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Conjectural variations as a reduced form

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Abstract

The conjectural variations solution is usually seen as the reduced form of the equilibrium of an (unmodeled) dynamic game. We show that, in linear oligopolies and for an open set of values of the discount factor, this correspondence holds exactly for a quantity-setting repeated game with minimax punishments during T periods.

Keywords: Conjectural variations; Oligopoly; Repeated games

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1. Introduction

The conjectural variations (CV) oligopoly solution, first introduced by Bowley (1924), has been a useful tool in both applied theoretic and empirical industrial organization. The basic idea is to consider a static quantity-setting oligopoly, together with the assumption that each firm expects a one-unit change in its quantity to lead to a change of $1 + \gamma$ in total output. That is, each firm conjectures that a one-unit change in its output leads to a variation of γ in the other firms' output. By varying the value of γ between -1 and $N - 1$ (where N is the number of firms), one obtains different solutions, from the most competitive to the most collusive one.

The CV solution is not entirely satisfactory from a game-theoretic point of view because it describes 'dynamics' based on a static model (cf. Tirole, 1989, pp. 244–245; Makowski, 1987). However, despite this theoretical shortcoming, many authors believe that the CV model, taken as a reduced-form model, can be a useful tool for many practical purposes. The idea is that conjectural variations are "best interpreted as reduced form parameters that summarize the intensity of rivalry that emerges from what may be complex patterns of behavior" (Schmalensee, 1989, p. 650); in particular, from "the equilibrium of an (unmodeled) dynamic oligopolistic game" (Farrell and Shapiro, 1990, p. 120, footnote 27).

In this paper, we attempt to formalize this idea by providing an explicit dynamic game of which the CV solution is an exact reduced form. The dynamic game we have in mind is a

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quantity-setting oligopoly supergame with an equilibrium of the following kind. In each period, each firm produces some designated quantity. If in any period one firm alone deviates from its designated action, then it is minimaxed for T periods (thus receiving a payoff of zero). For some given value of the discount factor δ, there exists a unique solution that maximizes total profits, the 'optimal' equilibrium. Our main result asserts that for each value of δ in a given open set, there exists a value of γ such that, *for any linear oligopoly structure*, each firm's quantity along the 'optimal' equilibrium path is equal to that firm's quantity in the CV solution.

Notice that the order of the quantifiers in the preceding section is crucial. To say that for any oligopoly structure and discount factor there exists a γ such that the CV solution replicates the dynamic solution would be a much weaker result. In particular, for the case of symmetric oligopolies, it would be a trivial result. The strength of our result lies in the fact that we may fix δ (and the corresponding γ), vary the oligopoly structure (among the set of linear oligopolies), and still maintain the exact correspondence. This is important insofar as it legitimizes comparative statics on changes in demand and costs based on an initial estimate of γ.

2. Main result

A linear oligopoly is defined by a linear demand function and a set of firms, each with its own linear cost function. Without loss of generality, we can denote a linear oligopoly by $\omega = (N, c)$, where N is the number of firms and c an N-dimensional vector with each firm's marginal cost, $c = [c_i]$. Units of c_i are chosen so that the demand function is given by $P = 1 - Q$.

The CV solution is given by a quantity-setting 'game' with the assumption that each firm expects a one-unit variation of its quantity to be 'followed' by a change of $1 + \gamma$ in total quantity, i.e. $\partial Q/\partial q_i = 1 + \gamma$, $i = 1, \ldots, N$.

Firm i's profits are given by

$$\Pi_i = q_i(1 - Q_{-i} - q_i - c_i) ,$$

where $Q_{-i} \equiv \Sigma_{j \neq i} q_j$. The 'equilibrium' quantities are then determined by the first-order conditions:

$$q_i = (1 - Q_{-i} - c_i)/(2 + \gamma) . \tag{1}$$

It is useful to think of the CV solution as a vector-valued function from the set of (linear) oligopolies to $\Re^N$, $f(\omega \mid \gamma): \Omega \rightarrow \Re^N$, giving each firm's equilibrium quantity; that is, the values $[q_i] = f(\omega \mid \gamma)$ that satisfy (1).

Now consider a repeated quantity-setting game with discount factor δ and the following class of equilibria. In each period, each firm chooses some designated quantity q_i. If in any period t firm i alone deviates from the prescribed action, then firm i is minimaxed in the following period, thus receiving a payoff of zero. After one period of punishment, the quantities corresponding to the equilibrium path are again played.

Equilibria of a similar type were studied by Abreu (1986), who concentrated on symmetric equilibria of symmetric oligopolies. In what follows, we will also consider the possibility of asymmetric oligopolies (and equilibria). In general, the conditions for the described strategies to form a Nash equilibrium are

$$\Pi^*(Q_{-i}) - \Pi(q_i, Q_{-i}) \leq \delta \Pi(q_i, Q_{-i}) , \tag{2}$$

where

$$\Pi^*(Q_{-i}) \equiv \max_q \, qP(q + Q_{-i}) - C(q) ,$$
$$\Pi(q_i, Q_{-i}) \equiv q_i P(q_i + Q_{-i}) - C(q_i) , \qquad i = 1, \ldots, N .$$

The left-hand side of (2) gives the increase in current period profits for a firm that deviates, while the right-hand side gives the loss in profits resulting from the minimax punishment in the following period.[1]

In the case of a linear oligopoly, we have

$$\Pi^*(Q_{-i}) = \tfrac{1}{4}(1 - Q_{-i} - c_i)^2 ,$$

$$\Pi(q_i, Q_{-i}) = q_i(1 - Q_{-i} - q_i - c_i) ,$$

and so the equilibrium conditions become

$$\tfrac{1}{4}(1 - Q_{-i} - c_i)^2 \leq + \delta) q_i(1 - Q_{-i} - q_i - c_i) . \tag{3}$$

There are, of course, many equilibria in this class. Of particular interest are what we call 'optimal equilibria', i.e. equilibria that maximize total profits $\Pi \equiv \Sigma_{i=1}^{N} q_i(1 - Q - c_i)$ in the given class of one-period minimax-punishment equilibria.[2]

By analogy with the CV solution, we can then define the function $g(\omega \mid \delta) : \Omega \rightarrow \Re^N$, which gives each firm's quantity along the equilibrium path in the optimal one-period minimax-punishment equilibrium; that is, the values $[q_i] = g(\omega \mid \delta)$, which maximize total profits subject to (3).

The main result in the paper concerns the equivalence between the above dynamic equilibria and the CV solution; more specifically, the equivalence between f and g.

Theorem 1. There exists an open set of values of δ such that

$$\gamma/2 = \delta + \sqrt{\delta(1 + \delta)} \Rightarrow f(\omega \mid \gamma) = g(\omega \mid \delta) , \quad \forall \omega .$$

[1] For simplicity, we only consider Nash equilibria at this stage. Below, we argue that for an open set of values of δ these equilibria are subgame perfect.

[2] Note that these equilibria are not optimal in a more general sense, since the punishments considered are not optimal themselves.

Proof. The proof consists of two steps. First, we show that the equilibrium necessary conditions (3) can be written as simple inequalities similar to the first-order conditions of the CV model. Second, we show that in the optimal equilibrium all these inequalities are binding.

Step 1. Solving (3) for q_i, we get

$$\begin{aligned}
&\tfrac{1}{4}(1-Q_{-i}-c_i)^2 \leq (1+\delta)q_i(1-Q_{-i}-q_i-c_i)\,,\\
&0 \geq (1+\delta)q_i^2-(1+\delta)(1-Q_{-i}-c_i)q_i+\tfrac{1}{4}(1-Q_{-i}-c_i)^2\,,\\
&q_i \geq \frac{(1+\delta)(1-Q_{-i}-c_i)}{2(1+\delta)} - \frac{\sqrt{(1+\delta)^2(1-Q_{-i}-c_i)^2-4(1+\delta)\frac{1}{4}(1-Q_{-i}-c_i)^2}}{2(1+\delta)}\\
&\quad = (1-Q_{-i}-c_i)\frac{1+\delta-\sqrt{\delta(1+\delta)}}{2(1+\delta)}\\
&\quad = \frac{1-Q_{-i}-c_i}{2+2(\delta+\sqrt{\delta(1+\delta)})}\,,
\end{aligned} \tag{4}$$

since optimality dictates we choose the lowest root. (That is, the second inequality resulting from the above quadratic inequality is redundant.)

Step 2. We now show that in the optimal equilibrium, the inequalities (4) are all binding. First note that if δ is zero, then all firms produce their Cournot quantities under the optimal equilibrium, and thus $Q=(N-\Sigma c_i)/(N+1)$. By continuity, for a small enough δ, $Q>(1-c_i)/2$. Suppose one of the constraints, say the ith, is not binding. If $c_i>c_j$ for some j, then we can easily increase total profits by decreasing q_i and increasing q_j by equal amounts, which contradicts the hypothesis of optimality. Let us therefore consider the case when c_i is the lowest of all costs. Consider a small decrease in q_i given by $\mathrm{d}q_i<0$. If we make $\mathrm{d}q_j=-\mathrm{d}q_i/(N+\gamma)$, then $\mathrm{d}Q_{-j}=\mathrm{d}q_i(1-(N-2)/(N+\gamma))=\mathrm{d}q_i(2+\gamma)/(N+\gamma)$ and so $\mathrm{d}q_j=-\mathrm{d}Q_{-j}/(2+\gamma)$. Therefore, all constraints (4) are still satisfied. What is the change in total profits? Since $\Pi=\Sigma\, q_i(1-Q-c_i)$, we have $\partial\Pi/\partial q_i=1-2Q-c_i$. Therefore,

$$\begin{aligned}
-\frac{\mathrm{d}\Pi}{\mathrm{d}q_i} &= -(1-2Q-c_i)+\frac{1}{N+\gamma}\sum_{j\neq i}(1-2Q-c_j)\\
&\geq -(1-2Q-c_i)+\frac{1}{N+\gamma}\sum_{j\neq i}(1-2Q-c_i)\\
&= \left(-1+\frac{N-1}{N+\gamma}\right)(1-2Q-c_i)\\
&>0\,,
\end{aligned}$$

which contradicts the hypothesis of optimality.

Finally, having established that in the optimal equilibrium all inequalities (4) hold as equalities, we have established the equivalence between this equilibrium and the CV solution, because substituting $2(\delta + \sqrt{\delta(1+\delta)})$ for γ in (1) we get (4) as equalities. □

3. Remarks

(1) In our main result we have applied the concept of Nash equilibrium. However, it can be shown that, for sufficiently high values of δ (but lower than the upper limit considered in the theorem), the equilibria considered are subgame perfect.

(2) Although we have only considered one-period punishments, our main result applied to T-period punishments as well, so long as the firm being punished is minimaxed in each of the T periods. In this case, the equilibrium conditions (3) would become

$$\tfrac{1}{4}(1 - Q_{-i} - c_i)^2 \leq \left(1 + \delta \frac{1-\delta^T}{1-\delta}\right) q_i (1 - Q_{-i} - q_i - c_i) .$$

(3) The equivalence between the static and the dynamic game was established through the values of γ and δ. Normally, we expect differences in δ across industries to result from differences in the length of each period (not the degree of impatience). The idea is then that the shorter the time before retaliation is possible, the greater the value of γ; that is, the more collusive the oligopoly solution is.

Notice, however, that the correspondence between models may also be established through the values of γ and T, holding δ fixed. In this case, the idea is that the longer the period of retaliation to a deviation, the greater the value of γ.

(4) A word should be said about econometric estimation. Although the CV model is a static one, it is typically estimated using time-series data for demand and cost parameters. How does the correspondence between the static and the dynamic models hold here? The answer is that if changes in demand and cost parameters are taken to be permanent changes, then econometric estimation will yield correct estimates of δ and correct predictions of q_i. Otherwise, expectations about future changes in the parameters have to be incorporated in the design of the optimal agreement, and the relation between the dynamic model and the CV model will only be approximate.

(5) Dockner (1992) presents a result similar to ours. He solves for the subgame perfect equilibrium of an infinite horizon adjustment cost model and shows that it coincides with the CV solution. The CV parameter is shown to be a continuous function of the discount factor as well as of adjustment costs. The main difference of our paper is that it is based on a repeated game rather than on a differential game. In this sense, our approaches are complementary; they present two different classes of dynamic models for which the CV solution results as a reduced form.

Dockner's approach has the advantage of not being restricted to linear oligopolies. Our approach has, in turn, an advantage in that adjustment to some exogenous shock (either to demand or to firms' costs) is processed immediately, whereas convergence to the new steady state of a differential game (like the one used by Dockner) will take a very long time. This implies that econometric estimation of the CV solution based on time-series data should yield better estimates of the repeated game model than of the differential game model.

4. Conclusion

In this paper, we show that, in linear oligopolies and for an open set of values of the discount factor, there exists an exact correspondence between the conjectural variations solution and the solution of a quantity-setting repeated game with minimax punishments during T periods.

The main result of the paper seems, therefore, to justify the use of the CV solution as the reduced form of the equilibrium of an (unmodeled) dynamic game; and the CV model as a means of estimating the degree of oligopoly power.

It should be noted, however, that the result only holds for the case of linear oligopolies and for a particular class of equilibria of the dynamic game. In other cases, the CV solution can only be taken as an approximate reduced form. It is an open question for empirical research whether it provides a good approximation in those cases.

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