I. Introduction

In the first module of this class we will study asset pricing implications of recursive contract theory. I will start by reviewing asset pricing in complete markets. Chapter 8 in Ljungqvist and Sargent (2004) provides background reading. The main topic of this module is asset pricing in an environment with limited commitment. As in the complete markets environment, agents can still trade a complete set of contingent claims. However, we assume that they can walk away from their debts. If they do so, they are excluded from trading forever. The inability of agents to commit leads to endogenous restrictions on trading. We will start by characterizing Pareto-efficient allocations. Then, we will study the Kehoe and Levine (1993) decentralization, where all trade takes place at time zero. Finally we will study a decentralization with sequential trade due to Alvarez and Jermann (2000).

Complete markets models imply perfect risk-sharing: An agent’s individual consumption growth does not depend on its individual income growth, only on aggregate consumption growth. However, both consumption data and asset pricing data reveal that households are unable to trade away all of their idiosyncratic risk. The limited commitment model reproduces this

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1 This is the standard punishment in the literature, see Kehoe and Levine (1993), Kehoe and Perri (2002), Krueger and Perri (2003), etc. Lustig (2003) and Lustig and VanNieuwerburgh (2004b) propose a different outside option, where agents retain access to credit markets but loose all collateral assets.

2 Papers find evidence at the household level (e.g. Cochrane (1991), Mace (1991), Nelson (1994), Krueger (2000), Blundell, Pistaferri and Preston (2002)), at the regional level (e.g. Hess and Shin (1998) and Lustig and VanNieuwerburgh (2004a)), and at the international level (e.g. Backus, Kehoe and Kydland (1992)).
II. Environment

A. Preferences and Endowments

- In each period there is a realization of an event \(s_t\). The history of events is denoted \(s^t = \{s_0, s_1, \ldots, s_t\}\).

- The conditional probability that a particular event \(s^t\) is realized is denoted \(\pi_t(s^t|s_0)\).

- There are \(I\) agents, \(i = 1 \ldots I\)

- Each agents owns a claim to a stochastic endowment \(y^i_t(s^t)\)

- Households purchase a consumption plan \(\{c^i_t(s^t)\}\)

- Households rank consumption streams according to

\[
U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c^i_t(s^t)) \pi_t(s^t|s_0)
\]

- Inada condition: Assume \(\lim_{c \to 0} u'(c) = \infty\)

- History: Endowments possibly depend on the entire history of shocks \(s^t\)

III. Full Commitment

In this first part, we assume that agents can fully commit to honoring their promises. We start by characterizing Pareto-efficient allocations. The first and second welfare theorem allow us to make the connection between the Pareto-efficient allocations and the Arrow-Debreu equilibria. In the AD equilibria, agents trade a complete set of claims to consumption whose delivery is contingent on a particular realization of the state of the world. Completeness means that consumption claims can be purchased that are contingent on any realization of the state of the world. One of the main results is that, in the case of additive utility, these Pareto-efficient allocations do not depend on the history of the economy. There is an alternative way of decentralizing
Pareto-efficient allocations by allowing agents to trade Arrow securities sequentially. In this decentralization, markets re-open each period, and agents re-trade every period.

**Definition 1.** A feasible allocation \( c \) satisfies
\[
\sum_i c_i^t(s^t) \leq \sum_i y_i^t(s^t), \text{ for all } t, s^t
\]

**A. Pareto Problem**

A benevolent planner produces a vector of weights \( \lambda \), one for each agent \( i \), to maximize a weighted sum of utilities
\[
W = \sum_i \lambda_i^t U(c_i^t)
\]
subject to the feasibility conditions
\[
\sum_i c_i^t(s^t) \leq \sum_i y_i^t(s^t), \forall t, s^t.
\]

**Definition 2.** A feasible allocation is efficient if it solves the planner problem for a strictly positive vector of weights \( \lambda \).

**B. Lagrangian**

We define a Lagrangian \( \mathcal{L} \)
\[
\mathcal{L} = \sum_i \lambda_i^t U(c_i^t) + \sum_{t=0}^s \sum_i \theta_t(s^t) \left[ \sum_i y_i^t(s^t) - \sum_i c_i^t(s^t) \right]
\]
where \( \theta_t(s^t) \) is the Lagrange multiplier resource constraint in node \( s^t \). This is a standard constrained optimization problem. The saddle point problem consists of maximizing the value of the objective function w.r.t. \( c \) and minimizing it w.r.t. \( \theta \):
\[
\max_c \min_\theta \mathcal{L}
\]

The first order conditions are necessary and sufficient. We derive the first order condition for consumption for agent \( i \) in node \( s^t \):
\[
\lambda_i^t \beta^t u'(c_i^t) \pi_t(s^t|s_0) = \theta_t(s^t). \tag{1}
\]
This implies that the ratio of first order conditions for two agents $i$ and $j$ with the same history $s^t$ is:

\[
\frac{u'(c^t_i(s^t))}{u'(c^t_j(s^t))} = \frac{\lambda^t_i}{\lambda^t_j}
\]  

(2)

Also note that the complementary slackness conditions need to be satisfied:

\[
\theta^t(s^t) \left[ \sum_i y^t_i(s^t) - \sum_i c^t_i(s^t) \right] = 0, \quad \forall s^t.
\]

**Proposition 3.** An efficient allocation is a function of the realized aggregate endowment only. It depends neither on the specific history $s^t$, nor on the realizations of the individual endowments.

**Proof.** To see this, then equation (2) for $j = 1$ implies that $c^t_i(s^t) = u'^{-1} \left( \frac{\lambda^t_i}{\lambda^t_j} u'(c^t_1(s^t)) \right)$. Because of feasibility and non-satiation, $\sum_i c^t_i(s^t) = \sum_i y^t_i(s^t)$. Substituting the expression for $c^t_i(s^t)$ in the resource constraint, we see that $c^t_1(s^t)$ only depends on the aggregate endowment in the current period, $\sum_i y^t_i(s^t)$. Because the weights $\lambda$ are constant, it follows that all agents’ consumption only depends on the aggregate endowment in the current period. In particular, consumption allocations do not depend on the individual income realizations. They are not history dependent, in the sense that any other state $s''$ that results in the same aggregate endowment gives rise to the same allocation.  

**C. Power Utility**

Consider the simplest case of power utility with coefficient of risk aversion $\gamma$, $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$. Verify that the following risk sharing rule satisfies (1) the condition on the ratio of marginal utilities and (2) feasibility:

\[
c^t_i(s^t) = \frac{\lambda^t_i}{\left( \sum_j \lambda^t_j \right)^{1/\gamma}} \sum_j y^t_i(s^t).
\]  

(3)

In particular, this rule implies perfect correlation between individual consumption and aggregate endowment! The consumption share of agent $i$, as a share of the aggregate, is fully pinned down by the its initial Pareto-weight $\lambda_i$, relative to the $1/\gamma$ moment of the cross-sectional distribution of weights. An important implication is that changes in the consumption share of an agent do not respond to changes in the individual income share. This is the nature of empirical tests in
the risk-sharing literature.

**Shadow Prices** How does the planner value a unit of consumption in different states of the world? Well, that information is embedded in \( \{ \theta_i(s') \} \). The Lagrangian multiplier on the resource constraint is:

\[
\theta_t(s') = \beta^t \pi_t(s'|s_0) \left( \frac{\sum_j y_j^t(s')}{{\sum_j} \lambda_j^{1/\gamma}} \right)^{-\gamma}
\]

The planner is willing to trade units of consumption in \( s^t \) for units of consumption at time 0 at the ratio:

\[
\tilde{q}_0^t(s') \equiv \frac{\theta_t(s')}{\theta_0(s'_0)} = \beta^t \pi_t(s'|s_0) \left( \frac{\sum_j y_j^t(s')}{{\sum_j} y_j^0(s'_0)} \right)^{-\gamma}
\]

We define \( \tilde{q}_i^t \) to be the time-zero shadow price of a unit of consumption to be delivered at time \( t \) in node \( s^t \).

The consumption trade-off between periods \( t-1 \) and \( t \) is analogous. The first order condition for agent \( i \)'s consumption (1) implies that the planner values resources in state \( s^t \) (in units of \( s^{t-1} \) consumption) as follows:

\[
\lambda^i \beta u'(c_i^t(s^t)) \frac{\lambda^i u'(c_i^{t-1}(s^{t-1}))}{\lambda^i u'(c_i^{t-1})} \pi_t(s^t|s_{t-1}) = \frac{\theta_t(s^t)}{\theta_{t-1}(s^{t-1})} \equiv \tilde{q}_{t-1}^i(s^t).
\]

Using the risk sharing rule in (3), this implies that, for any household \( i \):

\[
\tilde{q}_{t-1}^i(s^t) = \beta \pi_t(s^t|s_{t-1}) \left( \frac{\sum_j y_j^t(s^t)}{{\sum_j} y_j^{t-1}(s^{t-1})} \right)^{-\gamma}.
\]

The shadow price of aggregate consumption \( \tilde{q}_{t-1} \) increases in those states of the world in which aggregate endowment growth between \( t-1 \) and \( t \) is low. Because allocations are not history dependent, neither are shadow prices.

**D. Decentralization with Time Zero Trading**

If we let agents trade claims to consumption contingent on all states of the world \( s^t \), the equilibria that result are Pareto-efficient. Hence, this means these equilibrium allocations will not feature any history dependence. Here is the setup. Households trade history-contingent claims to consumption at time 0 after \( s^0 \) has been realized. Superscripts refer to the dates at which trades
occur, subscripts refer to the dates at which deliveries are to be made. At time \( t = 0 \) households can exchange claims on time \( t \)—consumption at prices \( q^0_t(s^t) \). Households face a single budget constraint:

\[
\sum_{t=0}^{\infty} \sum_{s^t} q^0_t(s^t) c^i_t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q^0_t(s^t) y^i_t(s^t)
\]

Attach a multiplier \( \mu^i \) to the household’s budget constraint. The first order condition for household \( i \) is:

\[
\beta^t \pi(s^t|s_0) u'(c^i_t(s^t)) = \mu^i q^0_t(s^t).
\]

**Definition 4.** A price system is a sequence of functions \( \{q^0_t(s^t)\} \) and an allocation is a list of sequences of functions \( \{c^i_t(s^t)\} \), one for each \( i \).

**Definition 5.** A competitive AD equilibrium is a feasible allocation and a price system such that given prices the household problem is solved for each \( i \) and the markets clear in each \( s^t \).

The first order condition implies that the ratio of marginal utilities satisfies:

\[
\frac{u'(c^i_t(s^t))}{u'(c^j_t(s^t))} = \frac{\mu^i}{\mu^j}, \forall s^t.
\]

Note that if we choose \( \mu^i = (\lambda^i)^{-1} \), this is the same condition as the one that characterized Pareto-efficiency. Furthermore \( \tilde{q}^0_t = q^0_t \).

**Remark 6.** A competitive AD equilibrium is a particular Pareto-efficient allocation, namely one that sets \( \mu^i = (\lambda^i)^{-1} \) for all \( i \).

**Stochastic Discount Factor** The stochastic discount factor or state price deflator is defined as:

\[
m_t(s^t) \equiv \frac{q^{t-1}_t(s^t)}{\pi_t(s^t|s_{t-1})} = \beta \frac{u'(c^i_t(s^t))}{u'(c^i_{t-1}(s^{t-1}))},
\]

\[
= \beta \left( \frac{\sum_j y^j_t(s^t)}{\sum_j y^j_{t-1}(s^{t-1})} \right)^{-\gamma}.
\]

This SDF is the one-period ahead pricing kernel \( q^{t-1}_t \), scaled by the corresponding state transition probability. In complete markets, it is the intertemporal marginal rate of substitution of the
representative household. Because of our risk-sharing rule, it is a function of the time discount factor \( \beta \) and the current growth rate of the aggregate endowment. In particular, the SDF is not a function of the history of aggregate endowment realizations.

An important property of the SDF is that it prices all claims in the economy. Consider an asset that pays a stream of dividends \( \{d_t(s^t)\} \). The price at time \( t - 1 \) of that claim is

\[
p_{t-1}(s^t) = \sum_{s^t} q^t_{t-1}(p_t(s^t) + d_t(s^t))
\]

Defining the one period return as \( R_t(s^t) \equiv p_t(s^t) + d_t(s^t) p_{t-1}(s^t-1) \), the previous equation implies that \( E_{t-1}[m_t R_t] = 1 \) must hold. This is a no-arbitrage condition. If it didn’t hold, agents in the economy could make unbounded profits by reconstructing the dividend sequence \( \{d_t(s^t)\} \) from the primitive Arrow-Debreu securities. Unconditional versions of this no-arbitrage condition form the basis of empirical tests in the asset pricing literature. See chapters 8 and 13 in Ljungqvist and Sargent (2004) and Duffie (2001) for more on the properties of stochastic discount factors.

E. Decentralization with Sequential Trading

There is another way of decentralizing Pareto-efficient allocations by allowing markets to re-open in each period. All we need is one-period state contingent claims. In this sequential trading environment, we have to resort to borrowing constraints to keep agents from running Ponzi schemes. We chose to use natural borrowing constraints. These constraints require the debt of a household to be smaller than what it could pay back if its consumption were zero from that period onwards in all future states. If we impose an Inada condition on the utility function, these borrowing constraints will never bind, because the marginal utility of consumption explodes as consumption tends to zero! These natural borrowing constraints are the weakest possible debt limits that suffice to implement the AD equilibrium allocations with sequential trading.

Denote household net wealth conditional on history \( s^t \) by

\[
\Upsilon_i(t)(s^t) = \sum_{t=\tau} \sum_{s^\tau | s^t} q^\tau_t(s^\tau) \left[ c^\tau_t(s^\tau) - y^\tau_t(s^\tau) \right],
\]

where \( \{q^\tau_t\} \) are prices that obtain when markets are re-opened at time \( t \). This is the value of all current and future net claims of household \( i \). The feasibility constraint at equality implies that:

\[
\sum_i \Upsilon_i(t)(s^t) = 0
\]
Debt Limits  We need some debt limits in sequential trading to prevent Ponzi schemes. The natural debt limit

\[ A^i_t(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau} q^i_\tau(s^\tau) y^i_\tau(s^\tau) \]

is the value of claim to household \( i \)'s endowment; it is the maximal amount that agent \( i \) can repay. At time \( t - 1 \) household \( i \) cannot promise to repay more in any state of the world tomorrow than the value of its labor income stream \( A^i_t(s^t) \).

Household Problem  Markets re-open each period. Let \( Q_t(s_{t+1}|s^t) \) be the price of one unit of consumption delivered contingent on the realization of \( s_{t+1} \) after history \( s^t \). At time \( t \) after history \( s^t \), household \( i \) can purchase a full set of Arrow securities, whose quantities are denoted by \( a^i_{t+1}(s_{t+1}, s^t) \). At time \( t \), the household chooses \( \{c^i_t(s^t), a^i_{t+1}(s_{t+1}, s^t)\} \) to maximize expected lifetime utility. The household faces a sequence of budget constraints:

\[ c^i_t(s^t) + \sum_{s_{t+1}} Q_t(s_{t+1}|s^t)a^i_{t+1}(s_{t+1}, s^t) \leq y^i_t(s^t) + a^i_t(s^t), \]

and a state-by-state borrowing constraint in node \( s^t \):

\[-a^i_{t+1}(s_{t+1}, s^t) \leq A^i_{t+1}(s_{t+1}^{t+1}), \forall s_{t+1}.\]

Because of the Inada condition, the natural debt limit will not be binding. In the absence of binding debt constraints, the first order condition implies

\[ \frac{\beta u'(c^i_{t+1}(s_{t+1}^{t+1}))}{u'(c^i_t(s^t))} \pi(s_{t+1}^{t+1}|s^t) = Q_t(s_{t+1}|s^t). \]  

(5)

Definition 7.  A distribution of wealth is a vector \( \bar{a}^i_t(s^t) = \{a^i_t(s^t)\}_{i=1}^{I}, \) satisfying \( \sum_i a^i_t(s^t) = 0. \)

Definition 8.  A sequential trading competitive equilibrium is an initial distribution of wealth \( \bar{a}_0(s^0) \), an allocation \( \{c^i_t(s^t)\}_{i=1}^{I} \) and pricing kernels \( Q_{t+1}(s_{t+1}|s^t) \) such that \( c^i \) solves the household problem, and \( \{c^i_t(s^t), a^i_{t+1}(s_{t+1}, s^t)\}_{i=1}^{I} \) satisfy for all \( s^t \)

\[ \sum_i c^i_t(s^t) = \sum_i y^i_t(s^t) \quad \text{and} \quad \sum_i a^i_t(s_{t+1}, s^t) = 0, \forall s_{t+1}. \]
F. Equivalence of Allocations

The AD equilibrium and the sequential equilibrium are equivalent. To show this, guess that for given AD prices \( \{q^0_t(s^t)\} \), we can recover \( Q_t(s_{t+1}|s^t) \) from the recursion

\[
q^0_{t+1}(s^{t+1}) = q^0_t(s^t)Q_t(s_{t+1}|s^t)
\]

If the pricing kernel satisfies this recursion, the first order condition for the AD problem

\[
\frac{\beta u'(c^i_{t+1}(s^{t+1}))}{u'(c^i_t(s^t))} \pi(s^{t+1}|s^t) = \frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)} = Q_t(s_{t+1}|s^t)
\]

coincides with the first order conditions for the sequential problem. Furthermore, by iterating forward on the sequential budget constraint and imposing a transversality condition (see Sargent (1984), chapter 8), we can match up the wealth of agent \( i \) in the sequential and in the AD economies:

\[
a^i_t(s^t) = \Upsilon^i_t(s^t)
\]

We conjecture that the initial wealth vector should be the null vector. Ljungqvist and Sargent (2004) show that this portfolio strategy is affordable and allows the financing of the AD equilibrium level of consumption. Furthermore, the household cannot increase consumption beyond this by lowering a component of the asset portfolio, lest it jeopardizes being able to finance the AD consumption in every state of the world tomorrow.

Recursive Competitive Equilibrium A special case arises when we impose a Markovian structure on the transition probabilities \( \pi \):

\[
\pi_t(s^t|s_0) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\ldots\pi(s_1|s_0)
\]

and assume that the individual endowment \( y^i_t(s^t) \) is only a function of the current state \( s_t \). Under these conditions equilibrium allocations and prices are only a function of the current state only:

\[
c^i_t(s^t) = c^i(s_t) \quad \text{and} \quad Q_t(s_{t+1}|s^t) = Q(s_{t+1}|s_t).
\]
IV. Limited Commitment

We now relax the assumption that agents can commit to honoring their promises. In order to induce continued adherence to the contract, we require that the consumption allocation satisfies a series of participation constraints:

\[ U(c^i)(s^i) \geq U^\text{aut}_i(s^i), \forall s^i, i, \]

where

\[ U^\text{aut}_i(s^i) = \sum_{t=\tau}^{\infty} \sum_{s^\tau|s^t} \beta^t u\left(y^i_{s^\tau}(s^\tau)\right) \pi_\tau(s^\tau|s_0) \]

The “outside” option is to revert permanently into autarchy.

**Definition 9.** A feasible allocation \( c \) satisfies

\[ \sum_i c^i_t(s^t) \leq \sum_i y^i_t(s^t), \text{ for all } t, s^t \]

and

\[ U(c^i)(s^i) \geq U^\text{aut}_i(s^i) \text{ in all } s^t, \text{ for all } i \]

**Theorem 10.** A necessary and sufficient condition for perfect risk sharing is that

\[ U\left(\frac{1}{T} \sum_i y^i_t(s^t)\right) \geq U^\text{aut}_i(s^t), \forall s^t, i. \]

A. Pareto Problem

A benevolent planner produces a vector of weights \( \lambda \), one for each agent \( i \), to maximize

\[ W = \sum_i \lambda^i U(c^i), \]

subject to the feasibility conditions

\[ \sum_i c^i_t(s^t) \leq \sum_i y^i_t(s^t), \forall t, s^t, \]

\[ U(c^i)(s^i) \geq U^\text{aut}_i(s^i), \forall s^t, i. \]
B. Lagrangian

We define a Lagrangian \( \mathcal{L} \)

\[
\mathcal{L} = \sum_i \left\{ \lambda^i U(c^i) + \sum_{t=0}^{\infty} \sum_{s^t} \mu^i_t(s^t) \left[ \sum_{\tau=0}^{\infty} \sum_{s^\tau|s^t} \beta^\tau u(c^i_{\tau}(s^\tau)) \pi^\tau(s^\tau|s^t) - U^{aut}_i(s^t) \right] \right\} \\
+ \sum_{t=0}^{\infty} \sum_{s^t} \theta_t(s^t) \left[ \sum_i g_t^i(s^t) - \sum_i c_t^i(s^t) \right]
\]

where \( \mu^i_t(s^t) \) is the multiplier on the participation constraint in node \( s^t \) and \( \theta_t(s^t) \) is the multiplier on the resource constraint. This is a standard optimization problem, except that we have an infinite number of constraints. The saddle point problem consists of maximizing the value of the objective function w.r.t. \( c \) and minimizing it w.r.t. \( \mu \) and \( \theta \):

\[
\max_c \min_{\mu, \theta} \mathcal{L}
\]

Marcet and Marimon (1999) noted that we can define cumulative multipliers that make the problem “recursive”:

\[
\xi^i_t(s^t) = \lambda^i + \sum_{\tau=0}^{t} \sum_{s^\tau|s^0} \mu^i_{\tau}(s^\tau)
\]

This is a recursive formulation, because it implies that :

\[
\xi^i_t(s^t) = \xi^i_{t-1}(s^{t-1}) + \mu^i_t(s^t), \quad \xi^i_0(s^0) = \lambda^i,
\]

The sequence \( \{\xi^i_t(s^t)\} \) is a non-decreasing stochastic process. This follows from the non-negativity of \( \mu^i_t(s^t) \).

Using the recursive formulation of \( \xi \) and Abel’s partial summation formula, we can restate the Lagrangian as:

\[
\mathcal{L} = \sum_i \left\{ \left[ \sum_{t=0}^{\infty} \sum_{s^t} \xi^i_t(s^t) \beta^t u(c^i_{\tau}(s^\tau)) \pi^\tau(s^\tau|s^t) \right] + \sum_{t=0}^{\infty} \sum_{s^t} \mu^i_t(s^t) \left[ -U^{aut}_i(s^t) \right] \right\} \\
+ \sum_{t=0}^{\infty} \sum_{s^t} \theta_t(s^t) \left[ \sum_i g_t^i(s^t) - \sum_i c_t^i(s^t) \right].
\]

The first order conditions are necessary and sufficient. We derive the first order condition
for consumption for agent $i$ in node $s^t$:

$$
\xi^t_i(s^t) \beta^t u' \left( c^t_i(s^t) \right) \pi_t(s^t|s_0) = \theta_t(s^t)
$$  \hspace{1cm} (6)

The ratio of first order conditions for two agents $i$ and $j$ with the same history $s^t$ is:

$$
\frac{u' \left( c^t_i(s^t) \right)}{u' \left( c^t_j(s^t) \right)} = \frac{\xi^t_i(s^t)}{\xi^t_j(s^t)}
$$  \hspace{1cm} (7)

As always, the complementary slackness conditions need to be satisfied:

$$
\mu^t_i(s^t) \left[ \sum_{t=\tau}^{\infty} \sum_{s^\tau|s^t} \beta^t u \left( c^t_i(s^\tau) \right) \pi_t(s^\tau|s^t) - U^\text{aut}_i(s^t) \right] = 0 \text{ for all } s^t, \text{ all } i
$$

$$
\theta_t(s^t) \left[ \sum_i y^t_i(s^t) - \sum_i c^t_i(s^t) \right] = 0 \text{ for all } s^t
$$

### C. Power utility

As in the case with full commitment, Pareto-efficient allocations take on an elegant form with power utility. We conjecture the following risk sharing rule:

$$
c^t_i(s^t) = \frac{\xi^t_i(s^t)^{1/\gamma}}{\sum_j \xi^t_j(s^t)^{1/\gamma}} \sum_j y^t_j(s^t)
$$  \hspace{1cm} (8)

It is easy to verify that this risk sharing rule satisfies (7) and market clearing (by construction).

**Proposition 11.** Amnesia property: A household’s consumption share decreases as long as it does not switch to a state with a binding constraint, but when it does, its consumption share increases to some cutoff level that does not depend on the history $(s^t)$ if the endowment process is first-order Markov.

**Proof.** The first part follows from the risk sharing rule and the fact that $\{\xi^t_i(s^t)\}$ is a non-decreasing process for all $i$. The second part follows from the complementary slackness condition, which says that, when the constraint binds:

$$
\left[ \sum_{t=\tau}^{\infty} \sum_{s^\tau|s^t} \beta^t u \left( c^t_i(s^\tau) \right) \pi_t(s^\tau|s_0) - U^\text{aut}_i(s^t) \right] = 0
$$

12
Now, if \( y \) is first-order Markov, then \( U^\text{aut}_i(s^t) \) only depends on \( s_t \). This implies \( c^t_i(s^t) \) cannot depend on \( s^t \), only on \( s_t \).

History dependence and time-varying Pareto-Negishi weights are signature of limited enforcement (and private information) problems.

**Shadow Prices** The first order condition for agent \( i \)'s consumption (6) implies that the planner values resources in state \( s^t \) (in units of \( s^{t-1} \) consumption) as follows:

\[

\frac{\xi^t_i(s^t)\beta u'(c^t_i(s^t))}{\xi^t_{t-1}(s^{t-1})u'(c^t_{t-1}(s^{t-1}))} \pi_t(s^t|s_{t-1}) = \frac{\theta_t(s^t)}{\theta_{t-1}(s^{t-1})} \equiv \tilde{q}^{t-1}_t(s^t).

\]

Using the risk sharing rule in (8), this implies that, for any household \( i \):

\[

\tilde{q}^{t-1}_t(s^t) = \beta \pi_t(s^t|s_{t-1}) \left( \frac{\sum_j y^t_j(s^t)}{\sum_j y^t_{t-1}(s^{t-1})} \right)^{-\gamma} \left( \frac{\sum_j \xi^t_j(s^t)^{1/\gamma}}{\sum_j \xi^t_{t-1}(s^{t-1})^{1/\gamma}} \right)^{\gamma}.
\]

The shadow price of aggregate consumption \( \tilde{q}^{t-1}_t(s^t) \) increases in those states of the world in which lots of agents are severely constrained, because in that case the aggregate weight shock \( g_t(s^t|s^{t-1}) \equiv \left( \frac{\sum_j \xi^t_j(s^t)^{1/\gamma}}{\sum_j \xi^t_{t-1}(s^{t-1})^{1/\gamma}} \right) \), would be large. In particular, \( g_t(s^t|s^{t-1}) > 1 \). Note that if no agent is constrained, \( g_t(s^t|s^{t-1}) = 1 \), and the last part simply drops out.

**D. Simple Recursive Characterization**

Actually solving that saddle point problem is computationally challenging. Instead we can try to use what we know about constrained efficient allocations to ease the burden of computing these allocations. We use consumption weights as state variables instead of cumulative multipliers because we want stationary state variables.

Define an \( I - 1 \times 1 \) vector of consumption shares \( \overrightarrow{\omega} \) where the \( i^{th} \) element is:

\[

\omega^i_{t-1} = \frac{c^t_{i-1}}{\sum_j y^t_{t-1}}, \text{ for } i = 2, \ldots, I.
\]

The consumption share of the first household is just the residual \( 1 - \sum_{i \neq 1} \omega^i_{t-1} \). A natural choice for the state variables is (\( \overrightarrow{\omega}, s \)). In the simplest case of two agents, we would keep track only of \( (\omega_1, s) \).
Cutoff Rule for Consumption  At the start of next period, we compare the agent’s consumption weight in the previous period $\omega_{t-1}^i$ to the cutoff value for the current state of the world: $\omega^i(\overrightarrow{\omega}, s)$. If the weight exceeds the cutoff weight, the consumption weight $\omega_{t-1}^i$ is left unchanged and the agent’s consumption weight in $t$ is:

$$\omega_t^i = \omega_{t-1}^i$$

If the consumption weight $\omega_{t-1}^i$ is smaller than the cutoff weight, the agent’s consumption weight in $t$ is:

$$\omega_t^i = \omega^i(\overrightarrow{\omega}, s)$$

Actual consumption is given by:

$$c_t^i = \frac{\omega_t^i}{\sum_j \omega_t^j} \sum_j y_t^j$$

for each agent $i$. The cutoff rule is determined such that the constraint binds exactly

$$U(c^i(s_t)) = U_{aut}^i(s_t)$$

when the consumption weight equals the cutoff weight $\omega_t^i = \omega^i$. 

At the end of each period we store the vector $\overrightarrow{\omega}$, which contains the consumption shares

$$\frac{\omega_t^i}{\sum \omega_t^i}$$

These consumption weights $\omega$ are exactly like the $\xi^{1/\gamma}$, but they are rescaled to make sure they sum to one at the end of each period. The optimality of this cutoff rule follows immediately from the first order conditions in the saddle point problem of the planner.

Two Agent Example  We consider the simplest example with two agents, two states and no aggregate uncertainty. Suppose there are two individual income states ($y_{lo}, y_{hi}$). In the first state agent 1 draws a low endowment, in the second state agent 1 draws a high endowment. The endowments sum to one in each state. Suppose the events $y$ are i.i.d., then the only state variable is the consumption share of the first agent. How do we solve for the constrained efficient allocations?
First, solve for the value of autarchy:

\[ U_{1}^{\text{aut}}(y_{lo}) = u(y_{lo}) + \beta \sum_{y'} \pi(y'|y) U_{1}^{\text{aut}}(y'), \]

\[ U_{1}^{\text{aut}}(y_{hi}) = u(y_{hi}) + \beta \sum_{y'} \pi(y'|y) U_{1}^{\text{aut}}(y'). \]

Using vector notation, the value of autarchy in each state \( y \) can be recovered from the following equation:

\[ U_{1}^{\text{aut}} = u(y)(I - \beta \Pi)^{-1}, \]

where \( I \) is the identity matrix. For the second household, we solve a similar equation:

\[ U_{2}^{\text{aut}} = u(1-y)(I - \beta \Pi)^{-1} \]

**Remark 12.** Necessary and sufficient condition for perfect risk sharing:

\[ U(1/2) \geq U_{1}^{\text{aut}}(y_{lo}) \text{ and } U(1/2) \geq U_{1}^{\text{aut}}(y_{hi}) \]

**Remark 13.** If the labor income process is too persistent or of agents are too impatient, perfect risk sharing is not feasible. As \( \beta \to 0 \) the inequality

\[ u(y) + \beta \sum_{y'} \pi(y'|y) U_{1}^{\text{aut}}(y') < u(1/2) + \beta \sum_{y'} \pi(y'|y) U(1/2) \]

cannot be satisfied because \( u(1/2) < u(hi) \). Similarly, as the persistence increases, \( \pi(hi|hi) \to 1 \), \( U_{1}^{\text{aut}}(hi) \to u(hi)/(1 - \beta) > u(1/2)/(1 - \beta) \).

Second, we solve for the four cutoff values

\[ (\omega_{1}(y_{lo}), \omega_{1}(y_{hi}), \overline{\omega}_{1}(y_{lo}), \overline{\omega}_{1}(y_{hi})) \]
from the following four non-linear equations:

\[
U_1(\omega_1(y)) = u(\omega_1(y)) + \beta \sum_{y'} \pi(y') U_1(\omega_1')
\]

\[
= U_1^{aut}(y), \forall y \in (y_{lo}, y_{hi})
\]

\[
U_2(\omega_1(y)) = u(1 - \omega_1(y)) + \beta \sum_{y'} \pi(y') U_2(\omega_1')
\]

\[
= U_2^{aut}(y), \forall y \in (y_{lo}, y_{hi}),
\]

where \(\omega_1'\) in the next period is found by applying the following rule, the analogue of the more general rule we described earlier:

- if \(\omega_1(y) < \omega_1 < \omega_1(y)\), \(\omega_1' = \omega_1\)
- if \(\omega_1(y) > \omega_1\), \(\omega_1' = \omega_1(y)\)
- if \(\omega_1(y) < \omega_1\), \(\omega_1' = \omega_1(y)\)

Perfect risk sharing is when both intervals \([\omega_1'(y_{lo}), \omega_1(y_{lo})]\) and \([\omega_1'(y_{hi}), \omega_1(y_{hi})]\) contain a consumption share of 1/2.

**Shadow Prices**  Go back to the expression for the shadow prices in this economy, and recall that the aggregate endowment is normalized to 1:

\[
q_{t-1}^{l-1}(s^t) = \frac{\theta_t(s^t)}{\theta_{t-1}(s^{t-1})} = \beta \pi_t(s^t|s_{t-1}) \left( \frac{\sum_{j=1}^{2} \xi_t^j(s^{t+1})^{1/\gamma}}{\sum_{j=1}^{2} \xi_{t-1}^j(s^{t-1})^{1/\gamma}} \right) \gamma
\]

\[
= \beta \pi_t(s^t|s_{t-1}) \left( \frac{\xi_t^i}{\xi_{t-1}^i} \right) \left( \frac{c_t^i}{c_{t-1}^i} \right)^{-\gamma}
\]

\[
= \beta \pi_t(s^t|s_{t-1}) \max_{i=1,2} \left( \frac{c_t^i}{c_{t-1}^i} \right)^{-\gamma}, \quad (9)
\]

where the last line follows from the fact that the unconstrained agent is the one with the highest IMRS. Remember that both agents cannot be constrained at the same time in equilibrium, otherwise markets don’t clear. We will come back to this later.
E. Decentralization with Time Zero Trading: Kehoe-Levine Equilibrium

Now, we actually let households trade. All trading occurs at time 0. Households trade history-contingent claims to consumption at time 0 after $s^0$ has been realized. Superscripts refer to the dates at which trades occur, subscripts refer to the dates at which deliveries are to be made. At time $t = 0$ households can exchange claims on time $t$—consumption at price $q^0_t(s^t)$. When trading at time 0, households face a single budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t} q^0_t(s^t)c^i_t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q^0_t(s^t)y^i_t(s^t)$$

and a series of participation constraints, one for each node $s^t$:

$$U(c^i_t)(s^t) \geq U^{aut}_i(s^t)$$

Definition 14. A price system is a sequence of functions $\{q^0_t(s^t)\}$ and an allocation is a list of sequences of functions $\{c^i_t(s^t)\}$, one for each $i$.

The Kehoe and Levine (1993) equilibrium concept constrains the feasible choice set using the participation constraints themselves. Households are not afforded the option of walking away from their debts.

Definition 15. A Kehoe-Levine equilibrium is a feasible allocation and a price system such that, given prices, the household problem is solved for each $i$ and the markets clear in each $s^t$.

This a standard saddle point problem. Construct the Lagrangian for household. Attach a multiplier $\mu^i$ to the household’s budget constraint and attach multipliers $\gamma^i_t(s^t)$ to the participation constraint in each node $s^t$. Derive the first order condition for the household $i$:

$$\zeta^i_t(s^t)\beta u'(c^i_t(s^t)) \tau_t(s^t|s_0) = \mu^i q^0_t(s^t), \quad (10)$$

where the cumulative multiplier $\zeta$ is defined recursively as follows

$$\zeta^i_t = \zeta^i_{t-1} + \gamma^i_t, \quad \zeta_0 = 1.$$  

The sequence $\{\zeta^i_t(s^t)\}$ is a non-decreasing process because $\gamma^i_t(s^t) \geq 0, \forall t, s^t$.  

17
The first order condition implies that the ratio of marginal utilities satisfies:

\[
q_i^{t-1}(s^t) = \frac{q_i^0(s^t)}{q_{t-1}^0(s^{t-1})} = \frac{\zeta_i^t(s^t)\beta u'(c_i^t(s^t)) \pi_t(s^t|s_0)}{\zeta_{t-1}^t(s^{t-1})\beta u'(c_{i-1}^t(s^{t-1})) \pi_{t-1}(s^{t-1}|s_0)}
\]

\[
= \frac{\zeta_i^t(s^t)}{\zeta_{t-1}^t(s^{t-1})} u'(c_i^t(s^t)) \pi_t(s^t|s_{t-1})
\]

\[
= \frac{\zeta_i^t(s^t)}{\zeta_{t-1}^t(s^{t-1})} u'(c_i^t(s^t)) \pi_t(s^t|s_{t-1})
\]

for all histories \(s^t\). In the last line we recover the first order condition of the Pareto problem for \(\xi_i^t = \frac{\zeta_i^t(s^t)}{\mu^i}\). There is a one-for-one mapping between the multipliers in the KL equilibrium \(\zeta\) and the multipliers in the Pareto problem \(\xi\): \(\frac{\xi_i^t(s^t)}{\xi_{t-1}^t(s^{t-1})} = \frac{\zeta_i^t(s^t)}{\zeta_{t-1}^t(s^{t-1})}\). Also, \(\bar{q}_i^0 = q_i^0\).

**Remark 16.** A Kehoe-Levine equilibrium is a particular constrained Pareto-efficient allocation, namely one that sets \(\mu^i = (\lambda^i)^{-1}\) for all \(i\).

**Stochastic Discount Factor** As in the complete markets economy, there exists a unique strictly positive SDF. It is defined as:

\[
m_t(s^t) = \frac{q_i^{t-1}(s^t)}{\pi_t(s^t|s_{t-1})} = \frac{\zeta_i^{t}(s^{t})\beta u'(c_i^{t}(s^{t}))}{\zeta_{t-1}^{t}(s^{t-1})\beta u'(c_{i-1}^{t}(s^{t-1}))} \pi_t(s^t|s_{t-1})
\]

\[
= \beta \left( \frac{\sum_j y_j^t(s^t)}{\sum_j y_{t-1}^j(s^{t-1})} \right)^{-\gamma} (g_t(s^t|s_{t-1}))^{-\gamma}.
\]

(11)

In the limited commitment economy, the SDF is a function of the time discount factor \(\beta\), the growth rate of the aggregate endowment, and the growth rate of the aggregate weight shock (third line). Note that if no agent is constrained between \(t\) and \(t + 1\), \(g_t = 1\) and the SDF \(m_t\) collapses to the SDF in complete markets (Breeden (1979), Lucas (1978)). If many agents are severely constrained, the aggregate weight shock \(g_t(s^t) \gg 1\).

The second line states that the SDF is the intertemporal marginal rate of substitution of each agent, weighted by the ratio of cumulative multipliers. For all unconstrained households \(k\), that ratio is one, because when \(\mu_k^t = 0\), \(\xi_k^t = \xi_{k-1}^t\). So, we have that the SDF equals the IMRS of the unconstrained households. Because the ratio of cumulative multipliers is strictly greater than one for all constrained households (recall that \(\{\xi_i^t\}\) is a non-decreasing stochastic process),
and because of the second line, their IMRS must be lower than the IMRS of the unconstrained households. This implies the SDF exceeds the IMRS of all agents, except for the “unconstrained” agents, more precisely those who did not enter a state with a binding constraint in period \( t \), node \( s' \):

\[
m_t = \max_i \beta \frac{u'(c_i^t)}{u'(c_{i-1}^t)}
\]

The SDF is maximum IMRS (across all households). In sum, the unconstrained agents are the agents with the highest IMRS (or the lowest consumption growth with power utility); their IMRS prices all the assets in this economy. The intuition for this result is that only the unconstrained agent can arbitrage when his IMRS is smaller than the state price of consumption in a particular state of the world. Furthermore, because of market clearing, there is always at least one unconstrained agent. However, this does not imply that the price of claim to a non-negative dividend stream equals the highest marginal valuation across all households. Before we move on, I want to stress this point.

Consider an asset that is a claim to a stream of non-negative dividends \( \{d\} \). Compute the marginal valuation of an agent:

\[
MV_i^0(s^t) = \sum_{\tau \geq t} \sum_{s^\tau} \pi(s^\tau|s_t) \beta u'(c_i^\tau) d_\tau(s^\tau) \leq q_t(s^t) \{d\}
\]

\[
\max_i MV_i^0(s^t) = \max_i \sum_{\tau \geq t} \sum_{s^\tau} \pi(s^\tau|s_t) \beta u'(c_i^\tau) d_\tau(s^\tau) \neq q_t(s^t) \{d\}
\]

There is not a single agent who prices the payoffs in the economy. Rather, the identity of the unconstrained household changes potentially in every node \( s' \). The lesson is that we cannot just use the IMRS of any household to price pay-outs, as we did in the perfect enforcement model, even though markets are ex ante complete.

**Proposition 17.** We can put bounds on the size of the aggregate weight shocks:

\[
1 \leq g_t(s^t) \leq \max \left( \frac{\hat{y}_t(s^t)}{\hat{y}_{t-1}(s^{t-1})} \right)^{-1},
\]

where \( \hat{y} \) denotes the labor income share.

**Proof.** The first inequality follows from the fact that the multipliers do not change if nobody is constrained. The second inequality follows from the fact that in autarchy the highest IMRS is for the household who switches from the highest to the lowest income share. If the state price
was higher than 

\[ m_{t+1} = \beta \left( \frac{\sum_j y_j(s^t)}{\sum_j y_j(s^{t-1})} \right)^{-\gamma} \left( \min \frac{\hat{y}_t(s^t)}{\hat{y}_{t-1}(s^{t-1})} \right)^{-\gamma} \]

there could be no trade, because nobody would be willing to buy contingent consumption claims at this high price. □

Remark 18. AD prices will be (weakly) higher than in the corresponding economy with perfect enforcement and interest rates will be (weakly) lower.

This follows from the fact that the aggregate weight shock sequence \( \{g_t(s^t)\} \geq 1 \). Hence the SDF is weakly higher and interest rates, \( r^f_t = E_t[m_{t+1}]^{-1} \), will be weakly lower.

F. Decentralization with Sequential Trading: Alvarez-Jermann Equilibrium

Recently, Alvarez and Jermann (2000) devised a more appealing decentralization which uses solvency constraints in a sequential trading environment. These constraints are portfolio constraints as opposed to direct restrictions on the consumption possibility set. They are judiciously chosen so that they are “not too tight”: They are tight enough to make sure the Kehoe and Levine (1993) participation constraints are always satisfied, but they do not bind when the corresponding participation constraints do not bind. This way, we still allow the maximum amount of risk sharing. The solvency constraints prevent default (reverting to autarchy) at the cost of reducing risk sharing.

We know from section II that we always need borrowing constraints if we allow sequential trading. Before we devised natural borrowing constraints that were “loose”; they were never binding under the Inada condition. The AJ-borrowing constraints will obviously be tighter and they may occasionally bind.

Environment and Trading   Markets re-open each period. Let \( Q_t(s_{t+1}|s^t) \) be the price of one unit of consumption delivered contingent on the realization of \( s_{t+1} \). At time \( t \), the household chooses \( \{c^t_i(s^t), a^t_i(s_{t+1}, s^t)\} \) to maximize expected utility. Agents face a sequence of budget constraints:

\[ c^t_i(s^t) + \sum_{s^t+1} Q_t(s_{t+1}|s^t) a^t_i(s_{t+1}, s^t) \leq y^t_i(s^t) + a^t_i(s^t) \]

Debt Limits   We need some debt limits in sequential trading to prevent Ponzi schemes. Now we have an endogenous debt limit that will replace the natural debt limit.
AJ impose a different state-by-state constraint on borrowing (or equivalently a lower bound on net wealth \( B \)), the so called solvency constraints:

\[-a_{t+1}^i(s_{t+1}, s') \leq -B_{t+1}^i(s^{t+1}), \forall s^{t+1}.

**Definition 19.** A distribution of wealth is a vector \( \vec{a}_t(s') = \{a_i^t(s')\}_{i=1}^I \) satisfying \( \sum_i a_i^t(s') = 0 \).

**Definition 20.** A sequential trading competitive equilibrium with solvency constraints \( \{B_i^t\}_{i=1}^I \) is an initial distribution of wealth \( \vec{a}_0(s^0) \), an allocation \( \{c_i^t(s'), a_{t+1}^i(s_{t+1}, s')\}_{i=1}^I \), and pricing kernels \( Q_{t+1}(s_{t+1}|s^t) \) such that, for each \( i \), the allocation solves the household problem:

\[
J_i^t(a_i^t, s^t) = \max_{c_i^t(s'), \{a_{t+1}^i(s_{t+1}, s')\}} u(c_i^t) + \beta \sum_{s'} \pi(s'|s) J_i^{t+1}(a_{t+1}^i, s^{t+1})
\]

subject to:

\[
c_i^t(s^t) + \sum_{s^{t+1}} Q_t(s_{t+1}|s^t) a_{t+1}^i(s_{t+1}, s^t) \leq y_i^t(s^t) + a_i^t(s^t)
\]

and

\[a_{t+1}^i(s_{t+1}, s^t) \geq B_{t+1}^i(s_{t+1}, s^t), \forall s_{t+1},
\]

and markets clear for all \( s^t \)

\[
\sum_i c_i^t(s^t) = \sum_i y_i^t(s^t) \text{ and } \sum_i a_i^t(s_{t+1}, s^t) = 0, \forall s_{t+1}.
\]

**Definition 21.** An equilibrium has solvency constraints \( \{B_{t+1}^i(s^{t+1})\}_{i=1}^I \) that are not too tight if

\[
J_{t+1}^i(B_{t+1}^i(s^{t+1}), s^{t+1}) = U_i^{aut}(s^{t+1}), \forall t, s^{t+1}.
\]

\( J_{t+1}^i(B_{t+1}^i(s^{t+1}), s^{t+1}) \) is the continuation utility of a household starting with assets \( B_{t+1}^i(s^{t+1}) \) in period \( t+1 \). Solvency constraints that satisfy this condition prevent default by prohibiting agents to accumulate more state contingent debt than they are willing to pay back. At the same time, they maximize the degree of risk-sharing.

The borrowing constraints are such that

\[
U(c^t)(s^t) \geq U_i^{aut}(s^t) \text{ and } U(c^t)(s^t) = U_i^{aut}(s^t) \Leftrightarrow B_i^t(s^t) = a_i^t(s^t)
\]
To enforce any “generic” solvency constraint, the agent’s entire portfolio must be known. But here we need a lot more information! Note that whoever is imposing these borrowing constraints needs information about households’ preferences and endowments to determine whether constraints are not too tight. In this sense, one could question whether this really is a decentralization.3

Sufficient conditions for a maximum are the Euler equation and the transversality condition:

\[ -u'(c^t_i(s^t))Q_t(s_{t+1}|s^t) + \beta \pi(s_{t+1}|s_t)u'(c^t_{i+1}(s^{t+1})) \leq 0 \]

\[ \lim_{t \to \infty} \beta^t \left[ a^t_i(s^t) - B^t_i(s^t) \right] u'(c^t_i(s^t)) \]

The unconstrained agents between \( t \) and \( t+1 \) have the highest IMRS; they are the ones pricing all the assets in the economy in period \( t \):

\[ Q_t(s_{t+1}|s^t) = \max_i \left\{ \beta \pi_{t+1}(s_{t+1}|s_t) \frac{u'(c^t_{i+1}(s^{t+1}))}{u'(c^t_i(s^t))} \right\} \]

AJ call this the high IMRS condition.

**Definition 22.** The implied interest rates are high if

\[ \sum_{t \geq 0} \sum_{s^t} q^0_t(s^t|s_0) \sum_i c^t_i(s^t) < +\infty \]

where \( \{q^0_t(s^t|s_0)\} \) is defined recursively as:

\[ q^0_t(s^t|s_0) = Q_t(s_t|s_{t-1})Q_t(s_{t-1}|s_{t-2}) \ldots Q_t(s_1|s_0). \]

G. The Welfare Theorems and the Relation between KL and AJ Equilibria

We are now in good shape to invoke the second welfare theorem: The Pareto optimal allocations can be implemented as an AJ equilibrium with solvency constraints:

**Proposition 23.** If an allocation \( \{c^t\} \) satisfies the resource constraint, the participation constraints, the high IMRS condition, and the high implied interest rate condition, then there exists solvency constraints \( \{B^t\} \) and an initial wealth distribution \( \tilde{a}_0(s^0) \), such that \( \{a^t, c^t\} \) are a

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3See Lustig (2003) and Lustig and VanNieuwerburgh (2004b) for a different outside option that circumvents the stringent informational requirements.
competitive equilibrium. The solvency constraints can be chosen such that they are not too tight (equation 12 holds for all $i$).

Proof. See Alvarez and Jermann (2000), pages 791-792. 

**Corollary 24.** Any constrained efficient allocation that has high implied interest rates can be decentralized as a competitive equilibrium with solvency constraints that are not too tight.

Intuitively, these AJ equilibrium allocations will be the constrained efficient allocations, because the solvency constraints in the sequential equilibrium problem serve the same purpose as the participation constraints in the Pareto problem.

Autarchy can always be decentralized with solvency constraints that are not too tight.

**Remark 25.** The autarchic allocations and prices are an equilibrium with solvency constraints that are not too tight:

$$c^i_t(s^t) = y^i_t(s^t); a^i_t(s^t) = 0 \text{ and } Q^\text{aut}_{t+1}(s_{t+1}|s_t) = \max_i \beta \pi_{t+1}(s_{t+1}|s_t) \frac{u'(y^i_{t+1}(s_{t+1}))}{u'(y^i_t(s^t))}$$

In general these are not constrained efficient, even though they can be decentralized! Also, $\{Q^\text{aut}\}$ may be so high (and implied interest rates so low) that the high implied interest rate condition may be violated in autarchy.

It turns out there is a tight relation between these AJ-equilibria in economies with solvency constraints that are not too tight and the K-L equilibria.

**Proposition 26.** Let $\{c, a, Q\}$ be an equilibrium with solvency constraints $\{B^i\}$ and initial wealth distribution $\overline{a_0}(s^0)$. If the solvency constraints are not too tight and the implied interest rates are high, then the consumption allocations and the implied AD prices are a KL equilibrium.


Since KL equilibria are standard AD equilibria, their allocations are Pareto efficient. This is the first welfare theorem. As a result of the previous proposition, the AJ equilibria with solvency constraints that are not too tight, are efficient.

**H. Asset Pricing in Economies with Solvency Constraints**

In this section, we zoom in on the asset pricing implications of the model with limited commitment. We start by allowing for stochastic growth in the aggregate endowment.
**Stationary Economy**  Suppose we are in a growing economy, where the growth rate $\lambda$ of the aggregate endowment $e$ depends on the current state of the world:

$$e_t(s^t) = e_{t-1}(s^{t-1})\lambda_t(s_t)$$

and the endowment share $\hat{y}$ only depends on the current state:

$$y^t_i(s^t) = e_t(s^t)\hat{y}^t_i(s_t).$$

Assume a constant discount factor $\beta$ and restate the state transition probabilities as

$$\hat{\pi}(s'|s) = \frac{\pi(s'|s)\lambda(s')^{1-\gamma}}{\sum \pi(s'|s)\lambda(s')^{1-\gamma}}$$

and $\hat{\beta}(s) = \beta \sum \pi(s'|s)\lambda(s')^{1-\gamma}$.

If the participation constraints are satisfied in the growing economy, they are satisfied in the economy with the unit aggregate endowment.

**Idiosyncratic Risk Independent of Aggregate State**  Denote the state $s_t = (x_t, z_t)$. Suppose that we can write $\hat{y}^t_i(x_t)$ and $\lambda_t(z_t)$ and suppose that aggregate shocks are i.i.d. over time:

$$\pi(z', x'| z, x) = \varphi(x'|x)\phi(z')$$

This implies we can state the value of autarchy only as a function of the current state $x$:

$$U^\text{aut}_i(x) = u(\hat{y}^t_i(x)) + \hat{\beta}\sum \phi(z') \hat{\pi}(x'|x)U^\text{aut}_i(x')$$

Neither the resource constraint, nor the participation constraints depend on the aggregate history $z^t$. The constrained efficient allocations imply consumption shares that are only a function of $x^t$: $\hat{c}^t_i(x^t)$. There is another way of thinking about this. The cutoff rule $\omega_i(x)$ will depend only on the current state $x$, but not on $z^t$. Allocations will depend on $y^t$, because your current consumption weight depends on $y^t$, but not on the history of aggregate shocks. In other words, we can write the second part of the SDF as a function only of the idiosyncratic history of shocks:

$$m_t = \beta\lambda(z_t)^{-\gamma}\phi(z_t) \left( \frac{\hat{c}^t_{t+1}(x^{t+1})}{\hat{c}^t_t(x^t)} \right)^{-\gamma} \varphi(x_t|x_{t-1})$$

**Proposition 27.** The risk premium on a one period strip is identical to that in a standard representative agent economy.
Proof.

\[
\frac{E_t R_{t,t+1}[c_{t+1}]}{E_t R_{t,t+1}[1]} = \frac{E_t \left[ c_{t+1} \right]}{E_t \left[ 1 \right]}
\]

\[
= \frac{\sum z_{t+1} \beta \phi(z_{t+1}) c_{t+1}(z_{t+1})}{\sum z_{t+1} \lambda(z_{t+1}) \varphi(x_{t+1}|x_t) c_{t+1}(z_{t+1})}
\]

\[
= \frac{\sum z_{t+1} \beta \phi(z_{t+1}) c_{t+1}(z_{t+1})}{\sum z_{t+1} \lambda(z_{t+1}) \varphi(x_{t+1}|x_t)}
\]

But this implies that the multiplicative risk premia is unchanged from the complete markets economy, and all this work was to no avail.

For asset pricing purposes, it is not sufficient simply to introduce some idiosyncratic risk. The idiosyncratic risk has to interact with the aggregate risk in some interesting way!

**Idiosyncratic Risk Dependent of Aggregate State** Suppose there are two aggregate states (recessions and expansions) \(z \in \{re, ex\}\) where \(\lambda(re) < \lambda(ex)\). Suppose there are two agents and two idiosyncratic endowment states, then we have \((y_{lo, re}, y_{hi, re}, y_{lo, ex}, y_{hi, ex})\). In the first state agent 1 draws a low endowment, in the second state agent 1 draws a high endowment. The endowment sum to one in each state. Suppose the events \(y\) are i.i.d. How do we solve for the constrained efficient allocations? First, solve for the value of autarchy:

\[
U_1^{aut}(lo, re) = u(y_{lo, re}) + \beta \sum_{z', x'} \pi(z', x'|lo, re) U_1^{aut}(x', z')
\]

\[
U_1^{aut}(hi, re) = u(y_{hi, re}) + \beta \sum_{z', x'} \pi(z', x'|hi, re) U_1^{aut}(x', z')
\]

Using vector notation, the value of autarchy in each state can be recovered from the following
equation:
\[ U_{aut}^1 = u(y)(I - \beta \Pi)^{-1} \]

For the second household, we solve the following equation:
\[ U_{aut}^2 = u(1-y)(I - \beta \Pi)^{-1} \]

The only state variables are the consumption share of the first agent and the current aggregate state. Next, we solve for the cutoff values
\[ U_1(\omega_1(s)) = u(\omega_1) + \hat{\beta}(s) \sum_{y'} \hat{\pi}(s'|s)U_1(\omega') = U_{aut}^1(s) \text{ for all } s \in (lo, re; lo, ex; hi, re; hi, ex) \]
\[ U_2(\omega_1(s)) = u(1-\omega_1) + \hat{\beta}(s) \sum_{y'} \hat{\pi}(s'|s)U_2(\omega') = U_{aut}^2(s) \text{ for all } s \in (lo, re; lo, ex; hi, re; hi, ex) \]

Claim 28. We get \( \omega_1^{hi, re} > \omega_1^{hi, ex} \) and \( \omega_1^{hi, re} < \omega_1^{hi, ex} \) if the cross-sectional dispersion of labor income shares increases in recessions, i.e. if
\[ \hat{y}(hi, re) > \hat{y}(hi, ex) \]

Assume one of the constraints binds in all four states of the world: that is agent one is always constrained in both of the high states and agent 2 is always constrained in both of the low states. So now, the ergodic set for the consumption shares consists only of
\[ (\omega_1(lo, re), \omega_1(lo, ex), \omega_1(hi, re), \omega_1(hi, ex)) \]

Claim 29. This implies that
\[ m(s, s') = \hat{\beta}(z)\lambda(z')^{-\gamma} \max\left( \frac{\omega'}{\omega}, \frac{1-\omega'}{1-\omega} \right)^{-\gamma} \]

Verify yourself that marginal utility growth increases in recessions because \( \max\left( \frac{\omega'}{\omega}, \frac{1-\omega'}{1-\omega} \right)^{-\gamma} \) is larger when \( z = re \).

This is a different version of an argument first made by Mankiw (1986) and elaborated upon by Constantinides and Duffie (1996). Idiosyncratic risk will not affect risk premia on stocks.
unless the risk itself is correlated with the aggregate state of the economy. Storesletten, Telmer and Yaron (2004) actually provide evidence that the conditional standard deviation idiosyncratic labor income risk more than doubles in recessions in the US. This finding provides an empirical underpinning for this specification of the labor income share process.

**Exercise 30.** Take \( \pi = \begin{bmatrix} .75 & .25 \\ .25 & .75 \end{bmatrix} \) and take \( y(lo) = .35 \) and \( y(hi) = .65 \). Set \( \beta = .65 \) and \( \gamma = 4 \) and compute the equilibrium allocations for the economy without aggregate uncertainty. Now assume that there is aggregate risk and that \( \lambda(ex) = 1.04, \lambda(re) = .96, y(lo, re) = .20, y(hi, re) = .80, y(lo, ex) = .34, \) and \( y(hi, ex) = .65 \). Again compute equilibrium allocations.
References


