Valuation in Over-the-Counter Markets*

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Abstract
We provide the impact on asset prices of search-and-bargaining frictions in over-the-counter markets. Under certain conditions, prices are lower and illiquidity discounts higher when counterparties are harder to find, when sellers have less bargaining power, when the fraction of qualified owners is smaller, or when risk aversion, volatility, or hedging demand are larger. If agents face risk limits, then higher volatility leads to greater difficulty locating unconstrained buyers, resulting in lower prices. We discuss a variety of financial applications and testable implications.

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Many assets, such as mortgage-backed securities, corporate bonds, emerging-market debt, bank loans, over-the-counter (OTC) derivatives, private equity, and real estate, are traded in OTC markets. Traders in these markets search for counterparties, incurring opportunity or other costs. When counterparties meet, their bilateral relationship is strategic; prices are set through a bargaining process that reflects each investor’s alternatives to immediate trade.

We provide a theory and applications of dynamic asset pricing in OTC markets, one that directly treats search and bargaining. We show how the explicitly calculated equilibrium allocations and prices depend on investors’ search abilities, bargaining powers, risk limits, and risk aversion, and discuss a variety of financial applications and testable implications.

Under certain conditions, illiquidity discounts are lower, and prices are higher, if investors can find each other more easily, if sellers have more bargaining power, if the fraction of qualified owners is greater, if volatility is lower, or if risk aversion is lower. If agents face risk limits, then higher volatility leads to greater difficulty locating unconstrained buyers, resulting in higher illiquidity discounts and lower prices. We also indicate situations in which search frictions can lead naturally to an increase in the price of the asset, conveying to it a search-induced scarcity value.

As a baseline model, we first consider the pricing of a consol bond traded by risk-neutral agents. Investors contact one another randomly at some mean rate $\lambda$, a parameter reflecting search ability. When two agents meet, they bargain over the terms of trade based on endogenously determined outside options. Gains from trade arise from heterogeneous costs or benefits of holding assets. We then extend this model to OTC markets with risky securities, incorporating the effects of risk aversion and risk limits.

In our OTC market model, a risk-averse asset owner searches for a potential buyer when the asset ceases to be a relatively good hedge of his endowment. We show how asset prices are affected by search frictions and demonstrate how they could magnify the effective risk premium due to incomplete risk sharing, beyond that of a liquid but incomplete-markets setting.

With the imposition of risk limits, we show how search-based liquidity frictions become endogenously dependent on volatility, in that higher volatility leads to smaller equilibrium holdings, resulting in more sellers and fewer available buyers. For sellers, this leads to longer search times and a relatively unfavorable bargaining position, which in turn implies higher illiquidity price discounts.

Weill (2002) and Vayanos and Wang (2002) have extended our baseline risk-neutral model in order to treat multiple assets in the same economy, obtaining cross-sectional restrictions on asset returns. In Duffie, Gârleanu, and Pedersen (2003), which focuses on the role of marketmakers, we show that search frictions have different implications for bid-ask spreads than do information frictions. Miao (2004) provides a variant of this model. Weill (2003) studies the implications of search frictions in an extension of our model in which marketmakers’ inventories “lean against” the outside order flow. Newman and Rierson (2003) present a model in which supply shocks temporarily depress prices across correlated assets, as providers of liquidity search for long-term investors, supported by empirical evidence of issuance impacts across the European telecommunications bond market. In Duffie, Gârleanu, and Pedersen (2002), we use the modeling framework introduced here to characterize the impact on asset prices and securities lending fees of the common institution by which would-be shortsellers must locate lenders of securities before being able to sell short. Difficulties in locating lenders of shares can allow for dramatic price imperfections, as, for example, in the case of the spinoff of Palm, Incorporated, documented by Mitchell, Pulvino, and Stafford (2002) and Lamont and Thaler (2003). Further discussion of implications for over-the-counter asset pricing is provided in Section 1.

Search models have been studied extensively in the context of labor economics, starting with the “coconuts” model of Diamond (1982), and in the context of monetary economics. Our search-and-bargaining structure, in particular, is similar to that of the monetary model of Trejos and Wright (1995), but our objectives and results are different.

The paper is organized as follows. Section 1 describes several important OTC markets and provides an overview of the model’s main empirical implications and financial applications. Section 2 lays out the baseline model and results, using risk-neutral agents. Section 3 treats hedging motives for trade under risk aversion, and Section 4 calibrates the model to the OTC market for corporate bonds. Section 5 characterizes the implications of risk limits on prices and trades. Proofs and supplementary results are relegated
1 Market Implications

Our framework has several asset pricing implications for over-the-counter (OTC) markets, that is, markets characterized by bilateral negotiation, delayed by search for suitable counterparties. These price effects may be relevant for private equity, real estate, and OTC-traded financial products such as interest-rate swaps and other OTC derivatives, mortgage-backed securities, corporate bonds, government bonds, emerging-market debt, and bank loans. Exemplifying the imperfect ability to match buyers and sellers in OTC markets, traders in the market for European corporate loans have ironically described\(^1\) trade in that market as “by appointment.”

Even in the most liquid OTC markets, the relatively small price effects arising from search frictions receive significant attention by economists. For example, the market for U.S. Treasury securities, an over-the-counter market considered to be a benchmark for high liquidity, is subject to widely noted illiquidity effects that differentiate the yields of on-the-run (latest-issue) securities from those of off-the-run securities. Positions in on-the-run securities are normally available in large amounts from relatively easily found traders such as hedge funds and government-bond dealers. Because on-the-run issues can be more quickly located by short-term investors such as hedgers and speculators, they command a price premium, even over a package of off-the-run securities of identical cash flows. Ironically, the importance ascribed to this relatively small premium is explained by the exceptionally high volume of trade in this market, and also by the importance of disentangling the illiquidity impact on measured Treasury interest rates for informational purposes elsewhere in the economy. Longstaff (2002) measures relatively larger illiquidity effects on government security prices during “flights to liquidity,” which he characterizes as periods during which a large demand for quick access to a safe haven causes Treasury prices to temporarily achieve markedly higher prices than equally safe government securities that are not as easily found.

Part of the price impact represented by the spread between on-the-run and off-the-run treasuries is conveyed by shortsellers who are willing to pay a lending premium to owners of relatively easily located securities. A search-based

\(^1\)See *The Financial Times*, November 19, 2003.
theory of securities lending is developed in Duffie, Gârleanu, and Pedersen (2002). Empirical evidence of the impact on treasury prices and securities-lending premia (“repo specials”) can be found in Duffie (1996), Jordan and Jordan (1997), and Krishnamurthy (2002). Fleming and Garbade (2003) document a new U.S. Government program to improve liquidity in treasury markets by allowing alternative types of treasury securities to be deliverable in settlement of a given repurchase agreement, mitigating the costs of search for a particular issue. Related effects in equity markets are measured by Geczy, Musto, and Reed (2002), D’Avolio (2002), and Jones and Lamont (2002). Difficulties in locating lenders of shares sometimes cause dramatic price imperfections, as was the case with the spinoff of Palm, Incorporated, one of a number of such cases documented by Mitchell, Pulvino, and Stafford (2002).

The potential for much larger price impacts in relatively less liquid OTC markets is exemplified in a study of Chinese equity prices by Chen and Xiong (2001). Certain Chinese companies have two classes of shares, one exchange traded, the other consisting of “restricted institutional shares” (RIS), which can be traded only privately. The two classes of shares are identical in every other respect, including their cash flows. Chen and Xiong (2001) find that RIS shares trade at an average discount of about 80% to the corresponding exchange-traded shares. Similarly, in a study involving U.S. equities, Silber (1991) compares the prices of “restricted stock” — which, for two years, can be traded only in private among a restricted class of sophisticated investors — with the prices of unrestricted shares of the same companies. Silber (1991) finds that restricted stocks trade at an average discount of 30%, and that the discount for restricted stock is increasing in the relative size of the issue. These price discounts can be captured in our search framework, but would be difficult to explain using standard models based on asymmetric information, given that the two classes of shares are claims to the same dividend streams, and given that the publicly-traded share prices are easily observable.

Our model can be used to predict the implications of a widespread shock to the abilities or incentives of traders to take asset positions.\(^2\) Such a “wealth shock” would lead to an increase in the number of sellers and a reduction in the number of buyers. As a result, the price drops in part because of the

\(^2\)Formally, using notation explained in Section 2, such a shock could be captured by a rise in \(\mu_a\) and a drop in \(\mu_{hn}\), and/or by a rise in \(\lambda_d\) and a drop in \(\lambda_u\), where the shock is temporary if and only if \(\lambda_d\) and \(\lambda_u\) are unchanged.
higher fraction of assets held by distressed traders, but, importantly, also by
the worsened bargaining position of sellers.

As we show in Sections 3 and 5, if agents are risk averse or have risk
limits, an increase in the risk of the asset has similar implications. Higher
risk (in the form of higher dividend volatility or higher correlation between
the dividends and an agent’s endowed income) leads to larger utility losses for
distressed agents. Agents can compensate for the increased risk by reduced
position limits, but then a larger fraction of the agents must hold the risky
asset, and liquidity is further reduced because finding a buyer becomes more
difficult. Hence, shocks to volatility can lead to increased liquidity frictions
and price drops, especially if risk-management practices imply a simultaneous
tightening of position limits.

Search frictions also help explain how the relative size of an asset in the
economy may affect its price (or price-dividend ratio). Proposition 1 of this
paper shows that if a higher fraction \( s \) of the agents must hold the asset,
then the price must fall. This resembles the usual effect of a downward-
sloping demand curve.

When comparing stocks cross-sectionally, a role for search frictions arises
from the additional implications of selection of participants into a market.
Investors prefer, all else equal, to participate in the market for larger stocks.
If, for instance, the number of investors participating in the market for a
firm’s shares is proportional to the size of the company, the corresponding
shorter mean delay for locating suitable counterparties for trades in the shares
of larger firms would lead higher price-dividend ratios. Such cross-sectional
asset-pricing results are studied more directly by Weill (2002) and Vayanos
and Wang (2002), who have extended our baseline model to the case of
multiple assets and shown, among other things, that securities with a larger
free float (shares available for trade) are more liquid and have lower expected
returns, and that concentrations of trade in a favored security, may explain
some of the price difference between on-the-run and off-the-run Treasury
securities.

A different set of search-based implications for financial markets is ob-
tained in Duffie, Gärleanu, and Pedersen (2003), which studies marketmak-
ers.\(^3\) Outside investors remain able to find other investors with some search
intensity \( \lambda \), but can also find marketmakers with some intensity \( \rho \). This
\(^3\)Other search-based models of intermediation include Rubinstein and Wolinsky (1987),
framework captures the feature that investors bargain sequentially with marketmakers. The price negotiation between a marketmaker and an investor reflects the investor’s outside options, including in particular the investor’s ability to meet and trade with other investors or marketmakers. It is shown that the marketmaker’s bid-ask spread is lower if the investor can find other investors on his own more easily. Further, the spread is lower if an investor can approach other marketmakers more easily. In other words, more “sophisticated” investors are quoted tighter spreads by marketmakers. Examples can be found in the typical hub-and-spoke structure of contact among marketmakers and their customers in OTC derivative markets. This distinguishes our search theory from traditional information-based theories that predict that more sophisticated (in this setting, more informed) investors are quoted wider spreads by marketmakers (Glosten and Milgrom (1985)).

Search and bargaining describes broker-dealer behavior in OTC markets for fixed-income derivatives. In these markets, a “sales trader” and an outside customer negotiate a price, implicitly including a dealer profit margin, that is based in part on the customer’s (perceived) outside option. In this setting, the risk that customers have superior information about future interest rates is often regarded as small. The customer’s outside option depends on how easily he can find a counterparty himself and how easily he can access other dealers. As explained by Commissioner of Internal Revenue (2001) (page 13) in recent litigation regarding the portion of dealer margins on interest-rate swaps that can be ascribed to profit, dealers typically negotiate prices with outside customers that reflect the customer’s relative lack of access to other market participants. In order to trade OTC derivatives with a bank, for example, a customer must have, among other arrangements, an account and a credit clearance. Smaller customers often have an account with only one, or perhaps a few, banks, and therefore have fewer search options. Hence, a testable implication of our search framework is that (small) investors with fewer search options receive less competitive prices. We note that these investors are less likely to be informed, so that models based on informational asymmetries alone would reach the opposite prediction.

Our results have been extended to illustrate that temporary external supply imbalances may have much bigger impacts on prices than would be the case with perfectly liquid markets, and that the degree of these price impacts can be mitigated by providers of liquidity such as underwriters, hedge funds, and marketmakers. Weill (2003) uses our approach to characterize the optimal behavior of marketmakers in absorbing supply shocks in order to mitigate
search frictions by “leaning against” the outside order flow. Newman and Rierson (2003) use our approach in a search-based model of corporate bond pricing, in which large issues of credit-risky bonds temporarily raise credit spreads throughout the issuer’s sector, because providers of liquidity such as underwriters and hedge funds bear extra risk as they search for long-term investors. They provide empirical evidence of temporary bulges in credit spreads across the European Telecom debt market during 1999-2002 in response to large issues by individual firms in this sector. Studying a different set of markets, Mikkelson and Partch (1985) find empirical support for “the notion that underwriting spreads are in part compensation for the selling effort.” In particular, they find that underwriting spreads are positively related to the size of the offering.

2 Basic Search Model of Asset Prices

This section introduces a baseline risk-neutral model of an over-the-counter market, that is, a market in which agents can trade only when they meet each other, and in which transaction prices are bargained. Some features of this baseline model and the calculations leading up to Proposition 1 are in common with Duffie, Gârleanu, and Pedersen (2003). In later sections of this paper, we introduce several salient features of financial markets, namely risk, risk aversion, and risk limits.

Agents are risk-neutral and infinitely lived, with a constant time-preference rate \( \beta > 0 \) for consumption of a single non-storable numeraire good.\(^4\)

An agent can invest in a bank account — which can also be interpreted as a “liquid” security — with a risk-free interest rate of \( r \). As a form of credit constraint that rules out “Ponzi schemes,” the agent must enforce some lower bound on the liquid wealth process \( W \). We take \( r = \beta \) in this baseline model, since agents are risk neutral.

Agents may trade a long-lived asset in an over-the-counter market. The asset can be traded only bilaterally, when in contact with a counterparty. We begin for simplicity by taking the OTC asset to be a consol, which pays one unit of consumption per unit of time. Later, when introducing the effects of risk limits, or risk aversion, we generalize to random dividend processes.

\(^4\)Specifically, an agent’s preferences among adapted finite-variation cumulative consumption processes are represented by the utility \( E \left( \int_0^\infty e^{-\beta t} \, dC_t \right) \) for a cumulative consumption process \( C \), whenever the integral is well defined.
An agent is characterized by an intrinsic preference for asset ownership that is “high” or “low.” A low-type agent, when owning the asset, has a holding cost of $\delta$ per time unit. A high-type agent has no such holding cost. We could imagine this holding cost to be a shadow price for ownership by low-type agents, due, for example, to (i) low personal liquidity, that is, a need for cash, (ii) high financing costs, (iii) adverse correlation of asset returns with endowments (formalized in Section 3), (iv) a relative tax disadvantage, as studied by Dai and Rydqvist (2003) in an empirical analysis of search-and-bargaining effects in the context of tax trading,\(^5\) or (v) a relatively low personal use for the asset, as may happen, for example, for certain durable consumption goods such as homes. The agent’s intrinsic type is a Markov chain, switching from low to high with intensity $\lambda_u$, and back with intensity $\lambda_d$. The intrinsic-type processes of any two agents are independent.\(^6\)

A fraction $s$ of agents are initially endowed with one unit of the asset. Investors can hold at most one unit of the asset and cannot shortsell. Because agents have linear utility, it is without much loss of generality that we restrict attention to equilibria in which, at any given time and state of the world, an agent holds either 0 or 1 unit of the asset. Hence, the full set of agent types is $T = \{ho, hn, lo, ln\}$, with the letters “h” and “l” designating the agent’s current intrinsic liquidity state as high or low, respectively, and with “o” or “n” indicating whether the agent currently owns the asset or not, respectively.

We suppose that there is a “continuum” (a non-atomic finite measure space) of agents, and let $\mu_\sigma(t)$ denote the fraction at time $t$ of agents of type $\sigma \in T$, so that

$$1 = \mu_{ho}(t) + \mu_{hn}(t) + \mu_{lo}(t) + \mu_{ln}(t). \quad (1)$$

Equating the per-capita supply $s$ with the fraction of owners gives

$$s = \mu_{ho}(t) + \mu_{lo}(t). \quad (2)$$

\(^5\)Dai and Rydqvist (2003) study tax trading between a small group of foreign investors and a larger group of domestic investors. They find that investors from the “long side of the market” get part of the gains from trade, under certain conditions, which they interpret as evidence of a search-and-bargaining equilibrium.

\(^6\)All random variables are defined on a probability space $(\Omega, \mathcal{F}, Pr)$ with corresponding filtration $\{\mathcal{F}_t : t \geq 0\}$ of sub-$\sigma$-algebras of $\mathcal{F}$ satisfying the usual conditions, as defined by Protter (1990). The filtration represents the resolution over time of information commonly available to investors.
An agent finds a counterparty with an intensity $\lambda$, reflecting the efficiency of the search technology. We assume the counterparty found is randomly selected from pool of other agents, so that the probability that the counterparty is of type $\sigma$ is $\mu_\sigma$. Thus, the intensity of finding a type-$\sigma$ investor is $\lambda \mu_\sigma$. Hence, assuming that the law of large numbers applies, $hn$ investors contact $lo$ investors at a total (almost sure) rate of $\lambda \mu_\sigma$ and, since $lo$ investors contact $hn$ investors at the same total rate, the total rate of such counterparty matchings is $2\lambda \mu_\sigma$. Duffie and Sun (2004) provide a discrete-time search-and-matching model in which the exact law of large numbers for a continuum of agents indeed applies in this sense.

To solve the model, we proceed in two steps. First, we use the insight that the only form of encounter that provides gains from trade is one in which low-type owners sell to high-type non-owners. From bargaining theory, we know (see Appendix A) that at these encounters, trade occurs immediately. We can therefore determine the steady-state asset allocations without reference to prices. Given the steady-state masses $\mu$, we consider an investor’s lifetime utility, depending on the investor’s type, the bargaining problem, and the resulting price.

In equilibrium, the rates of change of the fractions of the respective investor types are

$$
\dot{\mu}_{lo}(t) = -2\lambda \mu_{lo}(t)\mu_{ho}(t) - \lambda_u \mu_{lo}(t) + \lambda_d \mu_{ho}(t)
$$

$$
\dot{\mu}_{ho}(t) = 2\lambda \mu_{ho}(t)\mu_{lo}(t) - \lambda_u \mu_{ho}(t) + \lambda_u \mu_{lo}(t)
$$

$$
\dot{\mu}_{hn}(t) = 2\lambda \mu_{hn}(t)\mu_{lo}(t) - \lambda_d \mu_{hn}(t) + \lambda_u \mu_{ho}(t)
$$

$$
\dot{\mu}_{ln}(t) = 2\lambda \mu_{ln}(t)\mu_{lo}(t) - \lambda_u \mu_{ln}(t) + \lambda_d \mu_{hn}(t).
$$

The intuition for, say, the first equation in (3) is straightforward: Whenever an $lo$ agent meets an $hn$ investor, he sells his asset and is no longer an $lo$ agent. This (together with the law of large numbers) explains the first term on the right hand side of (3). The second term is due to intrinsic type changes in which $lo$ investors become $ho$ investors, and the last term is due to intrinsic type changes from $ho$ to $lo$.

Duffie, Gårleanu, and Pedersen (2003) show that there is a unique stable steady-state solution, that is, a constant solution with $\dot{\mu}(t) = 0$. The steady state is computed by using (1)–(2) and the fact that $\mu_{lo} + \mu_{hn} = \lambda_d / (\lambda_u + \lambda_d)$, in order to write the first equation in (3) as a quadratic equation in $\mu_{lo}$, given as Appendix equation (C.1).
Having determined the steady-state fractions of investor types, we compute the investors’ equilibrium intensities of finding counterparties of each type and, hence, their utilities for remaining lifetime consumption, as well as the bargained price $P$. For a particular agent, his utility depends on his current type, $\sigma(t) \in T$, and the wealth $W(t)$ in his bank account. Specifically, lifetime utility is $W(t) + V_{\sigma(t)}$, where, for each investor type $\sigma$ in $T$, $V_{\sigma}$ is a constant to be determined.

In steady state, the rate of growth of any agent’s expected indirect utility must be the discount rate $r$, which yields the steady-state equations

\[
\begin{align*}
0 &= rV_{lo} - \lambda_u(V_{ho} - V_{lo}) - 2\lambda\mu_{hn}(P - V_{lo} + V_{ln}) - (1 - \delta) \\
0 &= rV_{ln} - \lambda_u(V_{hn} - V_{ln}) \\
0 &= rV_{ho} + \lambda_d(V_{ho} - V_{lo}) - 1 \\
0 &= rV_{hn} + \lambda_d(V_{hn} - V_{ln}) - 2\lambda\mu_{lo}(V_{ho} - V_{hn} - P).
\end{align*}
\]

The price is determined through bilateral bargaining. A high-type non-owner pays at most his reservation value $\Delta V_h = V_{ho} - V_{hn}$ for obtaining the asset, while a low-type owner requires a price of at least $\Delta V_l = V_{lo} - V_{ln}$. Nash bargaining, or the Rubinstein-type game considered in Appendix A, implies that the bargaining process results in the price

\[
P = \Delta V_l(1 - q) + \Delta V_h q,
\]

where $q \in [0,1]$ is the bargaining power of the seller. The linear system (4)-(5) of equations has a unique solution, with

\[
P = \frac{1 - \delta}{r - \delta} \frac{r(1 - q) + \lambda_d + 2\lambda\mu_{lo}(1 - q) + \lambda_u + 2\lambda\mu_{hn}q}{1 - \lambda_d + 2\lambda\mu_{lo}(1 - q) + \lambda_u + 2\lambda\mu_{hn}q}.
\]

This price (6) is the present value, $1/r$, of dividends, reduced by an illiquidity discount. The price is lower and the discount is larger, ceteris paribus, if the distressed owner has less hope of switching type (lower $\lambda_u$), if it is more difficult for the owner to find other buyers (lower $\mu_{hn}$), if the buyer may more suddenly need liquidity himself (higher $\lambda_d$), if it is easier for the buyer to find other sellers (higher $\mu_{lo}$), or if the seller has less bargaining power (lower $q$).

These intuitive results are based on partial derivatives of the right-hand side of (6) — in other words, they hold when a parameter changes without influencing any of the others. We note, however, that the steady-state type fractions $\mu$ themselves depend on $\lambda_d$, $\lambda_u$, and $\lambda$. The following proposition offers a characterization of the equilibrium steady-state effect of changing each parameter.
Proposition 1 The steady-state equilibrium price $P$ is decreasing in $\delta$, $s$, and $\lambda_d$, and is increasing in $\lambda_u$ and $q$. Further, if $s < \lambda_u/(\lambda_u + \lambda_d)$, then $P \to 1/r$ as $\lambda \to \infty$, and $P$ is increasing in $\lambda$ for all $\lambda \geq \lambda^*$ depending on the other parameters of the model.

The condition that $s < \lambda_u/(\lambda_u + \lambda_d)$ means that, in steady state, there is less than one unit of asset per agent of high intrinsic type. While this corresponds to the intuitively anticipated increase in market value with increasing bilateral contact rate, the alternative is also possible. With $s > \lambda_u/(\lambda_u + \lambda_d)$, the marginal investor in perfect markets has the relatively lower reservation value, and search frictions lead to a “scarcity value.” For example, a high-type investor in an illiquid OTC market could pay more than the Walrasian price for the asset because it is hard to find, and given no opportunity to exploit the effect of immediate competition among many sellers.

It can be checked that the above results extend to risky dividends, for instance in the following ways: (i) If the cumulative dividend is risky with constant drift $\nu$, then the equilibrium is that for a consol bond with dividend rate of $\nu$; (ii) if the dividend rate and illiquidity cost are proportional to a process $X$ with $E_t[X(t+u)] = X(t)e^{\nu u}$, for some constant growth rate $\nu$, then the price and value functions are also proportional to $X$, with factors of proportionality given as above, with $r$ replaced by $r - \nu$; (iii) if the dividend-rate process $X$ satisfies $E_t[X(t+u)] = X(t) + mu$ for a constant drift $m$ (and if illiquidity costs are constant), then the continuation values are of the form $X(t)/r + v_\sigma$ for owners and $v_\sigma$ for non-owners, where the constants $v_\sigma$ are computed in a similar manner.

Next, we model risky dividends, using cases (i) and (iii) above, in the context of risk aversion and risk limits.

3 Risk-Aversion

This section provides a version of the asset-pricing model with risk aversion, in which the motive for trade between two agents is the different extent to which they derive hedging benefits from owning the asset. We provide a sense in which this economy can be interpreted in terms of the baseline economy of Section 2.

Agents have constant-absolute-risk-averse (CARA) additive utility, with a coefficient $\gamma$ of absolute risk aversion and with time preference at rate $\beta$. 

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An asset has a cumulative dividend process $D$ satisfying

$$dD(t) = m_D dt + \sigma_D dB(t),$$

where $m_D$ and $\sigma_D$ are constants, and $B$ is a standard Brownian motion with respect to the given probability space and filtration $\mathcal{F}_t$. Agent $i$ has a cumulative endowment process $\eta^i$, with

$$d\eta^i(t) = m_\eta dt + \sigma_\eta dB^i(t),$$

where the standard Brownian motion $B^i$ is defined by

$$dB^i(t) = \rho^i(t) dB(t) + \sqrt{1 - \rho^2(t)} dZ^i(t),$$

for a standard Brownian motion $Z^i$ independent of $B$, and where $\rho^i(t)$ is the “instantaneous correlation” between the asset dividend and the endowment of agent $i$. We model $\rho^i$ as a two-state Markov chain with states $\rho_h$ and $\rho_l > \rho_h$. The intrinsic type of an agent is identified with this correlation parameter. An agent $i$ whose intrinsic type is currently high (that is, with $\rho^i(t) = \rho_h$) values the asset more highly than does a low-intrinsic-type agent, because the increments of the high-type endowment have lower conditional correlation with the asset’s dividends. As in the baseline model of Section 2, agents’ intrinsic types are pairwise-independent Markov chains, switching from $l$ to $h$ with intensity $\lambda_u$, and from $h$ to $l$ with intensity $\lambda_d$. An agent owns either $\theta_n$ or $\theta_o$ units of the asset, where $\theta_n < \theta_o$. For simplicity, no other positions are permitted, which entails a loss in generality. Agents can trade the OTC security only when they meet, as previously. The agent type space is $T = \{lo, ln, ho, hn\}$. In this case, the symbols ‘o’ and ‘n’ indicate large and small owners, respectively. Given a total supply $\Theta$ of shares per investor, market clearing requires that

$$(\mu_{lo} + \mu_{ho})\theta_o + (\mu_{ln} + \mu_{hn})\theta_n = \Theta,$$

which, using (1), implies that the fraction of large owners is

$$\mu_{lo} + \mu_{ho} = s \equiv \frac{\Theta - \theta_n}{\theta_o - \theta_n}.$$
\( \sigma(t) \in \{ho, lo\} \) and otherwise \( \theta(t) = \theta_n \). We suppose that there is a perfectly liquid “money-market” asset with a constant risk-free rate of return \( r \), which, for simplicity, is assumed to be determined outside of the model, and with a perfectly elastic supply, as is typical in the literature treating multi-period asset-pricing models based on CARA utility, such as Wang (1994). The agent’s money-market wealth process \( W \) therefore satisfies

\[
dW(t) = (rW(t) - c(t)) \, dt + \theta(t) \, dD(t) + d\eta(t) - P \, d\theta(t),
\]

where \( c \) is the agent’s consumption process, \( \eta \) is the agent’s cumulative endowment process, and \( P \) is the asset price per share (which is constant in the equilibria that we examine). The last term thus captures payments in connection with trade. The consumption process is required to satisfy measurability, integrability, and transversality conditions stated in Appendix C.

We consider a steady-state equilibrium, and let \( J(w, \sigma) \) denote the indirect utility of an agent of type \( \sigma \in \{lo, ln, ho, hn\} \) with current wealth \( w \). Assuming sufficient differentiability, the Hamilton-Jacobi-Bellman (HJB) equation for an agent of current type \( lo \) is

\[
0 = \sup_{\tau \in \mathbb{R}} \left\{ -e^{-\gamma \tau} + J_w(w, lo)(rw - \tau + \theta_o m_D + m_\eta) \right. \\
+ \frac{1}{2} J_{ww}(w, lo)(\theta_o^2 \sigma_D^2 + \sigma_\eta^2 + 2 \rho_l \theta_o \sigma_D \sigma_\eta) - \beta J(w, lo) \\
+ \lambda_u [J(w, ho) - J(w, lo)] + 2 \lambda \mu_h [J(w + P \bar{\theta}, ln) - J(w, lo)],
\]

where \( \bar{\theta} = \theta_o - \theta_n \). The HJB equations for the other agent types are similar. Under technical regularity conditions found in Appendix C, we verify that

\[
J(w, \sigma) = -e^{-r \gamma (w + a_\sigma + \bar{a})},
\]

where

\[
\bar{a} = \frac{1}{r} \left( \frac{\log r}{\gamma} + m_\eta - \frac{1}{2} r \gamma \sigma_\eta^2 - \frac{r - \beta}{r \gamma} \right),
\]

and where, for each \( \sigma \), the constant \( a_\sigma \) is determined as follows. The first-order conditions of the HJB equation of an agent of type \( \sigma \) imply an optimal consumption rate of

\[
\bar{c} = -\frac{\log(r)}{\gamma} + r(w + a_\sigma + \bar{a}).
\]
Inserting this solution for $\bar{c}$ into the respective HJB equations yields a system of equations characterizing the coefficients $a_\sigma$.

The price $P$ is determined using Nash bargaining with seller bargaining power $q$, similar in spirit to the baseline model of Section 2. Given the reservation values of buyer and seller implied by $J(w, \sigma)$, the bargaining price satisfies $a_{lo} - a_{ln} \leq P\bar{\theta} \leq a_{ho} - a_{hn}$. The following result obtains.

**Proposition 2** In equilibrium, an agent’s consumption is given by (14), the value function is given by (12), and $(a_{lo}, a_{ln}, a_{ho}, a_{hn}, P) \in \mathbb{R}^5$ solve

$$0 = ra_{lo} + \lambda_u e^{-r\gamma(a_{ho} - a_{lo})} - \frac{1}{r^\gamma} + 2\lambda\mu_{hn} e^{-r\gamma(P\bar{\theta} + a_{ln} - a_{lo})} - \frac{1}{r^\gamma} - (\kappa(\theta_o) - \theta_o \bar{\delta})$$
$$0 = ra_{ln} + \lambda_u e^{-r\gamma(a_{ho} - a_{ln})} - \frac{1}{r^\gamma} - (\kappa(\theta_n) - \theta_n \bar{\delta})$$
$$0 = ra_{ho} + \lambda_d e^{-r\gamma(a_{lo} - a_{ho})} - \frac{1}{r^\gamma} - \kappa(\theta_o)$$
$$0 = ra_{hn} + \lambda_d e^{-r\gamma(a_{ln} - a_{hn})} - \frac{1}{r^\gamma} + 2\lambda\mu_{lo} e^{-r\gamma(-P\bar{\theta} + a_{ho} - a_{hn})} - \frac{1}{r^\gamma} - \kappa(\theta_n),$$

with

$$\kappa(\theta) = \theta m_D - \frac{1}{2} r^\gamma \left( \theta^2 \sigma_D^2 + 2\rho \theta \sigma_D \sigma_\eta \right)$$ (16)

$$\bar{\delta} = r^\gamma (\rho_l - \rho_h) \sigma_D \sigma_\eta > 0,$$ (17)

as well as the Nash bargaining equation,

$$q \left(1 - e^{r\gamma(P\bar{\theta} - (a_{lo} - a_{ln}))}\right) = (1 - q) \left(1 - e^{r\gamma(-P\bar{\theta} + a_{ho} - a_{hn})}\right).$$ (18)

A natural benchmark is the limit price associated with vanishing search frictions, characterized as follows.

**Proposition 3** If $s < \mu_{hn} + \mu_{ho}$, then, as $\lambda \to \infty$,

$$P \to \frac{\kappa(\theta_o) - \kappa(\theta_n)}{r\bar{\theta}},$$ (19)
In order to compare the equilibrium for this model to that of the baseline model, we use the linearization $e^z - 1 \approx z$, which leads to

$$
0 \approx ra_{lo} - \lambda_u(a_{ho} - a_{lo}) - 2\lambda u_{hn}(P\tilde{\theta} - a_{lo} + a_{ln}) - (\kappa(\theta_o) - \theta_o\tilde{\delta}) \\
0 \approx ra_{ln} - \lambda_u(a_{hn} - a_{ln}) - (\kappa(\theta_n) - \theta_n\tilde{\delta}) \\
0 \approx ra_{ho} - \lambda_d(a_{lo} - a_{ho}) - \kappa(\theta_o) \\
0 \approx ra_{hn} - \lambda_d(a_{in} - a_{hn}) - 2\lambda u_{lo}(a_{ho} - a_{hn} - P\tilde{\theta}) - \kappa(\theta_n) \\
P\tilde{\theta} \approx (1 - q)(a_{lo} - a_{ln}) + q(a_{ho} - a_{hn}).
$$

These equations are of the same form as those in Section 2 for the indirect utilities and asset price in an economy with risk-neutral agents, with dividends at rate $\kappa(\theta_o)$ for large owners and dividends at rate $\kappa(\theta_n)$ for small owners, and with illiquidity costs given by $\tilde{\delta}$ of (17). In this sense, we can view the baseline model as a risk-neutral approximation of the effect of search illiquidity in a model with risk aversion. The approximation error goes to zero for small agent heterogeneity (that is, small $\rho_l - \rho_h$). Solving specifically for the price $P$ in the associated linear model, we have

$$
P = \frac{\kappa(\theta_o) - \kappa(\theta_n)}{r\tilde{\theta}} - \frac{\tilde{\delta}}{r \frac{r(1 - q) + \lambda_d + 2\lambda u_{lo}(1 - q)}{r + \lambda_d + 2\lambda u_{lo}(1 - q) + \lambda_u + 2\lambda u_{hn}q}}. \tag{20}
$$

We see that the price is the perfect-market price from Proposition 3, less an illiquidity discount. The expression (17) for $\tilde{\delta}$ shows that the illiquidity cost in the baseline model can be interpreted as a hedging-based incentive to trade. This incentive is increasing in the risk aversion $\gamma$, the endowment-correlation difference $\rho_l - \rho_h$, and the volatilities of dividends and endowments.

### 4 Numerical Calibration

We select parameters for a numerical illustration of the implications of the model for a market roughly like that of the over-the-counter market for corporate bonds. Table 1 contains the exogenous parameters for the base-case risk-neutral model, and Table 2 contains the resulting steady-state fractions of each type and the price. As is seen in the Table 1, $\lambda = 625$ which implies

---

7The error introduced by the linearization is in $O((a_{ho} - a_{lo})^2 + (a_{hn} - a_{ln})^2 + (P\tilde{\theta} - a_{lo} + a_{ln})^2)$, which, by continuity, is in $O((\rho_l - \rho_h)^2)$ for a compact parameter space. Hence, if $\rho_l - \rho_h$ is small, then the approximation error is an order of magnitude smaller, of the order $(\rho_l - \rho_h)^2$. 

---
that an agent meets $2\lambda = 1250$ other agents each year, that is, $1250/250 = 5$ agents a day. Given the equilibrium mass of potential buyers, the average time needed to sell is $250 \times (2\lambda \mu_{hn})^{-1} = 1.8$ days. The switching intensities $\lambda_u$ and $\lambda_d$ mean that a high-type investor remains a high type for an average of 2 years, while an illiquid low type remains a low type for an average of 0.2 years. These intensities imply an annual turnover of $2\lambda \mu_{lo} \mu_{hn}/s = 49\%$ which roughly matches the median annual bond turnover of 51.7% reported by Edwards, Harris, and Piwowar (2004). The fraction of bond investors holding a bond is $s = 0.8$, the discount and interest rates are 5%, sellers and buyers each have half of the bargaining power $q = 0.5$, and the illiquidity cost is $\delta = 2.5$, as implied by the risk aversion parameters discussed below.

We see that only a small fraction of the asset, $\mu_{lo}/s = 0.0028/0.8 = 0.35\%$ of the total supply, is mis-allocated to low intrinsic types because of search frictions. The equilibrium asset price, 18.38, however, is substantially below the perfect market price of $1/r = 20$, reflecting a significant impact of illiquidity on the price, despite the relatively small impact on the asset allocation. Stated differently, the yield on the bond is its cashflow of 1 divided by the price, $1/18.38 = 5.44\%$, which is 44 basis points above the liquid-market yield $r$. This yield spread is of the order of magnitude of the corporate-bond liquidity spread estimated by Longstaff, Mithal, and Neis (2004), of between 9 and 65 basis points, depending on the specification and reference risk-free rate.

The base-case risk-neutral model specified in Table 1 corresponds to a model with risk-averse agents with additional parameters given in Table 3 in the following sense. First, the “illiquidity cost” $\delta = \overline{\delta} = 2.5$ of low-intrinsic-type is that implied by (17) from the hedging costs of the risk-aversion model.
Second, the total amount $\Theta$ of shares and the investor positions, $\theta_o$ and $\theta_n$, imply the same fraction $s = 0.8$ of the population holding large positions, using (11). The investor positions that we adopt for this calibration are realistic in light of the positions adopted by high and low type investors in the associated Walrasian (perfect) market with unconstrained trade sizes, which, as shown in Appendix B, has an equilibrium large-owner position size of 17,818 shares and a small-owner position size of $-2,182$ shares. Third, the certainty-equivalent dividend-rate per share, $(\kappa(\theta_o) - \kappa(\theta_n))/(\theta_o - \theta_n) = 1$, is the same as that of the baseline model. Finally, the mean parameter $\mu_D = 1$ mean and volatility parameter $\sigma_D = 0.5$ of the asset’s risky dividend implies that the standard deviation of yearly returns on the bond is approximately $\sigma_D/P = 2.75\%$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\rho_h$</th>
<th>$\rho_l$</th>
<th>$\mu_\eta$</th>
<th>$\sigma_\eta$</th>
<th>$\mu_D$</th>
<th>$\sigma_D$</th>
<th>$\Theta$</th>
<th>$\theta_o$</th>
<th>$\theta_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>-0.5</td>
<td>0.5</td>
<td>10000</td>
<td>10000</td>
<td>1</td>
<td>0.5</td>
<td>16000</td>
<td>20000</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3: Additional base-case parameters with risk-aversion.

In order to illustrate the sensitivity of the results to various parameters, we plot several graphs of the illiquidity discount. Figure 1 shows how prices are reduced by illiquidity as captured by the search intensity $\lambda$. The graph reflects the fact that, as the search intensity $\lambda$ becomes large, the allocation and price become equal to the perfect-market ones (Propositions 1 and 3).

Figures 2 and 3 show how prices are discounted for illiquidity, relative to the perfect-markets price, by an amount that depends on risk aversion and volatility. As we vary the parameters in these figures, we compute both the equilibrium solution of the risk-aversion model and the solution of the associated baseline risk-neutral model that is obtained by the linearization (20), taking $\delta$ from (17) case by case.

We see that the illiquidity discount increases with risk aversion and volatility, and that both effects are large for our benchmark parameters. The illiquidity discount ranges between $1\%$ and $40\%$, depending on the risk and risk aversion.

Also, these figures show that the equilibrium price of the OTC market model with risk aversion is generally well approximated by our closed-form expression (20).
5 Risk Limits and Endogenous Position Size

In this section, we consider the impact of risk limits and illiquidity on prices and on the equilibrium allocation of risky assets. Specifically, we consider explicit limits on the volatilities of agents’ positions, an idealization of risk limits imposed in practice,\(^8\) such as bounds on volatility or value at risk (VaR).

Consider the following variant of the baseline model of Section 2. Agents have the same preferences, including intrinsic-type processes, and the search technology of the baseline model. Rather than an asset paying a constant dividend rate, however, we suppose that the illiquid asset has a dividend-rate process \(X\) that is Lévy, meaning that it has independent and identically distributed increments over non-overlapping time periods of equal lengths. Examples include Brownian motions, simple and compound Poisson pro-

---

\(^8\)In practice, risk limits reflect agency costs, financial distress costs, and other costs that we do not model here.
Figure 2: Proportional price reduction relative to the perfect-market price, as a function of the investor risk aversion $\gamma$. The dashed line corresponds to the model with risk-averse agents (Equations (15)–(18)). The solid line corresponds to the linearized model (Equation (20)), in which the parameters $\bar{\delta}$ and $\kappa$ change with $\gamma$.

We consider economies in which counterparties choose to trade at a price $P(X(t))$ at time $t$, for some Lipschitz function $P(\cdot)$ that we shall calculate in equilibrium. The total gain in market value associated with holding one
Figure 3: Proportional price reduction relative to the perfect-market price, as a function of a volatility scaling factor that scales both $\sigma_\eta$ and $\sigma_D$. The dashed line corresponds to the model with risk-averse agents (Equations (15)–(18)). The solid line corresponds to the linearized model (Equation (20)), in which the parameters $\bar{\delta}$ and $\kappa$ change with $\sigma_\eta$ and $\sigma_D$.

The dividend process $X$ is integrable with respect to $t$ over compact time intervals since, without loss of generality, a Lévy process may be taken to be a right-continuous left-limits process.
policy limit $\sigma$, in that\(^{10}\)

$$\lim_{u \to 0^+} \frac{1}{u} \text{var}_t (\theta G_{t,u}) \leq \sigma^2,$$  
(24)

replacing the position limits of 0 and 1 used in the baseline model.

With only these adjustments of the baseline model, namely the introduction of risky dividends and risk limits on positions, we anticipate an equilibrium asset price per share of the form

$$P(X(t)) = \frac{X(t)}{r} + p,$$  
(25)

for a constant $p$ to be determined. The portion $X(t)/r$ of the price that depends on $X$ is the same as that in an economy with no liquidity effects, because illiquidity losses do not depend on $X(t)$.

The conjectured form (25) of the price process has constant volatility, so we conjecture an equilibrium in which agents are either long or short by a fixed position size $\theta > 0$ to be determined. These holdings are determined so that a high-type agent holds as large a long (positive) position as the risk limits allow, while a low-type agent holds as large a short position as allowed. (The model remains tractable if one also imposes a short-selling restriction or cost.) The total supply of shares per investor is some constant $\Theta$.

The masses of the four types of agents evolve according to (3). Equation (1) continues to hold, and market clearing implies that

$$(\mu_{lo} + \mu_{ho} - \mu_{ln} - \mu_{hn})\theta = \Theta,$$  
(26)

that is,

$$\mu_{lo} + \mu_{ho} = s = \frac{\Theta}{2\theta} + \frac{1}{2},$$  
(27)

where we have used (1). Hence, one can solve for the equilibrium masses by exploiting the solution obtained for the baseline model of Section 2, but, in this case, the fraction $s$ of long position holders is endogenous.\(^{11}\)

\(^{10}\)Because the cumulative dividend $\int_t^{t+u} X(s) \, ds$ is absolutely continuous with respect to $u$, this instantaneous volatility measure is determined by the limiting variance of $[P(X(t+u)) - P(X(t))]^2 / u$, and the dividend part of the gain plays no role in this restriction.

\(^{11}\)Parameter restrictions must be imposed to ensure that $s \leq 1$. In equilibrium, it is necessary and sufficient that $\Theta \sigma_X \leq r\bar{\sigma}$.  

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The steady-state equilibrium price is of the conjectured form (25), and the indirect utility of an investor of type $\sigma$ is of the form

$$V(X(t), \sigma) = \theta_{\sigma} \frac{X(t)}{r} + \theta v_{\sigma},$$

(28)

where $\theta_{\sigma}$ is either $\theta$ or $-\theta$, depending on the type, and where the type-dependent utility coefficient $v_{\sigma}$ is to be determined. The coefficients for the price and value functions are solved similarly to (4) and (5), in that

$$0 = r v_{lo} - \lambda_u (v_{ho} - v_{lo}) - 2\lambda \mu_{hn} (2p - v_{lo} + v_{ln}) - \left(\frac{m}{r} - \delta\right)$$

$$0 = r v_{ln} - \lambda_u (v_{hn} - v_{ln}) + \left(\frac{m}{r} - \delta\right)$$

$$0 = r v_{ho} - \lambda_d (v_{lo} - v_{ho}) - \frac{m}{r}$$

$$0 = r v_{hn} - \lambda_d (v_{ln} - v_{hn}) - 2\lambda \mu_{ho} (v_{ho} - v_{hn} - 2p) + \frac{m}{r}$$

$$2p = (v_{lo} - v_{ln}) (1 - q) + (v_{ho} - v_{hn}) q.$$  

In particular,

$$P(x) = \frac{1}{r} x + \frac{m}{r^2} \frac{\delta}{r} r (1 - q) + \lambda_d + 2\lambda \mu_{lo} (1 - q) + \frac{m}{r}$$

Thus, the volatility of the price is $\sigma_X/r$, so the largest admissible security position size is

$$\theta = \frac{r \bar{\sigma}}{\sigma_X}.$$  

(30)

Because of search delays, the equilibrium position size $\theta$ decreases with the volatility of the asset, which in turn implies the following impact of search with risk limits on the asset price.

**Proposition 4** For a given bargaining power $q$, fix the unique equilibria associated with two economies that differ only with respect to the dividend volatility coefficient, $\sigma_X$. The larger dividend volatility is associated with longer expected search times for selling, and a lower asset price.

This inverse dependence of the price on the volatility of the asset is a liquidity effect, brought about by a reduction in the risk-taking capacity of an investor relative to the total risk to be held. A larger volatility thus implies a smaller quantity of agents whose risk capacity qualifies them to buy the asset (that is, fewer liquid investors who do not already own the asset).
A Explicit Bargaining Game

The setting considered here is that of Section 2, with two exceptions. First, agents can interact only at discrete moments in time, $\Delta_t$ apart. Later, we return to continuous time by letting $\Delta_t$ go to zero. Second, the bargaining game is modeled explicitly.

We follow Rubinstein and Wolinsky (1985) and others in modeling an alternating-offers bargaining game, making the adjustments required by the specifics of our setup. When two agents are matched, one of them is chosen randomly — the seller with probability $\hat{q}$, the buyer with probability $1 - \hat{q}$ — to suggest a trading price. The other either rejects or accepts the offer, immediately. If the offer is rejected, the owner receives the dividend from the asset during the current period. At the next period, $\Delta_t$ later, one of the two agents is chosen at random, independently, to make a new offer. The bargaining may, however, break down before a counteroffer is made. A breakdown may occur because either of the agents changes valuation type, whence there are no longer gains from trade. A breakdown may also occur if one of the agents meets yet another agent, and leaves his current trading partner. The latter reason for breakdown is only relevant if agents are allowed to search while engaged in negotiation.

We consider first the case in which agents can search while bargaining. We assume that, given contact with an alternative partner, they leave the present partner in order to negotiate with the newly found one. The offerer suggests the price that leaves the other agent indifferent between accepting and rejecting it. In the unique subgame perfect equilibrium, the offer is accepted immediately (Rubinstein (1982)). The value from rejecting is associated with the equilibrium strategies being played from then onwards. Letting $P_o$ be the price suggested by the agent of type $\sigma$ with $\sigma \in \{lo, hn\}$, letting $\bar{P} = \hat{q}P_{lo} + (1 - \hat{q})P_{hn}$, and making use of the motion laws of $V_{lo}$ and $V_{hn}$, we have:

\[
\begin{align*}
  P_{hn} - \Delta V_t &= e^{-(r+\lambda_d+\lambda_u+2\lambda\mu_{lo}+2\lambda\mu_{hn})\Delta_t} (P - \Delta V_t) + O(\Delta^2_t) \\
  -P_{lo} + \Delta V_h &= e^{-(r+\lambda_d+\lambda_u+2\lambda\mu_{lo}+2\lambda\mu_{hn})\Delta_t} (-\bar{P} + \Delta V_h) + O(\Delta^2_t)
\end{align*}
\]

These prices, $P_{hn}$ and $P_{lo}$, have the same limit $P = \lim_{\Delta_t \to 0} P_{hn} = \lim_{\Delta_t \to 0} P_{lo}$. The two equations above readily imply that the limit price and limit value
functions satisfy
\[ P = \Delta V_l (1 - q) + \Delta V_h q, \tag{A.1} \]
with
\[ q = \hat{q}. \tag{A.2} \]

This result is interesting because it shows that the seller’s bargaining power, \( q \), does not depend on the parameters — only on the likelihood that the seller is chosen to make an offer. In particular, an agent’s intensity of meeting other trading partners does not influence \( q \). This is because one’s own ability to meet an alternative trading partner: (i) makes oneself more impatient, and (ii) also increases the partner’s risk of breakdown, and these two effects cancel out.

This analysis shows that the bargaining outcome used in our model can be justified by an explicit bargaining procedure. We note, however, that other bargaining procedures lead to other outcomes. For instance, if agents cannot search for alternative trading partners during negotiations, then the same price formula (A.1) applies with
\[ q = \frac{\hat{q}(r + \lambda_u + \lambda_d + 2\lambda \mu_l)}{\hat{q}(r + \lambda_u + \lambda_d + 2\lambda \mu_l) + (1 - \hat{q})(r + \lambda_u + \lambda_d + 2\lambda \mu_h)}. \tag{A.3} \]

This bargaining outcome would lead to a similar solution for prices, but the comparative-static results would change, since the bargaining power \( q \) would depend on the other parameters.

## B Walrasian Equilibrium with Risk Aversion

This section derives the competitive equilibrium with risk averse agents (as in Section 3) who can immediately trade any number of risky securities. We note that this is different from a competitive market with fixed exogenous position sizes, that is, it is different from the limit considered in Proposition 3.

Suppose that the Walrasian price is constant at \( P \), that is, agents can trade instantly at this price. An agent’s total wealth — cash plus the value of his position in risky assets — is denoted by \( \tilde{W} \). If an agent chooses to hold \( \theta(t) \) shares at any time \( t \), then the wealth-dynamics equation is
\[ d\tilde{W}_t = (r\tilde{W}_t - \theta_t P - c_t) dt + \theta_t dD_t + d\eta. \]
The HJB equation for an agent of intrinsic type \( \sigma \in \{h, l\} \) is

\[
0 = \sup_{\tau, \theta} \left\{ J_w(w, \sigma)(rw - \tau + \theta(m_D - rP) + m_\eta) 
+ \frac{1}{2} J_{ww}(w, \sigma)(\theta^2 \sigma_D^2 + \sigma_D^2 + 2 \rho_\sigma \theta \sigma_D \sigma_\eta) 
+ \lambda(\sigma, \sigma')[J(w, \sigma) - J(w, \sigma')] - e^{-\gamma t} - \beta J(w, \sigma) \right\},
\]

where \( \lambda(\sigma, \sigma') \) is the intensity of change of intrinsic type from \( \sigma \) to \( \sigma' \). Conjecturing the value function \( J(w, \sigma) = -e^{-r\gamma (w + a_\sigma + \bar{a})} \), optimization over \( \theta \) yields

\[
\theta_\sigma = \frac{m_D - rP - r\gamma \rho_\sigma \sigma_D \sigma_\eta}{r\gamma \sigma_D^2}.
\]  

(B.1)

Market clearing requires

\[
\mu_h \theta_h + \mu_l \theta_l = \Theta,
\]

with \( \mu_h = 1 - \mu_l = \lambda_u / (\lambda_u + \lambda_d) \), which gives the price

\[
P = \frac{m_D}{r} - \gamma \left( \Theta \sigma_D^2 + \frac{\sigma_D \sigma_\eta [\rho_l \lambda_d + \rho_h \lambda_u]}{\lambda_u + \lambda_d} \right).
\]  

(B.2)

Inserting this price into (B.1) gives the quantity choices

\[
\theta_h = \Theta + \frac{\sigma_\eta \lambda_d [\rho_l - \rho_h]}{\sigma_D(\lambda_u + \lambda_d)}, \quad \theta_l = \Theta - \frac{\sigma_\eta \lambda_u [\rho_l - \rho_h]}{\sigma_D(\lambda_u + \lambda_d)}.
\]  

(B.3)  

(B.4)

C Proofs

Proof of Proposition 1: The dependence on \( \delta \) and \( q \) is seen immediately, given that no other variable entering Equation (6) depends on either \( \delta \) or \( q \).

Viewing \( P \) and \( \mu_\sigma \) as functions of the parameters \( \lambda_d \) and \( s \), a simple differentiation exercise shows that the derivative of the price \( P \) with respect to \( \lambda_d \) is a positive multiple of

\[
(rq + \lambda_u + 2\lambda \mu_{hn} q) \left( 1 + 2\lambda \frac{\partial \mu_\sigma}{\partial \lambda_d} (1 - q) \right) 
- \left( r(1 - q) + \lambda_d + 2\lambda \mu_\sigma (1 - q) \right) \left( 2\lambda \frac{\partial \mu_{hn}}{\partial \lambda_d} q \right),
\]

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which is positive if $\frac{\partial \mu_u}{\partial \lambda_d}$ is positive and $\frac{\partial \mu_u}{\partial \lambda_u}$ is negative.

These two facts are seen as follows. From Equations (1)-(3) and the fact that $\mu_{lo} + \mu_{ln} = \lambda_d(\lambda_d + \lambda_u)^{-1} = 1 - y$, where

$$y = \frac{\lambda_u}{\lambda_u + \lambda_d},$$

it follows that $\mu_{lo}$ solves the equation

$$2\lambda \mu_{lo}^2 + (2\lambda(y - s) + \lambda_u + \lambda_d)\mu_{lo} - \lambda_d s = 0. \quad (C.1)$$

This quadratic equation has a negative root and a root in the interval $(0, 1)$, and this latter root is $\mu_{lo}$.

Differentiating (C.1) with respect to $\lambda_d$, one finds that

$$\frac{\partial \mu_{lo}}{\partial \lambda_d} = \frac{s - \mu_{lo} - 2\lambda \frac{\partial y}{\partial \lambda_d} \mu_{lo}}{2\lambda \mu_{lo} + 2\lambda(y - s) + \lambda_u + \lambda_d} > 0,$$

since $\frac{\partial y}{\partial \lambda_d} < 0$. Similar calculations show that

$$\frac{\partial \mu_{hn}}{\partial \lambda_d} = \frac{-\lambda_d + 2\lambda \frac{\partial y}{\partial \lambda_d} \mu_{hn}}{2\lambda \mu_{lo} + \lambda_u + \lambda_d} < 0,$$

which ends the proof of the claim that the price decreases with $\lambda_d$. Like arguments can be used to show that $\frac{\partial \mu_{lo}}{\partial \lambda_u} < 0$ and that $\frac{\partial \mu_{hn}}{\partial \lambda_u} > 0$, which implies that $P$ increases with $\lambda_u$.

Finally,

$$\frac{\partial \mu_{lo}}{\partial s} = \frac{\lambda_d + 2\lambda \mu_{lo}}{2\lambda \mu_{lo} + 2\lambda(y - s) + \lambda_u + \lambda_d} > 0$$

and

$$\frac{\partial \mu_{hn}}{\partial s} = \frac{-\lambda_u - 2\lambda \mu_{hn}}{2\lambda \mu_{lo} + \lambda_u + \lambda_d} < 0,$$

showing that the price decreases with the supply $s$.

In order to prove that the price increases with $\lambda$ for $\lambda$ large enough, it is sufficient to show that the derivative of the price with respect to $\lambda$ changes sign at most a finite number of times, and that the price tends to its upper bound, $1/r$, as $\lambda$ tends to infinity. The first statement is obvious, while the second one follows from Equation (6), given that, under the assumption $s < \lambda_u/(\lambda_u + \lambda_d)$, $\lambda \mu_{lo}$ stays bounded and $\lambda \mu_{hn}$ goes to infinity with $\lambda$. 

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Proof of Proposition 2: We impose on investors' choices of consumption and trading strategies the transversality condition that, for any initial agent type $\sigma_0$, $e^{-\beta T}E_0[J(W_T, \sigma_T)] \rightarrow 0$ as $T$ goes to infinity. Intuitively, the condition means that agents cannot consume large amounts forever by increasing their debt without restriction. We must show that our candidate optimal consumption and trading strategy satisfies that condition.

We conjecture that, for our candidate optimal strategy, $E_0[J(W_T, \sigma_T)] = e^{(\beta - r)T}J(W_0, \sigma_0)$. Clearly, this implies that the transversality condition is satisfied since $e^{-\beta T}E_0[J(W_T, \sigma_T)] = e^{-rT}J(W_0, \sigma_0) \rightarrow 0$. This conjecture is based on the insights that (i) the marginal utility, $u'(c_0)$, of time-0 consumption must be equal to the marginal utility, $e^{(r-\beta)T}u'(c_T)$, of time $T$ consumption; and (ii) the marginal utility is proportional to the value function in our (CARA) framework. (See Wang (2002) for a similar result.)

To prove our conjecture, we consider, for our candidate optimal policy, the wealth dynamics

$$
dW = \left( \frac{\log r}{\gamma} - ra_\sigma - r\bar{a} + \theta_\sigma m_D + m_\eta \right) dt + \theta_\sigma \sigma_D dB + \sigma_\eta dB^i - P d\theta_\sigma
$$

$$
= \left( -ra_\sigma + \theta_\sigma m_D + \frac{1}{2} r\gamma \sigma_\eta^2 + \frac{r - \beta}{r\gamma} \right) dt + \theta_\sigma \sigma_D dB + \sigma_\eta dB^i - P d\theta_\sigma
$$

$$
= M(\sigma) dt + \sqrt{\Sigma(\sigma)} d\hat{B} - P d\theta_\sigma,
$$

where $M$, $\Sigma$ and the standard Brownian motion $\hat{B}$ are defined by the last equation.

Define $f$ by

$$
f(W_t, \sigma_t, t) = E_t[J(W_T, \sigma_T)] = -E_t[e^{-r\gamma(W_T+a_{\sigma_T}+\bar{a})}].
$$

Then, by Ito's Formula,

$$
0 = f_t + f_w M(\sigma) + \frac{1}{2} f_{ww} \Sigma(\sigma)
$$

$$
+ \sum_{\{\sigma' : \sigma' \neq \sigma\}} \lambda(\sigma, \sigma') \left( f(w + z(\sigma, \sigma')P, \sigma', t) - f(w, \sigma, t) \right),
$$

where $\lambda(\sigma, \sigma')$ is the intensity of transition from $\sigma$ to $\sigma'$ and $z(\sigma, \sigma')$ is $-1$, $1$, or $0$, depending on whether the transition is, respectively, a buy, a sell, or an intrinsic-type change. The boundary condition is $f(w, \sigma, T) = -e^{-r\gamma(w+a_{\sigma_T}+\bar{a})}$.  

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The fact that \( f(w, \sigma, t) = e^{(\beta-r)(T-t)}J(w, \sigma) \) now follows from the facts that (i) this function clearly satisfies the boundary condition, and (ii) it solves (C.2), which is confirmed directly using (15) for \( a_{\sigma} \).

\[ \Box \]

**Proof of Proposition 3:** This result follows from Equations (15)–(18) as well as the fact that \( \lambda \mu_{hn} \to \infty \) and \( \lambda \mu_{lo} \) is bounded.

\[ \Box \]

**Proof of Proposition 4:** As stated formally by Equation (30), the position \( \theta \) decreases with the volatility \( \sigma_X \). As a consequence, the equilibrium agent masses change with an increase in \( \sigma_X \) in the same way as when the supply of the asset increases. That means, in particular, that \( \mu_{hn} \) decreases, which translates into longer search times for a seller (type \( lo \)). Proposition 1 establishes that the price decreases with the supply, whence also with the volatility \( \sigma_X \) of the dividends.

\[ \Box \]

**References**


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