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DEALERSHIP MARKET

Market-Making with Inventory*

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This study considers the problem of a price-setting monopolistic market-maker in a dealership market where the stochastic demand and supply are depicted by price-dependent Poisson processes [following Garman (1976)] The crux of the analysis is the dependence of the bid-ask prices on the market-maker's stock inventory position We derive the optimal policy and its characteristics and compare it to Garman's The results are shown to be consistent with some conjectures and observed phenomena, like the existence of a 'preferred' inventory position and the downward monotonicity of the bid-ask prices For linear demand and supply functions we derive the behavior of the bid-ask spread and show that the transaction-to-transaction price behavior is intertemporally dependent However, we prove that it is impossible to make a profit on this price dependence by trading against the market-maker Thus, in this situation, serially dependent price-changes are consistent with the market efficiency hypothesis

1. Introduction

The microstructure of non-Walrasian dealership markets is a subject of growing interest The main issue in question is the impact of the activities of a market-maker, acting on his own behalf (subject to institutional and ethical constraints) on the operational characteristics of the market

In a pioneering study, Garman (1976) presented a rigorous stochastic model of the dealership market This dealership market is entirely dominated by a centralized market-maker, who possesses a monopoly on all trading Being a price-setter, the market-maker quotes bid and ask prices that affect the stochastic mechanism which generates market sell and buy orders, respectively Garman introduced a stochastic analogue of the classic supply

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and demand functions which makes the operation of this mechanism tractable He suggested that the collective activity of the market agents can be characterized as a stochastic flow of market sell and buy orders whose mean rate per-unit-time is price-dependent This gives rise to a market supply (demand) curve which depicts the expected instantaneous arrival rates of incoming sell (buy) orders as a function of the quoted bid (ask) price

The possible temporal discrepancy between market buy and sell orders, and the obligation to maintain continuous trading, induce the market-maker to carry stock inventories, either positive (long position) or negative (short position) Garman studied the implications of some inventory-independent strategies, which are based on the selection of a fixed pair of bid–ask prices, and showed how they lead either to a sure failure or to a possible failure (see also in the next section) He suggested that 'the specialists must pursue a policy of relating their prices to their inventories in order to avoid failure' [Garman (1976, p 267)] This inventory-dependent policy is, in fact, the main issue of our paper [1]

In this study we derive the optimal pricing policy of the market-maker in a Garman-like dealership market, subject to constraints on his short and long stock inventory positions The crux of the analysis is the dependence of the quoted bid and ask prices on the market-maker's stock We derive the optimal policy and show that its characteristics are consistent with some conjectures and observed phenomena It is proved that the prices are monotone decreasing functions of the stock at hand, and that the resulting spread is always positive It is shown that the optimal policy implies the existence of a 'preferred' inventory position, as was suggested by Smidt (1971), Barnea and Logue (1975) and Stoll (1978a) We also obtain some noteworthy relations between Garman's model and ours concerning the objective function values and policy variables

Focusing on the case of linear demand and supply, we derive the explicit behavior of the bid–ask spread and the expected trading volume as functions of the inventory position Most importantly, we prove that the optimal pricing policy is consistent with the efficient market hypothesis in the sense that it is impossible to make a profit by speculating in the market (except, of course, the market-maker, who enjoys a monopolistic position) Thus, a transaction-to-transaction price behavior which lacks intertemporal independence may well be consistent with the market efficiency hypothesis

Recently, there has been a growing interest in the possibility of computerizing part of the market-maker's functions in the securities markets [e g, Beja and Hakansson (1979), this idea goes back to Fama (1970)] This

[1] Barnea and Logue (1975) also argued for an inventory-dependent pricing policy Stoll (1978a) derived the effect of inventory holding costs on the market-maker's quoted prices Beja and Hakansson (1979) consider the implications of some inventory-dependent trading rules for demand-smoothing

requires the specification of a transaction-by-transaction pricing policy for the market-maker We hope that our study is a step towards the feasible application of this idea

It has been suggested by Bagehot (1971) that the market-maker is faced with basically two kinds of traders the 'liquidity-motivated' transactors, who do not possess any information advantages, and insiders which are transactors with superior information He suggested that the market-maker gains from the former and loses to the latter, and the tradeoff between the two determines his spread It should be noted that our dealership market (and Garman's) is intended to describe the 'liquidity-motivated' transactions

It is worth mentioning some other avenues of research pursued in the study of market-maker impact on the securities markets The role of the market-maker as a price-setter and stabilizer in the stock market was presented by Baumol (1965, ch 3) who studied the effect of his monopolistic position on market prices and resource allocation In a recent overview of market mechanisms, Beja and Hakansson (1977) discussed some alternative means to facilitate trading in the securities markets The impact of trading mechanisms on various characteristics of price behavior in dealership markets was investigated in a series of studies by Cohen, Maier, Schwartz, Whitcomb, and Ness and Okuda In particular, they examined the effect of market thinness on the moments of stock returns in securities markets which operate with or without central market-makers [Cohen, Ness, Okuda, Schwartz and Whitcomb (1976), Cohen, Maier, Ness, Okuda, Schwartz and Whitcomb (1977)], furnished a theoretical framework for these phenomena [Cohen, Maier, Schwartz and Whitcomb (1978a)], and suggested policy implications [Cohen, Maier, Schwartz and Whitcomb (1977a)] Cohen, Maier, Schwartz and Whitcomb (1977b) explained the existence of serial correlation in the securities markets, even when the generated quotations have zero own- and cross-serial correlation, by the existence of bid–ask spread and non-simultaneous transactions in various stocks Beja and Goldman (1977) showed how the way in which expectations are formed may affect the rate of price convergence to the equilibrium price Beja and Goldman (1978) also showed how the impact of a specialist may lead to a serial correlation in security returns although the underlying process is a random walk

In a simulation study, Beja and Hakansson (1979) assessed the ramifications of various trading rules adopted by a (programmed) specialist whose role is to smooth the discrete demand function by buying or selling a sufficient number of shares to clear the market

Another group of studies [e g , Demsetz (1968), West and Tinic (1971), Tinic (1972), Benston and Hagerman (1974), Barnea and Logue (1975), Logue (1975), and Stoll (1978b)] focus on the determinants of the bid–ask spread charged by the market-maker They suggest that the bid–ask spread is a function of the cost of providing immediacy services, the risks to the market-maker inherent in the traded stock (in particular the specific risk due to insiders' trading) the degree of competition, the 'depth' of the market, the price per share and the volume of trading

The notion of a dynamic price-inventory adjustment policy was discussed by Smidt (1971, 1979), Barnea (1974), Barnea and Logue (1975) and Stoll (1976) They suggested that the market-maker has a preferred inventory position and when his realized inventory deviates from it — he will adjust the level, and possibly the spread, of bid–ask prices to restore that position Stoll (1976) presented and tested a model of dealer inventory response to past, current and future price changes and found that specialists tend to buy stocks on price declines and sell stocks on price increases In a later model, Stoll (1978a) considered an expected utility maximizing dealer whose quoted prices are a function of the cost of taking a position which deviates from his desired position, and derived the implication of his inventory policy on the bid–ask spread and on the structure of the dealership market

In what follows, we present the model in section 2, derive the optimal policy and its characteristics in section 3, and solve for the case of linear demand and supply in section 4 In section 5, we discuss some additional aspects and possible extensions

2. The model

The market considered in this study is similar to the dealership market introduced by Garman (1976)[1] The main features of this market concern the monopolistic position of the market-maker and the nature of the aggregate supply and demand functions Following Garman (1976, p 263), the underlying assumptions on the market are

(A) All exchanges are made through a single central market-maker, who possesses a monopoly on all trading No direct exchanges between buyers and sellers are permitted

(B) The market-maker is a price-setter He sets an ask price, P_a, at which he will fill a buy order for one unit, and a bid price, P_b, for a one-unit sell order

(C) Arrivals of buy and sell orders to the market are characterized by two independent Poisson processes, with arrival rates $D(P_a)$ and $S(P_b)$, respectively The stationary price-dependent rate functions $D(\ )$ and $S(\ )$ represent the market demand and supply, with $D'(\ )<0$, $S'(\ )>0$

(D) The objective of the market-maker is to maximize his expected average profit per unit-time Profit is defined as net cash inflow

These assumptions were used by Garman to explore the behavior of the market-maker He derived the probabilities of failure and the necessary conditions to avoid a sure failure Then he modified the assumptions to analyze two cases In the first, the market-maker sets bid and ask prices to maximize his expected profit per unit time subject to the constraint of no inventory drift In the second case, Garman assumed zero price-spread and analyzed the nature of failure and the duration of the process

Our model is a natural extension of Garman's As he suggested, we allow the prices set by the market-maker to depend on his stock inventory position, which leads to a dynamic pricing policy of the market-maker For that purpose we focus on the stochastic process which describes the development of inventory This process results from the arrival of market buy and sell orders whose rates are governed by the pricing decisions of the market-maker The underlying process of market-orders generation in our model is identical to that assumed by Garman The arrival processes are Poisson processes whose rates are price-dependent Yet, since our concern is with the development of inventory, we shall rewrite Garman's assumption (C) in a form suitable to our purpose

It is well known that the Poisson process is characterized by independent exponentially distributed interarrival times Thus, a given pair of prices, P_a and P_b, generates two competing exponential random variables τ_a with mean $1/D(P_a)$, and τ_b with mean $1/S(P_b)$ [see e g Howard (1971, pp 793–797)] The next arriving order will occur at time $\min\{\tau_a\ \tau_b\}$, being a buy order if $\tau_a < \tau_b$, or a sell order if $\tau_a > \tau_b$ It follows that assumption (C) may be written in the following equivalent form

(C′) For a given pair of prices, P_a and P_b, the next incoming order will be a buy order with probability $D(P_a)/(D(P_a)+S(P_b))$, or a sell order with probability $S(P_b)/(D(P_a)+S(P_b))$ The time until the next arriving order has an exponential distribution with mean $1/(D(P_a)+S(P_b))$

Formulation (C′) implies that the process of inventory development (due to the discrepancy between supply and demand) is in fact a birth and death process [see, e g, Howard (1971, pp 797–814)], whose parameters are controlled by the market-maker Thus we obtain a semi-Markov decision process [Howard (1971, ch 15)] where the state variable is the stock at hand, and the decision made for a given inventory level j is a pair of prices, P_{aj} and P_{bj}, which determine the respective demand and supply rates

The stock at hand is assumed to be bounded from above by some constant L and from below by $-K$, where K and L are integers and $-K < L$ This assumption reflects the limitations which are usually imposed on the market-maker's ability ot take long and short positions These limitations result from capital requirements or from administrative rules Note that the formulation of the model in terms of this finite state-space resolves the problem of possible ruin Formally, we assume

(E) The permissible stock inventory levels are $\{-K, -K+1, -K+2, \ldots, L-2, L-1, L\}$

For convenience of exposition, we re-number the states as $\{0,1,2, \ldots, M-1, M\}$, where $M=L+K$ We assume $M \geqq 3$. We also adopt the conventional terminology of birth and death processes, and let λ_k denote the birth rate in state k, and μ_k the corresponding death rate. We also define $\mu_0 = \lambda_M = 0$ Since $\lambda_k = S(P_{bk})$ is a monotone increasing function of P_{bk}, there is a one-to-one correspondence between λ_k and P_{bk} Similarly, μ_k is a monotone decreasing function of P_{ak}, with a one-to-one correspondence Thus, the transition rates λ_k and μ_k (rather than the corresponding prices) will be used as the decision variables in state k

The market demand and supply functions give rise to the market-maker's revenue and cost functions, respectively,

$$R(\mu) = \mu \cdot P_a(\mu) = \mu \cdot D^{-1}(\mu),$$

and

$$C(\lambda) = \lambda \cdot P_b(\lambda) = \lambda \cdot S^{-1}(\lambda)$$

$R(\mu)$ represents the expected sales revenue per unit time corresponding to demand rate μ (which is, in turn, a function of the pre-determined ask price P_a) Analogously, $C(\lambda)$ is the expected cash outlay per unit time, which represents the cost of inventory replenishment

We make the following regularity assumptions on $R(\cdot)$ and $C(\cdot)$

(F) The market-maker's revenue and cost functions, $R(\cdot)$ and $C(\cdot)$, respectively, are twice continuously differentiable with

(i) $R(\cdot)$ is strictly concave, i e, $R''(\mu)<0$,
(ii) $C(\cdot)$ is strictly convex, i e, $C''(\lambda)>0$,
(iii) $R'(0)>C'(0)$, $R'(\infty)<C'(\infty)$

Finally, we assume

(G) There are no transaction costs to the market-maker

Note that the existence of a transaction cost ξ per trade [see Garman (1976, p. 266)] paid, e g, by the market-maker, will simply shift the supply and demand functions by a constant, retaining the convexity and concavity properties of $C(\cdot)$ and $R(\cdot)$ Thus, there is no loss of generality in assuming $\xi=0$ (Obviously, other costs which are constant per unit time do not affect the optimal policy)

The problem is to find the optimal policy of the market-maker under assumptions (A)–(G) Following assumption (D), the objective function is [Howard (1971, p 868)]

$$g(\underline{\lambda}, \underline{\mu}) = \sum_{k=0}^{M} \phi_k q_k, \tag{1}$$

where

$$\underline{\lambda} = (\lambda_0, \quad, \lambda_{M-1}) \quad \text{and} \quad \underline{\mu} = (\mu_1, \quad, \mu_M)$$

q_k is the earning rate for a transition from state k, i e, the expected cash flow divided by the mean sojourn time In terms of our model, the expected cash flow per transition from state k is

$$\frac{\mu_k}{\lambda_k + \mu_k} P_a(\mu_k) - \frac{\lambda_k}{\lambda_k + \mu_k} P_b(\lambda_k) = \frac{R(\mu_k) - C(\lambda_k)}{\lambda_k + \mu_k},$$

and the expected sojourn time in state k is $(\lambda_k + \mu_k)^{-1}$, hence

$$q_k = R(\mu_k) - C(\lambda_k) \tag{2}$$

ϕ_k is the limiting probability of finding the process in state k It is well known that the stationary probabilities ϕ_k $(k = 0, 1, \quad, M)$ satisfy the relations

$$\lambda_k \phi_k = \mu_{k+1} \phi_{k+1}, \qquad k = 0, 1, \quad, M-1, \tag{3}$$

that is, the expected flow from state k to state $k+1$ equals the expected flow in the opposite direction It follows that

$$\phi_k = \phi_0 \frac{\lambda_0 \lambda_1 \quad \lambda_{k-1}}{\mu_1 \mu_2 \quad \mu_k} \tag{4}$$

Let

$$\bar{\lambda} = \sum_{k=0}^{M} \lambda_k \phi_k \quad \text{and} \quad \bar{\mu} = \sum_{k=0}^{M} \mu_k \phi_k, \tag{5}$$

be the mean rates of incoming sell and buy orders Then (3) implies

$$\bar{\lambda} = \bar{\mu}, \tag{6}$$

that is, the expected flows in both directions are equal Relations (3) and (4) are valid only when $\lambda_k > 0$ for $k = 0, 1, \quad, M-1$ and $\mu_k > 0$ for $k = 1, 2, \quad, M$

It will be proved in the sequel that under the optimal policy, the birth and death rates are indeed positive

3. Optimal market-maker behavior

In this section we first derive the optimality conditions and then study their implications on the behavior of the market-maker By a straightforward differentiation of the objective function (1) we obtain the following necessary conditions for optimality

$$\lambda_k \sum_{j=k+1}^{M} \phi_j [R(\mu_j) - C(\lambda_j)] - \lambda_k \phi_k C'(\lambda_k) = g(\underline{\lambda}, \underline{\mu}) \sum_{j=k+1}^{M} \phi_j \tag{7}$$

$$\mu_k \sum_{j=k}^{M} \phi_j [R(\mu_j) - C(\lambda_j)] - \mu_k \phi_k R'(\mu_k) = g(\underline{\lambda}, \underline{\mu}) \sum_{j=k}^{M} \phi_j \tag{8}$$

By subtracting the $(k+1)$st equation of (8) from the kth equation of (7) and using (3) we obtain

$$R'(\mu_{k+1}) = C'(\lambda_k), \qquad k = 0, 1, \quad , M-1, \tag{9}$$

which reminds of the ordinary optimality condition of a monopoly, except that here it relates to each pair of neighboring states Note that since

$$P_a(\mu_{k+1}) > R'(\mu_{k+1}) = C'(\lambda_k) > P_b(\lambda_k), \tag{10}$$

a purchase of one unit at state k and its sale at state $k+1$ always yields a profit [2] It follows that a loop of transitions starting from any state k, traversing other states and returning to state k yields a positive profit with probability one Thus, when the market-maker's initial resources (cash plus available credit) exceed $\Sigma_{k=0}^{M-1} P_{bk}$, the probability of cash failure [Garman (1976, p 263)] is zero, even in the worst possible case Since the probability of default by the market-maker is zero, initial credit (if needed) should be available

Subtraction of (7) from (8) yields

$$C(\lambda_0) - \lambda_0 C'(\lambda_0) + g(\underline{\lambda}, \underline{\mu}) = 0, \tag{11^0}$$

$$C(\lambda_k) - \lambda_k C'(\lambda_k) + g(\underline{\lambda}, \underline{\mu}) = R(\mu_k) - \mu_k R'(\mu_k), \tag{11^k}$$

$$k = 1, 2, \quad , M-1,$$

[2] A sale of a unit at state $k+1$ can always be attributed to a purchase at state k (except for the initial stock)

$$g(\underline{\lambda}, \underline{\mu}) = R(\mu_M) - \mu_M R'(\mu_M) \tag{11^M}$$

Eqs (11) give the basic relation between λ_k and μ_k, and thus the relation between the bid and ask prices for each inventory position k It follows that the optimal (λ_k, μ_k) are aligned along the curve defined by

$$[R(\mu) - \mu R'(\mu)] - [C(\lambda) - \lambda C'(\lambda)] = g, \tag{12}$$

[note that $(\lambda_0, 0)$ and $(0, \mu_M)$ are at the intersection of the curve with the positive semi-axes] Lemma 3 1 proves that the curve (12) depicts a downward sloping function on the positive quadrant

Lemma 3 1 *Along* (12), *for* $\lambda, \mu > 0$, $\mathrm{d}\mu/\mathrm{d}\lambda < 0$

Proof The proof follows since for $\lambda, \mu > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\mu}[R(\mu) - \mu R'(\mu)] > 0, \tag{13a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}[C(\lambda) - \lambda C'(\lambda)] < 0 \qquad \text{Q E D} \tag{13b}$$

The following theorem establishes that the optimal bid and ask prices are monotone decreasing functions of the stock at hand [This result is consistent with the dynamic pricing policy described by Smidt (1971, 1979), Barnea (1974) and Barnea and Logue (1975)]

Theorem 3 2 *Let* P_{bk} *and* P_{ak}, *respectively, be the optimal bid and ask prices at state* k *Then* $P_{b0} > P_{b1} > P_{b2} > \quad > P_{b,M-1}$ *and* $P_{a1} > P_{a2} > \quad > P_{aM}$ *Equivalently,* $\lambda_0 > \lambda_1 > \lambda_2 > \quad > \lambda_{M-1}$ *and* $\mu_1 < \mu_2 < \mu_3 < \quad < \mu_M$

Proof The proof is given in terms of $\underline{\lambda}$ and $\underline{\mu}$. The equivalent formulation in terms of price behavior follows from the monotonicity of the demand and supply functions

We first prove that $\lambda_0 > \lambda_1$ By subtracting eq (11^0) from (11^1) and noting that $R(\mu_1)/\mu_1 > R'(\mu_1)$, we obtain

$$C(\lambda_1) - \lambda_1 C'(\lambda_1) > C(\lambda_0) - \lambda_0 C'(\lambda_0)$$

The inequality now follows from (13b)

Next, $\lambda_0 > \lambda_1$ implies $C'(\lambda_0) > C'(\lambda_1)$, hence by (9), $R'(\mu_1) > R'(\mu_2)$ It follows that $\mu_1 < \mu_2$, and then we obtain by Lemma 3 1 that $\lambda_2 < \lambda_1$ The proof is completed by induction Q E.D

The following corollary shows that the bid–ask spread is always positive, as might be expected

Corollary 3 3 $P_{ak} - P_{bk} > 0$

Proof By (10) and Theorem 3 2 we obtain

$$P_{ak} = P_a(\mu_k) > P_a(\mu_{k+1}) > P_b(\lambda_k) = P_{bk} \qquad \text{Q E D}$$

We now turn to compare our results to a case treated by Garman (1976, pp 265–266) Consider a market-maker who wishes to prevent a drift in his expected inventory Thus he sets prices so as to equate the rates of incoming buy and sell orders, i e, $\mu = \lambda$, regardless of his inventory position This market-maker acts like an ordinary monopolist who equates marginal revenue to marginal cost

It might be argued that the profits of this monopolist, who restricts himself to $\mu = \lambda$, are lower than those of our market-maker who enjoys a greater flexibility in setting prices Yet, our market-maker has constraints on the long and short positions which he can take, whereas Garman's monopolist has no such constraints In fact, the following theorem proves that the profit of Garman's monopolist is an upper bound on $g(\underline{\lambda}, \underline{\mu})$

Theorem 3 4 *Let r^* be the (unique) solution of $R'(r^*) = C'(r^*)$ Then*

$$g(\underline{\lambda}, \underline{\mu}) < R(r^*) - C(r^*) \tag{14}$$

Proof The objective function of our market-maker is

$$g(\underline{\lambda}, \underline{\mu}) = \sum_{k=0}^{M} \phi_k [R(\mu_k) - C(\lambda_k)]$$

The function $R(\mu) - C(\lambda)$ is strictly concave on the (μ, λ)-plane, hence, by Jensen's inequality,

$$g(\underline{\lambda}, \underline{\mu}) < R(\bar{\mu}) - C(\bar{\lambda}),$$

where $\bar{\mu}$ and $\bar{\lambda}$ are given by (5) Furthermore, it follows from (6) that $\bar{\mu} = \bar{\lambda}$ is a subset of the constraints in our problem Now, Garman's problem can be written as

$$\max h(\lambda,\mu)=R(\mu)-C(\lambda)$$
$$\text{s t}\quad \mu=\lambda,$$

whose solution is $\lambda=\mu=r^*$ It follows that for all $\underline{\lambda},\underline{\mu}$,

$$g(\underline{\lambda},\underline{\mu})<R(\bar{\mu})-C(\bar{\lambda})\leqq h(r^*,r^*)=R(r^*)-C(r^*) \qquad \text{Q E D}$$

We now employ Theorem 3 4 to prove that it is not optimal to have a vanishing transition rate That is, the profit-maximizing market-maker will never choose to refrain from making buy or sell transactions (except for the case where he reaches his limiting positions) This also means that relaxing his constraints by expanding the allowed short or long positions strictly increases the market-maker's profits

Theorem 3 5 The optimal policy $(\underline{\lambda},\underline{\mu})$ satisfies $\lambda_k>0$ *for* $k=0,1,\quad,M-1$, *and* $\mu_k>0$ *for* $k=1,2,\quad,M$

Proof It is sufficient to prove that $\mu_1=0$ is not optimal [3] For that purpose we apply Howard's (1971, pp 983–1005) policy improvement procedure to show that a policy with $\mu_1>0$ is an improvement over the policy with $\mu_1=0$ Let the initial policy $(\underline{\lambda},\underline{\mu})$ be such that $\mu_1=0$ and $\lambda_0=r^*$ (note that since state 0 is transient, the value of λ_0 does not affect g), with relative state values v_k Consider an alternative policy $(\underline{\lambda}',\underline{\mu}')$ where $\underline{\lambda}'=\underline{\lambda}$, $\mu'_j=\mu_j$ for all $j\neq 1$, but $\mu'_1=r^*/2>0$ Let Γ_1 be the value of the test quantity [Howard (1971, p 986)] for evaluating the original policy at state 1, and Γ'_1 the corresponding test quantity for the alternative policy Then,

$$\Gamma_1=-C(\lambda_1)+(\lambda_1+0)[1\quad v_2-v_1],$$

and

$$\Gamma'_1=R(r^*/2)-C(\lambda_1)+(\lambda_1+r^*/2)\left[\frac{\lambda_1}{\lambda_1+r^*/2}v_2+\frac{r^*/2}{\lambda_1+r^*/2}v_0-v_1\right],$$

hence

$$\Gamma'_1-\Gamma_1=R(r^*/2)-r^*/2\quad(v_1-v_0)$$

[3]More generally, the proof implies that adding one state to a chain strictly increases the value of g (in that chain) Hence, the chain resulting from the optimal solution must contain all states

Now, $(v_1 - v_0)$ is found by performing the policy evaluation procedure for the initial policy at state 0,

$$v_0 + g/\lambda_0 = -C(\lambda_0)/\lambda_0 + v_1,$$

that is,

$$v_1 - v_0 = [g(\underline{\lambda}, \underline{\mu}) + C(r^*)]/r^*$$

By Theorem 3 4, $g(\underline{\lambda}, \underline{\mu}) < R(r^*) - C(r^*)$, hence $v_1 - v_0 < R(r^*)/r^*$, and

$$\Gamma_1' - \Gamma_1 > R(r^*/2) - R(r^*)/2 > 0 \qquad \text{Q E D}$$

We now proceed to study the characteristics of the stationary distribution $\{\phi_k\}_{k=0}^{M}$ under the optimal policy It follows from Theorem 3 5 that $\phi_k > 0$ for all $k = 0, 1, \quad, M$, so there is a positive probability of finding the market-maker in any of the allowed inventory positions In addition, other properties of $\{\phi_k\}_{k=0}^{M}$ are of interest What is the shape of this distribution? Is there a 'preferred' inventory position [Smidt (1971, 1979), Barnea (1974), Barnea and Logue (1975), Latané et al (1975), Stoll (1976)] If there is, what can be said about the 'preferred' rates and prices?

Since $\phi_{k+1}/\phi_k = \lambda_k/\mu_{k+1}$, the properties of $\{\phi_k\}_{k=0}^{M}$ may be obtained by considering the transition rates λ_k and μ_{k+1} The following lemma relates λ_k and μ_{k+1} to Garman's no-drift rate r^*

Lemma 3 6 *For all* $k = 0, 1, \quad, M-1$,

$$\min\{\lambda_k, \mu_{k+1}\} \leqq r^* \leqq \max\{\lambda_k, \mu_{k+1}\},$$

with equality if and only if $\lambda_k = r^* = \mu_{k+1}$

Proof (i) If $\mu_{k+1} < (=) r^*$, then

$$C'(\lambda_k) = R'(\mu_{k+1}) > (=) R'(r^*) = C'(r^*),$$

hence $\lambda_k > (=) r^*$, respectively

(ii) If $\mu_{k+1} > r^*$, then

$$C'(\lambda_k) = R'(\mu_{k+1}) < R'(r^*) = C'(r^*),$$

hence $\lambda_k < r^*$ Q E D

Lemma 3 6 states that r^* is always located between λ_k and μ_{k+1} This fact is illustrated in fig 1

Theorem 3 7 *The distribution* $\{\phi_k\}_{k=0}^{M}$ *is unimodal*[4] *The mode J is such that*

$$\lambda_k \geqq r^* \quad \text{for} \quad k<J, \qquad \lambda_k<r^* \quad \text{for} \quad k \geqq J, \tag{15}$$

$$\mu_k \leqq r^* \quad \text{for} \quad k \leqq J, \qquad \mu_k>r^* \quad \text{for} \quad k>J \tag{16}$$

Proof First, $\lambda_0 > r^*$ since otherwise we would have $\lambda_k < \lambda_0 \leqq r^* \leqq \mu_1 < \mu_{k+1}$ for all $k=1,2, \ ,M-1$ This would have implied $\bar{\lambda}<\bar{\mu}$, in contradiction to (6) A similar argument leads to $\mu_M > r^*$ It follows from Lemma 3 6 that

$$\phi_1/\phi_0 = \lambda_0/\mu_1 > 1 > \lambda_{M-1}/\mu_M = \phi_M/\phi_{M-1} \tag{17}$$

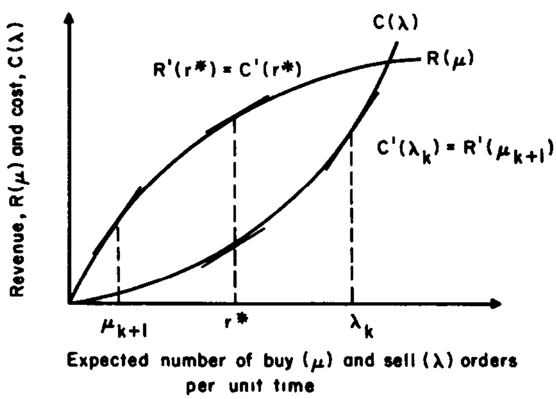

Fig 1 The market-maker's revenue as a function of the arrival rate of buy orders, $R(\mu)$, and his replenishment cost as a function of the arrival rate of sell orders, $C(\lambda)$ r^* is the arrival rate chosen by Garman's monopolist, maximizing $R(r)-C(r)$ λ_k and μ_{k+1} are related by (9) and thus the interval $[\mu_{k+1}, \lambda_k]$ embraces r^*

Next, $\phi_k/\phi_{k-1} = \lambda_{k-1}/\mu_k$, which — by Theorem 3 2 — is a strictly decreasing function of k Let

$$J = \max\{k \mid \lambda_{k-1}/\mu_k \geqq 1\}$$

Inequalities (17) imply that J is well defined and $0<J<M$ For $k \leqq J$, $\phi_k \geqq \phi_{k-1}$, whereas for $k>J$, $\phi_k < \phi_{k-1}$ It follows that $\phi_0 < \ < \phi_{J-2} < \phi_{J-1} \leqq \phi_J$, and $\phi_J > \phi_{J+1} > \phi_{J+2} > \ > \phi_M$ Now, (15) and (16) follow from Lemma 3 6 and from Theorem 3 2 Q E D

[4]This also includes the case of two consecutive modes, J and $J-1$

Theorem 3 7 suggests that the market-maker adopts a pricing policy which produces a 'preferred' inventory position J, located away from the limiting positions 0 and M The preference for position J is reflected in both the unimodality of the distribution and in the fact that the decline rate of ϕ_k as k withdraws from J, is faster than a geometric decline rate (since ϕ_k/ϕ_{k-1} is decreasing) Furthermore, when the market-maker finds himself in a position different from J, he will quote prices which will tend to bring him back to that position. That is, the probability that the next transition will be in the direction of J will exceed the probability of moving towards the extreme positions This aversion from the extremes follows since being there forces the market-maker to make transactions at unfavorable conditions Furthermore, at the 'preferred' inventory position J, λ and μ are approximately equal An exact equality between λ and μ is obtained at (λ^l, μ^l), where the curve (12) intersects with the 45°-line (see fig 2) The actual (λ_J, μ_J) is located in the neighborhood of (λ^l, μ^l), in the following sense

Let $a = \min\{\lambda_J, \mu_J\}$, and $A = \max\{\lambda_J, \mu_J\}$ Then,

$$(a, A) \subset (\mu_{J-1}, \lambda_{J-1}) \subset (\mu_{J-2}, \lambda_{J-2}) \subset \quad \subset (0, \lambda_0),$$

and

$$(a, A) \subset (\lambda_{J+1}, \mu_{J+1}) \subset (\lambda_{J+2}, \mu_{J+2}) \subset \quad \subset (0, \mu_M),$$

where $a \leqq \lambda^l = \mu^l \leqq A$

Thus, all the intervals between λ_k and μ_k straddle the interval between λ_J and μ_J, which in turn straddles $\lambda^l = \mu^l$ This gives (λ^l, μ^l) the interpretation of being the approximate 'likely' rates of the market-maker If some λ_k happens to equal λ^l, then $k = J$ is the preferred inventory position and $\lambda_J = \lambda^l = \mu^l = \mu_J$ are the exact modal rates

We finally relate the likely rates (λ^l, μ^l) to (r^*, r^*), the rates of Garman's monopolist

Theorem 3 8 $\lambda^l = \mu^l < r^*$

Proof Let $h(x) = [R(x) - xR'(x)] - [C(x) - xC'(x)]$ Now, $h'(x) > 0$ for $x > 0$, $h(r^*) = R(r^*) - C(r^*)$, and by (12), $h(\lambda^l) = g$ It follows from Theorem 3 4 that $h(\lambda^l) < h(r^*)$, hence $\lambda^l = \mu^l < r^*$ Q E D

The above relation, together with (15) and (16), are summarized in fig 2 Note that both (λ_J, μ_J) and (λ^l, μ^l) lie on the segment $s_1 s_2$ Furthermore, the only (λ_k, μ_k) contained in this segment are the modal rates More specifically, when $\lambda_{J-1}/\mu_J > 1$, we have $\lambda_{J-1} > r^* > \mu_J > \mu_{J-1}$ and $\lambda_{J+1} < \lambda_J < r^* < \mu_{J+1}$

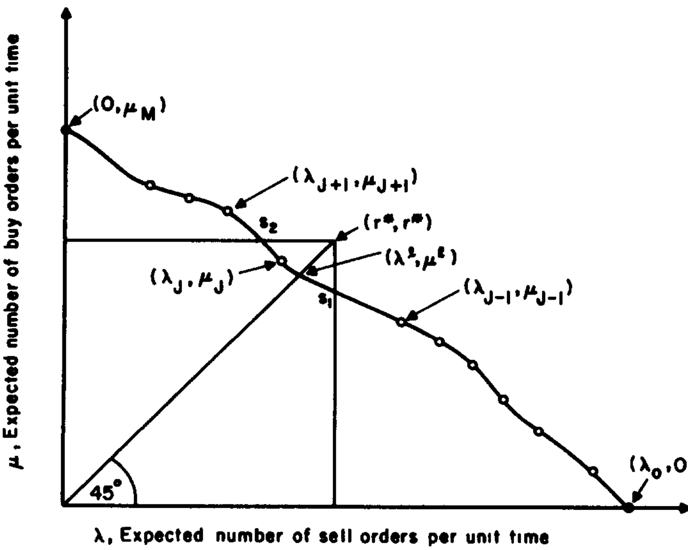

Fig 2 The curve $[R(\mu)-\mu R'(\mu)]-[C(\lambda)-\lambda C'(\lambda)]=g$, along which are aligned the optimal arrival rates of the public buy and sell orders (μ_k and λ_k, respectively, with the subscript denoting the related inventory position) The 'likely' rates satisfy $\lambda^l=\mu^l$ The actually preferred rates (λ_J, μ_J) are both less than or equal to Garman's rates (r^*, r^*), thus they are contained in the segment $s_1 s_2$

Hence, both modal rates are smaller than the rates of Garman's monopolist, i e, (λ_J, μ_J) is inside $s_1 s_2$, and for $k \neq J$, (λ_k, μ_k) lies outside $s_1 s_2$ In the case where $\lambda_{J-1}/\mu_J = 1$, $(\lambda_{J-1}, \mu_{J-1})$ and (λ_J, μ_J) coincide with the endpoints s_1 and s_2, respectively, in this case, both $J-1$ and J are the modes of the distribution $\{\phi_k\}_{k=0}^{M}$

The implication of the above results on the 'preferred' bid and ask prices charged by the market-maker is immediate $P_{aJ} \geq P_a(r^*) > P_b(r^*) > P_{bJ}$, and the 'preferred' bid–ask spread is always greater than the corresponding spread set by Garman's monopolist This also implies that the preferred bid–ask prices straddle the market-clearing price P^c at the intersection of the demand and supply curves P^c is the unique price at which the expected rate of buy orders equals the expected rate of sell orders ($\mu = \lambda$), thus clearing the market 'on the average' Garman (1976, p 266) suggested that P^c is the price set by a 'benevolent' zero-cost market-maker who provides a non-profit public service by filling all incoming orders from his inventory The choice of P^c thus guarantees no expected drift in the market-maker's inventory, together with zero expected profit [under assumption (G)] However, if there is a fixed cost ξ per transaction, then the spread net of transaction cost will be zero[5] Now, consider a market where there is a competition among market-makers The ordinary sufficient conditions for perfect competition in such a market are that entry is costless and free, that [see Fama (1970, p 387)] all available information is costlessly available to all market-makers, and that they all agree on the implication of current information for the current demand and supply functions There may be additional costs to the market-maker, notably the opportunity cost of his resources tied up in the dealership activity, and the actual cost of transacting As usual, competition leads to a pricing of the dealer's services so as to reflect the costs of providing these services If these costs were zero, competition would lead to a zero spread, and the assumed homogeneity of expectations would lead to $P_a = P_b = P^c$ (since any other price will leave the market uncleared on the average) The existence of positive costs of providing dealership services leads to a positive spread which straddles P^c[6] When the cost per transaction is a positive constant ξ [see discussion following assumption (G)], the competitive spread will equal $\xi > 0$

4. The case of linear demand and supply

In this section we apply our general results to the special case of linear demand and supply functions, $D(P_a) = \gamma - \delta P_a$, and $S(P_b) = \alpha + \beta P_b$ Then, $R(\mu) = (1/\delta)(\gamma\mu - \mu^2)$, and $C(\lambda) = (1/\beta)(\lambda^2 - \lambda\alpha)$

Here, the locus (12) of possible (λ_k, μ_k) is the ellipse

$$\mu^2/\delta + \lambda^2/\beta = g, \tag{18}$$

whose axes coincide with the coordinate axes The likely rates are given by the intersection of the 45°-line with the ellipse

$$\lambda^l = \mu^l = \sqrt{g\beta\delta/(\beta+\delta)} \tag{19}$$

The rates chosen by Garman's monopolist, (λ^m, μ^m), satisfy

$$\lambda^m = \mu^m = r^* = \frac{\gamma\beta + \alpha\delta}{2(\beta+\delta)}, \tag{20}$$

whereas the market-clearing rates, (λ^c, μ^c), at which the demand and supply

[5]A positive spread will exist whenever real resources are tied up in the market-making function

[6]For a discussion on the determination of the bid–ask spread on the basis of the dealer's cost, see Demsetz (1968), West and Tinic (1971)

functions intersect and $P_b = P_a$, are

$$\lambda^c = \mu^c = \frac{\gamma\beta + \alpha\delta}{(\beta+\delta)} = 2r^* \tag{21}$$

It follows that $\lambda^l < \lambda^m < \lambda^c$ (and $\mu^l < \mu^m < \mu^c$) (see also Theorem 3 8) The corresponding prices (using the respective upper-scripts) satisfy

$$P_b^l < P_b^m < P^c \quad \text{and} \quad P_a^l > P_a^m > P^c$$

It follows that the market-clearing price is contained in the interval $[P_b^l, P_a^l]$

It is of interest to investigate whether the bid and ask prices set by the market-maker always straddle the market-clearing price P^c This happens if and only if all λ_k, μ_k are not greater than $\lambda^c = \mu^c$ (otherwise, if some $\lambda_k > \lambda^c$ then $P_{ak} > P_{bk} > P^c$, if some $\mu_k > \mu^c$, then both prices are below P^c) This is equivalent to requiring that both λ_0, $\mu_M \leqq \lambda^c = \mu^c$ (since $\lambda_0 = \max_{k=0,1,\ldots,M} \lambda_k$ and $\mu_M = \max_{k=0,1,2,\ldots,M} \mu_k$)

In our case, a sufficient condition for the market-clearing price P^c to be straddled by all the bid–ask prices P_{bk}, P_{ak} $(k = 0, 1, \ldots, M)$ is

$$1/3 \leqq \delta/\beta \leqq 3 \tag{22}$$

The sufficiency of this condition follows since by Theorem 3 4 and the definitions of $R(\ )$ and $C(\ )$,

$$g < R(r^*) - C(r^*) = (r^*)^2/\delta + (r^*)^2/\beta, \tag{23}$$

hence

$$\lambda_0 = \sqrt{g\beta} < \lambda^c = 2r^* \quad \text{and} \quad \mu_M = \sqrt{g\delta} < \mu^c = 2r^*$$

Thus, when the (absolute values of the) slopes of the demand and supply curves are not grossly different, $P^c \subset [P_{bk}, P_{ak}]$ for all k

We now proceed to investigate the behavior of the bid–ask spread set by the market-maker According to the dynamic price/inventory adjustment theory suggested by Smidt (1971, 1979), Barnea (1974) and Barnea and Logue (1975), the spread should be minimal when the market-maker is at his preferred inventory level, and widens as his long or short position is undesirably high [7] Our model yields a similar behavioral pattern

[7] This phenomenon was attributed to the higher risk of positioning See also Latane et al (1975, pp 73–75), who in addition considered the effect of risk on the spread set by different specialists Benston and Hagerman (1974) and Stoll (1978b) provided a cross-sectional empirical study on the effect of the risks incurred by holding inventory on the bid–ask spread across shares

The bid–ask spread corresponding to supply and demand rates (λ, μ) is given by

$$\Delta(\lambda,\mu)=P_a(\mu)-P_b(\lambda)=(\gamma/\delta+\alpha/\beta)-(\mu/\delta+\lambda/\beta)$$

Along the ellipse, we have

$$\mathrm{d}\lambda/\mathrm{d}\mu=-\beta/\delta\ \ \mu/\lambda, \tag{24}$$

and

$$\mathrm{d}\Delta/\mathrm{d}\mu=(1/\delta)(\mu/\lambda-1)$$

The function $\Delta(\lambda,\mu)$ is thus minimized at (λ^l,μ^l), and increases as (λ,μ) approaches the limits $(\lambda_0, 0)$ and $(0,\mu_M)$ It follows (since μ_k increases with k) that the market-maker reduces the bid–ask spread as he approaches the likely inventory position This result and Theorem 3 2 yield a bid–ask price pattern which is illustrated in fig 3 Observe that this pattern resembles the one presented by Latané et al (1975, p 74, fig 4–1)

Next, we study the effect of the market-maker's inventory position on the total volume of transactions By setting bid and ask prices he determines the supply and demand rates, λ and μ, whose sum gives the expected number of transactions per unit time Thus, $\lambda+\mu$ represents the expected volume per unit time [Equivalently, $(\lambda+\mu)^{-1}$ is the expected inter-transaction time] Using (24) we obtain

$$(\mathrm{d}/\mathrm{d}\mu)(\lambda+\mu)=1-\beta/\delta\ \ \mu/\lambda$$

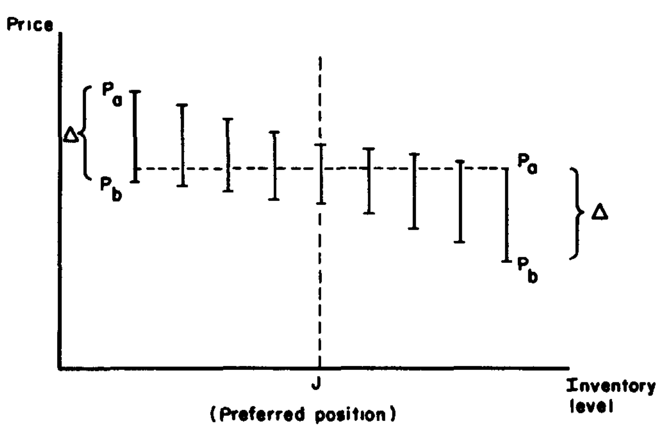

Fig 3 The bid–ask prices and the corresponding spread, Δ, as a function of the market-maker's inventory level

Thus, the volume is maximized at

$$\lambda=\beta\sqrt{g/(\beta+\delta)}=\sqrt{\beta/\delta}\,\lambda^{l}, \qquad \mu=\delta\sqrt{g/(\beta+\delta)}=\sqrt{\delta/\beta}\,\mu^{l}$$

As the inventory level approaches one of the short or long limits, the volume decreases as expected

We conclude the discussion of the linear model by examining the important issue of market efficiency In our dealership market, efficiency (in the 'fair game' sense) implies that it is impossible to make economic profits by trading against the market-maker It is well known that a sufficient condition for market efficiency is that market prices behave as a random walk However, the prices in our model do not adhere to this property since their distribution is mean-reverting [8] Thus, market agents might be tempted to form a trading rule based on past price behavior, by which they would profit through buying from the market-maker at low prices and selling back to him at higher prices However, we shall show that any trading rule which is based on monitoring the behavior of the market-maker is useless and is certain to produce a loss More specifically, we shall show that the pricing policy of the market-maker results in all ask prices being greater than all bid prices To show this, observe that by (23)

$$\sqrt{g}<r^{*}\sqrt{1/\beta+1/\delta}=r^{*}\frac{\sqrt{\beta+\delta}}{\sqrt{\beta\delta}}\,\frac{\sqrt{\beta+\delta}}{\sqrt{\beta+\delta}}<\frac{2r^{*}(\beta+\delta)}{\sqrt{\beta\delta}\,(\sqrt{\beta}+\sqrt{\delta})},$$

where the last inequality follows from

$$\sqrt{\beta+\delta}>\tfrac{1}{2}(\sqrt{\beta}+\sqrt{\delta})$$

[8]The model also gives us the transient price behavior which derives from the transient behavior of the inventory (recalling that quoted prices are one-to-one related to inventory) Starting from an initial state i at time $t=0$, the probability of finding the system in state j at time $t\geqq 0$ is given by the (i,j)-entry of the matrix e^{At} where A is the corresponding transition-rate matrix [see, e g, Howard (1971, ch 12)] The matrix e^{At} can be written as the sum of a matrix whose rows are all identical to the stationary probability vector $(\phi_0,\phi_1,\phi_2,\ \ ,\phi_M)$ (row identity reflects independence of the initial state), and M additional matrices which reflect dependence on the initial state The latter matrices are multiplied by coefficients which are exponentially decaying as a function of t Thus if we observe the system at state i at some time, the dependence of the state observed t time-units later on the initial state i diminishes to zero at an exponential rate (as $t\to\infty$) The corresponding relationship between observed quotations of bid and ask prices and any initial quoted prices readily follows Similarly, the observed transaction prices form a $2M$-state birth and death process with limiting probabilities $P\{\text{observed price}=P_{ak}\}=\phi_k\ \mu_k/(\mu_k+\lambda_k)$, and $P\{\text{observed price}=P_{bk}\}=\phi_k\ \lambda_k/(\mu_k+\lambda_k)$ The previous remarks regarding temporal dependence apply as well to this process (with a different transition-rate matrix)

Now, since $\lambda_0 = \sqrt{g\beta}$ and $\mu_M = \sqrt{g\delta}$, we have

$$\delta\lambda_0 + \beta\mu_M < 2r^*(\beta+\delta)$$

Using expression (20) for r^* yields

$$\lambda_0/\beta + \mu_M/\delta < \gamma/\delta + \alpha/\beta,$$

hence,

$$P_{b0} = (\lambda_0 - \alpha)/\beta < (\gamma - \mu_M)/\delta = P_{aM},$$

which implies, for all k, j,

$$P_{bk} \leqq P_{b0} < P_{aM} \leqq P_{aj} \qquad \text{Q E D}$$

It has long been noted that the random-walk property of prices is not a necessary condition for market efficiency in the 'fair game' sense [Fama (1970)] In fact, empirical studies have shown that serial dependence in price changes co-exists with market efficiency in the sense that it is impossible to make profit by use of publicly available information Our model provides a theoretical framework which implies a transaction-to-transaction price dependence, together with market efficiency As can be seen from fig 3, the systematic pattern of prices cannot be used to make a profit since all ask prices lie above all bid prices Therefore, any trading rule that attempts to profit from this price dependence is certain to produce a loss the bid-ask spread will wipe out any prospective profit

5. Concluding remarks and possible extensions

The last result implies that market traders can make no profitable use of information which is also available to the market-maker Even a knowledge of the market-maker's current inventory position and his pricing policy (derived from the demand and supply functions) cannot produce a profitable trading rule This result agrees with Bagehot's (1971, p 13) observation that 'the market-maker always gains in his transactions with liquidity-motivated transactors' Yet, there may be 'insiders', i e , transactors who possess special information which is not available to the market-maker, and can make a profitable use of it In other words, insiders have a more accurate assessment of the demand and supply functions than that of the market-maker, and they may use their superior information to make profit in excess of their cost implied in the bid–ask spread As Bagehot (1971, p 13) noted, the market-maker always loses to these insiders, and these losses represent an inventory holding cost to the market-maker [9] Clearly, our model is structured to treat the liquidity-motivated transactors and not the insiders' demand and supply A model which will take account of the insiders' trading in an explicit manner is a most important extension This model should also contain a learning mechanism by which the market-maker uses market information to update his assessment of the demand and supply [see Bagehot (1971, p 14)]

It is worth relating our objective function (i e, expected average profit per unit time) to value maximization, which implies here continuous discounting[10] of cash flows at some instantaneous discount rate α This gives rise to a rate-dependent transaction-to-transaction discount factor, which represents the present value of obtaining one-dollar at the next transaction It has long been known [see Jewel (1963, pp 956–957)] that for small discount rates (which are equivalent to close-to-unity transaction-to-transaction discount factors), the discounted value criterion turns out to be well approximated by our average profit criterion [11] It should also be noted that the relevance of discounting to existing dealership markets is limited when settlements take place a few days after the transactions, since then the actual timing of a transaction is of no importance If the underlying conditions in the market are such that discounting is of importance, it is a straightforward matter to formulate the problem as a discounted dynamic-programming problem Then, there is no direct analogy to our formulation of the objective function (1), and our closed-form results will be replaced by recursive relations Clearly, such a reformulation will be at the expense of the model's tractability

Considering the role of inventories in this paper, note that our market-maker's policy depends on his stock inventory position, whereas his cash flows appear only in the objective function The role of the cash position may be interesting in a combined cash-inventory dependent policy, where the market-maker maximizes his expected utility of consumption

It may also be of interest to study the sensitivity of the results of our model to the underlying assumptions on the order arrival process This can be done in several ways The order size may be assumed to be a random variable which represents orders of varying size Furthermore, the assumption of Poisson arrival may be extended to a more general renewal process This may raise the necessity for an empirical study of the interarrival time distribution [12]

[9] See also Stoll (1978a, p 1144)

[10] The need for continuous discounting results from the fact that the intertransaction times are not identically distributed

[11] If, for example, the yearly continuous discount rate is 14% (which is equivalent to 15% per annum) and the expected inter-transaction time is as large as an hour, the relevant discount factor is 0 999984

[12] A step in this direction is Garbade and Lieber (1977)

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