{t-\tau_i > y\}} \\ &= \left(\mathbf{1}_{\{t-\tau_i \leq y\}} \mathbf{1}_{\{\eta_i \leq t-\tau_i\}} + \mathbf{1}_{\{t-\tau_i > y\}} \mathbf{1}_{\{\eta_i \leq y\}}\right) + \mathbf{1}_{\{t-\tau_i > y\}} \mathbf{1}_{\{\eta_i > y\}}, \end{aligned} \tag{7}$$

Substituting (7) and (6) into (5) and summing over $A_0(y)$ and $E_t$ completes the proof. □

We now provide an intuitive explanation for the terms appearing in (4). The first term represents the number of customers in the system at time $0-$ that have been in the system for less than or equal to $y$ units of time; the second term represents the number of departures by time $t$ of those initial customers that had total service times less than or equal to $y$ units of time; and the third term represents the number of initial customers that had been in the system for less than or equal to $y$ units of time at time $0-$ but have been in the system for time greater than $y$ units of time at time $t$. The fourth, fifth and sixth terms represent similar quantities but for customers that arrive to the system after time $0-$.

Next, for $t, y \geq 0$, define $D^0 \equiv (D^0_t)_{t\geq 0} \in D([0,\infty), D)$ by

$$D^0_t(y) = \sum_{i=1}^{A_0} \left( \mathbf{1}_{\{\tilde{\eta}_i \leq t \wedge (y+\tilde{\tau}_i)\}} - \int_0^{\tilde{\eta}_i \wedge t \wedge (y+\tilde{\tau}_i)} h_{\tilde{\tau}_i}(u)\, du \right), \tag{8}$$

and define $D \equiv (D_t)_{t\geq 0} \in D([0,\infty), D)$ by

$$D_t(y) = \sum_{i=1}^{E_t} \left( \mathbf{1}_{\{\eta_i \leq (t-\tau_i)\wedge y\}} - \int_0^{\eta_i \wedge (t-\tau_i)\wedge y} h(u)\, du \right). \tag{9}$$

It then follows from (4) that

$$A_t(y) = A_0(y) + E_t - D_t^0(y) - D_t(y) - \sum_{i=1}^{A_0} \int_0^{\tilde{\eta}_i \wedge t \wedge (y+\tilde{\tau}_i)} h_{\tilde{\tau}_i}(u)\, du - \sum_{i=1}^{E_t} \int_0^{\eta_i \wedge (t-\tau_i) \wedge y} h(u)\, du$$
$$- \sum_{i=A_0(y-t)+1}^{A_0(y)} \mathbf{1}_{\{-\tilde{\tau}_i + \tilde{\eta}_i > y\}} - \sum_{i=1}^{E_{t-y}} \mathbf{1}_{\{\eta_i > y\}}. \tag{10}$$

To study the age process as a distribution-valued process, we use a Sobolev space of infinite order $\Phi_{\mathcal{A}}$ as the test function space, where $\Phi_{\mathcal{A}}$ is defined as in (1) with $\mu$ set to the excess of the service time distribution:

$$\mu(A) \equiv \int_A \bar{F}(y)\, dy, \quad \text{for} \quad A \in \mathcal{B}(\mathbb{R}_+) \tag{11}$$

Since $h \in C_b^\infty$, for each $\varphi \in \Phi_{\mathcal{A}}$ the integrals $\langle \mathcal{F}, \varphi \rangle$ and $\langle \mathcal{F}_e, \varphi \rangle$ both exist, where $\mathcal{F}_e$ denotes the distribution associated with $F_e$, the cdf of the stationary excess distribution of $F$, i.e. $F_e(y) = \int_0^y \bar{F}(x)\, dx$.

We associate with the process $A$ defined above the process $\mathcal{A}$ taking values in $\Phi_{\mathcal{A}}^{'}$, as defined in §1.1, such that for each $t \geq 0$ and $\varphi \in \Phi_{\mathcal{A}}$,

$$\langle \mathcal{A}_t, \varphi \rangle = \int_{\mathbb{R}} \varphi(y)\, dA_t(y). \tag{12}$$

We similarly define corresponding processes, $\mathcal{D}^0$ and $\mathcal{D}$, associated with $D^0$ and $D$, respectively. We also associate with $A_0$ a $\Phi_{\mathcal{A}}^{'}$-valued random variable $\mathcal{A}_0$. It is easy to see that for each $t \geq 0$, the quantities $\mathcal{A}_t$, $\mathcal{D}_t^0$ and $\mathcal{D}_t$ are well defined elements of $\Phi_{\mathcal{A}}^{'}$. It is also easy to see by right-continuity of the sample paths of $A$, $D^0$ and $D$ that $\mathcal{A}, \mathcal{D}^0, \mathcal{D} \in D([0,\infty), \Phi_{\mathcal{A}}^{'})$.

By integrating test functions $\varphi \in \Phi_{\mathcal{A}}$ with respect to each of the terms in (10) it follows that

$$\langle \mathcal{A}_t, \varphi \rangle = \langle \mathcal{A}_0, \varphi \rangle - \langle \mathcal{D}_t^0 + \mathcal{D}_t, \varphi \rangle - \sum_{i=1}^{A_0} \int_{-\tilde{\tau}_i}^{-\tilde{\tau}_i + (\tilde{\eta}_i \wedge t)} \varphi(y) h(y)\, dy - \sum_{i=1}^{E_t} \int_0^{\eta_i \wedge (t-\tau_i)} \varphi(y) h(y)\, dy$$
$$- \int_{\mathbb{R}_+} \varphi(y)\, d \left( \sum_{i=A_0(y-t)+1}^{A_0(y)} \mathbf{1}_{\{-\tilde{\tau}_i + \tilde{\eta}_i > y\}} + \sum_{i=1}^{E_{t-y}} \mathbf{1}_{\{\eta_i > y\}} \right). \tag{13}$$

We then have the following two propositions, which allow us to simplify (13):

**Proposition 2.2.** *For each $t \geq 0$,*

$$\sum_{i=1}^{A_0} \int_{-\tilde{\tau}_i}^{-\tilde{\tau}_i + (\tilde{\eta}_i \wedge t)} \varphi(y) h(y)\, dy + \sum_{i=1}^{E_t} \int_0^{\eta_i \wedge (t-\tau_i)} \varphi(y) h(y)\, dy = \int_0^t \langle \mathcal{A}_s, \varphi h \rangle\, ds.$$

*Proof.*

$$\sum_{i=1}^{A_0} \int_{-\tilde{\tau}_i}^{-\tilde{\tau}_i + (\tilde{\eta}_i \wedge t)} \varphi(y) h(y)\, dy + \sum_{i=1}^{E_t} \int_0^{\eta_i \wedge (t-\tau_i)} \varphi(y) h(y)\, dy$$
$$= \sum_{i=1}^{A_0} \int_0^t \mathbf{1}_{\{0 \leq s \leq \tilde{\eta}_i\}} \varphi(s - \tilde{\tau}_i) h(s - \tilde{\tau}_i)\, ds + \sum_{i=1}^{E_t} \int_0^t \mathbf{1}_{\{0 \leq s - \tau_i \leq \eta_i\}} \varphi(s - \tau_i) h(s - \tau_i)\, ds$$

$$= \int_0^t \left( \sum_{i=1}^{A_0} \mathbf{1}_{\{0 \le s \le \tilde{\eta}_i\}} \varphi(s-\tilde{\tau}_i) h(s-\tilde{\tau}_i) + \sum_{i=1}^{E_t} \mathbf{1}_{\{0 \le s-\tau_i \le \eta_i\}} \varphi(s-\tau_i) h(s-\tau_i) \right) ds$$
$$= \int_0^t \langle \mathcal{A}_s, \varphi h \rangle \, ds.$$

□

**Proposition 2.3.** *For each $t \ge 0$,*

$$-\int_{\mathbb{R}_+} \varphi(y) \, d \left( \sum_{i=A_0(y-t)+1}^{A_0(y)} \mathbf{1}_{\{-\tilde{\tau}_i+\tilde{\eta}_i > y\}} + \sum_{i=1}^{E_{t-y}} \mathbf{1}_{\{\eta_i > y\}} \right) = E_t \varphi(0) + \int_0^t \langle \mathcal{A}_s, \varphi^{'} \rangle \, ds.$$

*Proof.* Integrating by parts, we have that

$$\begin{aligned}
-E_t\varphi(0) - \int_{\mathbb{R}_+} \varphi(y) \, d \left( \sum_{i=A_0(y-t)+1}^{A_0(y)} \mathbf{1}_{\{-\tilde{\tau}_i+\tilde{\eta}_i > y\}} + \sum_{i=1}^{E_{t-y}} \mathbf{1}_{\{\eta_i > y\}} \right) \\
&= \int_{\mathbb{R}_+} \left( \sum_{i=A_0(y-t)+1}^{A_0(y)} \mathbf{1}_{\{-\tilde{\tau}_i+\tilde{\eta}_i > y\}} + \sum_{i=1}^{E_{t-y}} \mathbf{1}_{\{\eta_i \ge y\}} \right) \varphi'(y) \, dy \\
&= \int_{\mathbb{R}_+} \left( \sum_{i=1}^{A_0} \mathbf{1}_{\{\tilde{\tau}_i \ge -y, -\tilde{\tau}_i+\tilde{\eta}_i \ge y, \tilde{\tau}_i+y \le t\}} + \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i \ge y, \tau_i+y \le t\}} \right) \varphi'(y) \, dy \\
&= \sum_{i=1}^{A_0} \int_{\mathbb{R}_+} \mathbf{1}_{\{\tilde{\tau}_i \ge -y, -\tilde{\tau}_i+\tilde{\eta}_i \ge y, \tilde{\tau}_i+y \le t\}} \varphi'(y) \, dy + \sum_{i=1}^{E_t} \int_{\mathbb{R}_+} \mathbf{1}_{\{\eta_i \ge y, \tau_i+y \le t\}} \varphi'(y) \, dy \\
&= \sum_{i=1}^{A_0} \int_0^t \mathbf{1}_{\{0 \le s-\tilde{\tau}_i \le -\tilde{\tau}_i+\tilde{\eta}_i\}} \varphi'(s-\tilde{\tau}_i) \, ds + \sum_{i=1}^{E_t} \int_0^t \mathbf{1}_{\{0 \le s-\tau_i \le \eta_i\}} \varphi'(s-\tau_i) \, ds \\
&= \int_0^t \left( \sum_{i=1}^{A_0} \mathbf{1}_{\{0 \le s-\tilde{\tau}_i \le -\tilde{\tau}_i+\tilde{\eta}_i\}} \varphi^{'}(s-\tilde{\tau}_i) + \sum_{i=1}^{E_t} \mathbf{1}_{\{0 \le s-\tau_i \le \eta_i\}} \varphi^{'}(s-\tau_i) \right) ds \\
&= \int_0^t \left( \int_{\mathbb{R}_+} \varphi^{'}(u) Q_s(du) \right) ds \\
&= \int_0^t \left\langle \mathcal{A}_s, \varphi^{'} \right\rangle ds.
\end{aligned}$$

□

Combining Propositions 2.2 and 2.3 with system equation (13), we arrive at

$$\langle \mathcal{A}_t, \varphi \rangle = \langle \mathcal{A}_0, \varphi \rangle + \langle \mathcal{E}_t - \mathcal{D}_t^0 - \mathcal{D}_t, \varphi \rangle - \int_0^t \langle \mathcal{A}_s, h\varphi \rangle \, ds + \int_0^t \langle \mathcal{A}_s, \varphi^{'} \rangle \, ds, \tag{14}$$

where we define the $\Phi_{\mathcal{A}}^{'}$-valued process $\mathcal{E} \equiv E.\delta_0$ so that $\langle \mathcal{E}_t, \varphi \rangle = E_t\varphi(0)$ for each $\varphi \in \Phi_{\mathcal{A}}$ and $t \ge 0$. In general, we refer to (14) as the semi-martingale decomposition of $\mathcal{A}$ for reasons that become clear in §4.

## 2.2 Residuals

We next move on to the residual service time process $\mathcal{R}$. As in §2, customers arrive to the system according to a general arrival process $(E_t)_{t\geq 0} \in D([0,\infty),\mathbb{R})$ and we denote by $\tau_i$ and $\eta_i$ the arrival time and service time, respectively, of the $i$th customer to arrive to the system after time $0-$. Customer service times are iid with cdf $F$. We assume the service-time distribution has a bounded hazard rate function, but here we make no assumptions on the smoothness of the hazard rate function as we did in §2.1. Assuming the boundedness of the hazard rate is helpful in defining our space of test functions. In general, we could drop all assumptions on the service time distribution, but this would require us to restrict our space of test functions.

Let $R_t(y)$ denote the number of customers at time $t \geq 0$ that have less than or equal to $y \in \mathbb{R}$ units of service time remaining. Note that as in [8], we allow $y \leq 0$ so that in addition to keeping track of customers present in the system at time $t$, we also keep track of customers who have already departed the system. We assume that at time $0-$ there are $R_0(y)$ customers in the system that have less than or equal to $y \geq 0$ units of service time remaining. We let $R_0 \equiv R_0(\infty)$ denote the total number of customers present in the system at time $0-$. By first principles, it then follows that

$$R_t(y) = R_0(t+y) + \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i-(t-\tau_i)\leq y\}}. \tag{15}$$

The following Proposition presents an alternative form of (15).

**Proposition 2.4.** *For each $t \geq 0$ and $y \in \mathbb{R}$,*

$$R_t(y) = R_0(y) + (R_0(t+y) - R_0(y)) + \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i\leq y\}} + \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i>y,(\tau_i+\eta_i)-t\leq y\}}. \tag{16}$$

*Proof.* By (15),

$$R_t(y) = R_0(t+y) + \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i-(t-\tau_i)\leq y\}} = R_0(y) + (R_0(t+y) - R_0(y)) + \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i-(t-\tau_i)\leq y\}}. \tag{17}$$

However,

$$\mathbf{1}_{\{\eta_i-(t-\tau_i)\leq y\}} = \mathbf{1}_{\{\eta_i\leq y\}} + \mathbf{1}_{\{\eta_i>y,\eta_i-(t-\tau_i)\leq y\}}. \tag{18}$$

Substituting (18) into (17) and summing over $E_t$, completes the proof. □

We now give an explanation for each of the terms appearing in (16). The first term represents the number of customers in the system at time $0-$ with less than or equal to $y$ units of service time remaining. The second term represents the number of customers that arrived to the system with greater than $y$ units of total service time but at time $t$ have less than or equal to $y$ units of service time remaining. The third and fourth terms can be explained analogously but for customers that arrive to the system after time $0-$.

For each $y \geq 0$ and $t \geq 0$, define $G \equiv (G_t)_{t\geq 0} \in D([0,\infty), D)$ by

$$G_t(y) = \sum_{i=1}^{E_t} \left(\mathbf{1}_{\{\eta_i\leq y\}} - F(y)\right).$$

By Proposition (2.4), $R_t(y)$ may then be rewritten as

$$R_t(y) = R_0(y) + (R_0(t+y) - R_0(y)) + G_t(y) + E_t F(y) + \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i > y, (\tau_i+\eta_i)-t \le y\}}. \tag{19}$$

To study this residual process as a distribution-valued process, we will use another Sobolev space of infinite order $\Phi_{\mathcal{R}}$ as the test function space, where $\Phi_{\mathcal{R}}$ here is defined as in (1) with $\mu \equiv \mu_- + \mu_+$, where

$$\mu_+(A) \equiv \int_A \bar{F}(y)\, dy, \quad \text{for} \quad A \in \mathcal{B}(\mathbb{R}_+), \tag{20}$$

and $\mu_-$ is Lebesgue measure on $\mathbb{R}_-$. Notice $\mu_+$ here is defined just as $\mu$ in (11). The assumption that $h$ is bounded implies that for each $\varphi \in \Phi_{\mathcal{R}}$ the integrals $\langle \mathcal{F}, \varphi \rangle$ and $\langle \mathcal{F}_e, \varphi \rangle$ both exist.

Now associating $\Phi_{\mathcal{R}}^{'}$-valued processes $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{F}$ to $R$, $G$ and $F$, respectively, as in (12), and plugging in test functions and integrating each of the terms in (19), we get that for each $\varphi \in \Phi_{\mathcal{R}}$,

$$\begin{aligned}\langle \mathcal{R}_t, \varphi \rangle = \langle \mathcal{R}_0, \varphi \rangle &+ \int_{\mathbb{R}} \varphi(y)\, d(R_0(t+y) - R_0(y)) \\ &+ \langle \mathcal{G}_t, \varphi \rangle + E_t \langle \mathcal{F}, \varphi \rangle + \int_{\mathbb{R}} \varphi(y)\, d\left( \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i > y, (\tau_i+\eta_i)-t \le y\}} \right). \end{aligned} \tag{21}$$

The following proposition now allows us to simplify the form of (21):

**Proposition 2.5.** *For each $t \ge 0$,*

$$\int_{\mathbb{R}} \varphi(y)\, d(R_0(t+y) - R_0(y)) + \int_{\mathbb{R}} \varphi(y)\, d\left( \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i > y, (\tau_i+\eta_i)-t \le y\}} \right) = -\int_0^t \langle \mathcal{R}_s, \varphi^{'} \rangle\, ds. \tag{22}$$

*Proof.* The proof parallels the proof of Proposition 2.3. Integrating by parts, we have that

$$\begin{aligned}
&-\int_{\mathbb{R}} \varphi(y)\, d(R_0(t+y) - R_0(y)) - \int_{\mathbb{R}} \varphi(y)\, d\left( \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i > y, (\tau_i+\eta_i)-t \le y\}} \right) \\
&\qquad = \int_{\mathbb{R}} \left( \sum_{i=1}^{R_0} \mathbf{1}_{\{y \le \tilde{\eta}_i \le t+y\}} \right) \varphi^{'}(y)\, dy + \int_{\mathbb{R}} \left( \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i > y, (\tau_i+\eta_i)-t \le y\}} \right) \varphi^{'}(y)\, dy \\
&\qquad = \sum_{i=1}^{R_0} \int_{\mathbb{R}} \mathbf{1}_{\{y \le \tilde{\eta}_i \le t+y\}} \varphi^{'}(y)\, dy + \sum_{i=1}^{E_t} \int_{\mathbb{R}} \mathbf{1}_{\{\eta_i > y, (\tau_i+\eta_i)-t \le y\}} \varphi^{'}(y)\, dy \\
&\qquad = \sum_{i=1}^{R_0} \int_{\mathbb{R}} \mathbf{1}_{\{\tilde{\eta}_i - t \le y \le \tilde{\eta}_i\}} \varphi^{'}(y)\, dy + \sum_{i=1}^{E_t} \int_{\mathbb{R}} \mathbf{1}_{\{(\tau_i+\eta_i)-t \le y < \eta_i\}} \varphi^{'}(y)\, dy \\
&\qquad = \sum_{i=1}^{R_0} \int_0^t \varphi^{'}(\tilde{\eta}_i - s)\, ds + \sum_{i=1}^{E_t} \int_0^t \mathbf{1}_{\{\tau_i \le s\}} \varphi^{'}(\tau_i + \eta_i - s)\, ds \\
&\qquad = \int_0^t \left( \sum_{i=1}^{R_0} \varphi^{'}(\tilde{\eta}_i - s) + \sum_{i=1}^{E_t} \mathbf{1}_{\{\tau_i \le s\}} \varphi^{'}(\tau_i + \eta_i - s) \right) ds \\
&\qquad = \int_0^t \left( \int_{\mathbb{R}_+} \varphi^{'} R_s(du) \right) ds
\end{aligned}$$

$$= \int_0^t \langle \mathcal{R}_s, \varphi' \rangle \, ds.$$

□

Substituting (22) into (21),

$$\langle \mathcal{R}_t, \varphi \rangle = \langle \mathcal{R}_0, \varphi \rangle + \langle \mathcal{G}_t, \varphi \rangle + E_t \langle \mathcal{F}, \varphi \rangle - \int_0^t \langle \mathcal{R}_s, \varphi' \rangle \, ds. \tag{23}$$

We refer to (23) as the semi-martingale decomposition of $\mathcal{R}$. In §4, we will prove that the process $\mathcal{G}$ in (23) is a martingale. Note the similarity of (23) with (4) of [8].

## 3 Regulator Map Result

In this section we show that given a Sobolev space of infinite order $\Phi$, the integral equation associated with $(\nu_0, \mu) \in \Phi' \times D([0,\infty), \Phi')$,

$$\langle \nu_t, \varphi \rangle = \langle \nu_0, \varphi \rangle + \langle \mu_t, \varphi \rangle + \int_0^t \langle \nu_s, B\varphi \rangle \, ds, \tag{24}$$

for $B$ a continuous linear operator on $\Phi$ and $\varphi \in \Phi$, defines a continuous function $\Psi_B : \Phi' \times D([0,\infty), \Phi') \to D([0,\infty), \Phi')$ mapping $(\nu_0, \mu)$ to $\nu$, under some mild restrictions on $B$.

Before we prove our result we need the following definition from [21]:

**Definition 3.1.** *A family $(S_t)_{t\geq 0}$ of linear operators on $\Phi$ is said to be a $(C_0, 1)$* ***semi-group*** *if*

1. *$S_0 = I$, where $I$ is the identity operator, and for all $s, t \geq 0$, $S_s S_t = S_{s+t}$.*
2. *The map $t \to S_t\varphi$ is continuous for each $\varphi \in \Phi$.*
3. *For each $q \geq 0$ there exist numbers $M_q$, $\sigma_q$ and $p \geq q$ such that*

$$\|S_t\varphi\|_q \leq M_q e^{\sigma_q t} \|\varphi\|_p,$$

*for all $\varphi \in \Phi$, $t \geq 0$.*

**Theorem 3.2.** *Let $B$ be the infinitesimal generator of a $(C_0, 1)$ semi-group $(S_t)_{t\geq 0}$. Then for each $(\nu_0, \mu) \in \Phi' \times D([0,\infty), \Phi')$, the equation (24) has a unique solution given by*

$$\langle \nu_t, \varphi \rangle = \langle \nu_0, S_t\varphi \rangle + \langle \mu_t, \varphi \rangle + \int_0^t \langle \mu_s, S_{t-s} B\varphi \rangle \, ds. \tag{25}$$

*Furthermore, (24) defines a continuous function $\Psi_B : \Phi' \times D([0,\infty), \Phi') \to D([0,\infty), \Phi')$ mapping $(\nu_0, \mu)$ to $\nu$.*

*Proof.* That $\Psi_B$ is a well-defined function from $D([0,\infty), \Phi')$ to $D([0,\infty), \Phi')$ and the form of the solution (25) follow from Steps 1-3 of the proof of Theorem 2.1 of [21] (see also Corollary 2.2).

To show continuity we adapt the argument in the proof of Proposition 3 of [8] (see also [22]). By the form of (25), it suffices to show that for each $T > 0$ the function mapping $\Phi'$ to $D([0,\infty), \Phi')$ defined by $\nu_0 \mapsto S^*_{\cdot}\nu_0$ and the function mapping $D([0,\infty), \Phi')$ to $D([0,T], \mathbb{R})$ defined by $\mu \mapsto \int_0^{\cdot} B^* S^*_{\cdot - s} \mu_s \, ds$, where $B^*$ and $S^*_t$ denote the adjoint operators of $B$ and $S_t$, respectively,

are continuous. The desired result then follows from Theorem 1.1 and the fact that the addition map on $D([0,T],\mathbb{R}) \times D([0,T],\mathbb{R})$ is continuous at continuous limits (Theorem 4.1 of [32]).

Let $(\nu_0^n)_{n\geq 1}$ be a sequence in $\Phi'$ converging to $\nu_0$. Then by Proposition 0.6.7 of [30], for each precompact set $K \subset \Phi$ we have

$$\sup_{\varphi \in K} |\langle \nu_0^n - \nu_0, \varphi \rangle| \to 0 \quad \text{as} \quad n \to \infty. \tag{26}$$

Since the map $t \to S_t\varphi$ is continuous for each $\varphi \in \Phi$ by the definition of $(C_0, 1)$ semigroup and $[0,T]$ is compact in $\mathbb{R}$, the set $\{S_u B\varphi,\ u \in [0,T]\}$ is compact in $\Phi$. Thus, applying (26) to the compact set $K \equiv \{S_u B\varphi,\ u \in [0,T]\}$ gives us

$$\sup_{u \in [0,T]} |\langle \nu_0^n - \nu_0, S_u\varphi \rangle| \to 0 \quad \text{as} \quad n \to \infty.$$

By Theorem 1.1, this proves $S_\cdot^* \nu_0^n \Rightarrow S_\cdot^* \nu$ in $D([0,T],\mathbb{R})$ as $n \to \infty$ and thus that the map $\nu_0 \mapsto S_\cdot^* \nu_0$ is continuous.

Now let $(\mu^n)_{n\geq 1}$ be a sequence in $D([0,\infty), \Phi')$ converging to $\mu$. Then there exist increasing homeomorphisms $(\lambda^n)_{n\geq 1}$ of the interval $[0,T]$ such that for each $\varphi \in \Phi$

$$\left\| \left\langle \mu_\cdot^n - \mu_{\lambda^n(\cdot)}, \varphi \right\rangle \right\|_T \to 0 \quad \text{and} \quad \|\lambda^n - e\|_T \to 0 \quad \text{as} \quad n \to \infty,$$

where $e$ denotes the identity mapping on $[0,T]$. Again, by Proposition 0.6.7 of [30], for each precompact set $K \subset \Phi$, we have

$$\sup_{t \in [0,T]} \sup_{\varphi \in K} \left| \left\langle \mu_t^n - \mu_{\lambda^n(t)}, \varphi \right\rangle \right| \to 0 \quad \text{as} \quad n \to \infty, \tag{27}$$

Just as above, the set $\{S_u B\varphi,\ u \in [0,T]\}$ is compact in $\Phi$. Thus, applying (27) to the compact set $K \equiv \{S_u B\varphi,\ u \in [0,T]\}$ gives us

$$\sup_{t \in [0,T]} \sup_{u \in [0,T]} \left| \left\langle \mu_t^n - \mu_{\lambda^n(t)}, S_u B\varphi \right\rangle \right| \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore,

$$\begin{aligned}
\sup_{t \in [0,T]} \left| \left\langle \int_0^t B^* S_{t-s}^* \left(\mu_s^n - \mu_{\lambda^n(s)}\right) ds, \varphi \right\rangle \right| &= \sup_{t \in [0,T]} \left| \int_0^t \left\langle \mu_s^n - \mu_{\lambda^n(s)}, S_{t-s} B\varphi \right\rangle ds \right| \\
&\leq \sup_{t \in [0,T]} \int_0^t \left| \left\langle \mu_s^n - \mu_{\lambda^n(s)}, S_{t-s} B\varphi \right\rangle \right| ds \\
&\leq \sup_{t,u \in [0,T]} \int_0^t \left| \left\langle \mu_s^n - \mu_{\lambda^n(s)}, S_u B\varphi \right\rangle \right| ds \\
&\leq T \sup_{s,u \in [0,T]} \left| \left\langle \mu_s^n - \mu_{\lambda^n(s)}, S_u B\varphi \right\rangle \right| \to 0,
\end{aligned}$$

as $n \to \infty$. By Theorem 1.1, this proves that the map $\mu \mapsto \int_0^\cdot B^* S_{\cdot - s}^* \mu_s \, ds$ is continuous and completes our proof. □

### 3.1 Ages

If we define the linear operator $B^{\mathcal{A}}$ on $\Phi_{\mathcal{A}}$ so that $B^{\mathcal{A}}\varphi = \varphi' - h\varphi$ for $\varphi \in \Phi_{\mathcal{A}}$, we can write (14) as

$$\mathcal{A} = \Psi_{B^{\mathcal{A}}}(\mathcal{A}_0, \mathcal{E} - \mathcal{D}^0 - \mathcal{D}). \tag{28}$$

We now verify that $B^{\mathcal{A}}$ generates a $(C_0, 1)$ semi-group so that Theorem 3.2 will apply to (14).

**Proposition 3.3.** *$B^{\mathcal{A}}$ generates a $(C_0, 1)$ semi-group $(S_t^{\mathcal{A}})_{t\geq 0}$ defined by*

$$S_t^{\mathcal{A}}\varphi = \bar{F}^{-1}\tau_{-t}\left(\bar{F}\varphi\right) \quad \textit{for} \quad \varphi \in \Phi_{\mathcal{A}}. \tag{29}$$

*Proof.* First we check that $B^{\mathcal{A}}$ is the infinitesimal generator of the semigroup given by (29). For all $\varphi \in \Phi_{\mathcal{A}}$ we have

$$\begin{aligned}\lim_{h\to 0}\frac{S_h^{\mathcal{A}}\varphi - \varphi}{h} &= \lim_{h\to 0}\frac{\bar{F}^{-1}\bar{F}(\cdot + h)\varphi(\cdot + h) - \varphi}{h} = \bar{F}^{-1}\lim_{h\to 0}\frac{\bar{F}(\cdot + h)\varphi(\cdot + h) - \bar{F}\varphi}{h} \\ &= \bar{F}^{-1}\left(\bar{F}\varphi\right)' = \bar{F}^{-1}\left(\bar{F}\varphi' - f\varphi\right) = \varphi' - h\varphi = B^{\mathcal{A}}\varphi.\end{aligned}$$

Now we check that the semigroup $(S_t^{\mathcal{A}})_{t\geq 0}$ is a $(C_0, 1)$ semigroup. Part 1. of the definition of $(C_0, 1)$ semigroup is clearly satisfied. Part 2. follows from Lemma 2 of [8]. For Part 3., by the definition of the seminorms inducing the topology on $\Phi_{\mathcal{A}}$ (1), it is enough to show that for each $n \geq 0$ there exists an $M_n$ such that for each $\varphi \in \Phi_{\mathcal{A}}$ and $s \geq 0$,

$$\|(S_t^{\mathcal{A}}\varphi)^{(n)}\|_{L_2} \leq M_n \sum_{i=0}^{n} \|\varphi^{(i)}\|_{L_2}. \tag{30}$$

Using the chain rule and triangle inequality, for each $n \geq 0$ we have

$$\begin{aligned}\|(S_t^{\mathcal{A}}\varphi)^{(n)}\|_{L_2} &= \left\|\sum_{i=0}^{n}\binom{n}{i}\left(\frac{\bar{F}(\cdot + t)}{\bar{F}}\right)^{(n-i)}\varphi^{(i)}(\cdot + t)\right\|_{L^2} \\ &\leq \sum_{i=0}^{n}\binom{n}{i}\left\|\left(\frac{\bar{F}(\cdot + t)}{\bar{F}}\right)^{(n-i)}\varphi^{(i)}(\cdot + t)\right\|_{L^2} \\ &= \sum_{i=0}^{n}\binom{n}{i}\left\|(h - h(\cdot + t))^{n-i}\frac{\bar{F}(\cdot + t)}{\bar{F}}\varphi^{(i)}(\cdot + t)\right\|_{L^2} \\ &\leq \sum_{i=0}^{n}\binom{n}{i}2\|h\|_\infty^{n-i}\left\|\frac{\bar{F}(\cdot + t)}{\bar{F}}\varphi^{(i)}(\cdot + t)\right\|_{L^2},\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the sup norm. Now focusing on the $L^2$ norm in the expression above,

$$\begin{aligned}\left\|\frac{\bar{F}(\cdot + t)}{\bar{F}}\varphi^{(i)}(\cdot + t)\right\|_{L^2}^2 &= \int_{\mathbb{R}_+}\left(\frac{\bar{F}(x + t)}{\bar{F}(x)}\varphi^{(i)}(x + t)\right)^2\bar{F}(x)\,dx \\ &= \int_{\mathbb{R}_+}\frac{\bar{F}(x + t)}{\bar{F}(x)}\varphi^{(i)}(x + t)^2\bar{F}(x + t)\,dx \\ &\leq \int_{\mathbb{R}_+}\varphi^{(i)}(x + t)^2\bar{F}(x + t)\,dx \\ &\leq \|\varphi^{(i)}\|_{L^2}^2,\end{aligned}$$

so that finally we have (30) with $M_n \equiv \max_{0\leq i\leq n}\left\{\binom{n}{i}2\|h\|_\infty^{n-i}\right\}$. □

### 3.2 Residuals

If we define the linear operator $B^{\mathcal{R}}$ on $\Phi_{\mathcal{R}}$ so that $B^{\mathcal{R}}\varphi \equiv -\varphi^{'}$ for $\varphi \in \Phi_{\mathcal{R}}$, then we can write (23) as

$$\mathcal{R} = \Psi^{\mathcal{R}}(\mathcal{R}_0, \mathcal{G} + E\mathcal{F}), \tag{31}$$

We now verify that $B^{\mathcal{R}}$ generates a $(C_0, 1)$ semi-group so that Theorem 3.2 applies to (23).

**Proposition 3.4.** *$B^{\mathcal{R}}$ generates the $(C_0, 1)$ semi-group $(\tau_t)_{t\geq 0}$.*

*Proof.* First we check that $B^{\mathcal{R}}$ generates the semigroup $(\tau_t)_{t\geq 0}$. For each $\varphi \in \Phi_{\mathcal{R}}$ we have

$$\lim_{h\to 0} \frac{\tau_h\varphi - \varphi}{h} = \lim_{h\to 0} \frac{\varphi(\cdot - h) - \varphi}{h} = -\varphi^{'}.$$

We now check that $(\tau_t)_{t\geq 0}$ is $(C_0, 1)$ semigroup. It clearly satisfies Part 1. of the definition of $(C_0, 1)$ semigroup. Part 2. again follows from Lemma 2 of [8]. To show Part 3. note that it suffices to show that for all $n \geq 1$

$$\|\varphi^{(n)}(\cdot - t)\|_{L^2} \leq \|\varphi^{(n)}\|_{L_2}.$$

Then, we have

$$\begin{aligned}
\|\varphi^{(n)}(\cdot - t)\|_{L^2}^2 &= \int_{\mathbb{R}_-} \varphi^{(n)}(x-t)^2\, dx + \int_{\mathbb{R}_+} \varphi^{(n)}(x-t)^2 \bar{F}(x)\, dx. \\
&\leq \int_{\mathbb{R}_-} \varphi^{(n)}(x)^2\, dx + \int_{\mathbb{R}_+} \varphi^{(n)}(x)^2 \bar{F}(x+t)\, dx. \\
&\leq \int_{\mathbb{R}_-} \varphi^{(n)}(x)^2\, dx + \int_{\mathbb{R}_+} \varphi^{(n)}(x)^2 \bar{F}(x)\, dx. \\
&\leq \|\varphi^{(n)}\|_{L^2}^2.
\end{aligned}$$

□

## 4 Martingale Results

In this section we show that the processes $\mathcal{D}^0 + \mathcal{D}$ and $\mathcal{G}$ defined in §2 are $\Phi_{\mathcal{A}}^{'}$ and $\Phi_{\mathcal{R}}^{'}$-valued martingales, respectively. That fact that $\mathcal{D}^0 + \mathcal{D}$ is a martingale will be used with the martingale functional central limit theorem and the continuous mapping theorem in §5.1 and §6.1 to prove fluid and diffusion limits, respectively, for our age process. The fact that $\mathcal{G}$ is a martingale will not be needed to prove limit theorems for our residuals process but can be used to show that the diffusion limit for our residuals process is a Markov process (see §7) and possibly in other future work.

### 4.1 Ages

First we first show that the process $\mathcal{D} + \mathcal{D}^0$ defined in §2.1 is a martingale with respect to the filtration $(\mathcal{F}_t^{\mathcal{A}})_{t\geq 0}$ defined by

$$\mathcal{F}_t^{\mathcal{A}} = \sigma\{\mathbf{1}_{\{\eta_i=0\}}, \mathbf{1}_{\{\eta_i\leq s-\tau_i\}}, s \leq t,\ i = 1, 2, ..., E_t\} \vee \sigma\{E_s,\ s \leq t\} \vee \mathcal{N}.$$

**Proposition 4.1.** *The process $\mathcal{D}^0+\mathcal{D}$ is a $\Phi'_{\mathcal{A}}$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale with tensor quadratic variation process given for all $\varphi, \psi \in \Phi_{\mathcal{A}}$ by*

$$<< \mathcal{D}^0 + \mathcal{D} >>_t (\varphi, \psi) = \sum_{i=1}^{A_0} \int_0^{\tilde{\eta}_i \wedge t} \varphi(x - \tilde{\tau}_i)\psi(x - \tilde{\tau}_i) h_{\tilde{\tau}_i}(x)\, dx + \sum_{i=1}^{E_t} \int_0^{\eta_i \wedge (t-\tau_i)^+} \varphi(x)\psi(x)h(x)\, dx. \tag{32}$$

*Proof.* We first analyze $\mathcal{D}$. Note that by (9) for $t \geq 0$, we have that

$$D_t(y) = \sum_{i=1}^{\infty} D_t^i(y), \quad y \geq 0$$

where for each $i \geq 1$

$$D_t^i(y) \equiv \mathbf{1}_{\{\eta_i \leq (t-\tau_i)^+ \wedge y\}} - \int_0^{\eta_i \wedge (t-\tau_i) \wedge y} h(u)\, du. \tag{33}$$

We will show for the associated $\Phi'_{\mathcal{A}}$-valued processes $\mathcal{D}^i$, $i \geq 1$:

1. For each $i \geq 1$, $\mathcal{D}^i$ is a $\Phi'_{\mathcal{A}}$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale.
2. For $i \neq j$, $\mathcal{D}^i$ and $\mathcal{D}^j$ are orthogonal.
3. For each $i \geq 1$, the tensor quadratic variation of $\mathcal{D}^i$ is given for all $\varphi, \psi \in \Phi_{\mathcal{A}}$ by

$$<< \mathcal{D}^i_\cdot >>_t (\varphi, \psi) = \int_0^{\eta_i \wedge (t-\tau_i)^+} \varphi(x)\psi(x)h(x)\, dx.$$

If we can then show that $\sum_{i=1}^k \langle \mathcal{D}_t^i, \varphi \rangle$ is dominated by an integrable random variable uniformly over $k \geq 0$, it will then follow by Lesbegue's dominated convergence theorem for conditional expectations and 1. above that

$$\mathbb{E}\left[\langle \mathcal{D}_t, \varphi \rangle | \mathcal{F}_s\right] = \mathbb{E}\left[\sum_{i=1}^{\infty} \langle \mathcal{D}_t^i, \varphi \rangle \middle| \mathcal{F}_s\right] = \sum_{i=1}^{\infty} E\left[\langle \mathcal{D}_t^i, \varphi \rangle | \mathcal{F}_s\right] = \sum_{i=1}^{\infty} \langle \mathcal{D}_s^i, \varphi \rangle = \langle \mathcal{D}_s, \varphi \rangle$$

and hence $\mathcal{D}$ will be a $\Phi'_{\mathcal{A}}$-valued $\mathcal{F}_t$-martingale. Note that for each $k \geq 1$, we have that

$$\begin{aligned}
\left|\sum_{i=1}^{k} \langle \mathcal{D}_t^i, \varphi \rangle\right| &= \left|\sum_{i=1}^{k} \int_0^{(t-\tau_i)^+} \varphi(x)\, d\left(\mathbf{1}_{\{\eta_i \leq x\}} - \int_0^{\eta_i \wedge x} h(u)\, du\right)\right| \\
&\leq \sum_{i=1}^{\infty} \sup_{0 \leq s \leq t} |\varphi(s)| \left(\mathbf{1}_{\{\eta_i \leq (t-\tau_i)^+\}} + \int_0^{\eta_i \wedge (t-\tau_i)^+} h(u)\, du\right) \\
&\leq E_t \sup_{0 \leq s \leq t} |\varphi(s)|(1 + t\|h\|_\infty).
\end{aligned}$$

However, clearly $\mathbb{E}[E_t \sup_{0 \leq s \leq t} |\varphi(s)|(1 + t\|h\|_\infty)|\mathcal{F}_s] = \sup_{0 \leq s \leq t} |\varphi(s)|(1 + t\|h\|_\infty)\mathbb{E}[E_t|\mathcal{F}_s] < \infty$, which completes the proof that $\mathcal{D}$ is a $\Phi'_{\mathcal{A}}$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale.

Further, a similar dominated convergence argument along with 2. and 3. above shows that the quadratic variation of the $\Phi'_{\mathcal{A}}$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale $\mathcal{D}$ is given for all $\varphi, \psi \in \Phi_{\mathcal{A}}$ by

$$<< \mathcal{D}_\cdot >>_t (\varphi, \psi) = \sum_{i=1}^{\infty} << \mathcal{D}^i_\cdot >>_t (\varphi, \psi) = \sum_{i=1}^{E_t} \int_0^{\eta_i \wedge (t-\tau_i)^+} \varphi(x)\psi(x)h(x)\, dx. \tag{34}$$

It remains to show 1.-3. We begin with 1. It suffices to show that for each $\varphi \in \mathcal{S}$, $(\langle \mathcal{D}^i_t, \varphi\rangle)_{t\geq 0}$ is a real-valued $\mathcal{F}^{\mathcal{A}}_t$- martingale. First, $(\langle \mathcal{D}^i_t, \varphi\rangle)_{t\geq 0}$ is clearly $\mathcal{F}^{\mathcal{A}}_t$-adapted. To show that the martingale property holds for $(\langle \mathcal{D}^i, \varphi\rangle)_{t\geq 0}$, it suffices to show that the martingale property holds for $D^i(y)$ for each fixed $y \geq 0$. It will then follow that for $s \leq t$,

$$\begin{aligned}
\mathbb{E}\left[\left\langle \mathcal{D}^i_t, \varphi\right\rangle | \mathcal{F}^{\mathcal{A}}_s\right] &= \mathbb{E}\left[\int_{\mathbb{R}_+} D^i_t(y)\varphi^{'}(y)\, dy | \mathcal{F}^{\mathcal{A}}_s\right] \\
&= \int_{\mathbb{R}_+} \mathbb{E}\left[D^i_t(y)|\mathcal{F}^{\mathcal{A}}_s\right]\varphi^{'}(y)\, dy \\
&= \int_{\mathbb{R}_+} D^i_s(y)\varphi^{'}(y)\, dy, \\
&= \left\langle \mathcal{D}^i_s, \varphi\right\rangle
\end{aligned}$$

and so $\mathcal{D}^i$ will be a $\Phi^{'}_{\mathcal{A}}$-valued $\mathcal{F}^{\mathcal{A}}_t$-martingale. Since $y \wedge (t-\tau_i) = (t \wedge (\tau_i + y) - \tau_i)$, we have

$$D^i_t(y) = D^i_{t\wedge(\tau_i+y)}(y) = D^i_{t\wedge(\tau_i+y)}(\infty).$$

We know by the proof of Lemma 3.5 of [24] that $D^i(\infty)$ is an $\mathcal{F}^{\mathcal{A}}_t$-martingale and it is easy to see that $\tau_i + y$ is a $\mathcal{F}^{\mathcal{A}}_t$-stopping time. Therefore, the stopped process $(D^i_{\cdot\wedge(\tau_i+y)}(\infty))_{t\geq 0}$, is a $\mathcal{F}^{\mathcal{A}}_t$-martingale.

We now focus on 2. To prove orthogonality of $\mathcal{D}^i$ and $\mathcal{D}^j$ for $i \neq j$, it suffices to show that for all $\varphi, \psi \in \Phi_{\mathcal{A}}$ the process $(\langle \mathcal{D}^i_t, \varphi\rangle\langle \mathcal{D}^j_t, \psi\rangle)_{t\geq 0}$ is a $\mathcal{F}^{\mathcal{A}}_t$-martingale. As in 1., it suffices to prove that for each fixed $y \geq 0$ the martingale property holds for the process $(D^i_t(y)D^j_t(y))_{t\geq 0}$. Again, this follows from the fact that $(D^i_t(\infty)D^j_t(\infty))_{t\geq 0}$ is a martingale, which we know from the proof of Lemma 3.5 of [24].

We now calculate the tensor-quadratic variation for $\mathcal{D}^i$ to prove 3. First by (33), for all $\varphi \in \Phi_{\mathcal{A}}$ we have

$$\langle \mathcal{D}^i_t, \varphi\rangle = \int_{\mathbb{R}_+} \varphi(x)\, dD^i_t(x) = \int_0^{(t-\tau_i)^+} \varphi(x)\, d\left(\mathbf{1}_{\{\eta_i \leq x\}} - \int_0^{\eta_i\wedge x} h(u)\, du\right). \tag{35}$$

Therefore, as on page 259 of [24],

$$<< \langle \mathcal{D}^i_t, \varphi\rangle >>_t = \int_0^{(t-\tau_i)^+} \varphi(x)\, d\left\langle \mathbf{1}_{\{\eta_i \leq x\}} - \int_0^{\eta_i\wedge x} h(u)\, du\right\rangle = \int_0^{\eta_i\wedge(t-\tau_i)^+} \varphi(x)^2 h(x)\, dx, \tag{36}$$

so that

$$\begin{aligned}
<< \mathcal{D}^i >>_t (\varphi, \psi) &=< \langle \mathcal{D}^i_\cdot, \varphi\rangle, \langle \mathcal{D}^i_\cdot, \psi\rangle >_t \\
&= \frac{1}{4}\left(<< \langle \mathcal{D}^i_\cdot, \varphi+\psi\rangle >>_t - << \langle \mathcal{D}^i_\cdot, \varphi-\psi\rangle >>_t\right) \\
&= \frac{1}{4}\left(\int_0^{\eta_i\wedge(t-\tau_i)^+} (\varphi(x)+\psi(x))^2\, h(x)\, dx - \int_0^{\eta_i\wedge(t-\tau_i)^+} (\varphi(x)-\psi(x))^2 h(x)\, dx\right) \\
&= \int_0^{\eta_i\wedge(t-\tau_i)^+} \varphi(x)\psi(x)h(x)\, dx
\end{aligned}$$

where the second equality follows from polarization and the third equality follows from (36).

We now analyze $\mathcal{D}^0$. First note that

$$D_t^0(y) = \sum_{i=1}^{A_0} D_t^{0,i}(y), \tag{37}$$

where for each $i \geq 1$,

$$D_t^{0,i}(y) = \mathbf{1}_{\{\tilde{\eta}_i \leq t \wedge (y+\tilde{\tau}_i)\}} - \int_0^{\tilde{\eta}_i \wedge t \wedge (y+\tilde{\tau}_i)} h_{\tilde{\tau}_i}(u)\, du.$$

Replicating the analysis for $\mathcal{D}$ above, we can show that for each $i \geq 1$, $\mathcal{D}^{0,i} \equiv (\mathcal{D}_t^{0,i})_{t\geq 0}$ is a $\Phi_{\mathcal{A}}^{'}$-valued martingale and that for $i \neq j$, $\mathcal{D}^{0,i}$ and $\mathcal{D}^{0,j}$ are orthogonal. We now calculate the tensor-quadratic variation for $D^0$. First, for all $\varphi \in \Phi_{\mathcal{A}}$, we have

$$\langle \mathcal{D}_t^{0,i}, \varphi \rangle = \int_{\mathbb{R}_+} \varphi(x)\, dD_t^{0,i}(x) = \int_0^t \varphi(x - \tilde{\tau}_i)\, d\left(\mathbf{1}_{\{\tilde{\eta}_i \leq x\}} - \int_0^{\tilde{\eta}_i \wedge x} h_{\tilde{\tau}_i}(u) du\right). \tag{38}$$

This gives us

$$<< \langle \mathcal{D}_t^{0,i}, \varphi \rangle >>= \int_0^{\tilde{\eta}_i \wedge t} \varphi^2(x - \tilde{\tau}_i) h_{\tilde{\tau}_i}(x)\, dx,$$

Using the polarization identity as in the analysis of $\mathcal{D}$, it then follows that

$$<< D_\cdot^{0,i} >>_t (\varphi, \psi) = \int_0^{\tilde{\eta}_i \wedge t} \varphi(x - \tilde{\tau}_i)\psi(x - \tilde{\tau}_i) h_{\tilde{\tau}_i}(x)\, dx.$$

Summing the quadratic variations of each of the terms in (37) and noting the orthogonality of the martingales in the sum,

$$<< \mathcal{D}_\cdot^0 >>_t (\varphi, \psi) = \sum_{i=1}^{A_0} << \mathcal{D}_\cdot^{0,i} >>_t (\varphi, \psi) = \sum_{i=1}^{A_0} \int_0^{\tilde{\eta}_i \wedge t} \varphi(x - \tilde{\tau}_i)\psi(x - \tilde{\tau}_i) h_{\tilde{\tau}_i}(x)\, dx. \tag{39}$$

Now since $\mathcal{D}$ and $\mathcal{D}^0$ are both $\mathcal{F}_t^{\mathcal{A}}$-martingales, $\mathcal{D}^0 + \mathcal{D}$ is a $\mathcal{F}_t^{\mathcal{A}}$-martingale. Furthermore, $\mathcal{D}$ and $\mathcal{D}^0$ are orthogonal since they are independent. Therefore, $<< \mathcal{D}^0 + \mathcal{D} >>=<< \mathcal{D}^0 >> + << \mathcal{D} >>$. Plugging (34) and (39) into this equality gives us (32). □

## 4.2 Residuals

We now show that the process $\mathcal{G}$ defined in §2.2 is a martingale. This will be useful in future work where we wish to show that the residual service time process is a martingale. Let $\mathcal{F}_t^{\mathcal{G}}$ be the natural filtration generated by $\mathcal{G}$. We then have the following result.

**Proposition 4.2.** *The process $\mathcal{G}$ is a $\Phi_{\mathcal{R}}^{'}$-valued $\mathcal{F}_t^{\mathcal{G}}$-martingale with tensor optional quadratic variation process given for all $\varphi, \psi \in \Phi_{\mathcal{R}}$ by*

$$[\mathcal{G}]_t(\varphi, \psi) = \sum_{i=1}^{E_t} \varphi(\eta_i)\psi(\eta_i). \tag{40}$$

*Proof.* We first prove the martingale property. Define the filtration $(\mathcal{H}_k)_{k\geq 1}$ by $\mathcal{H}_k \equiv \sigma\{E_t, t \geq 0\} \vee \sigma\{\eta_1, \eta_2, ..., \eta_k\} \vee \mathcal{N}$. Define the discrete-time $D$-valued process $(G^i)_{i\geq 1}$ by

$$G^k(y) = \sum_{i=1}^{k} \left(\mathbf{1}_{\{\eta_i \leq y\}} - F(y)\right), \tag{41}$$

and let $(\mathcal{G}^k)_{k\geq 1}$ be the associated $\Phi_{\mathcal{R}}$-valued process. It is then clear by the independence of the service times from the arrival process that for each $\varphi \in \Phi_{\mathcal{R}}$, the process $(\langle \mathcal{G}^k, \varphi\rangle)_{k\geq 1}$ is an $\mathcal{H}_k$-martingale. However, since for each $t \geq 0$ we have that $E_t$ is a stopping time with respect to the filtration $(\mathcal{H}_k)_{k\geq 1}$, it follows that the filtration $(\mathcal{H}_{E_t})_{t\geq 0}$ is well-defined and, furthermore, by the optional sampling theorem, we have that for each $\varphi \in \Phi_{\mathcal{R}}$, $(\langle \mathcal{G}_t, \varphi\rangle)_{t\geq 0} = (\langle \mathcal{G}^{E_t}, \varphi\rangle)_{t\geq 0}$ is a $\mathcal{H}_{E_t}$-martingale. The result now follows since any martingale is a martingale relative to its natural filtration.

The form of the tensor optional quadratic variation (40) is immediate by Theorem 3.3 of [28]. □

# 5 Fluid Limits

In this section, we begin proving our weak convergence results. We consider a sequence of $G/GI/\infty$ queues indexed by $n \geq 1$, each following the assumptions of §2. We assume that the service time distribution is held fixed across the systems. We add a superscript $n \geq 1$ to all processes and quantities defined in §2 to indicate association to the $n$th queue in the sequence. We focus on fluid limits for the age and residual processes in §5.1 and §5.2, respectively. We move on to diffusion limits in §6.

## 5.1 Ages

We start with the age processes of §2.1. Define

$$\bar{\mathcal{A}}^n \equiv \frac{\mathcal{A}^n}{n}, \quad \bar{\mathcal{A}}_0^n \equiv \frac{\mathcal{A}_0^n}{n}, \quad \bar{E}^n \equiv \frac{E^n}{n}, \quad \bar{\mathcal{D}}^n \equiv \frac{\mathcal{D}^n}{n}, \quad \bar{\mathcal{D}}^{0,n} \equiv \frac{\mathcal{D}^{0,n}}{n}, \tag{42}$$

and $\bar{\mathcal{E}}^n \equiv \bar{E}^n \delta_0$ for $n \geq 1$. Then by (28) for $n \geq 1$ we have

$$\bar{\mathcal{A}}^n = \Psi_{B^{\mathcal{A}}}(\bar{\mathcal{A}}_0^n, \bar{\mathcal{E}}^n - \bar{\mathcal{D}}^{0,n} - \bar{\mathcal{D}}^n).$$

We now prove convergence of $\bar{\mathcal{D}}^{0,n} + \bar{\mathcal{D}}^n$ jointly with $(\bar{\mathcal{A}}_0^n, \bar{E}^n)$:

**Proposition 5.1.** *If $(\bar{\mathcal{A}}_0^n, \bar{E}^n) \Rightarrow (\bar{\mathcal{A}}_0, \bar{E})$ in $\Phi_{\mathcal{A}}' \times D$ as $n \to \infty$, then*

$$\left(\bar{\mathcal{A}}_0^n, \bar{\mathcal{E}}^n, \bar{\mathcal{D}}^{0,n} + \bar{\mathcal{D}}^n\right) \Rightarrow \left(\bar{\mathcal{A}}_0, \bar{\mathcal{E}}, 0\right) \quad \text{in} \quad \Phi_{\mathcal{A}}' \times D([0,\infty), \Phi')^2 \quad \text{as} \quad n \to \infty.$$

*Proof.* We first prove

$$\bar{\mathcal{D}}^{0,n} + \bar{\mathcal{D}}^n \Rightarrow 0 \quad \text{in} \quad D([0,\infty), \Phi') \quad \text{as} \quad n \to \infty. \tag{43}$$

For each $\varphi, \psi \in \Phi_{\mathcal{A}}$, $T > 0$ and $0 \le t \le T$ we have

$$
\begin{aligned}
&<< \bar{\mathcal{D}}^{0,n} + \bar{\mathcal{D}}^{n} >>_t (\varphi, \psi) \\
&\quad = \frac{1}{n^2} \left( \sum_{i=0}^{A_0^n} \int_0^{\tilde{\eta}_i \wedge t} \varphi(x - \tilde{\tau}_i^n) \psi(x - \tilde{\tau}_i^n) h_{\tilde{\tau}_i^n}(x)\, dx + \sum_{i=0}^{E_t^n} \int_0^{\eta_i \wedge (t - \tau_i^n)^+} \varphi(x)\psi(x)h(x)\, dx \right) \\
&\quad \le \frac{1}{n^2} \left( \sum_{i=0}^{A_0^n} \int_0^{T} |\varphi(x)\psi(x)h(x)|\, dx + \sum_{i=0}^{E_T^n} \int_0^{T} |\varphi(x)\psi(x)h(x)|\, dx \right) \\
&\quad \le \frac{1}{n} \left( \bar{A}_0^n + \bar{E}_T^n \right) T \sup_{0 \le s \le T} |\varphi(s)| \sup_{0 \le s \le T} |\psi(s)| \|h\|_\infty \Rightarrow 0,
\end{aligned}
$$

in $D$ as $n \to \infty$ by the assumed convergence of $\bar{\mathcal{A}}_0^n$ and $\bar{E}^n$. Therefore, (43) follows by the martingale FCLT. Then joint convergence holds by virtue of Theorem 11.4.5 of [34] and the fact that the limit in (43) is deterministic. □

We then have

**Theorem 5.2.** *If $(\bar{\mathcal{A}}_0^n, \bar{E}^n) \Rightarrow (\bar{\mathcal{A}}_0, \bar{E})$ in $\Phi_{\mathcal{A}}' \times D$ as $n \to \infty$, then*

$$\bar{\mathcal{A}}^n \Rightarrow \bar{\mathcal{A}} \quad \text{in} \quad D([0,\infty), \Phi') \quad \text{as} \quad n \to \infty,$$

*where $\bar{\mathcal{A}}$ satisfies the deterministic integral equation*

$$\langle \bar{\mathcal{A}}_t, \varphi \rangle = \langle \bar{\mathcal{A}}_0, \varphi \rangle + \bar{E}_t \varphi(0) - \int_0^t \langle \bar{\mathcal{A}}_s, h\varphi \rangle\, ds + \int_0^t \langle \bar{\mathcal{A}}_s, \varphi' \rangle\, ds, \tag{44}$$

*for all $\varphi \in \Phi_{\mathcal{A}}$.*

*Proof.* By the assumption and Proposition 5.1, we have

$$\left( \bar{\mathcal{A}}_0^n, \bar{\mathcal{E}}^n, \bar{\mathcal{D}}^{0,n} + \bar{\mathcal{D}}^n \right) \Rightarrow \left( \bar{\mathcal{A}}_0, \bar{\mathcal{E}}, 0 \right) \quad \text{in} \quad \Phi_{\mathcal{A}}' \times D([0,\infty), \Phi')^2 \quad \text{as} \quad n \to \infty.$$

Then, since $\bar{\mathcal{A}}^n = \Psi_{B^{\mathcal{A}}}(\bar{\mathcal{A}}_0^n, \bar{\mathcal{E}}^n - \bar{\mathcal{D}}^{0,n} - \bar{\mathcal{D}}^n)$ and $\Psi_{B^{\mathcal{A}}}$ is continuous by Theorem 3.2 and Proposition 3.3, the result follows from the continuous mapping theorem (see [2] and [34]) with $\Psi_{\mathcal{A}}$ and the addition map. The addition map is convergence preserving here by Theorem 4.1 of [32] since the limits of $\bar{\mathcal{D}}^{0,n}$ and $\bar{\mathcal{D}}^n$ as $n \to \infty$ are continuous. □

**Remark 5.3.** *Note that we can now use Theorem 5.2 along with Theorem 3.2 to write an explicit expression for $\bar{\mathcal{A}}$. Similarly, we can get explicit expressions for $\bar{\mathcal{R}}$, $\hat{\mathcal{A}}$ and $\hat{\mathcal{R}}$ in Theorems 5.5, 6.5 and 6.7 below.*

## 5.2 Residuals

We now proceed to prove a fluid limit for the residual processes of §2.2. Let

$$\bar{\mathcal{R}}^n \equiv \frac{\mathcal{R}^n}{n}, \qquad \bar{\mathcal{R}}_0^n \equiv \frac{\mathcal{R}_0^n}{n}, \qquad \bar{\mathcal{G}} \equiv \frac{\mathcal{G}^n}{n},$$

for $n \ge 1$. Then by (23) we have

$$\bar{\mathcal{R}}^n = \Psi_{B^{\mathcal{R}}}(\bar{\mathcal{R}}_0^n, \bar{\mathcal{G}}^n + \bar{\mathcal{E}}^n \mathcal{F}).$$

We first prove convergence of $\bar{\mathcal{G}}^n$ jointly with $(\bar{\mathcal{R}}_0^n, \bar{E}^n)$:

**Proposition 5.4.** *If* $(\bar{\mathcal{R}}_0^n, \bar{E}^n) \Rightarrow (\bar{\mathcal{R}}_0, \bar{E})$ *in* $\Phi_{\mathcal{R}}^{'} \times D$ *as* $n \to \infty$*, then*

$$\left(\bar{\mathcal{R}}_0^n, \bar{\mathcal{E}}^n, \bar{\mathcal{G}}^n\right) \Rightarrow \left(\bar{\mathcal{R}}_0, \bar{\mathcal{E}}, 0\right) \quad in \quad \Phi_{\mathcal{R}}^{'} \times D([0,\infty), \Phi^{'})^2 \quad as \quad n \to \infty. \tag{45}$$

*Proof.* Notice that for each $\varphi \in \Phi_{\mathcal{R}}$, $\langle \bar{\mathcal{G}}^n, \varphi \rangle$ can be written as

$$\langle \bar{\mathcal{G}}^n, \varphi \rangle = \left( \frac{1}{n} \sum_{i=1}^{\lfloor n \cdot \rfloor} \int_{\mathbb{R}_+} \varphi(x)\, d\left(\mathbf{1}_{\{\eta_i \le x\}} - F(x)\right) \right) \circ \bar{E}^n.$$

The first term above converges to the 0 function by the functional weak law of large numbers. Thus, by continuity of the composition map at continuous limit points (see page 145 of [2] or Theorem 13.2.1 of [34]), we have $\langle \bar{\mathcal{G}}^n, \varphi \rangle \Rightarrow 0$ in $D$ as $n \to \infty$. Since this limit is deterministic, we get the full joint convergence (45) by Theorem 11.4.5 of [34]. □

We then have

**Theorem 5.5.** *If* $(\bar{\mathcal{R}}_0^n, \bar{E}^n) \Rightarrow (\bar{\mathcal{R}}_0, \bar{E})$ *in* $\Phi_{\mathcal{R}}^{'} \times D$ *as* $n \to \infty$*, then*

$$\bar{\mathcal{R}}^n \Rightarrow \bar{\mathcal{R}} \quad in \quad D([0,\infty), \Phi^{'}) \quad as \quad n \to \infty,$$

*where* $\bar{\mathcal{R}}$ *satisfies the deterministic integral equation*

$$\langle \bar{\mathcal{R}}_t, \varphi \rangle = \langle \bar{\mathcal{R}}_0, \varphi \rangle + \bar{E}_t \langle \mathcal{F}, \varphi \rangle - \int_0^t \langle \bar{\mathcal{R}}_s, \varphi^{'} \rangle\, ds, \tag{46}$$

*for all* $\varphi \in \Phi_{\mathcal{R}}$*.*

*Proof.* By the assumption and Proposition 5.4, we have

$$\left(\bar{\mathcal{R}}_0^n, \bar{\mathcal{E}}^n, \bar{\mathcal{G}}^n\right) \Rightarrow \left(\bar{\mathcal{R}}_0, \bar{\mathcal{E}}, 0\right) \quad \text{in} \quad \Phi_{\mathcal{R}}^{'} \times D([0,\infty), \Phi^{'})^2 \quad \text{as} \quad n \to \infty.$$

Then, since $\bar{\mathcal{R}}^n = \Psi_{B^{\mathcal{R}}}(\bar{\mathcal{R}}_0^n, \bar{\mathcal{E}}^n \mathcal{F} + \bar{\mathcal{G}}^n)$ and $\Psi_{B^{\mathcal{R}}}$ is continuous by Theorem 3.2 and Proposition 3.4, the result follows from the continuous mapping theorem with $\Psi_{B^{\mathcal{R}}}$ and the addition map. The addition map is convergence preserving here by Theorem 4.1 of [32] since the limit of $\bar{\mathcal{G}}^n$ as $n \to \infty$ is continuous. □

# 6 Diffusion Limits

We now move on to the diffusion limits. First we define generalized $\Phi^{'}$-valued Wiener process and generalized $\Phi^{'}$-valued Ornstein-Uhlenbeck process as in [3]. These notions will be used to characterize our diffusion limits for the age and residual processes.

**Definition 6.1.** *A continuous* $\Phi^{'}$*-valued Gaussian process* $W \equiv (W_t)_{t \ge 0}$ *is called a* ***generalized*** $\Phi^{'}$***-valued Wiener process*** *with* ***covariance functional*** $K(s, \varphi; t, \psi) \equiv \mathbb{E}\left[\langle W_s, \varphi \rangle \langle W_t, \psi \rangle\right]$ *if it has continuous trajectories and for each* $s, t \ge 0$ *and* $\varphi, \psi \in \Phi$*,* $K(s, \varphi; t, \psi)$ *has the form*

$$K(s, \varphi; t, \psi) = \int_0^{s \wedge t} \langle Q_u \varphi, \psi \rangle\, du,$$

*where the operators* $Q_u : \Phi \to \Phi^{'}$*,* $u \ge 0$*, have the properties:*

1. *$Q_u$ is linear, continuous, symmetric and positive for each $u \geq 0$, and*

2. *the function $u \mapsto \langle Q_u \varphi, \psi \rangle$ is in $D$ for each $\varphi, \psi \in \Phi$.*

*If $Q_u$ does not depend on $u \geq 0$, then the process is a $\Phi'$**-valued Wiener process**.*

**Definition 6.2.** *A $\Phi'$-valued process $X \equiv (X_t)_{t \geq 0}$ is called a **(generalized) $\Phi'$-valued Ornstein-Uhlenbeck process** if for each $\varphi \in \Phi$ and $t \geq 0$,*

$$\langle X_t, \varphi \rangle = \langle X_0, \varphi \rangle + \int_0^t \langle X_u, A\varphi \rangle \, du + \langle W_t, \varphi \rangle,$$

*where $W \equiv (W_t)_{t \geq 0}$ is a (generalized) $\Phi'$-valued Wiener process and $A : \Phi \to \Phi$ is a continuous operator.*

## 6.1 Ages

Define

$$\hat{\mathcal{A}}^n \equiv \sqrt{n}\left(\bar{\mathcal{A}}^n - \bar{\mathcal{A}}\right), \; \hat{\mathcal{A}}_0^n \equiv \sqrt{n}\left(\bar{\mathcal{A}}_0^n - \bar{\mathcal{A}}\right)_0, \; \hat{E}^n \equiv \sqrt{n}\left(\bar{E}^n - \bar{E}\right), \; \hat{\mathcal{D}}^n \equiv \sqrt{n}\bar{\mathcal{D}}^n, \; \hat{\mathcal{D}}^{0,n} \equiv \sqrt{n}\bar{\mathcal{D}}^{0,n},$$

and $\hat{\mathcal{E}}^n \equiv \hat{E}^n \delta_0$ for $n \geq 1$. Then, centering the system equation (14) by the fluid limit of Theorem 5.2, for $n \geq 1$ we have

$$\hat{\mathcal{A}}^n = \Psi_{B^{\mathcal{A}}}(\hat{\mathcal{A}}_0^n, \hat{\mathcal{E}}^n - \hat{\mathcal{D}}^{0,n} - \hat{\mathcal{D}}^n).$$

We now use the following result to approximate $\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n$ by a process that is independent of $(\hat{\mathcal{A}}_0^n, \hat{E}^n)$ for each $n \geq 1$. This is used to prove the required joint convergence in Proposition 6.4.

**Lemma 6.3.** *Define $\check{D}^{0,n} \equiv (\check{D}_t^{0,n})_{t \geq 0} \in D([0,\infty), D)$ so that*

$$\check{D}_t^{0,n}(y) \equiv \sum_{i=1}^{n\bar{A}_0} \left( \mathbf{1}_{\{\tilde{\eta}_i \leq t \wedge (y + \check{\tilde{\tau}}_i)\}} - \int_0^{\tilde{\eta}_i \wedge t \wedge (y + \check{\tilde{\tau}}_i)} h_{\check{\tilde{\tau}}_i}(u) \, du \right). \tag{47}$$

*and define $\check{D}^n \equiv (\check{D}_t^n)_{t \geq 0} \in D([0,\infty), D)$ so that*

$$\check{D}_t^n(y) \equiv \sum_{i=1}^{n\bar{E}_t} \left( \mathbf{1}_{\{\eta_i \leq (t - \check{\tau}_i) \wedge y\}} - \int_0^{\eta_i \wedge (t - \check{\tau}_i) \wedge y} h(u) \, du \right), \tag{48}$$

*for $t \geq 0$, $y \geq 0$, where*

$$-\check{\tilde{\tau}}_i^n \equiv \inf\left\{ s | n\bar{A}_0(s) \geq i \right\},$$
$$\check{\tau}_i^n \equiv \inf\left\{ s | n\bar{E}_s \geq i \right\},$$

*for $i \geq 0$. Let $\check{\mathcal{D}}^{n,0}$ and $\check{\mathcal{D}}^n$ be the $D([0,\infty), \Phi')$-valued processes associated with $\check{D}^{0,n}$ and $\check{D}^n$, respectively, for $n \geq 1$. Then, if $(\hat{\mathcal{A}}_0^n, \hat{E}^n) \Rightarrow (\hat{\mathcal{A}}_0, \hat{E})$ in $\Phi'_{\mathcal{A}} \times D$ as $n \to \infty$, then*

$$\left( \hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n, \check{\mathcal{D}}^{0,n} + \check{\mathcal{D}}^n \right) \Rightarrow \left( \hat{\mathcal{D}}^0 + \hat{\mathcal{D}}, \hat{\mathcal{D}}^0 + \hat{\mathcal{D}} \right) \quad \text{in} \quad D([0,\infty), \Phi')^2 \quad \text{as} \quad n \to \infty, \tag{49}$$

*where $\hat{\mathcal{D}}^0 + \hat{\mathcal{D}}$ is a generalized $\Phi'_{\mathcal{A}}$-valued Wiener process with covariance functional given for each $\varphi, \psi \in \Phi_{\mathcal{A}}$ and $s, t \geq 0$ by*

$$K_{\hat{\mathcal{D}}^0 + \hat{\mathcal{D}}}(s, \varphi; t, \psi) = \int_0^{s \wedge t} \langle \bar{\mathcal{A}}_u h, \varphi\psi \rangle \, du. \tag{50}$$

*Proof.* See appendix. □

**Proposition 6.4.** *If* $(\hat{\mathcal{A}}_0^n, \hat{E}^n) \Rightarrow (\hat{\mathcal{A}}_0, \hat{E})$ *in* $\Phi'_{\mathcal{A}} \times D$ *as* $n \to \infty$*, then*

$$\left(\hat{\mathcal{A}}_0^n, \hat{\mathcal{E}}^n, \hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n\right) \Rightarrow \left(\hat{\mathcal{A}}_0, \hat{\mathcal{E}}, \hat{\mathcal{D}}^0 + \hat{\mathcal{D}}\right) \quad in \quad \Phi'_{\mathcal{A}} \times D([0,\infty), \Phi')^2 \quad as \quad n \to \infty, \tag{51}$$

*where* $\hat{\mathcal{D}}^0 + \hat{\mathcal{D}}$ *is given in Proposition* 6.3 and is independent of $\hat{E}$.

*Proof.* By Lemma 6.3 we have the joint convergence

$$\left(\hat{\mathcal{A}}_0^n, \hat{\mathcal{E}}^n, \check{\mathcal{D}}^{0,n} + \check{\mathcal{D}}^n\right) \Rightarrow \left(\hat{\mathcal{A}}_0, \hat{\mathcal{E}}, \hat{\mathcal{D}}^0 + \hat{\mathcal{D}}\right) \quad \text{in} \quad \Phi'_{\mathcal{A}} \times D([0,\infty), \Phi')^2 \quad \text{as} \quad n \to \infty \tag{52}$$

since each of the component processes in the prelimit above are independent of each other. Also by Lemma 6.3 we have,

$$\left(\hat{\mathcal{A}}_0^n, \hat{E}^n, \hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n\right) - \left(\hat{\mathcal{A}}_0^n, \hat{E}^n, \check{\mathcal{D}}^{0,n} + \check{\mathcal{D}}^n\right) \Rightarrow (0,0,0) \quad \text{in} \quad \Phi'_{\mathcal{A}} \times D([0,\infty), \Phi')^2 \quad \text{as} \quad n \to \infty. \tag{53}$$

Combining (53) with (52) gives us our result. □

We then have

**Theorem 6.5.** *If* $(\hat{\mathcal{A}}_0^n, \hat{E}^n) \Rightarrow (\hat{\mathcal{A}}_0, \hat{E})$ *in* $\Phi'_{\mathcal{A}} \times D$ *as* $n \to \infty$*, then*

$$\hat{\mathcal{A}}^n \Rightarrow \hat{\mathcal{A}} \text{ in } D([0,\infty), \Phi') \text{ as } n \to \infty,$$

*where* $\hat{\mathcal{A}}$ *satisfies the stochastic integral equation*

$$\langle \hat{\mathcal{A}}_t, \varphi \rangle = \langle \hat{\mathcal{A}}_0, \varphi \rangle + \hat{E}_t \varphi(0) - \langle \hat{\mathcal{D}}^0 + \hat{\mathcal{D}}, \varphi \rangle - \int_0^t \langle \hat{\mathcal{A}}_s, h\varphi \rangle \, ds + \int_0^t \langle \hat{\mathcal{A}}_s, \varphi' \rangle \, ds. \tag{54}$$

*for all* $\varphi \in \Phi_{\mathcal{A}}$*. If, in addition,* $\hat{E}$ *is Brownian motion with diffusion coefficient* $\sigma$*, then* $\hat{\mathcal{A}}$ *is a generalized* $\Phi'_{\mathcal{A}}$*-valued Ornstein-Uhlenbeck process driven by a generalized* $\Phi'_{\mathcal{A}}$*-valued Wiener process with covariance functional*

$$K_{\hat{\mathcal{E}} - (\hat{\mathcal{D}}^0 + \hat{\mathcal{D}})}(s, \varphi; t, \psi) = \int_0^{s \wedge t} \langle \sigma^2 \delta_0 + \bar{\mathcal{A}}_u h, \varphi\psi \rangle \, du. \tag{55}$$

*Proof.* Since $\hat{\mathcal{A}}^n = \Psi_{B^{\mathcal{A}}}(\hat{\mathcal{A}}_0^n, \hat{\mathcal{E}}^n - \hat{\mathcal{D}}^{0,n} - \hat{\mathcal{D}}^n)$ and $\Psi_{B^{\mathcal{A}}}$ is continuous by Theorem 3.2 and Proposition 3.3, the convergence follows from Proposition 6.4 and the continuous mapping theorem with $\Psi_{B^{\mathcal{A}}}$ and the addition map. The subtraction map is convergence preserving here by Theorem 4.1 of [32] since the limits of $\hat{\mathcal{D}}^{0,n}$ and $\hat{\mathcal{D}}^n$ as $n \to \infty$ are continuous.

If $\hat{E}$ is Brownian motion with diffusion coefficient $\sigma$, then $\hat{\mathcal{E}}$ is a generalized $\Phi'_{\mathcal{A}}$-valued Wiener process with covariance functional $K_{\hat{\mathcal{E}}}(s, \varphi; t, \psi) = \sigma^2 (s \wedge t) \varphi(0) \psi(0)$. Combining this with (50) and the fact that $\hat{\mathcal{D}}^0 + \hat{\mathcal{D}}$ and $\hat{E}$ are independent from Proposition 6.4 gives us (55). Thus, $\hat{\mathcal{A}}$ is a generalized $\Phi'_{\mathcal{A}}$ valued Ornstein-Uhlenbeck process. □

### 6.2 Residuals

Define

$$\hat{\mathcal{R}}^n \equiv \sqrt{n}\left(\bar{\mathcal{R}}^n - \bar{\mathcal{R}}\right), \qquad \hat{\mathcal{R}}_0^n \equiv \sqrt{n}\left(\bar{\mathcal{R}}_0^n - \bar{\mathcal{R}}\right), \qquad \hat{\mathcal{G}}^n \equiv \sqrt{n}\bar{\mathcal{G}}^n,$$

for $n \geq 1$. Centering the system equation (23) by the fluid limit of Theorem 5.5 we have for all $n \geq 1$,

$$\hat{\mathcal{R}}^n = \Psi_{B^{\mathcal{R}}}(\hat{\mathcal{R}}_0^n, \hat{\mathcal{G}}^n + \hat{\mathcal{E}}^n\mathcal{F}).$$

**Proposition 6.6.** *If $(\hat{\mathcal{R}}_0^n, \hat{E}^n) \Rightarrow (\hat{\mathcal{R}}_0, \hat{E})$ in $\Phi_{\mathcal{R}}' \times D$ as $n \to \infty$, then*

$$\left(\hat{\mathcal{R}}_0^n, \hat{\mathcal{E}}^n, \hat{\mathcal{G}}^n\right) \Rightarrow \left(\hat{\mathcal{R}}_0, \hat{\mathcal{E}}, \hat{\mathcal{G}}\right) \quad \text{in} \quad \Phi_{\mathcal{R}}' \times D([0,\infty), \Phi')^2 \quad \text{as} \quad n \to \infty, \tag{56}$$

*where $\hat{\mathcal{G}}$ is a $\Phi_{\mathcal{R}}'$-valued Wiener process with covariance functional*

$$K_{\hat{G}}(s, \varphi; t, \psi) = (\bar{E}_s \wedge \bar{E}_t)\mathrm{Cov}(\varphi(\eta), \psi(\eta)), \tag{57}$$

*where $\eta$ is a random variable with cdf $F$.*

*Proof.* Notice that for each $\varphi \in \Phi_{\mathcal{R}}$, $\langle \hat{\mathcal{G}}^n, \varphi \rangle$ can be written as

$$\langle \hat{\mathcal{G}}^n, \varphi \rangle = \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor n \cdot \rfloor} \int_{\mathbb{R}_+} \varphi(x)\, d\left(\mathbf{1}_{\{\eta_i \leq x\}} - F(x)\right) \right) \circ \bar{E}^n.$$

By the functional central limit theorem (see Theorem 16.1 of [2]), the first term above converges to a Brownian motion with diffusion coefficient $\sqrt{\mathrm{Var}(\varphi(\eta))}$. Thus, our result follows by the random time-change theorem (see [11]). □

We then have

**Theorem 6.7.** *If $(\hat{\mathcal{R}}_0^n, \hat{E}^n) \Rightarrow (\hat{\mathcal{R}}_0, \hat{E})$ in $\Phi_{\mathcal{R}}' \times D$ as $n \to \infty$, then*

$$\hat{\mathcal{R}}^n \Rightarrow \hat{\mathcal{R}} \quad \text{in} \quad D([0,\infty), \Phi') \quad \text{as} \quad n \to \infty,$$

*where $\hat{\mathcal{R}}$ satisfies the stochastic integral equation*

$$\langle \hat{\mathcal{R}}_t, \varphi \rangle = \langle \hat{\mathcal{R}}_0, \varphi \rangle + \langle \hat{\mathcal{G}}_t, \varphi \rangle + \hat{E}_t \langle \mathcal{F}, \varphi \rangle - \int_0^t \langle \hat{\mathcal{R}}_s, \varphi' \rangle\, ds, \tag{58}$$

*for all $\varphi \in \Phi_{\mathcal{R}}$. If, in addition, $\hat{E}$ is Brownian motion with diffusion coefficient $\sigma$, then $\hat{\mathcal{R}}$ is a generalized $\Phi_{\mathcal{R}}'$-valued Ornstein-Uhlenbeck process driven by a generalized $\Phi_{\mathcal{R}}'$-valued Wiener process with covariance functional*

$$K_{\hat{E}\mathcal{F}+\hat{G}}(s, \varphi; t, \psi) = \sigma^2(s \wedge t)\mathbb{E}[\varphi(\eta)]\mathbb{E}[\psi(\eta)] + \left(\bar{E}_s \wedge \bar{E}_t\right)\mathrm{Cov}(\varphi(\eta), \psi(\eta)), \tag{59}$$

*where $\eta$ is a random variable with cdf $F$.*

*Proof.* Since $\hat{\mathcal{R}}^n = \Psi_{B^{\mathcal{R}}}(\hat{\mathcal{R}}_0^n, \hat{\mathcal{E}}^n\mathcal{F} + \hat{\mathcal{G}}^n)$, and $\Psi_{B^{\mathcal{R}}}$ is continuous by Theorem 3.2 and Proposition 3.4, the convergence follows from Proposition 6.6 and the continuous mapping theorem with $\Psi_{B^{\mathcal{R}}}$ and the addition map. The subtraction map is convergence preserving here by Theorem 4.1 of [32] since the limit of $\hat{\mathcal{G}}$ as $n \to \infty$ is continuous.

If $\hat{E}$ is Brownian motion with diffusion coefficient $\sigma$, it is easily checked that $\hat{E}\mathcal{F}$ is a $\Phi_{\mathcal{R}}'$-valued Wiener process with covariance functional

$$K_{\hat{E}\mathcal{F}}(s, \varphi; t, \psi) = \sigma^2(s \wedge t)\mathbb{E}[\varphi(\eta)]\mathbb{E}[\psi(\eta)]. \tag{60}$$

Combining (60) with (57) gives us (59). □

**Remark 6.8.** *Notice that in the special case when the arrival process to the nth system is Poisson with rate $\lambda n$, $\bar{E} = \lambda\cdot$ and $\hat{E}$ is Brownian motion with diffusion coefficient $\lambda$. Thus, $K_{\hat{E}\mathcal{F}}(s,\varphi;t,\psi) = \lambda(s\wedge t)\mathbb{E}[\varphi(\eta)\psi(\eta)]$ and Theorem 6.7 gives us a version of Theorem 3 of [8].*

# 7 Markov Process Results

In this section we prove that under extra conditions on the arrival process and the fluid limit of the initial conditions, the limiting age process $\hat{\mathcal{A}}$ of Theorem 6.5 is a time-homogeneous Markov process. We then identify the generator of $\hat{\mathcal{A}}$, which enables us to determine its transient and stationary distributions using results from Markov process theory (see [11]). One could also follow the same program to analyze the diffusion limit of the residual-service time process in Theorem 6.7.

We begin with the following result about the stationary solution to the fluid equation (44) when $\bar{E} = \lambda\cdot$. This assumption holds, for example, when the arrival process is a renewal process (as is the case for that $GI/GI/\infty$ queue).

**Proposition 7.1.** *If $\bar{E} = \lambda\cdot$, then $\bar{\mathcal{A}}_\cdot = \lambda\mathcal{F}_e$ is a stationary solution to the fluid equation (44).*

*Proof.* Plugging $\bar{\mathcal{A}}_\cdot = \lambda\mathcal{F}_e$ and $\bar{E} = \lambda\cdot$ into (44) we see that it suffices to verify that

$$\lambda\int_{\mathbb{R}_+}\varphi(y)\,dF_e(y) = \lambda\int_{\mathbb{R}_+}\varphi(y)\,dF_e(y) + \lambda t\varphi(0) - \lambda t\int_{\mathbb{R}_+} h(y)\varphi(y) - \varphi^{'}(y)\,dF_e(y).$$

But this follows since

$$\begin{aligned}\lambda t\int_{\mathbb{R}_+} h(y)\varphi(y) - \varphi^{'}(y)\,dF_e(y) &= \lambda t\int_{\mathbb{R}_+} h(y)\varphi(y) - \varphi^{'}(y)\bar{F}(y)\,dy\\ &= \lambda t\int_{\mathbb{R}_+} f(y)\varphi(y) - \bar{F}(y)\varphi^{'}(y)\,dy\\ &= -\lambda t\int_{\mathbb{R}_+}\left(\bar{F}(y)\varphi(y)\right)^{'}\,dy\\ &= \lambda t\varphi(0).\end{aligned}$$

□

The next proposition shows that $\hat{\mathcal{A}}$ has a simpler form when $\bar{E} = \lambda\cdot$, $\hat{E}$ is Brownian motion and $\bar{\mathcal{A}}_0 = \lambda\mathcal{F}_e$. For each $\varphi \in \Phi_{\mathcal{A}}$, define the function $F_\varphi : \Phi_{\mathcal{A}} \to \mathbb{C}$ by

$$F_\varphi(\mu) = e^{i\langle\mu,\varphi\rangle} \quad \text{for} \quad \mu \in \Phi_{\mathcal{A}}^{'},$$

and define the set $E(\Phi_{\mathcal{A}})$ to be the smallest algebra containing the set $\{F_\varphi,\ \varphi \in \Phi_{\mathcal{A}}\}$. We will use $E(\Phi_{\mathcal{A}})$ to determine the generator of $\hat{\mathcal{A}}$. Let $C_b(\Phi_{\mathcal{A}}^{'},\mathbb{C})$ denote the space of bounded continuous functions from $\Phi_{\mathcal{A}}^{'}$ to $\mathbb{C}$. We then have the following result.

**Lemma 7.2.** *$E(\Phi_{\mathcal{A}})$ is dense in $C_b(\Phi_{\mathcal{A}}^{'},\mathbb{C})$.*

*Proof.* The result follows by an application of the Stone-Weierstrass theorem for complex-valued functions (see Theorem 4.51 of [13]). □

**Proposition 7.3.** *If $\bar{E} = \lambda\cdot$, $\hat{E}$ is Brownian motion with diffusion coefficient $\sigma$, and $\bar{\mathcal{A}}_0 = \lambda\mathcal{F}_e$, then $\hat{\mathcal{A}}$ is a $\Phi'_{\mathcal{A}}$-valued Ornstein-Uhlenbeck process driven by a $\Phi'_{\mathcal{A}}$-valued Wiener process with covariance functional*

$$K_{\hat{\mathcal{E}}-(\hat{\mathcal{D}}^0+\hat{\mathcal{D}})}(s,\varphi;t,\psi) = (s\wedge t)\langle\sigma^2\delta_0 + \lambda\mathcal{F}, \varphi\psi\rangle. \tag{61}$$

*Furthermore, $\hat{\mathcal{A}}$ is a Markov process with generator $\mathcal{G}_{\mathcal{A}}$ satisfying*

$$\left(\mathcal{G}_{\mathcal{A}}F_\varphi\right)(\mu) = \left(i\langle\mu,\varphi' - h\varphi\rangle - \frac{\sigma^2}{2}\varphi^2(0) - \frac{\lambda}{2}\langle\mathcal{F},\varphi^2\rangle\right)F_\varphi(\mu), \tag{62}$$

*for each $\varphi \in \Phi_{\mathcal{A}}$, $\mu \in \Phi'_{\mathcal{A}}$.*

*Proof.* Since $\hat{E}$ is Brownian motion with diffusion coefficient $\sigma$, the covariance functional of $\hat{\mathcal{E}}$ is given by

$$K_{\hat{\mathcal{E}}}(s,\varphi;t,\psi) = (s\wedge t)\langle\sigma^2\delta_0, \varphi\psi\rangle. \tag{63}$$

We now show

$$K_{\hat{\mathcal{D}}^0+\hat{\mathcal{D}}}(s,\varphi;t,\psi) = \lambda(s\wedge t)\langle\mathcal{F},\varphi\psi\rangle. \tag{64}$$

By Proposition 7.1, since $\bar{\mathcal{A}}_0 = \lambda\mathcal{F}_e$, $\bar{\mathcal{A}}_\cdot = \lambda\mathcal{F}_e$ solves the fluid equation (44). Therefore, by Proposition 6.4, for each $\varphi,\psi \in \Phi_{\mathcal{A}}$, we have

$$\begin{aligned}
K_{\hat{\mathcal{D}}^0+\hat{\mathcal{D}}}(s,\varphi;t,\psi) &= \int_0^{s\wedge t} \left\langle \bar{\mathcal{A}}_u, \varphi\psi h\right\rangle du \\
&= \int_0^{s\wedge t}\int_{\mathbb{R}_+} \varphi(x)\psi(x)h(x)\, d\bar{A}_u(x)\, du \\
&= \lambda(s\wedge t)\int_{\mathbb{R}_+} \varphi(x)\psi(x)h(x)\bar{F}(x)\, dx \\
&= \lambda(s\wedge t)\int_{\mathbb{R}_+} \varphi(x)\psi(x)f(x)\, dx.
\end{aligned}$$

(61) then follows by combining (63) and (64) and using the fact that $\hat{\mathcal{E}}$ and $\hat{\mathcal{D}}^0+\hat{\mathcal{D}}$ are independent by Proposition 6.4.

Since $\hat{E}\varphi(0)$ is a Brownian motion with infinitesimal variance $\sigma^2\varphi(0)^2$ and $\langle\hat{\mathcal{D}}^0+\hat{\mathcal{D}},\varphi\rangle$ is a Brownian motion with infinitesimal variance $\lambda\langle\mathcal{F},\varphi^2\rangle$, it follows that $\langle\hat{\mathcal{A}}_t,\varphi\rangle$ is a semimartingale and hence by Itô's formula we have for each $\varphi \in \Phi_{\mathcal{A}}$ and $t \geq 0$,

$$\begin{aligned}
e^{i\langle\hat{\mathcal{A}}_t,\varphi\rangle} = e^{i\langle\hat{\mathcal{A}}_0,\varphi\rangle} + i\varphi(0)\int_0^t e^{i\langle\hat{\mathcal{A}}_s,\varphi\rangle}\, d\hat{E}_s - i\int_0^t e^{i\langle\hat{\mathcal{A}}_s,\varphi\rangle}\, d\langle\hat{\mathcal{D}}^0_s + \hat{\mathcal{D}}_s,\varphi\rangle + i\int_0^t e^{i\langle\hat{\mathcal{A}}_s,\varphi\rangle}\langle\hat{\mathcal{A}}_s,\varphi' - h\varphi\rangle\, ds \\
- \frac{\sigma^2}{2}\varphi^2(0)\int_0^t e^{i\langle\hat{\mathcal{A}}_s,\varphi\rangle}\, ds - \frac{\lambda}{2}\int_0^t e^{i\langle\hat{\mathcal{A}}_s,\varphi\rangle}\langle\mathcal{F},\varphi^2\rangle\, ds.
\end{aligned} \tag{65}$$

Now, plugging in (62) we can write (65) for each $\varphi \in \Phi_{\mathcal{A}}$ and $t \geq 0$,

$$F_\varphi(\mathcal{A}_t) - F_\varphi(\hat{\mathcal{A}}_0) - \int_0^t \mathcal{G}_{\mathcal{A}}F_\varphi(\hat{\mathcal{A}})ds = \int_0^t F_\varphi(\hat{\mathcal{A}})d\langle\hat{\mathcal{E}}_s - \hat{\mathcal{D}}^0_s - \hat{\mathcal{D}}_s,\varphi\rangle. \tag{66}$$

Since $\hat{\mathcal{E}}-\hat{\mathcal{D}}^0-\hat{\mathcal{D}}$ is a martingale, the stochastic integral on the right-hand side of (66) is a martingale. Thus the expression on the left-hand side of (66) is a martingale for each $\varphi \in \Phi_{\mathcal{A}}$. Since every element of $E(\Phi_{\mathcal{A}})$ is a linear combination of elements of $\{F_\varphi,\ \varphi \in \Phi_{\mathcal{A}}\}$ and (66) holds for every $\hat{\mathcal{A}}_0 \in \Phi'_{\mathcal{A}}$, Lemma 7.2 then implies that $\hat{\mathcal{A}}$ satisfies the martingale problem for $\mathcal{G}_{\mathcal{A}}$. Applying Theorem 4.4.1 of [11] then gives us our result. □

We now wish to calculate the stationary distribution of $\hat{\mathcal{A}}$. Assuming $\hat{\mathcal{A}}$ has a stationary distribution, denote it by $\pi_{\hat{\mathcal{A}}}$ and let $\hat{\mathcal{A}}_\infty$ denote a random variable with distribution $\pi_{\hat{\mathcal{A}}}$. Recall that by the basic adjoint relationship (see Proposition 4.9.2 of [11]) the stationary distribution $\pi_{\hat{\mathcal{A}}}$ of $\hat{\mathcal{A}}$ is uniquely determined by the equations,

$$\int_{\Phi'_{\mathcal{A}}} \mathcal{G}_{\mathcal{A}} F_\varphi(\mu)\, \pi_{\hat{\mathcal{A}}}(d\mu) = 0, \tag{67}$$

for $F_\varphi \in E(\Phi_{\mathcal{A}})$.

**Proposition 7.4.** *Under the assumptions of Proposition 7.3, $\hat{\mathcal{A}}_\infty$ is a $\Phi'_{\mathcal{A}}$-valued Gaussian random variable with mean 0 and covariance functional given by*

$$\mathbb{E}[\langle \hat{\mathcal{A}}_\infty, \varphi\rangle \langle \hat{\mathcal{A}}_\infty, \psi\rangle] = \langle \mathcal{F}_e, \left(\lambda F + \sigma^2 \bar{F}\right) \varphi\psi\rangle,$$

*for $\varphi, \psi \in \Phi_{\mathcal{A}}$.*

*Proof.* By Proposition 7.3, Lemma 7.2 and (67) it suffices to show that for each $\varphi \in \Phi_{\mathcal{A}}$

$$i \int_{\Phi'_{\mathcal{A}}} F_\varphi(\mu) \langle \mu, \varphi' - h\varphi\rangle\, \pi_{\hat{\mathcal{A}}}(d\mu) = \left(\frac{\sigma^2}{2}\varphi^2(0) + \frac{\lambda}{2}\langle \mathcal{F}, \varphi^2\rangle\right) \int_{\Phi'_{\mathcal{A}}} F_\varphi(\mu)\, \pi_{\hat{\mathcal{A}}}(d\mu). \tag{68}$$

Notice the left hand side of (68) is of the form $\mathbb{E} i e^{iX} Y$, where $(X, Y)$ is bivariate normal with mean $(0, 0)$ and covariance matrix $\Sigma = (\Sigma_{ij})_{i,j=1,2}$ with

$$\Sigma_{11} = \int_{\mathbb{R}_+} \varphi(x)^2 \left(\lambda \bar{F}(x) F(x) + \sigma^2 \bar{F}(x)^2\right)\, dx \tag{69}$$

$$\Sigma_{12} = \Sigma_{21} = \int_{\mathbb{R}_+} \varphi(x) \left(\varphi'(x) - h(x)\varphi(x)\right) \left(\lambda \bar{F}(x) F(x) + \sigma^2 \bar{F}(x)^2\right)\, dx \tag{70}$$

$$\Sigma_{22} = \int_{\mathbb{R}_+} \psi(x)^2 \left(\lambda \bar{F}(x) F(x) + \sigma^2 \bar{F}(x)^2\right)\, dx.$$

Thus, if we denote the characteristic function of this bivariate distribution by $\phi(\mathbf{t}) \equiv \mathbb{E} e^{i\mathbf{t}\cdot(X,Y)} = e^{-\frac{1}{2}\mathbf{t}^T \Sigma \mathbf{t}}$, then we can write the left hand side of (68) as

$$\mathbb{E} i e^{iX} Y = \frac{\partial}{\partial t_2} \phi(1, 0) = -\Sigma_{12} e^{-\frac{1}{2}\Sigma_{11}}. \tag{71}$$

Plugging (69) and (70) into (71) gives us

$$\begin{aligned} & i \int_{\Phi'_{\mathcal{A}}} F_\varphi(\mu) \langle \mu, \varphi' - h\varphi\rangle\, \pi_{\hat{\mathcal{A}}}(d\mu) \\ & \qquad = \int_{\mathbb{R}_+} \varphi(x) \left(h(x)\varphi(x) - \varphi'(x)\right) \left(\lambda \bar{F}(x) F(x) + \sigma^2 \bar{F}(x)^2\right)\, dx \int_{\Phi'_{\mathcal{A}}} F_\varphi(\mu)\, \pi_{\hat{\mathcal{A}}}(d\mu). \end{aligned} \tag{72}$$

Finally,

$$\begin{aligned}
&\int_{\mathbb{R}_+} \varphi(x)\left(h(x)\varphi(x)-\varphi^{'}(x)\right)\left(\lambda\bar{F}(x)F(x)+\sigma^2\bar{F}(x)^2\right)\,dx \\
&\quad= \int_{\mathbb{R}_+} \varphi^2(x)f(x)\left(\lambda F(x)+\sigma^2\bar{F}(x)\right)\,dx - \int_{\mathbb{R}_+} \varphi(x)\varphi^{'}(x)\left(\lambda F(x)+\sigma^2\bar{F}(x)\right)\bar{F}(x)\,dx \\
&\quad= \int_{\mathbb{R}_+} \varphi^2(x)f(x)\left(\lambda F(x)+\sigma^2\bar{F}(x)\right)\,dx - \frac{1}{2}\int_{\mathbb{R}_+} \left(\varphi(x)^2\right)^{'}\left(\lambda F(x)+\sigma^2\bar{F}(x)\right)\bar{F}(x)\,dx \\
&\quad= \frac{\sigma^2}{2}\varphi^2(0) + \frac{1}{2}\int_{\mathbb{R}_+} \varphi^2(x)f(x)\left(\lambda F(x)+\sigma^2\bar{F}(x)\right)\,dx + \frac{1}{2}\int_{\mathbb{R}_+} \varphi(x)^2(\lambda-\sigma^2)f(x)\bar{F}(x)\,dx \\
&\quad= \frac{\sigma^2}{2}\varphi^2(0) + \frac{\lambda}{2}\langle\mathcal{F},\varphi^2\rangle. \qquad (73)
\end{aligned}$$

The second equality above follows by integrating the second integral on the left-hand side by parts. Combining (72) and (73) gives us (68) and completes the proof. □

We can also verify the transient distributions of $\hat{\mathcal{A}}$ using $\mathcal{G}_{\mathcal{A}}$. Instead of using (67), one may use the following generalization (see Proposition 4.9.18 of [11]). Let $\mathbb{P}_t$ denote the distribution of $\hat{\mathcal{A}}_t$ for $t \geq 0$. Then, $(\mathbb{P}_t)_{t\geq 0}$ is uniquely determined by the equations

$$\int_{\Phi_{\mathcal{A}}^{'}} F_\varphi(\mu)\,\mathbb{P}_t[d\mu] - \int_{\Phi_{\mathcal{A}}^{'}} F_\varphi(\mu)\,\mathbb{P}_0[d\mu] = \int_0^t \int_{\Phi_{\mathcal{A}}^{'}} \mathcal{G}_{\mathcal{A}}F_\varphi(\mu)\,\mathbb{P}_s[d\mu]\,ds, \qquad (74)$$

for $F_\varphi \in E(\Phi_{\mathcal{A}})$ and $t \geq 0$. We then have the following result:

**Proposition 7.5.** *Under the assumptions of Proposition 7.3, for each $t \geq 0$, $\hat{\mathcal{A}}_t$ is a $\Phi_{\mathcal{A}}^{'}$-valued Gaussian random variable with mean*

$$\mathbb{E}[\langle\hat{\mathcal{A}}_t,\varphi\rangle] = \left\langle \mathcal{A}_0, \bar{F}^{-1}\tau_{-t}\left(\varphi\bar{F}\right)\right\rangle, \qquad (75)$$

*for $\varphi \in \Phi_{\mathcal{A}}$ and covariance functional given by*

$$\mathbb{E}[\langle\hat{\mathcal{A}}_t,\varphi\rangle\langle\hat{\mathcal{A}}_t,\psi\rangle] = \lambda\left\langle \mathcal{F}_e, \bar{F}^{-1}\tau_{-t}\left(\varphi\psi\bar{F}\right)\left(1-\bar{F}^{-1}\tau_{-t}\bar{F}\right)\right\rangle + \int_0^t \varphi(u)\psi(u)\left(\lambda F(u)+\sigma^2\bar{F}(u)\right)\bar{F}(u)du, \qquad (76)$$

*for $\varphi,\psi \in \Phi_{\mathcal{A}}$.*

*Proof.* We verify the Gaussian distributions defined by (75) and (76) solve the equation (74) for each $F_\varphi$. Plugging the proposed transient distributions into the left-hand side of (74) and using the characteristic function of a Gaussian distribution gives us

$$e^{i\left\langle \mathcal{A}_0,\bar{F}^{-1}\tau_{-t}\left(\varphi\bar{F}\right)\right\rangle - \frac{\lambda}{2}\left\langle \mathcal{F}_e,\bar{F}^{-1}\tau_{-t}\left(\varphi\psi\bar{F}\right)\left(1-\bar{F}^{-1}\tau_{-t}\bar{F}\right)\right\rangle - \frac{1}{2}\int_0^t \varphi^2(y)\left(\lambda F(y)\,dy+\sigma^2\bar{F}(y)\right)\bar{F}(y)\,dy} - e^{i\langle\mathcal{A}_0,\varphi\rangle} \qquad (77)$$

Plugging the generator (62) into the right-hand side of (74) gives us

$$\int_0^t \int_{\Phi_{\mathcal{A}}^{'}} \left(i\langle\mu,\varphi^{'}-h\varphi\rangle - \frac{\sigma^2}{2}\varphi^2(0) - \frac{\lambda}{2}\langle\mathcal{F},\varphi^2\rangle\right) e^{i\langle\mu,\varphi\rangle}\,\mathbb{P}_s[d\mu]\,ds. \qquad (78)$$

Next, plugging the proposed transient distributions into (78) and using the characteristic function of a bivariate Gaussian distribution as in the proof of Proposition 7.4 gives us

$$\int_0^t \Big( i \left\langle \hat{\mathcal{A}}_0, \bar{F}^{-1}\tau_{-s}\left(\varphi^{'}\bar{F} - \varphi f\right)\right\rangle - \lambda \left\langle \mathcal{F}_e, \bar{F}^{-1}\tau_{-s}\left(\varphi^{'}\bar{F} - \varphi f\right)\left(1 - \bar{F}\tau_{-s}\bar{F}\right)\right\rangle \tag{79}$$
$$- \int_0^s \varphi(u)\left(\varphi^{'}(y)\bar{F}(y) - \varphi(u)f(y)\right)\left(\lambda F(y) + \sigma^2\bar{F}(y)\right)du - \frac{\sigma^2}{2}\varphi^2(0) - \frac{\lambda}{2}\langle \mathcal{F}, \varphi^2\rangle\Big)$$
$$e^{i\langle \mathcal{A}_0, \bar{F}^{-1}\tau_{-s}(\varphi\bar{F})\rangle - \frac{\lambda}{2}\left\langle \mathcal{F}_e, \bar{F}^{-1}\tau_{-s}(\varphi\psi\bar{F})\left(1-\bar{F}^{-1}\tau_{-s}\bar{F}\right)\right\rangle - \frac{1}{2}\int_0^s \varphi^2(y)\left(\lambda F(y)+\sigma^2\bar{F}(y)\right)\bar{F}(y)\,dy}\,ds.$$

It now suffices to show that the first factor in the integral of (79) is the derivative of the power of the exponential factor. The desired equality (74) will then follow by the fundamental theorem of calculus.

It is easy to see by inspection that the derivative of the first term in the power of the exponential in (79) gives us the first term in the first factor of (79). The derivative of the rest of the power of the exponential is given by

$$-\lambda \int_{\mathbb{R}_+} \varphi(s+y)\left(\varphi^{'}(s+y)\bar{F}(s+y) - \varphi(s+y)f(s+y)\right)\left(1 - \frac{\bar{F}(s+y)}{\bar{F}(y)}\right)dy \tag{80}$$
$$+ \frac{\lambda}{2}\int_{\mathbb{R}_+} \varphi^2(s+y)f(s+y)\,dy - \frac{1}{2}\varphi^2(s)\left(\lambda F(s) + \sigma^2\bar{F}(s)\right)\bar{F}(s)$$

Using the fundamental theorem of calculus, we can write the last term of (80) as

$$-\int_0^s \varphi(y)\left(\varphi^{'}\bar{F}(y) - \varphi(y)f(y)\right)\left(\lambda F(y) + \sigma^2\bar{F}(y)\right)dy - \frac{\sigma^2}{2}\varphi^2(0) - \frac{\lambda}{2}\int_0^s \varphi^2(y)f(y)\,dy. \tag{81}$$

Combining (80) and (81) gives us the rest of the first factor of the integrand of (79) and concludes the verification of (74).

□

# A Proof of Lemma 6.3

*Proof.* We first prove

$$\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n \Rightarrow \hat{\mathcal{D}}^0 + \hat{\mathcal{D}} \quad \text{in} \quad D([0,\infty), \Phi^{'}) \quad \text{as} \quad n \to \infty. \tag{82}$$

By Proposition 4.1 we have

$$\begin{aligned}
&<< \hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n >>_t (\varphi, \psi) \\
&\quad = \frac{1}{n}\left(\sum_{i=1}^{A_0^n}\int_0^{\tilde{\eta}_i^n \wedge t} \varphi(u - \tilde{\tau}_i^n)\psi(u - \tilde{\tau}_i^n)h_{\tilde{\tau}_i^n}(u)\,du + \sum_{i=1}^{E_t^n}\int_0^{\eta_i \wedge (t - \tau_i^n)} \varphi(u)\psi(u)h(u)\,du\right) \\
&\quad = \int_0^t \left\langle \bar{\mathcal{A}}_s^n, \varphi\psi h\right\rangle ds \Rightarrow \int_0^t \left\langle \bar{\mathcal{A}}_s, \varphi\psi h\right\rangle ds
\end{aligned}$$

in $D$ as $n \to \infty$. The third equality follows from Proposition 2.2 with $y > T$ and $\varphi$ there set to $\varphi\psi h$ and the convergence follows from Theorem 5.2 and continuity of the integral mapping on $D$. Now by the martingale FCLT, $\hat{\mathcal{D}}^0 + \hat{\mathcal{D}}$ is a Gaussian martingale with quadratic variation

$$<< \hat{\mathcal{D}}^0 + \hat{\mathcal{D}} >>_t (\varphi, \psi) = \int_0^t \left\langle \bar{\mathcal{A}}_s h, \varphi\psi\right\rangle ds. \tag{83}$$

Then, (50) can be proven using (83) and the fact that $\langle \hat{\mathcal{D}}^0 + \hat{\mathcal{D}}, \varphi \rangle$ has independent increments for each $\varphi \in \Phi_{\mathcal{A}}$.

Repeating the argument for $\check{\mathcal{D}}^{0,n} + \check{\mathcal{D}}^n$ shows that

$$\check{\mathcal{D}}^{0,n} + \check{\mathcal{D}}^n \Rightarrow \hat{\mathcal{D}}^0 + \hat{\mathcal{D}} \quad \text{in} \quad D([0,\infty), \Phi')^2 \quad \text{as} \quad n \to \infty.$$

It remains to show that the joint convergence (49) holds. To show (49), it is sufficient to show that for each $\varphi \in \Phi_{\mathcal{A}}$ and $t \geq 0$,

$$\mathbb{E}\left|\left\langle \hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n_t, \varphi \right\rangle - \left\langle \check{\mathcal{D}}^{0,n} + \check{\mathcal{D}}^n_t, \varphi \right\rangle\right|^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{84}$$

This is because since $\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n \Rightarrow \hat{\mathcal{D}}^0 + \hat{\mathcal{D}}$ and $\check{\mathcal{D}}^{0,n} + \check{\mathcal{D}}^n \Rightarrow \check{\mathcal{D}}^0 + \hat{\mathcal{D}}$ individually in $D([0,\infty), \Phi')$ as $n \to \infty$, the sequences $(\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n)_{n\geq 1}$ and $(\check{\mathcal{D}}^{0,n} + \check{\mathcal{D}}^n)_{n\geq 0}$ are both tight, so that $((\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n) - (\check{\mathcal{D}}^{0,n} + \check{\mathcal{D}}^n))_{n\geq 0}$ is tight. Then, by Chebyshev's inequality (84) implies convergence of finite-dimensional distributions to 0 so that

$$(\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n) - (\check{\mathcal{D}}^{0,n} + \check{\mathcal{D}}^n) \Rightarrow 0 \quad \text{in} \quad D([0,\infty), \Phi') \quad \text{as} \quad n \to \infty. \tag{85}$$

Combining (85) with (82) and using Theorem 11.4.7 of [34] gives us (49).

We will now show that

$$\mathbb{E}\left|\left\langle \hat{\mathcal{D}}^n_t, \varphi \right\rangle - \left\langle \check{\mathcal{D}}^n_t, \varphi \right\rangle\right|^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{86}$$

Then, the limit

$$\mathbb{E}\left|\left\langle \hat{\mathcal{D}}^{0,n}_t, \varphi \right\rangle - \left\langle \check{\mathcal{D}}^{0,n}_t, \varphi \right\rangle\right|^2 \to 0 \quad \text{as} \quad n \to \infty \tag{87}$$

can be proven analogously and the previous two limits imply (84).

Just as in the proof of Proposition 4.1, for $t \geq 0$, we have that

$$D^n_t(y) = \sum_{i=1}^{E^n_t} D^{i,n}_t(y), \quad y \geq 0,$$

where for each $n, i \geq 1$

$$D^{i,n}_t(y) \equiv \mathbf{1}_{\{\eta_i \leq (t-\tau^n_i)^+ \wedge y\}} - \int_0^{\eta_i \wedge (t-\tau^n_i)\wedge y} h(u)\, du,$$

We then have

$$\begin{aligned}
\mathbb{E}\left|\left\langle \hat{\mathcal{D}}^n_t, \varphi \right\rangle - \left\langle \check{\mathcal{D}}^n_t, \varphi \right\rangle\right|^2 &= \frac{1}{n}\mathbb{E}\left(\sum_{i=1}^{E^n_t} \int_0^{t-\tau^n_i} \varphi(x)\, d\left(\mathbf{1}_{\{\eta_i \leq x\}} - \int_0^{\eta_i \wedge x} h(u)\, du\right)\right. \\
&\qquad \left. - \sum_{i=1}^{\lfloor n\bar{E}_t \rfloor} \int_0^{t-\check{\tau}^n_i} \varphi(x) d\left(\mathbf{1}_{\{\eta_i \leq x\}} - \int_0^{\eta_i \wedge x} h(u)\, du\right)\right)^2 \\
&= \frac{1}{n}\mathbb{E}\left(\sum_{i=\lceil n\bar{E}_t \wedge E^n_t \rceil}^{\lfloor n\bar{E}_t \vee E^n_t \rfloor} \left\langle \mathcal{D}^{n,i}_t, \varphi \right\rangle\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{i=1}^{\lfloor n\bar{E}_t \rfloor} \int_{t-\tau_i^n}^{t-\check{\tau}_i^n} \varphi(x) d\left(\mathbf{1}_{\{\eta_i \le x\}} - \int_0^{\eta_i \wedge x} h(u)\, du\right)\Bigg)^2 \\
&\le \frac{1}{n}\mathbb{E}\left(\sum_{i=\lceil n\bar{E}_t \wedge E_t^n \rceil}^{\lfloor n\bar{E}_t \vee E_t^n \rfloor} \left\langle \mathcal{D}_t^{n,i}, \varphi \right\rangle\right)^2 \\
&\quad + \frac{1}{n}\mathbb{E}\left(\sum_{i=1}^{\lfloor n\bar{E}_t \rfloor} \int_{t-\tau_i^n}^{t-\check{\tau}_i^n} \varphi(x)\, d\left(\mathbf{1}_{\{\eta_i \le x\}} - \int_0^{\eta_i \wedge x} h(u)\, du\right)\right)^2 \qquad (88)
\end{aligned}
$$

Focusing on the first term in (88), we have

$$
\begin{aligned}
\frac{1}{n}\mathbb{E}\left(\sum_{i=\lceil n\bar{E}_t \wedge E_t^n \rceil}^{\lfloor n\bar{E}_t \vee E_t^n \rfloor} \left\langle \mathcal{D}_t^{n,i}, \varphi \right\rangle\right)^2 &= \frac{1}{n}\mathbb{E}\left[<< \sum_{i=\lceil n\bar{E}_t \wedge E_t^n \rceil}^{\lfloor n\bar{E}_t \vee E_t^n \rfloor} \left\langle \mathcal{D}_t^{n,i}, \varphi \right\rangle >>\right] \\
&= \frac{1}{n}\mathbb{E}\left[\sum_{i=\lceil n\bar{E}_t \wedge E_t^n \rceil}^{\lfloor n\bar{E}_t \vee E_t^n \rfloor} << \left\langle \mathcal{D}_t^{n,i}, \varphi \right\rangle >>\right] \\
&= \frac{1}{n}\mathbb{E}\left[\sum_{i=\lceil n\bar{E}_t \wedge E_t^n \rceil}^{\lfloor n\bar{E}_t \vee E_t^n \rfloor} \int_0^{\eta_i \wedge (t-\tau_i)^+} \varphi(x)^2 h(x)\, dx\right] \\
&\le \frac{1}{n}\mathbb{E}\left[\sum_{i=\lceil n\bar{E}_t \wedge E_t^n \rceil}^{\lfloor n\bar{E}_t \vee E_t^n \rfloor} \int_0^{t} \varphi(x)^2 h(x)\, dx\right] \\
&= \int_0^t \varphi(x)^2 h(x)\, dx \mathbb{E}\left[|\bar{E}_t^n - \bar{E}_t|\right] \to 0,
\end{aligned}
$$

as $n \to \infty$. Focusing on the second term in (88), we have

$$
\begin{aligned}
&\frac{1}{n}\mathbb{E}\left(\sum_{i=1}^{\lfloor n\bar{E}_t \rfloor} \int_{t-\tau_i^n}^{t-\check{\tau}_i^n} \varphi(x)\, d\left(\mathbf{1}_{\{\eta_i \le x\}} - \int_0^{\eta_i \wedge x} h(u)\, du\right)\right)^2 \\
&\quad = \frac{1}{n}\mathbb{E}\left[<< \sum_{i=1}^{\lfloor n\bar{E}_t \rfloor} \int_{t-\tau_i^n}^{t-\check{\tau}_i^n} \varphi(x)\, d\left(\mathbf{1}_{\{\eta_i \le x\}} - \int_0^{\eta_i \wedge x} h(u)\, du\right) >>\right] \\
&\quad = \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{\lfloor n\bar{E}_t \rfloor} << \int_{t-\tau_i^n}^{t-\check{\tau}_i^n} \varphi(x)\, d\left(\mathbf{1}_{\{\eta_i \le x\}} - \int_0^{\eta_i \wedge x} h(u)\, du\right) >>\right] \\
&\quad = \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{\lfloor n\bar{E}_t \rfloor} \int_{\eta_i \wedge (t-\tau_i^n)}^{\eta_i \wedge (t-\check{\tau}_i^n)} \varphi(x)^2 h(x)\, dx\right] \\
&\quad \le \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{\lfloor n\bar{E}_t \rfloor} \|\varphi^2 h\|_t |\tau_i^n - \check{\tau}_i^n|\right] \\
&\quad \le \bar{E}_t \|\varphi^2 h\|_t \mathbb{E}\left[\sup_{0 \le i \le \lfloor n\bar{E}_t \rfloor} |\tau_i^n - \check{\tau}_i^n|\right].
\end{aligned}
$$

Now, noticing that

$$\tau_i^n = \inf\left\{s|\bar{E}_s^n \geq \frac{i}{n}\right\} = \left(\bar{E}^n\right)^{-1}\left(\frac{i}{n}\right),$$
$$\check{\tau}_i^n = \inf\left\{s|\bar{E}_s \geq \frac{i}{n}\right\} = \bar{E}^{-1}\left(\frac{i}{n}\right),$$

we have

$$\begin{aligned}\sup_{0\leq i\leq\lfloor n\bar{E}_t\rfloor}|\tau_i^n - \check{\tau}_i^n| &= \sup_{0\leq i\leq\lfloor n\bar{E}_t\rfloor}\left|\left(\bar{E}^n\right)^{-1}\left(\frac{i}{n}\right) - \bar{E}^{-1}\left(\frac{i}{n}\right)\right| \\ &\leq \sup_{0\leq\frac{i}{n}\leq\bar{E}_t}\left|\left(\bar{E}^n\right)^{-1}\left(\frac{i}{n}\right) - \bar{E}^{-1}\left(\frac{i}{n}\right)\right| \\ &\leq \sup_{0\leq u\leq\bar{E}_t}|\left(\bar{E}^n\right)^{-1}(u) - \bar{E}^{-1}(u)| \to 0,\end{aligned}$$

by Theorem 13.7.2 of [34]. Since $\{\sup_{0\leq i\leq\lfloor n\bar{E}_t\rfloor}|\tau_i^n - \check{\tau}_i^n|\}_{n\geq 1}$ is uniformly integrable since it is bounded by $t$, we have proven our result. □

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