Queues in Tandem with Customer Deadlines and Retrials

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Abstract

We study queues in tandem with customer deadlines and retrials. We first consider a 2-queue Markovian system with blocking at the second queue, analyze it, and derive its stability condition. We then study a non-Markovian setting and derive the stability condition for an approximating diffusion, showing its similarity to the former condition. In the Markovian setting, we use probability generating functions and matrix analytic techniques. In the diffusion setting, we consider expectations of the first hitting times of compact sets.

Keywords: Tandem queues; deadlines; retrials; blocking; diffusion approximation.

1 Introduction

In this paper we study a system of queues in tandem with customer deadlines and retrials. Networks of queues in tandem or more elaborate topologies have long been the subject of many articles in the literature (see e.g., the famous linear Jackson network [13, 14]). Tandem queues with finite buffers and blocking, causing retrials, have recently been studied and applied to Internet data traffic (see e.g., [6, 7]). There is also a vast literature on retrial queues (see e.g., [3, 9, 11, 30] and references therein). Queues with impatient customers and abandonments have also been studied, see for example [1, 31]. Many authors have also recently studied many-server queues with abandonments [2, 12, 19, 20].

We are motivated by a scenario in which data packets are being sent through several routers. Each packet has a deadline time by which it must arrive to its final destination. If the packet is still in the system when its deadline expires, then it is removed from the system and in its place a new packet is entered into the system. The new packet is assigned a new deadline time as well.

Consequently, we consider a system comprised of two queues in tandem, where the first queue has an unlimited buffer capacity and the second queue has a finite buffer capacity. Each arriving job (customer) carries with it a deadline time such that if its processing at the first queue does not start by the time its deadline expires, the job is fed back to the end of the queue. Upon completion of service at the first queue the job procceeds to the second queue. If it is blocked there, because the buffer is full, the job is fed back to the end of the first queue. The same applies if the job is admitted to the second queue but its waiting time there exceeds its deadline.

We first study a Markovian system with two queues in tandem and with exponential deadlines in each queue. We use both a Probability Generating Function (PGF) approach, as well as Matrix Geometric analysis. We obtain the condition for stability and give it a probabilistic interpretation. Based on this interpretation we consider a more general 2-queue system with general arrival and service processes and apply a diffusion approximation to obtain the stability condition of the system.

The structure of the paper is as follows. The Markovian model is presented in Section 2. Balance equations are derived in Section 3. PGFs are applied in Section 4, while the calculation of the so called 'boundary probabilities' is discussed in Section 5. A theorem on the roots of a polynomial related to the set of PGFs is presented, from which the stability condition of the system is derived. Marginal probabilities are discussed in Section 6. The Matrix Geometric method is used in Section 7 and a stability condition is obtained. It is shown that this condition is equivalent to the stability condition derived in Section 5. A probabilistic interpretation is discussed in Section 8. In Section 9 we consider the non-Markovian setting and present a diffusion approximation in Section 10. In Section 11 we present a stability result regarding our diffusion approximation and the subsequent proof may be found in Sections 12 and 13.

2 The Model

We consider a system comprised of two Markovian queues in tandem. The first queue (Q_1) is an unlimited-buffer $M/M/1$-type queue with homogeneous Poisson arrivals at rate λ. The service time for each individual customer at station 1 is exponentially distributed with mean $1/\mu_1$. Each customer in Q_1 has a deadline on her waiting time. If service does not start before the customer's deadline runs out, the customer reneges from her position in the waiting line and goes to the end of the queue, activating a new deadline, independent of the previous deadlines. We assume that the deadline time is a random variable, exponentially distributed, with mean $1/\gamma$. Upon completion of service in Q_1, a customer immediately moves to queue 2 (Q_2), which is a limited-buffer $\cdot/M/1/N$ queue with service rate μ_2. Here again, there is a deadline on a customer's queueing time. However, if the deadline, exponentially distributed with mean $1/\gamma$, expires before the customer starts service in Q_2, the customer moves all the way back to the end of the first queue, Q_1. Moreover, when a customer completes service in Q_1 and finds that there are N customers present in Q_2 ($N-1$ waiting and one being served), she is fedback all the way to the end of the first queue. The system is depicted in Figure 1.

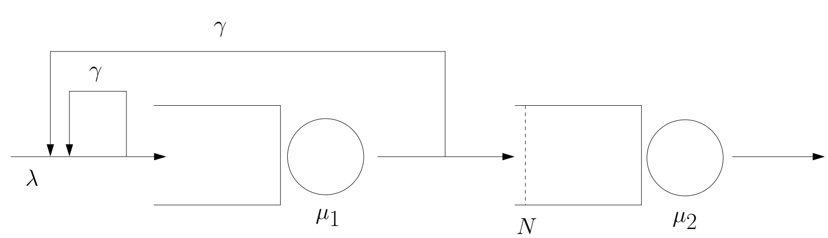

Figure 1: Two queues in tandem with deadlines, blocking and retrials.

Our aim is to analyze this "deadline-with-blocking and retrials" system, find its steady-state 2-

dimensional distribution function, reveal the system's stability condition, and give it a probabilistic interpretation.

3 Balance Equations

Consider the system in steady state. Let L_i denote the total number of customers (waiting and being served) in Q_i for $i = 1, 2$. Then (L_1, L_2) is a Markov process with transition-rate diagram as depicted in Figure 2. In this section, we derive the balance equations for (L_1, L_2) in stationarity.

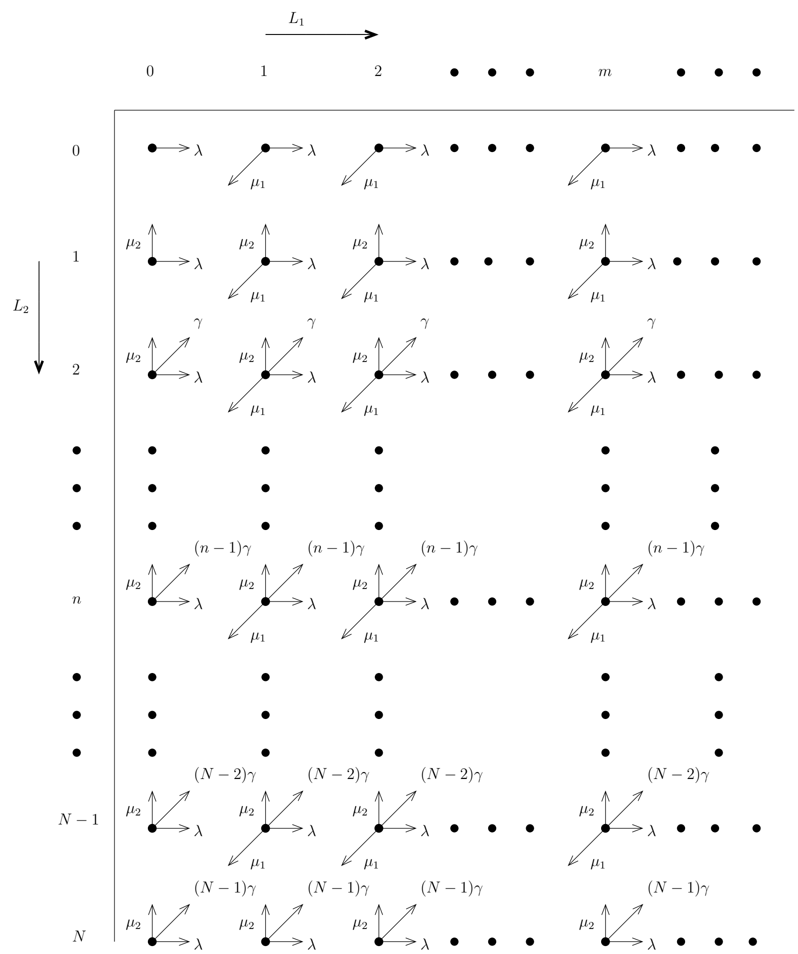

Figure 2: Transition-rate diagram for (L_1, L_2).

Define the steady-state probabilities of the system's states

$$P_{mn} = P(L_1 = m, L_2 = n) \quad m = 0, 1, 2, ...; n = 0, 1, ..., N.$$

Then, we can write the balance equations as follows:

(i) For $L_2 = 0$,

$$L_1 = 0: \qquad \lambda P_{00} = \mu_2 P_{01} \tag{1}$$
$$L_1 = m: \qquad (\lambda + \mu_1) P_{m0} = \mu_2 P_{m1} + \lambda P_{m-1,0} \quad (m = 1, 2, 3 \ldots), \tag{2}$$

(ii) For $L_2 = n, \quad (1 \le n \le N-1)$,

$$L_1 = 0: \qquad (\lambda + \mu_2 + (n-1)\gamma) P_{0n} = \mu_2 P_{0,n+1} + \mu_1 P_{1,n-1} \tag{3}$$
$$\begin{aligned} L_1 = m: \qquad & (\lambda + \mu_2 + (n-1)\gamma + \mu_1) P_{mn} \\ & = \mu_2 P_{m,n+1} + \mu_1 P_{m+1,n-1} + n\gamma P_{m-1,n+1} + \lambda P_{m-1,n}), \quad m \ge 1, \end{aligned} \tag{4}$$

(iii) For $L_2 = N$,

$$L_1 = 0: \qquad (\lambda + \mu_2 + (N-1)\gamma) P_{0N} = \mu_1 P_{1,N-1} \tag{5}$$
$$L_1 = m: \qquad (\lambda + \mu_2 + (N-1)\gamma) P_{mN} = \mu_1 P_{m+1,N-1} + + \lambda P_{m-1,N} \quad (m \ge 1). \tag{6}$$

4 Generating Functions

Define $N+1$ probability generating functions (PGFs) as follows:

$$G_n(z) \quad = \quad \sum_{m=0}^{\infty} P_{mn} z^m \qquad 0 \le n \le N.$$

Then, for $L_2 = 0$, multiplying each equation in (2) by z^m and summing all resulting equations, including equation (1), leads to

$$\lambda G_0(z) + \mu_1 (G_0(z) - P_{00}) \quad = \quad \mu_2 G_1(z) + \lambda z G_0(z).$$

Arranging terms we have

$$[\lambda(1-z) + \mu_1] G_0(z) - \mu_2 G_1(z) \quad = \quad \mu_1 P_{00}. \tag{7}$$

Similarly, for $L_2 = n$ $(1 \le n \le N-1)$, using equations (3) and (4) results in

$$\begin{aligned} & [\lambda + \mu_2 + (n-1)\gamma] G_n(z) + \mu_1 [G_n(z) - P_{0n}] \\ = \quad & \mu_2 G_{n+1}(z) + \frac{\mu_1}{z} [G_{n-1}(z) - P_{0,n-1}] + n\gamma z G_{n+1}(z) + \lambda z G_n(z). \end{aligned}$$

Arranging terms we get

$$\begin{aligned} & z[\lambda(1-z) + \mu_1 + \mu_2 + (n-1)\gamma] G_n(z) - \mu_1 G_{n-1}(z) - z[\mu_2 + n\gamma z] G_{n+1}(z) \\ = \quad & \mu_1 z P_{0,n} - \mu_1 P_{0,n-1} \quad (1 \le n \le N-1). \end{aligned} \tag{8}$$

Finally, for $L_2 = N$, considering equations (5) and (6), we obtain

$$[\lambda + \mu_2 + (N-1)\gamma] G_N(z) \quad = \quad \frac{\mu_1}{z} [G_{N-1}(z) - P_{0,N-1}] + \lambda z G_N(z),$$

or

$$z[\lambda(1-z) + \mu_2 + (N-1)\gamma] G_N(z) - \mu_1 G_{N-1}(z) \quad = \quad -\mu_1 P_{0,N-1}. \tag{9}$$

Equations (7), (8) and (9) define a linear set of equations in the unknown PGFs $G_0(z), G_1(z), \ldots, G_N(z)$, depending on the N unknown boundary probabilities $P_{00}, P_{01}, \ldots, P_{0,N-1}$.

5 Solving for the Boundary Probabilities $P_{00}, P_{01}, ..., P_{0,N-1}$,

Define the $(N+1)$-dimensional column vector $\underline{G}(z) = (G_0(z), G_1(z), ..., G_N(z))^T$, and the column vector

$$\underline{b}(z) \quad = \quad \mu_1[P_{00}, zP_{01} - P_{00}, zP_{02} - P_{01}, ..., zP_{0,N-1} - P_{0,N-2}, -P_{0,N-1}].$$

Then, equations (7), (8) and (9) can be written in a matrix form as

$$A(z)\underline{G}(z) \quad = \quad \underline{b}(z), \tag{10}$$

where the $(N+1)$-dimensional square matrix $A(z)$ is given by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & N-2 & N-1 & N \end{array} \\ \begin{pmatrix} [\lambda(1-z)+\mu_1] & -\beta_0(z) & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\mu_1 & \alpha_1(z) & -z\beta_1(z) & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\mu_1 & \alpha_2(z) & -z\beta_2(z) & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\mu_1 & \alpha_3(z) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\mu_1 & \alpha_{N-1}(z) & -z\beta_{N-1}(z) \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\mu_1 & z[\lambda(1-z)+\mu_2+(N-1)\gamma] \end{pmatrix} \end{array},$$

with

$$\alpha_n(z) \quad = \quad z[\lambda(1-z) + \mu_1 + \mu_2 + (n-1)\gamma], \quad 1 \leq n \leq N-1$$

and

$$\beta_n(z) \quad = \quad \mu_2 + n\gamma z, \quad n = 0, 1, 2, ..., N-1.$$

By Cramer's rule, each generating function can be calculated as

$$G_n(z) \quad = \quad \frac{|A_n(z)|}{|A(z)|}, \quad n = 0, 1, 2, ..., N, \tag{11}$$

where $A_n(z)$ is obtained from $A(z)$ by replacing it's nth column with the RHS vector $b(z)$ of equation (10), and $|A(z)|$ is the determinant of the matrix $A(z)$.

Thus, if we know the N unknown boundary probabilities appearing in the vector $\underline{b}(z)$, each generating function $G_n(z), n = 0, 1, 2, ..., N$, is fully determined by (11). Now, since $G_n(z)$ is analytic within $-1 \leq z \leq 1$, if there is a root $\hat{z}$ in that interval such that $|A(\hat{z})| = 0$, then the same root applies to $A_n(z)$ such that $|A_n(\hat{z})| = 0$ as well. Moreover, $|A_n(\hat{z})| = 0$ gives an equation involving the N unknown boundary probabilities, appearing in $\underline{b}(z)$. In order to solve for those N probabilities we claim the following.

Theorem 5.1. *For any $\lambda > 0$, $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\gamma \geq 0$, and $N = 2k_N$, $N = 2k_N + 1$, $k_N = 0, 1, 2, ..., N \geq 1$, $|A(z)|$ is a polynomial of degree $2N+1$, possessing a root of multiplicity k_N at $z_0 = 0, N - k_N - 1$ distinct roots in the open interval $(0, 1)$, a single root at $z = 1$ and N roots in the open interval $(1, \infty)$. If the condition*

$$\lambda \quad > \quad \mu_1 \left(1 + \sum_{k=1}^{N-1} \frac{\mu_1^k}{\prod_{j=1}^{k}(\mu_2 + j\gamma)}\right) \left(1 + \sum_{k=1}^{N} \frac{\mu_1^k}{\prod_{j=0}^{k-1}(\mu_2 + j\gamma)}\right)^{-1}$$

holds, then another root exists in the open interval $(0,1)$*. However, if*

$$\lambda \;<\; \mu_1 \left(1+\sum_{k=1}^{N-1}\frac{\mu_1^k}{\prod_{j=1}^{k}(\mu_2+j\gamma)}\right)\left(1+\sum_{k=1}^{N}\frac{\mu_1^k}{\prod_{j=0}^{k-1}(\mu_2+j\gamma)}\right)^{-1}, \tag{12}$$

the other root exists in the open interval $(1,\infty)$*, while if equality holds, the root is simply* $z=1$*.*

Proof. The line of reasoning of the tedious proof is similar to the proofs given in [18] and [23] and therefore will be omitted. □

Now, if (12) holds, we have k_N roots at $z_0=0$ and $N-k_N-1$ roots in $(0,1)$ for a total of $N-1$ roots in $[0,1)$. Each root yields an equation involving the N unknown probabilities. Together, with equation (1) we have N equations determining the probabilities in $\underline{b}(z)$, as needed.

It follows that condition (12) is the condition for stability of the system, namely for (L_1,L_2) to be positive recurrent. If condition (12) is reversed we have an extra root, yielding another equation in the unknown probabilities, leading to an un-solved set of equations, i.e. (L_1,L_2) is transient. If (12) is an equality, then (L_1,L_2) is null-recurrent. In the sections that follow, we provide a meaningful probabilistic (and intuitive) interpretation of the condition (12).

6 Marginal Probabilities

Define the marginal probabilities for L_2 :

$$P_{\bullet n} \;=\; P(L_2=n) \;=\; \sum_{m=0}^{\infty} P_{mn}, \quad n=0,1,2,...,N.$$

Then, by setting $z=1$ in equations (7), (8) and (9) we get, respectively

$$\mu_1 P_{\bullet 0}-\mu_2 P_{\bullet 1} \;=\; \mu_1 P_{00} \tag{13}$$

$$(\mu_1+\mu_2+(n-1)\gamma)P_{\bullet n}-\mu_1 P_{\bullet,n-1}-(\mu_2+n\gamma)P_{\bullet,n+1} \;=\; \mu_1(P_{0n}-P_{0,n-1}) \quad (1\le n\le N-1) \tag{14}$$

$$(\mu_2+(N-1)\gamma)P_{\bullet N}-\mu_1 P_{\bullet,N-1} \;=\; -\mu_1 P_{0,N-1}. \tag{15}$$

Adding equation (13) to equation (14) for $n=1$ yields

$$\mu_1(P_{\bullet 1}-P_{01}) \;=\; (\mu_2+\gamma)P_{\bullet 2}.$$

Continuing, we obtain

$$\mu_1(P_{\bullet n}-P_{0n}) \;=\; (\mu_2+n\gamma)P_{\bullet,n+1} \quad (0\le n\le N-1). \tag{16}$$

Now, once the boundary probabilities $(P_{00},P_{01},...,P_{0,N-1})$ are determined, the set of equations (16) together with the normalizing condition

$$\sum_{n=0}^{N} P_{\bullet n} \;=\; 1$$

provide a unique solution to the marginal probabilities $\{P_{\bullet n}\}$.

Remark. Equations (16) can also be obtained by considering horizontal 'cuts' in Figure 2 between "levels" $L_2 = n$ and $L_2 = n+1$, $(n = 0, 1, 2, ..., N-1)$.

Define now the marginal probabilities for $L_1 = m$:

$$p_{m\bullet} \;=\; \sum_{n=0}^{N} P_{mn}, \quad m = 0, 1, 2, ...$$

Then, by taking vertical 'cuts' between columns m and $m+1$ in Figure 2, we get

$$\lambda P_{m\bullet} + \gamma \sum_{n=1}^{N} (n-1) P_{mn} \;=\; \mu_1 P_{m+1,\bullet}, \quad m = 0, 1, 2, ... \tag{17}$$

Summing (17) over m yields

$$\lambda + \gamma \sum_{n=1}^{N} (n-1) P_{\bullet n} \;=\; \mu_1 (1 - P_{0\bullet}).$$

That is,

$$\lambda + \gamma E[\tilde{Q}_2] \;=\; \mu_1 (1 - P_{0\bullet}), \tag{18}$$

where $E[\tilde{Q}_2]$ denotes the mean queue size (customers waiting for service to start) of Q_2. In (18), the LHS gives the total effective arrival rate to the server at Q_1, composed of the original rate λ and the fedback customers that didn't meet their deadline in Q_2 (the fedback customers from Q_1 do not affect L_1 as they renege before their service starts). This effective arrival rate equals the effective service rate at Q_1, being $\mu_1(1 - P_{0\bullet})$. Finally, once $P_{0\bullet}$ is determined, $E[\tilde{Q}_2]$ is easily calculated, and $\gamma E[\tilde{Q}_2]$ gives the rate of customers not meeting their deadline in Q_2.

7 Matrix Geometric Method

Consider again the state space $\{m, n\}$ denoting m customers in Q_1 and n customers in Q_2, $m \geq 0$, $0 \leq n \leq N$. When $L_2 = n$ we say that the system is in level, or in phase, n. We construct a quasi birth-and-death (QBD) process with generator Q, satisfying

$$\underline{P} Q \;=\; \underline{0},$$

where

$$\underline{P} \;=\; (\underline{P}_0, \underline{P}_1, \underline{P}_2, ...)$$

with

$$\underline{P}_m \;=\; (P_{m0}, P_{m1}, ..., P_{mN}), \quad m = 0, 1, 2, ...$$

$\underline{P}_m$ denotes the $(N+1)$-dimensional vector of probabilities that the system is in state m and level n, $(n = 0, 1, 2, ..., N)$. The generator Q is given by

$$Q = \begin{pmatrix} A_1^0 & A_0^0 & 0 & \cdots & \cdots & \cdots & \cdots \\ A_2 & A_1 & A_0 & 0 & \cdots & \cdots & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{pmatrix},$$

where the $(N+1) \times (N+1)$ square matrices A_1^0, A_0^0, A_2, A_1 and A_0 are given by

$$A_1^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \mu_2 & -(\lambda+\mu_2) & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mu_2 & -(\lambda+\mu_2+\gamma) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \mu_2 & -(\lambda+\mu_2+(N-1)\gamma) \end{pmatrix},$$

$$A_0^0 = \begin{pmatrix} \lambda & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \lambda & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \gamma & \lambda & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & (N-2)\gamma & \lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & (N-1)\gamma & \lambda \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & \mu_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mu_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mu_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$
A_1 = \begin{pmatrix}
-(\lambda+\mu_1) & 0 & 0 & \cdots & 0 & 0 & 0 \\
\mu_2 & a_0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \mu_2 & a_1 & \cdots & 0 & 0 & 0 \\
0 & 0 & \mu_2 & a_2 & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \mu_2 & a_{N-2} & 0 \\
0 & 0 & 0 & \cdots & 0 & \mu_2 & -(\lambda+\mu_2+(N-1)\gamma)
\end{pmatrix},
$$

with $a_n = -(\lambda + \mu_1 + \mu_2 + n\gamma)$, $n = 0, 1, 2, ..., N-2$, and

$$
A_0 = \begin{pmatrix}
\lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \lambda & 0 & \cdots & 0 & 0 & 0 \\
0 & \gamma & \lambda & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \vdots & (N-2)\gamma & \lambda & 0 \\
0 & \gamma & \lambda & \cdots & 0 & (N-1)\gamma & \lambda
\end{pmatrix}.
$$

Consider now the matrix $A = A_0 + A_1 + A_2$, which is given by

$$
A = \begin{pmatrix}
-\mu_1 & \mu_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\mu_2 & -(\mu_1+\mu_2) & \mu_1 & 0 & \cdots & 0 & 0 & 0 \\
0 & (\mu_2+\gamma) & -(\mu_1+\mu_2+\gamma) & \mu_1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & (\mu_2+(N-2)\gamma) & -(\mu_1+\mu_2+(N-2)\gamma) & \mu_1 \\
0 & 0 & 0 & 0 & \cdots & 0 & (\mu_2+(N-1)\gamma) & -(\mu_2+(N-1)\gamma)
\end{pmatrix}.
$$

The matrix A represents the generator of a limited-buffer $M/M/1/N + M$-type queue with constant arrival rate μ_1, service rate μ_2 and individual reneging (abandonment) rate γ, such that if the number of customers present is $L = j$, the instantaneous departure rate from the system A is $\mu_2 + (j-1)\gamma$. This system is depicted in Figure 3.

This queue has a stationary distribution function $\underline{\pi} = (\pi_0, \pi_1, ..., \pi_N)$, where $\pi_k = P(L = k)$, $k = 0, 1, 2, ..., N$. We have

$$
\underline{\pi} A = \underline{0} \quad \text{and} \quad \underline{\pi} \cdot \underline{e} = 1, \quad \text{where} \quad \underline{e} = (1, 1, ..., 1)^T.
$$

We readily obtain that

$$
\pi_0 = \left[1 + \sum_{k=1}^{N} \frac{\mu_1^k}{\prod_{j=0}^{k-1}(\mu_2 + j\gamma)} \right]^{-1}
$$

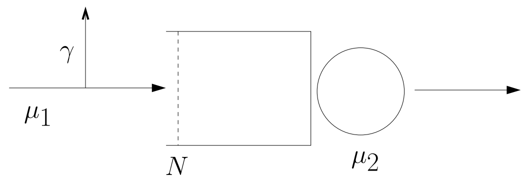

Figure 3: The system given by the generator A.

and

$$\pi_k = \frac{\mu_1^k}{\prod_{j=0}^{k-1}(\mu_2 + j\gamma)}\pi_0, \quad k = 1, 2, ..., N.$$

The stability for the QBD process with the generator Q is given by (see [22])

$$\underline{\pi}A_0\underline{e} < \underline{\pi}A_2\underline{e}. \tag{19}$$

Now, denoting by $L_q(A)$ the queue size in the system represented by the matrix A, we have

$$\underline{\pi}A_0\underline{e} = \lambda + \gamma\sum_{k=2}^{N}(k-1)\pi_k = \lambda + \gamma E[L_q(A)],$$

and

$$\underline{\pi}A_2\underline{e} = \mu_1(1 - \pi_N).$$

Thus, the stability condition (19) for the QBD process Q is

$$\lambda + \gamma E[L_q(A)] < \mu_1(1 - \pi_N). \tag{20}$$

We remark that it can be shown, following tedious algebraic calculations, that condition (20) is exactly condition (12) given in Theorem 5.1.

If $N = 1$, $E[L_q(A)] = 0$, $\pi_0 = \mu_2/(\mu_1 + \mu_2)$ and $\pi_1 = \mu_1/(\mu_1 + \mu_2)$. Thus, condition (20) translates into

$$\lambda < \frac{\mu_1\mu_2}{\mu_1 + \mu_2},$$

independent of γ. If $\mu_1 = \mu_2$, then condition (20) is further reduced to $\lambda < \mu_1/2$. Finally, the steady-state probability vectors $\underline{P}_m$ are given by

$$\begin{aligned} \underline{P}_0 A_1^0 + \underline{P}_1 A_2 &= \underline{0} \\ \underline{P}_0 A_0^0 + \underline{P}_1 A_1 + \underline{P}_2 A_2 &= \underline{0} \\ \underline{P}_{m-1} A_0 + \underline{P}_m A_1 + \underline{P}_{m+1} A_2 &= \underline{0}, \quad m = 2, 3, ... \end{aligned}$$

with

$$\underline{P}_m = \underline{P}_{m-1}R = \underline{P}_1 R^{m-1}, \quad (m \geq 2).$$

The matrix R is the minimal non-negative solution of the matrix equation

$$A_0 + RA_1 + R^2 A_2 = 0.$$

The normalizing condition is

$$\begin{aligned}\sum_{m=0}^{\infty} \underline{P}_m \underline{e} &= \underline{P}_0 \underline{e} + \sum_{m=1}^{\infty} \underline{P}_m \underline{e} \\ &= (\underline{P}_0 + \underline{P}_1 \sum_{m=0}^{\infty} R^m) \underline{e} \\ &= (\underline{P}_0 + \underline{P}_1 (I - R)^{-1}) \underline{e} \\ &= 1.\end{aligned}$$

Now, the mean number of customers in the first queue, Q_1, is given by

$$E[L_1] = \sum_{m=1}^{\infty} m \underline{P}_m \underline{e} = \underline{P}_1 \sum_{m=1}^{\infty} m R^{m-1} \underline{e} = \underline{P}_1 (I - R)^{-2} \cdot \underline{e}.$$

8 Probabilistic Interpretation

We now provide a probabilistic interpretation for the stability conditions (12) and (20). As indicated, these conditions can be shown to be equivalent to one another. We begin with an explanation of sufficiency. Consider the righthand side of (20). $\mu_1(1 - \pi_N)$ represents the maximum possible rate at which customers may be processed at station 1. Next, consider the lefthand side of (20). In particular, consider $\gamma E[L_q(A)]$. This represents the maximum possible rate at which customers may abandon from station 2 back to station 1. We say total maximum since in the generator A we assume that customers arrive to station 2 at the maximum possible rate μ_1. The maximum possible rate at which customers can arrive to station 1 is therefore $\lambda + \gamma E[L_q(A)]$ and so if this is less than $\mu_1(1 - \pi_N)$, the system will be stable.

Suppose on the other hand that the inequality (20) is reversed and that initially at time zero there are a large number of customers at station 1. While processing this initial set of customers, the departure rate from station 1 will be μ_1. Moreover, up until the final customer is processed, the departure process will behave as a Poisson process with rate μ_1. This will then cause station 2 to behave as the process described by the generator A and consequently customers will abandon back to station one at approximately rate $\gamma E[L_q(A)]$. Thus, if $\lambda + \gamma E[L_q(A)] > \mu_1(1 - \pi_N)$, station 1 will not be able to finish processing its initial set of customers before receiving an additional round of customers. In fact, the number of customers at station 1 will grow larger while it is processing its initial set customers and so the system will be unstable.

The type of reasoning used above is made rigorous in the following section where we provide an approximating stability condition for the non-Markovian setting.

9 Non-Markovian Setting

We now consider the system depicted in Figure 1 but with general interarrival and service time distributions and with unlimited buffer space at station two ($N = \infty$). We suppose that the external arrival process to the system $(A(t), t \geq 0)$ is a renewal process where

$$A(t) = \max \left\{ n : \sum_{i=1}^{n} u_i \leq \lambda t \right\},$$

where $\{u_i, i \geq 1\}$ is an i.i.d. sequence of random variables with mean one and variance a_1 and that $\lambda > 0$. We also assume that the number of customers served by station i, $i = 1, 2$, in its first t units of processing time is given by

$$S_i(t) = \max \left\{ n : \sum_{i=1}^{n} v_i \leq \mu_i t \right\},$$

where $\{v_i, i \geq 1\}$ is an i.i.d. sequence of random variables with mean one and variance b_i and $\mu_i > 0$. Finally, we assume that customers abandon station 2 according to an exponential distribution and return to station 1 at a Poissonian rate γ.

Unlike the exact stability condition for the Markovian case given by (12) and (20), the exact stability condition in this non-Markovian setting appears to be difficult to determine. However, the type of network described above falls into the class of networks referred to as generalized Jackson networks in [26]. In particular, Theorem 1 of [26] provides a heavy-traffic diffusion approximation to the queue length process for this network. Our approach therefore is to first use Theorem 1 of [26] in order determine the proper heavy-traffic diffusion approximation for this system and then to determine the stability condition for the diffusion approximation.

10 Diffusion Approximation

As in Section 3, let $(L_1, L_2) = ((L_1(t), L_2(t)), t \geq 0)$ be the two-dimensional process representing the number of customers present at stations 1 and 2, respectively, at time t. Our main result in this section is to show that (L_1, L_2) may be approximated by a two-dimensional diffusion approximation known as a reflected Ornstein-Uhlenbeck (RO-U) process. We therefore begin with the following definition of a RO-U process taken from [26]. Let $d \geq 1$ and $\theta \in \mathbb{R}^d$ and let $C, M, \Gamma \in \mathbb{R}^{d \times d}$, where C is a variance-covariance matrix.

Definition 10.1. *A d-dimensional RO-U process confined to $\mathbb{R}^d_+$ with parameters (θ, C, M, Γ) and initial position $Z(0) \in \mathbb{R}^d_+$ is defined to be the process $Z = (Z(t), t \geq 0) \in D([0, \infty), \mathbb{R}^d_+)$ satisfying in vector notation*

$$Z(t) = Z(0) + B(t) + \theta t - \int_0^t \Gamma Z(s) ds + MY(t), \quad t \geq 0,$$

where $B = (B(t), t \geq 0)$ is a d-dimensional Brownian motion with variance-covariance matrix C, $Z(t) \in \mathbb{R}^d_+$ for $t \geq 0$, and $Y = (Y(t), t \geq 0) \in D([0, \infty), \mathbb{R}^d_+)$ is such that for each $i = 1, ..., d$,

1. $Y_i(0) = 0$,
2. $Y_i = (Y_i(t), t \geq 0)$ *is non-decreasing,*
3. $$\int_0^\infty 1\{Z_i(s) > 0\} dY_i(s) = 0.$$

We remark that as is shown in Proposition 2 of [26], when the matrix M has positive diagonal elements, non-positive off-diagonal elements and a non-negative inverse, then there exists a unique, strong solution Z in Definition 10.1 above.

We now introduce the heavy-traffic regime of [26] in which (L_1, L_2) may be approximated by a two-dimensional RO-U. We consider a sequence of systems indexed by $n \geq 1$. For the nth system, we denote the system parameters by $\lambda^n, \mu_1^n, \mu_2^n$ and γ^n and we assume that as $n \to \infty$,

$$n^{-1}\lambda^n \to \lambda, \quad n^{-1}\mu_1^n \to \mu_1, \quad \text{and} \quad n^{-1}\mu_2^n \to \mu_2 \tag{21}$$

and

$$\sqrt{n}(n^{-1}\lambda^n - n^{-1}\mu_1^n) \to \theta_1, \quad \text{and} \quad \sqrt{n}(n^{-1}\mu_1^n - n^{-1}\mu_2^n) \to \theta_2, \tag{22}$$

for some $(\theta_1, \theta_2) \in \mathbb{R}^2$. We also assume that

$$\gamma^n = \gamma, \quad n \geq 1. \tag{23}$$

Now, for each $n \geq 1$, let (L_1^n, L_2^n) be the two-dimensional process representing the number of customers present at stations 1 and station 2, respectively, at time t in the nth system and let $(\tilde{L}_1^n, \tilde{L}_2^n) = (1/\sqrt{n})(L_1^n, L_2^n)$ be the diffusion scaled queue length process. The following is then a direct result of Theorem 1 of [26].

Proposition 10.2. *Under assumptions (21)-(23), if $(\tilde{L}_1^n(0), \tilde{L}_2^n(0)) \Rightarrow (\tilde{L}_1(0), \tilde{L}_2(0))$ as $n \to \infty$, then $(\tilde{L}_1^n, \tilde{L}_2^n) \Rightarrow \tilde{L} = (\tilde{L}_1, \tilde{L}_2)$ as $n \to \infty$, where $(\tilde{L}_1, \tilde{L}_2)$ is a two-dimensional RO-U process confined to $\mathbb{R}_+^2$ with parameters (θ, C, M, Γ) and initial position $(\tilde{L}_1(0), \tilde{L}_2(0))$, where $\theta = (\theta_1, \theta_2)$,*

$$C = \begin{bmatrix} \lambda a_1 + \mu_1 b_1 & -\mu_1 b_1 \\ -\mu_1 b_1 & \mu_1 b_1 + \mu_2 b_2 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \textit{and} \quad \Gamma = \begin{bmatrix} 0 & \gamma \\ 0 & -\gamma \end{bmatrix}.$$

We remark that the matrix M given above satisfies the criteria for uniqueness of the RO-U given by Proposition 2 of [26] and hence the above limit is unique in law. One may also consult Figure 4 below for a depiction of the state-space and directions of reflection for $\tilde{L}$.

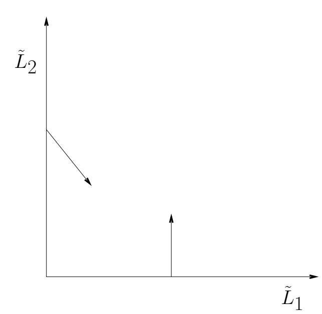

Figure 4: The state space and directions of reflection for $\tilde{L}$.

11 Stability Condition for RO-U Approximation

The following is our main result of this section. It provides a necessary and sufficient condition for the limit process $\tilde{L} = (\tilde{L}_1, \tilde{L}_2)$ in Proposition 10.2 to be positive recurrent. Let $\mathcal{N}(\mu, \sigma^2)$ be a normal random variable with mean μ and variance σ^2.

Theorem 11.1. *The limit process $\tilde{L}$ in Proposition 10.2 is positive recurrent if and only if*

$$\gamma \cdot E\left[\mathcal{N}\left(\frac{\theta_2}{\gamma}, \frac{\mu_1 b_1 + \mu_2 b_2}{2\gamma}\right) \mid \mathcal{N}\left(\frac{\theta_2}{\gamma}, \frac{\mu_1 b_1 + \mu_2 b_2}{2\gamma}\right) > 0\right] \quad < \quad -\theta_1. \tag{24}$$

We remark that the probabilistic interpretation of condition (24) is similar to that of (20) for the Markovian setting in Section 8. In particular, we make the following observations. The quantity on the lefthand side of (24), after multiplying by $\sqrt{n}$, approximately represents the maximum possible steady-state rate of abandonment from station 2 in the nth system, assuming that station 1 is never idle. That is, we approximately have that

$$\gamma \frac{E[L_2^n(\infty)]}{\sqrt{n}} \quad < \quad \gamma \cdot E\left[\mathcal{N}\left(\frac{\theta_2}{\gamma}, \frac{\mu_1 b_1 + \mu_2 b_2}{2\gamma}\right) \mid \mathcal{N}\left(\frac{\theta_2}{\gamma}, \frac{\mu_1 b_1 + \mu_2 b_2}{2\gamma}\right) > 0\right]. \tag{25}$$

On the other hand, by (22),

$$\theta_1 \quad \approx \quad \frac{\lambda^n}{\sqrt{n}} - \frac{\mu_1^n}{\sqrt{n}}. \tag{26}$$

Thus, using (25) and (26), one sees that (24) is approximately the same as the condition

$$\lambda^n + \gamma E[L_2^n(\infty)] \quad < \quad \mu_1^n,$$

which is similar to (20).

12 Proof of Necessity

In this section, we provide the proof of the necessity of (24) in order for $\tilde{L}$ to be positive recurrent. In Section 13, we provide the proof of sufficiency.

Proof of necessity of condition (24). Let $\hat{L}_2$ be a $(\theta_2, \mu_1 b_1 + \mu_2 b_2, 1, \gamma)$ 1-d RO-U confined to $\mathbb{R}_+$, with initial condition $\hat{L}_2(0)$. In other words, $\hat{L}_2$ is given by the unique, strong solution to

$$\hat{L}_2(t) \quad = \quad \hat{L}_2(0) + \tilde{X}_2(t) + \theta_2 t - \gamma \int_0^t \hat{L}_2(s) ds + \hat{Y}_2(t), \tag{27}$$

for $t \geq 0$, where $\tilde{X}_2 = (\tilde{X}_2(t), t \geq 0)$ is a Brownian motion with infinitesimal variance $\mu_1 b_1 + \mu_2 b_2$ and $\hat{Y}_2$ satisfies Conditions 1-3 of Definition 10.1. Since $\gamma > 0$, it is well known (see for instance [27]) that $\hat{L}_2(t) \Rightarrow \hat{L}_2(\infty)$ as $t \to \infty$, where $\hat{L}_2(\infty)$ is distributed as

$$\mathcal{N}\left(\frac{\theta_2}{\gamma}, \frac{\mu_1 b_1 + \mu_2 b_2}{2\gamma}\right) \mid \mathcal{N}\left(\frac{\theta_2}{\gamma}, \frac{\mu_1 b_1 + \mu_2 b_2}{2\gamma}\right) > 0.$$

That is, $\hat{L}_2(\infty)$ is distributed as a normal random variable conditioned to be positive.

Now note that one may view $\hat{L}_2$ as a regenerative process with regeneration point 0. In particular, let $\delta > 0$ and set

$$\vartheta(\delta) \;=\; \inf\{t > 0 : \hat{L}_2(t) = \delta\} \;\text{ and }\; \beta_0(\delta) \;=\; \inf\{t > \vartheta(\delta) : \hat{L}_2(t) = 0\}$$

and define $\vartheta(\delta) + \beta_0(\delta)$ to be a regeneration cycle started from 0. By Theorem 2 of [27], it follows that $\mathbb{E}[\vartheta(\delta)|\hat{L}_2(0) = 0] < \infty$. Next, let $\vartheta(0) = \inf\{t > 0 : \hat{L}_2(t) = 0\}$. By Proposition 4 of [27], the fact that $\hat{L}_2$ is a strong Markov process and the P-a.s. continuity of the sample-paths of $\hat{L}_2$, one has that

$$\mathbb{E}[\beta_0(\delta) - \vartheta_\delta|\hat{L}_2(0) = 0] \;=\; \mathbb{E}[\vartheta(0)|\hat{L}_2(0) = \delta] \;<\; \infty.$$

Hence, $\mathbb{E}[\beta_0(\delta) - \vartheta_\delta|\hat{L}_2(0) = 0] < \infty$ and so $\hat{L}_2$ has finite expected regeneration times. Next, note, again using the fact that $\hat{L}_2$ is a strong Markov process and the P-a.s. continuity of the sample-paths of $\hat{L}_2$, that

$$\begin{aligned} & E\left[\int_0^{\vartheta(\delta)+\beta_0(\delta)} \hat{L}_2(u)du|\hat{L}_2(0) = 0\right] \qquad (28) \\ = \;& E\left[\int_0^{\vartheta(\delta)} \hat{L}_2(u)du|\hat{L}_2(0) = 0\right] + E\left[\int_0^{\vartheta(0)} \hat{L}_2(u)du|\hat{L}_2(0) = \delta\right] \\ \leq \;& \delta E\left[\vartheta(\delta)|\hat{L}_2(0) = 0\right] + E\left[\int_0^{\vartheta(0)} \hat{L}_2(u)du|\hat{L}_2(0) = \delta\right]. \end{aligned}$$

By Theorem 2 of [27], $\delta E\left[\vartheta(\delta)|\hat{L}_2(0) = 0\right] < \infty$. Next, by the definition of $\hat{L}_2$ in (27), the fact that $\vartheta(0)$ is a stopping time and the optional sampling theorem [16], it follows that

$$0 \;=\; \delta + \theta_2 E\left[\vartheta(0)|\hat{L}_2(0) = \delta\right] - \gamma E\left[\int_0^{\vartheta(0)} \hat{L}_2(u)du|\hat{L}_2(0) = \delta\right] + E\left[\hat{Y}_2(\vartheta(0))|\hat{L}_2(0) = \delta\right].$$

By Condition 3 of Definition 10.1,

$$E\left[\hat{Y}_2(\vartheta(0))|\hat{L}_2(0) = \delta\right] \;=\; 0.$$

Moreover, by Proposition 4 of [27], $E\left[\vartheta(0)|\hat{L}_2(0) = \delta\right] < \infty$, so that

$$E\left[\int_0^{\vartheta(0)} \hat{L}_2(u)du|\hat{L}_2(0) = \delta\right] \;<\; \infty,$$

which, by (28), implies

$$E\left[\int_0^{\vartheta(\delta)+\beta_0(\delta)} \hat{L}_2(u)du|\hat{L}_2(0) = 0\right] \;<\; \infty.$$

It now follows by Theorem 3.1 of [4], that for each $z_2 \geq 0$,

$$P\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t \hat{L}_2(s)ds = E\left[\mathcal{N}\left(\frac{\theta_2}{\gamma}, \frac{\mu_1 b_1 + \mu_2 b_2}{2\gamma}\right) | \; \mathcal{N}\left(\frac{\theta_2}{\gamma}, \frac{\mu_1 b_1 + \mu_2 b_2}{2\gamma}\right) > 0\right] |\hat{L}_2(0) = z_2\right) \quad (29)$$

equals one.

Now assume that condition (24) does not hold (assume that the inequality is reversed) and let $\varepsilon, \delta > 0$ be such that $\varepsilon < 1$ and that there exists a $v_{\varepsilon,\delta}$ such that

$$P\left(\gamma \int_0^u \hat{L}_2(s)ds > (-\theta_1 + \delta)u \;\; \text{for} \;\; u \geq v_{\delta,\varepsilon} | \hat{L}_2(0) = z_2\right) \;\; > \;\; 1 - \varepsilon. \tag{30}$$

Such a triplet $(\varepsilon, \delta, v_{\delta,\varepsilon})$ may always be found by virture of (30) and the assumption that the inequality in (24) is reversed.

Next, let $\tilde{X}_1 = (\tilde{X}_1(t), t \geq 0)$ be a Brownian motion with infinitesimal variance $\lambda a_1 + \mu_1 b_1$ and note that by the strong law of large numbers for Brownian motion [15]

$$P\left(\lim_{t\to\infty} \frac{\tilde{X}_1(t)}{t} \;=\; 0\right) \;\;=\;\; 1.$$

Thus, there exists a $w_{\varepsilon,\delta}$ such that

$$P\left(\tilde{X}_1(u) > -\frac{\delta}{2}u \;\; \text{for} \;\; u \geq w_{\delta,\varepsilon}\right) \;\; > \;\; 1 - \varepsilon. \tag{31}$$

Finally, let $z_1 > 0$ be sufficiently large such that

$$P\left(\inf_{0 \leq u \leq v_{\varepsilon,\delta} \vee w_{\varepsilon,\delta}} \tilde{X}_1(u) + \theta_1 u > -z_1/2\right) \;\; > \;\; 1 - \varepsilon. \tag{32}$$

Now note that by Proposition 10.2 and by Definition 10.1, $(\tilde{L}_1, \tilde{L}_2)$ may be written as the unique, strong solution to the stochastic differential equation

$$\tilde{L}_1(t) \;\;=\;\; \tilde{L}_1(0) + \tilde{X}_1(t) + \theta_1 t + \gamma \int_0^t \tilde{L}_2(s)ds + \tilde{Y}_1(t) \tag{33}$$

$$\tilde{L}_2(t) \;\;=\;\; \tilde{L}_2(0) + \tilde{X}_2(t) + \theta_2 t - \gamma \int_0^t \tilde{L}_2(s)ds - \tilde{Y}_1(t) + \tilde{Y}_2(t), \tag{34}$$

for $t \geq 0$, subject to $(\tilde{L}_1(t), \tilde{L}_2(t)) \in \mathbb{R}_+^2$ and $(\tilde{Y}_1, \tilde{Y}_2)$ adhering to Conditions 1-3 of Definition 10.1. It will also be assumed that $(\tilde{L}_1, \tilde{L}_2)$ is defined on the same probability space as $\hat{L}_2$ in (27). In particular, the process $\tilde{X}_2$ is shared by both $(\tilde{L}_1, \tilde{L}_2)$ and $\hat{L}_2$.

Now let z_1, z_2 be as defined above and fix $(\tilde{L}_1(0), \tilde{L}_2(0)) = (z_1, z_2) \in \mathbb{R}_+^2$. Next, define the set

$$\begin{aligned} \Xi \;\;=\;\; & \left\{\gamma \int_0^u \hat{L}_2(s)ds > (-\theta_1 + \delta)u \;\; \text{for} \;\; u \geq v_{\delta,\varepsilon}\right\} \\ & \cap \left\{\tilde{X}_1(u) > -\frac{\delta}{2}u \;\; \text{for} \;\; u \geq w_{\delta,\varepsilon}\right\} \\ & \cap \left\{\inf_{0 \leq u \leq v_{\varepsilon,\delta} \vee w_{\varepsilon,\delta}} \tilde{X}_1(u) + \theta_1 u > -z_1/2\right\}. \end{aligned}$$

By (30),(31) and (32), it is clear that $\mathbb{P}(\Xi) > 1 - 3\varepsilon$. Moreover, we claim that on the set Ξ one has that

$$\tilde{L}_1(t) \;\;=\;\; z_1 + \tilde{X}_1(t) + \theta_1 t + \gamma \int_0^t \hat{L}_2(s)ds, \tag{35}$$

for $t \geq 0$. By the definition of Ξ, this implies that on Ξ one has that for $u \geq v_{\varepsilon,\delta} \vee w_{\varepsilon,\delta}$,

$$\tilde{L}_1(u) \quad > \quad z_1 + \frac{\delta}{2}u \;\rightarrow\; \infty \text{ as } u \rightarrow \infty.$$

Thus,

$$\mathbb{P}\left(\lim_{u\rightarrow\infty} \tilde{L}_1(u) = \infty | (\tilde{L}_1(0), \tilde{L}_2(0)) = (z_1, z_2)\right) > 1 - 3\varepsilon,$$

which implies that $(\tilde{L}_1, \tilde{L}_2)$ cannot be positive recurrent as desired.

In order to complete the proof, it now suffices to show that (35) holds. First note that by the definition of Ξ,

$$\Xi \quad \subseteq \quad \left\{\inf_{0\leq u\leq v_{\varepsilon,\delta}\vee w_{\varepsilon,\delta}} \tilde{X}_1(u) + \theta_1 u > -z_1/2\right\},$$

and so it follows by (33) and the positivity of γ that on Ξ one has $\tilde{L}_1(t) > 0$ for $0 \leq t \leq v_{\varepsilon,\delta} \vee w_{\varepsilon,\delta}$. By Condition 3 of Definition 10.1, this then implies that $\tilde{Y}_1(t) = 0$ for $0 \leq t \leq v_{\varepsilon,\delta} \vee w_{\varepsilon,\delta}$, which, by (27) and (33), implies that $\tilde{L}_2(t) = \hat{L}_2(t)$ for $0 \leq t \leq v_{\varepsilon,\delta} \vee w_{\varepsilon,\delta}$. Thus, (35) holds on Ξ for $0 \leq t \leq v_{\varepsilon,\delta} \vee w_{\varepsilon,\delta}$. Next, note that

$$\Xi \quad \subseteq \quad \left\{\gamma \int_0^u \hat{L}_2(s)ds > (-\theta_1 + \delta)u \;\text{ for }\; u \geq v_{\delta,\varepsilon}\right\} \cap \left\{\tilde{X}_1(u) > -\frac{\delta}{2}u \;\text{ for }\; u \geq w_{\delta,\varepsilon}\right\}.$$

Thus, on Ξ, one has that for $u > v_{\varepsilon,\delta} \vee w_{\varepsilon,\delta}$,

$$z_1 + \tilde{X}_1(u) + \theta_1 u + \gamma \int_0^u \hat{L}_2(s)ds \quad > \quad z_1 + \frac{\delta}{2}u \;>\; 0. \tag{36}$$

We now claim that (36) implies that $\tilde{Y}_1(u) = 0$ for $u > v_{\varepsilon,\delta} \vee w_{\varepsilon,\delta}$, which, using (27), (33) and (34) implies (35) on Ξ for $t \geq v_{\varepsilon,\delta} \vee w_{\varepsilon,\delta}$, thus completing the proof. Suppose that $\tilde{Y}_1(u) > 0$ for some $u > v_{\varepsilon,\delta} \vee w_{\varepsilon,\delta}$. By Condition 3 of Definition 10.1, this necessarily implies that $\tilde{L}_1(u) = 0$ for some $u > v_{\varepsilon,\delta} \vee w_{\varepsilon,\delta}$. However, since $\hat{L}_2(u) = \tilde{L}_2(u)$ up until to the first time that $\tilde{L}_1$ hits zero, by (33) this then implies that

$$z_1 + \tilde{X}_1(u) + \theta_1 u + \gamma \int_0^u \hat{L}_2(s)ds \quad = \quad 0 \tag{37}$$

for some $u > v_{\varepsilon,\delta} \vee w_{\varepsilon,\delta}$. However, (37) is in direct contradiction to (36), which completes the proof. □

13 Proof of Sufficiency

In this section, we provide the proof of the sufficiency of condition (24). For each $z = (z_1, z_2) \in \mathbb{R}^2_+$, let $\tilde{L}^z = (\tilde{L}^z_1, \tilde{L}^z_2)$ be the limit process in Proposition 10.2 started from z. That is, $\tilde{L}^z$ is the unique, strong solution to the stochastic differential equation

$$\tilde{L}^z_1(t) \quad = \quad z_1 + \tilde{X}_1(t) + \theta_1 t + \gamma \int_0^t \tilde{L}^z_2(s)ds + \tilde{Y}^z_1(t) \tag{38}$$

$$\tilde{L}^z_2(t) \quad = \quad z_2 + \tilde{X}_2(t) + \theta_2 t - \gamma \int_0^t \tilde{L}^z_2(s)ds - \tilde{Y}^z_1(t) + \tilde{Y}^z_2(t), \tag{39}$$

for $t \geq 0$, subject to $(\tilde{L}_1^z(t), \tilde{L}_2^z(t)) \in \mathbb{R}_+^2$ and $(\tilde{Y}_1^z, \tilde{Y}_2^z)$ adhering to Conditions 1-3 of Definition 10.1. All relevant quantities in (38)-(39) are superscripted by z in order to emphasize their dependence on the initial state z. Note also that it is not necessary to superscript $\tilde{X}$ since $\tilde{X}$ is always a Brownian motion started from the origin.

Next, for each $z = (z_1, z_2) \in \mathbb{R}_+^2$, define the norm $|z| = |z_1| + |z_2|$ and, for each $\varepsilon > 0$, let $\mathcal{B}_\varepsilon$ be the compact set

$$\mathcal{B}_\varepsilon \;=\; \{z \in \mathbb{R}_+^2 : |z| \leq \varepsilon\}.$$

Let us also define the stopping time

$$\tau_\varepsilon^z \;=\; \inf\{t \geq 0 : \tilde{L}^z(t) \in \mathcal{B}_\varepsilon\}.$$

Our main result in this section is the following.

Proposition 13.1. *If (24) holds, then, for each $z \in \mathbb{R}_+^2$ and $\varepsilon > 0$, $\mathbb{E}[\tau_\varepsilon^z] < \infty$. Moreover, for each compact set $\mathcal{C} \subset \mathbb{R}_+^2$,*

$$\sup_{z \in \mathcal{C}} \mathbb{E}[\tau_\varepsilon^z] \;<\; \infty.$$

Using a standard argument such as that provided by proof of Theorem 4.1 of [5], one may then show that Proposition 13.1 implies Theorem 11.1. The details are omitted in the present paper.

Our approach to proving Proposition 13.1 is to first study a coupled process which on a sample-path basis is P-a.s. larger than $\tilde{L}^z$. For each $z = (z_1, z_2) \in \mathbb{R}_+^2$, let $\hat{L}^z = ((\hat{L}_1^z(t), \hat{L}_2^z(t)), t \geq 0)$ be the solution to the stochastic differential equation

$$\hat{L}_1^z(t) \;=\; z_1 + \tilde{X}_1(t) + \theta_1 t + \gamma \int_0^t \hat{L}_2^z(s)ds + \hat{Y}_1^z(t) \tag{40}$$

$$\hat{L}_2^z(t) \;=\; z_2 + \tilde{X}_2(t) + \theta_2 t - \gamma \int_0^t \hat{L}_2^z(s)ds + \hat{Y}_2^z(t), \tag{41}$$

for $t \geq 0$, subject to $(\hat{L}_1^z(t), \hat{L}_2^z(t)) \in \mathbb{R}_+^2$ and $(\hat{Y}_1^z, \hat{Y}_2^z)$ adhering to Conditions 1-3 of Definition 10.1. Note that the process $\tilde{X}$ is the same for both $\tilde{L}^z$ and $\hat{L}^z$ implying that $\tilde{L}^z$ and $\hat{L}^z$ are defined on the same probability space.

The following is our first result and it shows that $\hat{L}^z$ dominates $\tilde{L}^z$ on a sample-path basis. We assume in the proofs that follow that $e = (t, t \geq 0)$ is the identity process.

Lemma 13.2. *For each $(y_1, y_2), (z_1, z_2) \in \mathbb{R}_+^2$ such that $y_1 \leq z_1$ and $y_2 \leq z_2$, $\tilde{L}_1^y(t) \leq \hat{L}_1^z(t)$ and $\tilde{L}_2^y(t) \leq \hat{L}_2^z(t)$ for $t \geq 0$, P-a.s.*

Proof. Let $(y_1, y_2), (z_1, z_2) \in \mathbb{R}_+^2$ such that $y_1 \leq z_1$ and $y_2 \leq z_2$. Then, by (39), (41) and the fact that $\tilde{Y}_1^y$ is a non-decreasing process, it follows by Proposition 2.2 of [25] that $\tilde{L}_2^y(t) \leq \hat{L}_2^z(t)$, P-a.s., for each $t \geq 0$.

Next, note that from (38) and Condition 1-3 of Definition 10.1, we may write

$$\tilde{L}_1^y \;=\; \Psi\left(y_1 + \tilde{X}_1 + \theta_1 e + \gamma \int_0^e \tilde{L}_2^y(s)ds\right), \tag{42}$$

where $\Psi : D([0,\infty),\mathbb{R}) \mapsto D([0,\infty),\mathbb{R})$ is the standard one-dimensional regulator map [28] and, from (40) and Conditions 1-3 of Definition 10.1, we may write

$$\hat{L}_1^z \quad = \quad \Psi\left(z_1 + \tilde{X}_1 + \theta_1 e + \gamma \int_0^e \hat{L}_2^z(s)ds\right). \tag{43}$$

From the first portion of the proof $\hat{L}_2^z(t) \geq \tilde{L}_2^y(t) \geq 0$, P-a.s., which implies that for each $t \geq 0$,

$$\gamma \int_0^t \hat{L}_2^z(s)ds \quad \geq \quad \gamma \int_0^t \tilde{L}_2^y(s)ds, \tag{44}$$

P-a.s.. Thus, using (42), (43), (44) and the fact that $y_1 \leq z_1$, it follows by standard monotonicity results for Ψ that for each $t \geq 0$, $\tilde{L}_1^y(t) \leq \hat{L}_1^z(t)$, P-a.s. This completes the proof. □

We now study the process $\hat{L}^z$. Recall that for a generic vector $z = (z_1, z_2) \in \mathbb{R}_+^2$, we define $|z| = |z_1| + |z_2|$. Also, in the next result we assume that $z = (z_1, z_1)$ where $z_1 \in \mathbb{R}_+$.

Lemma 13.3. *If (24) holds, then there exists a $\delta > 0$ such that for each $\varepsilon > 0$,*

$$\mathbb{P}\left(\frac{1}{|z|} \cdot |\hat{L}^z(|z|\delta)| > \varepsilon\right) \quad \to \quad 0 \quad \textit{as} \quad z_1 \to \infty. \tag{45}$$

Proof. In order to show (45), we show that there exists a $\delta > 0$ such that for each $\varepsilon > 0$,

$$\mathbb{P}\left(\frac{1}{|z|} \cdot |\hat{L}_1(|z|\delta)| > \varepsilon\right) \quad \to \quad 0 \quad \text{as} \quad z_1 \quad \to \quad \infty \tag{46}$$

and

$$\mathbb{P}\left(\frac{1}{|z|} \cdot |\hat{L}_2(|z|\delta)| > \varepsilon\right) \quad \to \quad 0 \quad \text{as} \quad z_1 \quad \to \quad \infty. \tag{47}$$

Let $\delta > 0$ and $z = (z_1, z_1) \in \mathbb{R}_+^2$. By (40), after some algebra may write

$$\frac{1}{|z|}\hat{L}_2^z(|z|\delta) \quad = \quad \frac{1}{2} + \frac{\tilde{X}_2(|z|\delta)}{|z|} + \theta_2\delta - \gamma \int_0^\delta \hat{L}_2^z(|z|s)ds + \frac{1}{|z|} \cdot \hat{Y}_2^z(|z|\delta).$$

Since $\hat{Y}_2^z$ satisfies Conditions 1-3 of Definition 10.1, it then follows that we may write

$$\frac{1}{|z|}\hat{L}_2^z(|z|\delta) \quad = \quad \Psi\left(\frac{1}{2} + \frac{\tilde{X}_2(|z|e)}{|z|} + \theta_2 e - \gamma \int_0^e \hat{L}_2^z(|z|s)ds\right)(\delta), \tag{48}$$

where $\Psi : D([0,\infty),\mathbb{R}) \mapsto D([0,\infty),\mathbb{R})$ is the standard one-dimensional regulator map [28], which is well known to be continuous.

Next, let $c = \theta_2/\gamma$ and set

$$\tau_{c,0}^z \quad = \quad \inf\{t \geq 0 : \hat{L}_2^z(t) = c \vee 0\}.$$

Since $\hat{L}_2^z(t) \geq 0$ for each $t \geq 0$, we then have that for each $t \geq 0$,

$$\begin{aligned} \gamma \int_0^t \hat{L}_2^z(|z|s)ds &= \frac{\gamma}{|z|} \int_0^{|z|t} \hat{L}_2^z(s)ds \\ &= \frac{\gamma}{|z|} \int_0^{\tau_{c,0}^z \wedge |z|t} \hat{L}_2^z(s)ds + \frac{\gamma}{|z|} \int_{\tau_{c,0}^z \wedge |z|t}^{|z|t} \hat{L}_2^z(s)ds \\ &\geq \frac{\gamma}{|z|} \int_{\tau_{c,0}^z \wedge |z|t}^{|z|t} \hat{L}_2^z(s)ds. \end{aligned} \tag{49}$$

Thus, using (48) and (49), it follows by standard monotonicity results for the map Ψ that

$$\frac{1}{|z|}\hat{L}_2^z(|z|\delta) \leq \Psi\left(\frac{1}{2} + \frac{\tilde{X}_2(|z|e)}{|z|} + \theta_2 e - \frac{\gamma}{|z|}\int_{\tau_{c,0}^z \wedge |z|e}^{|z|e} \hat{L}_2^z(s)ds\right)(\delta). \tag{50}$$

Next, let

$$\kappa = E\left[\mathcal{N}\left(\frac{\theta_2}{\gamma}, \frac{\mu_1 b_1 + \mu_2 b_2}{2\gamma}\right) \mid \mathcal{N}\left(\frac{\theta_2}{\gamma}, \frac{\mu_1 b_1 + \mu_2 b_2}{2\gamma}\right) > 0\right]$$

and note that $\kappa > \min\{0, \theta_2/\gamma\}$. We then claim that

$$\frac{1}{2} + \frac{\tilde{X}_2(|z|e)}{|z|} + \theta_2 e - \frac{\gamma}{|z|}\int_{\tau_{c,0}^z \wedge |z|e}^{|z|e} \hat{L}_2^z(s)ds \Rightarrow \frac{1}{2} + \theta_2 e - \gamma\kappa e. \tag{51}$$

Using Doob's martingale inequality [15], it is straightforward to show that

$$\frac{\tilde{X}_2(|z|e)}{|z|} \Rightarrow 0 \text{ as } z_1 \to \infty.$$

Therefore, in order to show (51), it remains to show that

$$\frac{1}{|z|}\int_{\tau_{c,0}^z \wedge |z|e}^{|z|e} \hat{L}_2(s)ds \Rightarrow \kappa e \text{ as } z_1 \to \infty. \tag{52}$$

However, note that since

$$\frac{1}{|z|}\int_{\tau_{c,0}^z \wedge |z|e}^{|z|e} \hat{L}_2(s)ds$$

is a non-decreasing process, in order to show (52) it suffices to show that for each $t \geq 0$,

$$\frac{1}{|z|}\int_{\tau_{c,0}^z \wedge |z|t}^{|z|t} \hat{L}_2(s)ds \Rightarrow \kappa t \text{ as } z_1 \to \infty. \tag{53}$$

We begin by evaluating $\mathbb{E}[\tau_{c,0}^z]$. Suppose first that $c \geq 0$ and that z_1 is sufficiently large so that $z_1 \geq c$. That is, $0 \leq c \leq z_1$. It then follows that $\tau_{c,0}^z$ is equal in law to the first hitting time of c by an unreflected O-U process started at the level z_1. Using the representation for the distribution of $\tau^z_{c,0}$ found in [24], one may write

$$\mathbb{E}[\tau^z_{c,0}] \quad = \quad \int_0^\infty t \cdot \frac{|\xi|}{\sqrt{2\pi}} \left(\frac{\gamma}{\sinh(\gamma t)} \right)^{3/2} \exp\left(-\frac{\gamma \xi^2 e^{-\gamma t}}{2 \sinh(\gamma t)} + \frac{\gamma t}{2} \right) dt, \tag{54}$$

where

$$\xi \quad = \quad \frac{z_1}{\mu_1 b_1 + \mu_2 b_2} - \frac{\theta_2}{\gamma(\mu_1 b_1 + \mu_2 b_2)}. \tag{55}$$

Using (54) and (55), it is then straightforward to show that

$$\frac{\mathbb{E}[\tau^z_{c,0}]}{|z|} \;\Rightarrow\; 0 \;\text{ as }\; z_1 \to \infty. \tag{56}$$

Now suppose that $c < 0$ and let $\tilde{\tau}^z_c$ be equal in law to the first hitting time of c by an unreflected O-U process started at the level z_1. Recall also that for $z_1 \geq c$, $\tau^z_{c,0}$ is equal in the law to the first hitting time of an unreflected O-U process to the level $c \vee 0$. Thus, $\tau^z_{c,0} \leq_{\text{st}} \tilde{\tau}^z_c$. Moreover, formula (54) continues to hold for $c < 0$ and so

$$\frac{\mathbb{E}[\tau^z_{c,0}]}{|z|} \quad \leq \quad \frac{\mathbb{E}[\tilde{\tau}^z_c]}{|z|} \;\Rightarrow\; 0 \;\text{ as }\; z_1 \to \infty. \tag{57}$$

Next note that using straightforward algebra, for each $t \geq 0$ we may write

$$\frac{1}{|z|t} \int_{\tau^z_{c,0} \wedge |z|t}^{|z|t} \hat{L}^z_2(s) ds \quad = \quad \left(1 - \left(\frac{\tau^z_{c,0}}{|z|t} \wedge 1 \right) \right) \cdot \frac{1}{|z|t - \tau^z_{c,0} \wedge |z|t} \int_{\tau^z_{c,0} \wedge |z|t}^{|z|t} \hat{L}^z_2(s) ds. \tag{58}$$

(56) and (57) and the fact that $\tau^z_{c,0} \geq 0$, imply that

$$\left(1 - \left(\frac{\tau^z_{c,0}}{|z|t} \wedge 1 \right) \right) \quad \Rightarrow \quad 1 \;\text{ as }\; z_1 \to \infty \tag{59}$$

and also that for each $b > 0$,

$$P(|z|t - \tau^z_{c,0} \wedge |z|t > b) \quad \to \quad 1 \;\text{ as }\; z_1 \to \infty. \tag{60}$$

Recall next by [27] that $\hat{L}^z_2(t) \Rightarrow \tilde{L}^z_2(\infty)$ as $t \to \infty$, where $\hat{L}^z_2(\infty)$ has the distribution

$$\mathcal{N}\left(\frac{\theta_2}{\gamma}, \frac{\mu_1 b_1 + \mu_2 b_2}{2\gamma} \right) \mid \mathcal{N}\left(\frac{\theta_2}{\gamma}, \frac{\mu_1 b_1 + \mu_2 b_2}{2\gamma} \right) > 0.$$

Thus, since $\hat{L}^z_2$ is a strong Markov process and $\tau^z_{c,0}$ is a stopping time for $\hat{L}^z_2$, (60) and Theorem 3.1 of [4] may be used to show that

$$\frac{1}{|z|t - \tau^z_{c,0} \wedge |z|t} \int_{\tau^z_{c,0} \wedge |z|t}^{|z|t} \hat{L}^z_2(s) ds \quad \Rightarrow \quad \kappa \;\text{ as }\; z_1 \to \infty. \tag{61}$$

(58), (59) and (61) now imply (52), which implies (51)

Now note that by (48), (51) and the continuous mapping theorem [8],

$$\Psi\left(\frac{1}{2}+\frac{\tilde{X}_2(|z|e)}{|z|}+\theta_2 e-\frac{\gamma}{|z|}\int_{\tau^z_{c,0}\wedge|z|e}^{|z|e}\hat{L}^z_2(s)ds\right)(\delta) \quad\Rightarrow\quad \Psi\left(\frac{1}{2}+\theta_2 e-\gamma\kappa e\right)(\delta),\quad \delta\geq 0.$$

However, since $\kappa > \min\{0,\theta_2/\gamma\}$, it follows that $(1/2)+\theta_2 e-\gamma\kappa e$ is a strictly decreasing, linear process and so we may select δ_2 large enough so that $\Psi\left(1/2+\theta_2 e-\gamma\kappa e\right)(\delta)=0$ for $\delta\geq\delta_2$. Using (50), this then implies (47).

We next proceed to show that (46) holds. Using (40), for each $\delta>0$ we may write

$$\frac{1}{|z|}\hat{L}^z_1(|z|\delta) \;=\; \Psi\left(\frac{1}{2}+\frac{\tilde{X}_1(|z|e)}{|z|}+\theta_1 e+\gamma\int_0^e\hat{L}^z_2(|z|s)ds\right)(\delta), \tag{62}$$

where $\Psi: D([0,\infty),\mathbb{R})\mapsto D([0,\infty),\mathbb{R})$ is the standard one-dimensional regulator map, which is a continuous map [28]. Next note that since $\hat{L}^z_2(t)\geq 0$ for each $t\geq 0$, it follows that for each $t\geq 0$ we may write

$$\begin{aligned}\gamma\int_0^t\hat{L}^z_2(|z|s)ds &= \frac{\gamma}{|z|}\int_0^{|z|t}\hat{L}^z_2(s)ds \qquad (63)\\ &= \frac{\gamma}{|z|}\int_0^{\tau^z_{c,0}\wedge|z|t}\hat{L}^z_2(s)ds+\frac{\gamma}{|z|}\int_{\tau^z_{c,0}\wedge|z|t}^{|z|t}\hat{L}^z_2(s)ds\\ &\leq \frac{\gamma}{|z|}\int_0^{\tau^z_{c,0}}\hat{L}^z_2(s)ds+\frac{\gamma}{|z|}\int_{\tau^z_{c,0}\wedge|z|t}^{|z|t}\hat{L}^z_2(s)ds.\end{aligned}$$

Thus, using (62) it follows by standard monotonicity results for the map Ψ that

$$\frac{1}{|z|}\hat{L}^z_1(|z|\delta) \;\leq\; \Psi\left(\frac{1}{2}+\frac{\tilde{X}_1(|z|e)}{|z|}+\theta_1 e+\frac{\gamma}{|z|}\int_0^{\tau^z_{c,0}}\hat{L}^z_2(s)ds+\frac{\gamma}{|z|}\int_{\tau^z_{c,0}\wedge|z|t}^{|z|t}\hat{L}^z_2(s)ds\right)(\delta). \tag{64}$$

We now show that

$$\frac{1}{2}+\frac{\tilde{X}_1(|z|e)}{|z|}+\theta_1 e+\frac{\gamma}{|z|}\int_0^{\tau^z_{c,0}}\hat{L}^z_2(s)ds+\frac{\gamma}{|z|}\int_{\tau^z_{c,0}\wedge|z|t}^{|z|t}\hat{L}^z_2(s)ds \;\Rightarrow\; \frac{3}{2}+\theta_1 e-\gamma\kappa e, \tag{65}$$

as $z_1\to\infty$.

Using Doob's martingale inequality [15], it is straightforward to show that

$$\frac{\tilde{X}_1(|z|e)}{|z|} \;\Rightarrow\; 0 \;\text{ as }\; z_1\to\infty.$$

Next, by (58),

$$\frac{1}{|z|t}\int_{\tau^z_{c,0}\wedge|z|t}^{|z|t}\hat{L}^z_2(s)ds \;\Rightarrow\; \kappa \;\text{ as }\; z_1\to\infty.$$

Thus, in order to show (65), it now suffices to show that

$$\frac{\gamma}{|z|}\int_0^{\tau^z_{c,0}}\hat{L}^z_2(s)ds \;\Rightarrow\; \frac{1}{2} \;\text{ as }\; z_1\to\infty. \tag{66}$$

Note that by Conditions 1-3 of Definition 10.1, $\hat{Y}_2^z(\tau_{c,0}^z) = 0$, P-a.s. Hence, by [15], we obtain that

$$\frac{\gamma}{|z|}\int_0^{\tau_{c,0}^z} \hat{L}_2^z(s)ds \;=\; \frac{1}{2} + \frac{\tilde{X}_2(\tau_{c,0}^z)}{|z|} + \theta_2 \frac{\tau_{c,0}^z}{|z|} - \frac{c \vee 0}{|z|}.$$

By (56) and (57),

$$\frac{\mathbb{E}[\tau_{c,0}^z]}{|z|} \;\Rightarrow\; 0 \;\;\text{as}\;\; z_1 \to \infty. \tag{67}$$

Using Theorem 3.3.28 of [15], (67) then implies

$$\frac{\tilde{X}_2(\tau_{c,0}^z)}{|z|} \;\Rightarrow\; 0 \;\;\text{as}\; z_1 \to \infty.$$

Moreover, since $\tau_{c,0}^z \geq 0$, (67) also implies that $\tau_{c,0}^z/|z| \Rightarrow 0$ as $z_1 \to \infty$. Finally, clearly $(c \vee 0)/|z| \to 0$ as $z_1 \to \infty$. It now follows that (66) holds, which implies (65).

It now follows by (65) and the continuous mapping theorem [8] that for each $\delta > 0$,

$$\begin{aligned} &\Psi\left(\frac{1}{2} + \frac{\tilde{X}_1(|z|e)}{|z|} + \theta_1 e + \frac{\gamma}{|z|}\int_0^{\tau_{c,0}^z} \hat{L}_2^z(s)ds + \frac{\gamma}{|z|}\int_{\tau_{c,0}^z \wedge |z|t}^{|z|t} \hat{L}_2^z(s)ds\right)(\delta) \\ \Rightarrow\;\; &\Psi\left(\frac{3}{2} + \theta_1 e - \gamma\kappa e\right)(\delta), \end{aligned}$$

as $z_1 \to \infty$. However, by assumption (24), $(3/2) + \theta_1 e - \gamma\kappa e$ is a decreasing process and so there exists a $\delta_1 > 0$ such that

$$\Psi\left(\frac{3}{2} + \theta_1 e - \gamma\kappa e\right)(\delta) \;=\; 0,$$

for all $\delta \geq \delta_1$. By (64), this then implies (46), which completes the proof.

□

We now strengthen the result of Lemma 13.3 by upgrading the convergence in (45) to convergence in expectation. We assume in the following that $z = (z_1, z_1)$ where $z_1 \in \mathbb{R}$.

Lemma 13.4. *If (24) holds, then there exists a $\delta > 0$ such that*

$$\mathbb{E}\left[\frac{1}{|z|} \cdot |\hat{L}^z(|z|\delta)|\right] \;\to\; 0 \;\;\textit{as}\;\; z_1 \to \infty. \tag{68}$$

Proof. By Lemma 13.3 and (3.18) of [8], it suffices to show the uniform integrability condition

$$\sup_{z_1 > 0} \mathbb{E}\left[\left(\frac{1}{|z|} \cdot |\hat{L}^z(|z|\delta)|\right)^2\right] \;<\; \infty.$$

However, note that since

$$\left(\frac{1}{|z|} \cdot |\hat{L}^z(|z|\delta)|\right)^2 \;=\; \left(\frac{1}{|z|}\right)^2 (|\hat{L}_1^z(|z|\delta)| + |\hat{L}_2(|z|\delta)|)^2 \;\leq\; 4 \cdot \left(\frac{1}{|z|}\right)^2 (|\hat{L}_1^z(|z|\delta)|^2 + |\tilde{L}_2^z(|z|\delta)|^2),$$

it suffices to show both

$$\sup_{z_1>0} \mathbb{E}\left[\left(\frac{1}{|z|}|\hat{L}_1^z(|z|\delta)|\right)^2\right] \quad < \quad \infty \tag{69}$$

and

$$\sup_{z_1>0} \mathbb{E}\left[\left(\frac{1}{|z|}|\hat{L}_2^z(|z|\delta)|\right)^2\right] \quad < \quad \infty. \tag{70}$$

We begin with (70). As in the proof of Lemma 13.3, define the hitting time of the origin by $\hat{L}_2^z$,

$$\tau_0^z \quad = \quad \inf\{t \geq 0 : \hat{L}_2^z(t) = 0\}.$$

Next, recall that $\hat{L}_2^z$ possess the strong Markov property and note that τ_0^z is a stopping time for $\hat{L}_2^z$ and so

$$\begin{aligned} & \mathbb{E}[(\hat{L}_2^z(|z|\delta))^2] \\ = \; & \mathbb{E}[(\hat{L}_2^z(|z|\delta))^2 1\{\tau_0^z \leq |z|\delta\}] + \mathbb{E}[(\hat{L}_2^z(|z|\delta))^2 1\{\tau_0^z > |z|\delta\}] \\ = \; & \int_0^{|z|\delta} \mathbb{E}[(\hat{L}_2^0(|z|\delta - s))^2] P(\tau_0^z \in ds) + \mathbb{E}[(\hat{L}_2^z(|z|\delta))^2 1\{\tau_0^z > |z|\delta\}]. \end{aligned} \tag{71}$$

We treat each term on the righthand side of the final equality above separately. We begin with the integral term. As in the proof of Lemma 13.3, let $c = \gamma/\theta_2$ and set

$$\tau_{c,0}^z \quad = \quad \inf\{t \geq 0 : \hat{L}_2^z(t) = c \vee 0\}$$

If $c < 0$, then $\tau_{c,0}^z = \tau_0^z$ and so, as in the proof of Lemma 13.3, it follows that

$$\frac{\mathbb{E}[\tau_0^z]}{|z|} \quad = \quad \frac{\mathbb{E}[\tau_{c,z}^0]}{|z|} \to 0 \text{ as } \; z_1 \to \infty. \tag{72}$$

On the other hand, suppose that $c > 0$. By [29], $\mathbb{E}[\tau_0^c] < \infty$. Thus, since $\hat{L}_2^z$ is a strong Markov process and since $\tau_{c,0}^z$ is a stopping time, it follows using (56) in the proof of Lemma 13.3 that for $z_1 \geq c$,

$$\frac{\mathbb{E}[\tau_0^z]}{|z|} \quad = \quad \frac{\mathbb{E}[\tau_{c,0}^z]}{|z|} + \frac{\mathbb{E}[\tau_0^c]}{|z|} \to 0 \;\text{ as }\; z_1 \to \infty. \tag{73}$$

In summary, by (72) and (73),

$$\frac{\mathbb{E}[\tau_0^z]}{|z|} \quad \to \quad 0 \text{ as } \; z_1 \to \infty, \tag{74}$$

which, since $\tau_z^0 \geq 0$, also implies that

$$\frac{\tau_0^z}{|z|} \quad \Rightarrow \quad 0 \text{ as } \; z_1 \to \infty. \tag{75}$$

Now consider the integral term

$$\int_0^{|z|\delta} \mathbb{E}[(\hat{L}_2^0(|z|\delta - s))^2] P(\tau_0^z \in ds).$$

By Proposition 3.3 of [25],

$$\mathbb{E}[(\hat{L}_2^0(t))^2] \quad \to \quad \mathbb{E}[(\hat{L}_2(\infty))^2] \; < \; \infty \quad \text{as} \quad t \to \infty. \tag{76}$$

Hence, using (75), it is straightforward to show that

$$\int_0^{|z|\delta} \mathbb{E}[(\hat{L}_2^0(|z|\delta - s))^2] \mathbb{P}(\tau_0^z \in ds) \quad \to \quad \mathbb{E}[(\hat{L}_2(\infty))^2] \quad \text{as} \quad z_1 \to \infty. \tag{77}$$

Next, consider $\mathbb{E}[(\hat{L}_2^z(|z|\delta))^2 1\{\tau_0 > |z|\delta\}]$. First note the equality

$$\mathbb{E}[(\hat{L}_2^z(|z|\delta))^2 1\{\tau_0^z > |z|\delta\}] \quad = \quad \mathbb{E}[(\check{L}_2^z(|z|\delta))^2 1\{\check{\tau}_0^z > |z|\delta\}], \tag{78}$$

where $\check{L}_2^z$ is an unreflected O-U process started from z_1 and $\check{\tau}_0^z$ is its first hitting time of zero. That is, $\check{L}_2$ is the unique, strong solution to

$$\check{L}_2^z(t) \quad = \quad z_1 + \tilde{X}_2(t) + \theta_2 t - \gamma \int_0^t \check{L}_2^z(s) ds, \quad t \geq 0, \tag{79}$$

and

$$\check{\tau}_0^z \quad = \quad \inf\{t \geq 0 : \check{L}_2^z(t) = 0\}. \tag{80}$$

Using the explicit form of the solution to (79) (see, for instance, [15]), it is straightforward to show that for each $\delta > 0$,

$$\mathbb{E}[(\check{L}_2^z(\delta|z|))^2] \quad \to \quad \mathbb{E}[(\check{L}_2(\infty))^2] \; < \; \infty \quad \text{as} \quad z_1 \to \infty. \tag{81}$$

Next, note that since $\check{\tau}_0^z$ is equal in distribution to τ_0^z, it follows using (74) that

$$\frac{\mathbb{E}[\check{\tau}_0^z]}{|z|} \; = \; \frac{\mathbb{E}[\tau_0^z]}{|z|} \quad \to \quad 0 \quad \text{as} \quad z_1 \to \infty,$$

which, since $\check{\tau}_0^z \geq 0$, P-a.s., implies that

$$\frac{\check{\tau}_0^z}{|z|} \quad \Rightarrow \quad 0 \quad \text{as} \quad z_1 \to \infty. \tag{82}$$

Now note that we may write

$$\mathbb{E}[(\check{L}_2^z(\delta|z|))^2] \quad = \quad \mathbb{E}[(\check{L}_2^z(\delta|z|))^2 1\{\check{\tau}_0^z \leq \delta|z|\}] + \mathbb{E}[(\check{L}_2^z(\delta|z|))^2 1\{\check{\tau}_0^z > \delta|z|\}]. \tag{83}$$

However, by (81),

$$\mathbb{E}[(\check{L}_2^z(\delta|z|))^2] \quad \to \quad \mathbb{E}[(\check{L}_2(\infty))^2] \; < \; \infty \quad \text{as} \quad z_1 \to \infty \tag{84}$$

and, by (81) and (82),

$$\mathbb{E}[(\check{L}_2^z(\delta|z|))^2 1\{\check{\tau}_0^z \leq \delta|z|\}] \;\; \rightarrow \;\; \mathbb{E}[(\check{L}_2(\infty))^2] \;<\; \infty \;\; \text{as} \;\; z_1 \rightarrow \infty. \tag{85}$$

Thus, by (83),

$$\mathbb{E}[(\check{L}_2^z(\delta z))^2 1\{\check{\tau}_0 > \delta z\}] \;\; \rightarrow \;\; 0 \;\; \text{as} \; z_1 \rightarrow \infty. \tag{86}$$

Using (71), (77) (78) and (86), it now follows that (70) holds.

Next, consider (69). First note that by the basic inequality

$$(x_1 + ... + x_I)^2 \;\; \leq \;\; 2^I (x_1^2 + ... + x_I^2)$$

and (40), it follows that for each $\delta > 0$,

$$\begin{aligned} & \frac{1}{2^5} \cdot \left(\frac{1}{|z|} \cdot \hat{L}_1(|z|\delta) \right)^2 \qquad (87) \\ \leq \;\; & \frac{1}{4} + \left(\frac{\tilde{X}_1(|z|\delta)}{|z|} \right)^2 + \theta_1^2 \delta^2 + \gamma^2 \left(\frac{1}{|z|} \cdot \int_0^{|z|\delta} \hat{L}_2^z(s) ds \right)^2 + \left(\frac{\hat{Y}_1^z(|z|\delta)}{|z|} \right)^2. \end{aligned}$$

Now note that

$$\mathbb{E}\left[\left(\frac{\tilde{X}_1(|z|\delta)}{|z|} \right)^2 \right] \;\; = \;\; \frac{\delta^2}{|z|}(\lambda a_1 + \mu_1 b_1) \;\rightarrow\; 0 \;\; \text{as} \;\; z_1 \rightarrow \infty.$$

Next, using (40) and the explicit solution to the one-sided regulator map Ψ, one has that

$$\begin{aligned} \frac{\tilde{Y}_1^z(|z|\delta)}{|z|} \;\; &= \;\; -\sup_{0 \leq s \leq |z|\delta} \min\left\{ 0, \left(\frac{1}{2} + \frac{\tilde{X}_1(s)}{|z|} + \theta_1 \frac{s}{|z|} + \frac{\gamma}{|z|} \int_0^s \hat{L}_2^z(s) ds \right) \right\} \qquad (88) \\ &\leq \;\; -\sup_{0 \leq s \leq |z|\delta} \min\left\{ 0, \left(\frac{1}{2} + \frac{\tilde{X}_1(s)}{|z|} + \theta \frac{s}{|z|} \right) \right\} \\ &\leq \;\; \frac{1}{2} + \theta\delta + \sup_{0 \leq s \leq |z|\delta} \left| \frac{\tilde{X}_1(s)}{|z|} \right|. \end{aligned}$$

Using the the expression for the distribution of the running maximum of Brownian motion [15], it is straightforward to show that

$$\sup_{z_1 > 0} E\left[\sup_{0 \leq s \leq |z|\delta} \left| \frac{\tilde{X}_1(s)}{|z|} \right|^2 \right] \;\; < \;\; \infty,$$

and so from (88) it follows that

$$\sup_{z_1 > 0} \mathbb{E}\left[\left(\frac{\tilde{Y}_1^z(|z|\delta)}{|z|} \right)^2 \right] \;\; < \;\; \infty.$$

Hence, by (87), in order to complete the proof it suffices to show that

$$\sup_{z_1>0} \mathbb{E}\left[\left(\frac{1}{|z|}\cdot\int_0^{|z|\delta} \hat{L}_2^z(s)ds\right)^2\right] \;\; < \;\; \infty. \tag{89}$$

First note that

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{|z|}\cdot\int_0^{|z|\delta} \hat{L}_2^z(s)ds\right)^2\right] &= \mathbb{E}\left[\frac{1}{|z|^2}\cdot\int_0^{|z|\delta}\int_0^{|z|\delta} \hat{L}_2^z(s)\hat{L}_2^z(u)dsdu\right] \\
&= \frac{1}{|z|^2}\cdot\int_0^{|z|\delta}\int_0^{|z|\delta} \mathbb{E}\left[\hat{L}_2^z(s)\hat{L}_2^z(u)\right]dsdu.
\end{aligned} \tag{90}$$

Next, by the Cauchy-Schwartz inequality [17],

$$\mathbb{E}\left[\hat{L}_2^z(s)\hat{L}_2^z(u)\right] \;\; \leq \;\; \sqrt{\mathbb{E}\left[(\hat{L}_2^z(s))^2\right]}\cdot\sqrt{\mathbb{E}\left[(\hat{L}_2^z(u))^2\right]}.$$

Substituting into (90), one then obtains

$$\mathbb{E}\left[\left(\frac{1}{|z|}\cdot\int_0^{|z|\delta} \hat{L}_2^z(s)ds\right)^2\right] \;\; \leq \;\; \left(\frac{1}{|z|}\cdot\int_0^{|z|\delta}\sqrt{\mathbb{E}\left[(\hat{L}_2^z(s))^2\right]}ds\right)^2. \tag{91}$$

Now note that

$$\begin{aligned}
\mathbb{E}\left[(\hat{L}_2^z(s))^2\right] &= \mathbb{E}\left[(\hat{L}_2^z(s))^2 1\{\tau_0^z \leq s\}\right] + \mathbb{E}\left[(\hat{L}_2^z(s))^2 1\{\tau_0^z > s\}\right] \\
&= \mathbb{E}\left[(\hat{L}_2^z(s))^2 1\{\tau_0^z \leq s\}\right] + \mathbb{E}\left[(\check{L}_2^z(s))^2 1\{\check{\tau}_0^z > s\}\right] \\
&\leq \mathbb{E}\left[(\hat{L}_2^z(s))^2 1\{\tau_0^z \leq s\}\right] + \mathbb{E}\left[(\check{L}_2^z(s))^2\right],
\end{aligned}$$

where $\check{L}_2^z$ is the unreflected O-U process given by (79) and $\check{\tau}_0^z = \inf\{t \geq 0 : \check{L}_2^z(t) = 0\}$ as in (80). Thus,

$$\sqrt{\mathbb{E}\left[(\hat{L}_2^z(s))^2\right]} \;\; \leq \;\; \sqrt{\mathbb{E}\left[(\hat{L}_2^z(s))^2 1\{\tau_0^z \leq s\}\right]} + \sqrt{\mathbb{E}\left[(\check{L}_2^z(s))^2\right]}$$

and so by (91),

$$\begin{aligned}
\sqrt{\mathbb{E}\left[\left(\frac{1}{|z|}\cdot\int_0^{|z|\delta} \hat{L}_2^z(s)ds\right)^2\right]} \;\; &\leq \;\; \frac{1}{|z|}\cdot\int_0^{|z|\delta}\sqrt{\mathbb{E}\left[(\hat{L}_2^z(s))^2 1\{\tau_0^z \leq s\}\right]}ds \\
&\quad + \frac{1}{|z|}\cdot\int_0^{|z|\delta}\sqrt{\mathbb{E}\left[(\check{L}_2^z(s))^2\right]}ds.
\end{aligned} \tag{92}$$

Now consider each of the terms on the righthand side of (92). Conditioning on τ_0^z as in (71) and using (76) it follows that

$$\sup_{z_1, s\geq 0}\sqrt{\mathbb{E}\left[(\hat{L}_2^z(s))^2 1\{\tau_0^z \leq s\}\right]} \;\; < \;\; \infty,$$

from which one obtains

$$\sup_{z_1 \geq 0} \frac{1}{|z|} \cdot \int_0^{|z|\delta} \sqrt{\mathbb{E}\left[(\hat{L}_2^z(s)) 1\{\tau_0^z \leq s\}\right]} ds \quad < \quad \infty. \tag{93}$$

Also, using the explicit solution for $\hat{L}_2^z$ (see, for instance, [15]) it is straightforward to show that

$$\sqrt{\mathbb{E}\left[(\check{L}_2^z(s))^2\right]} \quad \leq \quad \kappa_1 z_1 e^{-\kappa_2 s} + \kappa_3,$$

where

$$\kappa_1 = 8, \quad \kappa_2 = 2\gamma \quad \text{and} \quad \kappa_3 = 8\left(\frac{\theta_2^2}{\gamma^2} + \frac{(\mu_1 b_1 + \mu_2 b_2)^2}{2\gamma}\right).$$

Hence, since κ_1, κ_2 and κ_3 are independent of z, it follows that

$$\begin{aligned} \sup_{z_1 > 0} \frac{1}{|z|} \cdot \int_0^{|z|\delta} \sqrt{\mathbb{E}\left[(\check{L}_2^z(s))^2\right]} ds \quad &\leq \quad \sup_{z_1 > 0} \frac{1}{|z|} \cdot \int_0^{|z|\delta} (\kappa_1 z_1 e^{-\kappa_2 s} + \kappa_3) ds \\ &= \quad \kappa_1/\kappa_2 + \delta\kappa_3 \\ &< \quad \infty. \end{aligned} \tag{94}$$

(92), (93) and (94) now show (89), which completes the proof. ☐

We next show that Lemma 13.4 implies the following stronger result. Note also that in the following z is allowed to be an arbitrary element of $\mathbb{R}_+^2$.

Lemma 13.5. *If* (24) *holds, then there exists a* $\delta > 0$ *such that*

$$\mathbb{E}\left[\frac{1}{|z|} \cdot |\tilde{L}^z(|z|\delta)|\right] \quad \to \quad 0 \quad \text{as} \quad z \to \infty.$$

Proof. Note that by the definition of the norm $|z| = |z_1| + |z_2|$, it suffices to prove that

$$\mathbb{E}\left[\frac{1}{|z|} \cdot |\tilde{L}_1^z(|z|\delta)|\right] \quad \to \quad 0 \quad \text{as} \quad z \to \infty \tag{95}$$

and

$$\mathbb{E}\left[\frac{1}{|z|} \cdot |\tilde{L}_2^z(|z|\delta)|\right] \quad \to \quad 0 \quad \text{as} \quad z \to \infty. \tag{96}$$

Suppose that $|z| = \varepsilon$. Then, by the definition of the norm $|\cdot|$, $z_1, z_2 \leq \varepsilon$ and so by Lemma 13.2, $\tilde{L}_1^z(t) \leq \tilde{L}_1^x(t)$ and $\tilde{L}_2^z(t) \leq \tilde{L}_2^x(t)$ for $t \geq 0$, where $x = (\varepsilon, \varepsilon)$. Thus, $\tilde{L}_1^z(2|z|\delta) \leq \tilde{L}_1^x(|x|\delta)$ and $\tilde{L}_2^z(2|z|\delta) \leq \tilde{L}_2^x(|x|\delta)$ for each $\delta > 0$ and (95) and (96) now follow by Lemma 13.4. ☐

We are now in a position to provide the proof of Proposition 13.1.

Proof of Proposition 13.1: We follow the proof of Theorem 3.1 of [10]. Let $0 < \varepsilon < 1$ and note that by Lemma 13.5 there exists a $\kappa \geq 1$ such that

$$\mathbb{E}\left[\frac{1}{|z|} \cdot |\tilde{L}^z(|z|\delta)|\right] \quad \leq \quad 1 - \varepsilon$$

for all $z \in \mathbb{R}^2_+$ such that $|z| \geq \kappa$. Moreover, following the same reasoning as in the proof of Lemma 13.5, there exists a $b > 0$ such that

$$\sup_{z \in \mathcal{B}_\kappa} \mathbb{E}\left[|\tilde{L}^z(|z|\delta)|\right] \quad \leq \quad b. \tag{97}$$

Thus, we may write

$$\mathbb{E}\left[|\tilde{L}^z(|z|\delta)|\right] \quad \leq \quad (1-\varepsilon)|z| + b1\{z \in \mathcal{B}_\kappa\}.$$

Now let $n(z) = |z|\delta$ if $z \notin \mathcal{B}_\kappa$ and let $n(z) = \delta$ if $z \in \mathcal{B}_\kappa$. Note that $n(z) \geq 1$ for all $z \in \mathbb{R}^2_+$ and so it follows from (97) that

$$\mathbb{E}\left[|\tilde{L}^z(n(z))|\right] \quad \leq \quad |z| - \frac{\varepsilon}{\delta} n(z) + \tilde{b}1\{z \in \mathcal{B}_\kappa\}$$

for some $\tilde{b} > 0$ and all $z \in \mathbb{R}^2_+$. Therefore, proceeding exactly as in the proof of Theorem 2.1(ii) of [21], it follows that for each $z \in \mathbb{R}^2_+$,

$$\mathbb{E}\left[\tau^z_\kappa\right] \quad \leq \quad \frac{\delta}{\varepsilon}(|z| + \tilde{b}) \; < \; \infty,$$

which completes the proof. □

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