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ASYMMETRIC EQUILIBRIA IN SYMMETRIC GAMES WITH MANY PLAYERS

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We show that in symmetric games with many players, an asymmetric pure-strategy equilibrium can be thought of as the approximate outcome of the play of a specific symmetric mixed-strategy equilibrium.

1. Introduction

It is often the case in symmetric games in normal form (i.e., games with a symmetric payoff matrix), that the only existing pure-strategy equilibria are asymmetric. Examples of this are models of entry into an industry [cf., Dixit and Shapiro (1986)], models of price-dispersion [cf. Salop and Stiglitz (1976)], and models of information about prices [cf. Grossman and Stiglitz (1976)].

If there is an asymmetric equilibrium for a model with N 'equal' players, then there are multiple equilibria, only differing on the 'name' of the players 'assigned' to each one of the actions which together form an equilibrium. A natural question to ask is, then, how to select among these equilibria. If there are few players, one can assume that there is some communication and coordination mechanism which will lead the players to a specified equilibrium. [See for example Farrell (1987).] However, if there are many players, communication and coordination are not so simple, and another mechanism should be found.

In this note, we show that in symmetric games with many players, an asymmetric pure-strategy equilibrium can be thought of as the approximate outcome of the play of a specific symmetric mixed-strategy equilibrium. In this mixed-strategy equilibrium, each player choses action a_i with probability close to the fraction of players choosing that action in the asymmetric pure-strategy equilibrium. The idea is that with large numbers ex-ante probability and ex-post frequency are approximately the same. Schneidler (1973) presents a result (Theorem 2) similar to the one in this note, but for the case of non-atomic games, whereas we deal with games with a finite number of players.

2. The theorem

The result presented below is only valid for a specific class of models, which we will descibe next. While the assumptions made may seem too restrictive, they turn out to be satisfied by the examples

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referred to before. Furthermore, the result can be extended to a broader class of models, but at the cost of a more complicated proof.

We consider a game with complete information and a finite set of players $N = \{1, \ldots, N\}$. The set of pure-strategies of each player is given by $S = \{0, 1\}$. The payoff for choosing action i is assumed to depend only on the fraction of players choosing each action. Without loss of generality, we can write $\Pi_i(p)$ $(i = 1, 2)$ for the payoff of choosing action i given that a fraction p of players choose '1'. We assume that Π_0 is increasing, and Π_1 decreasing with p. (Therefore $\Delta\Pi \equiv \Pi_0 - \Pi_1$ is increasing with p.) Define $\Pi \equiv (\Pi_0, \Pi_1)$.

To simplify notation, we write '$x(N) \to y$' to imply that $\lim_{N \to \infty} x(N) = y$ and '$x(N) \overset{p}{\to} y$' to imply that $\lim_{N \to \infty} P[\,|x(N) - y| > \epsilon] = 0$ (convergence in probability). We can then state the following result.

Theorem. Suppose there is a sequence of games (N, S, Π), $N = N_1, N_2, \ldots, N \to +\infty$, *such that for each* N_i *there exists a unique (asymmetric) pure-strategy equilibrium with* Np^* *players choosing action* '1' *and* $N(1 - p^*)$ *choosing action* '0'. *Then,*

(i) *there exists a Nash mixed-strategy equilibrium in which each player chooses strategy* '1' *with probability* $\hat{p}(N)$;

(ii) *the outcome of each play is, with probability greater than* $1 - \delta(N)$, *a Nash* ϵ*-equilibrium with* $\epsilon = \epsilon(N)$ *and a fraction* $\bar{p}(N)$ *of players choosing strategy* '1';

(iii) $\hat{p}(N) \to p^*$, $\bar{p}(N) \overset{p}{\to} p^*$, $\delta(N) \to 0$, *and* $\epsilon(N) \to 0$.

Proof. Suppose each player chooses a mixed-strategy in which he or she plays '1' with probability $\hat{p}$. A necessary and sufficient condition for $\hat{p}$ to be an optimal strategy is that the expected payoffs of playing '0' and '1' are the same:

$$\mathrm{E}[\Pi_0[\bar{p}(N-1)/N]] = \mathrm{E}[\Pi_1[(\bar{p}(N-1)+1)/N]], \tag{1}$$

where $\bar{p}$ is the fraction of *other* players *actually* choosing '1', given that each chooses '1' with probability $\hat{p}$. The number of *other* players choosing '1', n, has a binomial distribution:

$$n : B(N-1, \hat{p}) \qquad (n = 0, 1, \ldots, N-1),$$

with

$$\mathrm{E}(n) = \hat{p}(N-1) \quad \text{and} \quad V(n) = \hat{p}(1-\hat{p})(N-1) = \sigma(N-1),$$

where $\sigma = \hat{p}(1 - \hat{p})$. Therefore, $\bar{p}$ has also a binomial distribution, with

$$\mathrm{E}(\bar{p}) = \hat{p} \quad \text{and} \quad V(\bar{p}) = \sigma/(N-1).$$

An increase in $\hat{p}$ shifts the distribution of $\bar{p}$ to the right, in a continuous way, in the sense of first-order stochastic dominance. Given the monotonicity of $\Pi_i(p)$, we conclude that $\mathrm{E}[\Pi_0(\bar{p})]$ is continuous and increasing, and $\mathrm{E}[\Pi_1(\bar{p})]$ continuous and decreasing with $\hat{p}$. Clearly, if $\hat{p}$ equals zero (one), so will $\bar{p}$ equal zero (one). Again, given the monotonicity of $\Pi_i(p)$ and the fact that there is an interior Nash equilibrium, $\Pi_0(0) \le \Pi_1(1/N)$ and $\Pi_0((N-1)/N) \ge \Pi_1(N)$. Together, these facts imply that there exists a unique solution to (1). Let us refer to it as $\hat{p} = \hat{p}(N)$. Also, define $\sigma(N) = \hat{p}(N)(1 - \hat{p}(N))$.

Now suppose all players choose action '1' with probability $\hat{p}(N)$. Then, we can apply Chebyshev's inequality to get

$$P\left[p^{L}(N) \le \bar{p}(N) \le p^{H}(N)\right] \ge 1 - 1/r^{2}, \tag{2}$$

where

$$p^{L}(N) = \hat{p}(N) - r^{2}\sigma(N)/N$$

and

$$p^{H}(N) = \hat{p}(N) + r^{2}\sigma(N)/N,$$

for any given positive r.

Define

$$\epsilon(N) = \max\{\Pi_1(p^{L}) - \Pi_0(p^{L}), \Pi_0(p^{H}) - \Pi_1(p^{H})\}, \tag{3}$$

$$\delta(N) = 1/r^{2}, \tag{4}$$

$$r = N^{\gamma}, \quad \text{where} \quad 0 < \gamma < 1/2. \tag{5}$$

By (2) and definitions (3)–(4), the second part of the theorem follows. Clearly, $r \to +\infty$, and thus $\delta(N) \to 0$.

On the other hand, given (5) and the fact that $\sigma(N)$ is bounded, the order of magnitude of $r^{2}\sigma(N)/N$ is less than zero. Therefore,

$$p^{L}(N) \to \hat{p}(N) \quad \text{and} \quad p^{H}(N) \to \hat{p}(N), \tag{6}$$

which shows that $\bar{p} \xrightarrow{P} \hat{p}$. Since payoffs are bounded and continuous, we have, by Slutsky's theorem [see Greenberg and Webster (1983, p. 8)],

$$\mathrm{E}[\Pi_0[\bar{p}(N)(N-1)/N]] \xrightarrow{P} \Pi_0[\hat{p}(N)(N-1)/N] \to \Pi_0[\hat{p}(N)]$$

and

$$\mathrm{E}[\Pi_1[(\bar{p}(N)(N-1)+1)/N] \xrightarrow{P} \Pi_1[(\hat{p}(N)(N-1)+1)/N \to \Pi_1[(\hat{p}(N)],$$

which implies that

$$\Delta\Pi[\hat{p}(N)] = \Pi_1[(\hat{p}(N)] - \Pi_0[\hat{p}(N)] \xrightarrow{P} 0. \tag{7}$$

Given that (N, p^{*}, Π) is a Nash equilibrium,

$$\Pi_0[p^{*}] \ge \Pi_1[(p^{*}N+1)/N]$$

and

$$\pi_1[p^*] \geq \Pi_0[(p^*N-1)/N],$$

i.e., players choosing '0' or '1' have no incentive to deviate. Therefore,

$$\Delta\Pi(p^*) = \Pi_1(p^*) - \Pi_0(p^*) \to 0. \tag{8}$$

Putting together (7) and (8), and recalling that $\Delta\Pi$ is monotonic, we conclude that

$$p(N) - p^* \overset{\mathrm{p}}{\to} 0.$$

Finally, (3), (6) and (7) imply that $\epsilon(N) \to 0$. Q.E.D.

3. Remarks

(1) There are several alternative ways of interpreting the result presented above. One is the following. In general, mixed-strategy equilibria imply ex-post regret: after having observed the other players' actions, some players will regret the choice they have made. With a large number of players, however, the outcome of each play (of a mixed-strategy equilibrium) is close to an ϵ-Nash equilibrium, and therefore it implies no ex-post regret (in an approximate sense).

(2) A second alternative interpretation of the theorem is that, with a large number of players, mixed-strategy equilibria are close to correlated equilibria. Recall Harsanyi's 'purification' argument: a mixed-strategy equilibrium can be interpreted as the reduced form of a game with incomplete information. Each player observes the realization of random shocks which affect their own payoffs, and which are each player's private information. If these disturbances are independent across players, then as the number of players increases players can 'know' (in a statistical sense) the other players' random shocks, and act as if they were correlating their strategies.

(3) Finally, note that with a large number of players, the sets of equilibrium payoffs of mixed- and pure-strategy equilibria are the same. This may have interesting implications in the area of repeated games [cf. Abreu, Pearce, and Stacchetti (1986)].

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