A two-asset, intertemporal portfolio selection model is formulated incorporating proportional transaction costs. The demand for assets is shown to be sensitive to these costs. However, transaction costs have only a second-order effect on the liquidity premia implied by equilibrium asset returns: the derived utility is insensitive to deviations from the optimal portfolio proportions, and investors accommodate large transaction costs by drastically reducing the frequency and volume of trade. A single-period model with an appropriately chosen length of period does not imply the same liquidity premium as the intertemporal model because the appropriate length of the time period is asset specific.

I. Introduction

Transaction costs are an essential feature of some economic theories, such as the transactions demand for money. They are, however, an inessential nuisance in the real asset pricing theory of Sharpe (1964) and Lintner (1965) and its intertemporal extensions. Although inessential, transaction costs may be safely ignored in the real asset pricing theory only if it can be shown that they induce merely second-order effects on the theory’s empirically testable implications.

In a two-asset intertemporal model without transaction costs and with isoelastic utility of consumption, the optimal investment policy is characterized by one number, the ratio of the two asset values in the stock.
portfolio. When proportional transaction costs are introduced, a simple investment policy is characterized by a region of no transactions, which is an interval on the real line: an investor refrains from transacting as long as the ratio of asset values lies in this interval. One finds that the region of no transactions is wide, and, therefore, an investor’s demand for the assets is sensitive to the current composition of the portfolio. Furthermore, the average (over time) demand for one of the two assets, which is subject to transaction costs, is substantially reduced. Therefore, transaction costs have a first-order effect on the assets’ demand. This is the bad news.

The good news and the primary result of this paper is that transaction costs have only a second-order effect on equilibrium asset returns: investors accommodate large transaction costs by drastically reducing the frequency and volume of trade. It turns out that an investor’s expected utility of the future consumption stream is insensitive to deviations of the asset proportions from those proportions that are optimal in the absence of transaction costs. Therefore, a small liquidity premium is sufficient to compensate an investor for deviating significantly from the target portfolio proportions.

In this paper trades are generated by the endogenously determined adjustments of the portfolio assets in a way that maximizes the investor’s expected utility of the infinite lifetime’s consumption stream. This contrasts with models in which investors randomly arrive in the market to trade and with single-period models in which investors must liquidate their assets and consume the entire proceeds at the end of the period.

To illustrate the significance of endogenous trading, consider two risky assets with perfectly correlated rates of return and the same variance of their rate of return. The first is traded without transaction costs. The second is subject to proportional transaction costs, where $k$ is the one-way transaction cost rate. The liquidity premium is defined as the difference in these two assets’ annual expected rates of return such that an investor is indifferent between holding the one or the other. Suppose that the investor is endowed with the optimal portfolio proportions. In a single-period model in which the period length is 1 year, the liquidity premium is $k$ per year. By contrast, in the infinite-horizon model with proportional transaction costs the investor trades infrequently. For a stock with the standard deviation of the annual rate of return .20, the liquidity premium is only about .15$k$ per year; that is, it is smaller by one order of magnitude (see table 1).

Transaction costs may play a significant role in dissipating arbitrage profits, that is, preventing an investor from purchasing an underpriced asset with the intention of selling it a few months later and realizing a profit net of transaction costs. I find, however, that transac-
tion costs do not explain systematic and significant deviations of assets' expected returns adjusted for risk premia.

The starting point of the analysis is Merton's (1973) intertemporal consumption and investment model, outlined in Section II. I keep the discussion tractable by allowing for only two assets in the economy, one riskless and one risky.

In Section III, I introduce proportional transaction costs and define the set of simple investment and consumption policies. I argue that the optimal investment policy is simple if one assumes that it exists and that the derived utility function is differentiable. The optimal consumption policy is not in the set of simple consumption policies, but it is argued in Section III and in the Appendix that this is not a major issue. I solve for the optimal simple consumption and investment policy. I correct a technical error in Magill and Constantinides (1976), pointed out by Duffie (1983), but, more to the point, I provide numerical solutions that quantify the effect of transaction costs on the optimal simple investment and consumption policy (see tables 1, 2, and 3 and fig. 1).

Two measures of the liquidity premium are defined in Section IV and are estimated for a wide range of parameter values (see tables 1, 2, and 3). These estimates are obtained under the constraint that the optimal policy is simple and, therefore, represent an upper bound to the true liquidity premium. These upper bounds substantiate the main claim of this paper, that even large transaction costs give rise to a modest liquidity premium.

I find that the liquidity premium is strongly positively related to the variance of the asset's rate of return. By contrast, a single-period model with proportional transaction costs states that the liquidity premium is independent of the variance.

Finally, in Section V, I outline extensions and provide concluding remarks.

II. The Model

I consider an exchange economy with a single consumption good as the numeraire. There exist two securities only, with prices \( P_0(t) \) and \( P_1(t) \) at time \( t \). The investor takes the prices as given and may trade continuously at these prices. The shares of the securities are infinitely divisible. Short sales are permitted with full use of the proceeds. Taxes on capital gains are zero. The securities pay no dividend, and the capital gains are as follows:

\[
dP_0(t) = P_0(t)rdt
\]  

(1a)
and

$$dP_1(t) = P_1(t) [\mu dt + \sigma dw(t)], \quad (1b)$$

where $r$, $\mu$, and $\sigma$ are constants and $dw(t)$ is the increment of a Wiener process in $R^1$.

The investor has wealth $W(t)$ at time $t$, denominated in units of the consumption good. The investor consumes $c(t)dt$ over $[t, t + dt]$ and invests fraction $\alpha(t)$ of the wealth in the risky asset and the remaining fraction $1 - \alpha(t)$ in the riskless asset. When zero transaction costs and zero labor income are assumed, the wealth dynamics is

$$dW(t) = \{[(\mu - r)\alpha(t) + r]W(t) - c(t)]dt + \sigma \alpha(t)W(t)dw(t). \quad (2)$$

The investor makes sequential consumption, $c(t)$, and investment, $\alpha(t)$, decisions with the objective to maximize expected utility subject to the wealth dynamics (2) and given initial endowment $W(0) = W_0$. The expected utility is

$$E_0 \int_0^\infty e^{-\rho t} \gamma^{-1} c^\gamma(t) dt \quad (\equiv J[W_0]), \quad (3)$$

where $E_0$ is the expectation at time zero over the Wiener process $w(t)$. The impatience factor $\rho$ is a constant. The relative risk aversion coefficient, $1 - \gamma$, is assumed to be positive.\(^1\) A solution exists provided that

$$0 < \left(\frac{1}{1 - \gamma}\right) \left[\rho - \gamma r - \frac{(\mu - r)^2}{2(1 - \gamma)\sigma^2}\right] \quad (\equiv h) \quad (4)$$

and is given in Merton (1973):

$$\frac{c^*(t)}{W(t)} = h, \quad (5a)$$

$$\alpha^*(t) = \frac{\mu - r}{(1 - \gamma)\sigma^2}, \quad (5b)$$

and

$$J(W_0) = \frac{h^{\gamma - 1} W_0^\gamma}{\gamma}. \quad (5c)$$

\(^1\) Given the assumption that one of the assets is riskless, it is straightforward to generalize the utility function $c^\gamma/\gamma$ to $[c - \bar{c}(t)]^{\gamma}/\gamma$; the investor invests $\int_0^\infty e^{-\rho t} \bar{c}(t) dt$ in the riskless asset and allocates the remaining wealth between the risky and riskless assets by maximizing the expected utility as given by eq. (3). The same transformation applies to the problem with proportional transaction costs defined in the following section. In the remainder of the paper I set $\bar{c}(t) = 0$ without loss of generality.
In the next section I reexamine this problem in the presence of proportional transaction costs.

III. Proportional Transaction Costs

Prior to a transaction at time \( t \), the investor’s holdings in the riskless and risky securities are \( x(t) \) and \( y(t) \), respectively, denominated in units of the consumption good. If the investor increases (or decreases) the holding of the risky asset to \( y(t) + \nu(t) \), the holding of the riskless asset decreases (or increases) to \( x(t) - \nu(t) - |\nu(t)|k \). The proportional transaction cost rate, \( k \), is a given constant. I employ the convention that transaction costs and consumption deplete the riskless, but not the risky, asset.\(^2\)

An investment policy is defined as *simple* if it is characterized by two reflecting barriers, \( \bar{\lambda}, \bar{\lambda}, \lambda \leq \bar{\lambda} \), such that the investor refrains from transacting as long as the ratio \( y(t)/x(t) \) lies in the interval \( [\lambda, \bar{\lambda}] \) and transacts to the closest boundary, \( \lambda \) or \( \bar{\lambda} \), of the region of no transactions, \( [\lambda, \bar{\lambda}] \), whenever the ratio \( y(t)/x(t) \) lies outside this interval. In a discrete-time version of the proportional transaction costs model above, Constantinides (1979) proves that the optimal investment policy is simple. In the continuous-time framework characterized by equations (1a) and (1b), Takar, Klass, and Assaf (1983) assume that the investor does not consume but maximizes the expected rate of growth of wealth. They prove that the optimal investment policy is simple.

For the problem at hand, with asset price dynamics given by (1a) and (1b), the objective given by (3), and proportional transaction costs, results on the existence and form of the optimal consumption and investment policy have not been derived.\(^3\) I therefore confine my attention to the set of simple investment policies, as defined above, and to the set of simple consumption policies, defined by the property that the consumption rate is a constant fraction of the holding in the riskless asset; that is, \( \beta \equiv c(t)/x(t) \) is independent of \( x(t), y(t), \) and \( t \).\(^4\)

\(^2\) The assumption that the transaction costs are charged to the riskless asset instead of to the risky asset can be changed. Also the assumption that the transaction cost rate for increasing the holding of the risky asset equals the rate for decreasing it is innocuous and can be relaxed. Finally, the assumption that consumption depletes the riskless rather than the risky asset can also be relaxed.

\(^3\) The difficult task is proving the existence of an optimal policy. If the existence of an optimal policy and differentiability of \( J \) are assumed, one can prove that an optimal investment policy is simple. However, the optimal consumption policy is not, in general, simple, and this issue is taken up below and in the App.

\(^4\) An alternative approach is to replace the continuous-time problem at hand by a discrete-time problem, in which the decision interval is \( h \). Equation (1a) is replaced by \( P_0(t+h) = e^{\theta h}P_0(t) \). Equation (1b) is replaced by a binomial process \( P_1(t+h) = aP_1(t) \)
An optimal simple policy is defined as a simple policy that maximizes expected utility among all simple policies.

The primary goal of this paper is to show that transaction costs have only a second-order effect on the liquidity premium of an asset’s rate of return. In Section IV the liquidity premium is defined and is shown to be a decreasing function of the maximized expected utility of consumption in the presence of transaction costs. By limiting the investor’s consumption and investment policy to the set of simple policies, I underestimate the maximized expected utility of consumption and, therefore, overestimate the liquidity premium. The finding that the overestimated liquidity premium is small strengthens the claim that the actual liquidity premium is small. The sensitivity of my results to the assumed simple consumption policy is further discussed in the Appendix.

A given simple investment policy is characterized by the two parameters $\lambda, \tilde{\lambda}$, and a given simple consumption policy is characterized by the parameter $\beta, \beta = c(t)/x(t)$. The expected derived utility function of $x(t)$ and $y(t)$ is defined as

$$J[x(t), y(t); \beta, \lambda, \tilde{\lambda}] = E_t \int_{t}^{\infty} e^{-\rho(\sigma - \lambda)\tau} c^\gamma(\tau) d\tau,$$

where $E_t$ is the expectation at time $t$ over the Wiener process $w(\tau)$. In the region of no transactions the dynamics of $x(t)$ and $y(t)$ are given by

$$dx(t) = rx(t)dt - c(t)dt \tag{7a}$$

and

$$dy(t) = \mu y(t)dt + \sigma y(t)dw(t). \tag{7b}$$

The derived utility satisfies the Bellman equation

$$\frac{c^\gamma}{\gamma} + (rx - c) f_x + \mu y f_y + \frac{\sigma^2}{2} y^2 f_{yy} - \rho J = 0, \quad \lambda \leq \frac{y}{x} \leq \tilde{\lambda}. \tag{8}$$

Subscripts on $J$ denote derivatives, and the arguments of $x, y, c,$ and $J$ are suppressed.

---

with probability $p$ and $P_1(t + h) = a^{-1}P_1(t)$ with probability $1 - p$. Equation (2) is replaced by its obvious discrete-time counterpart, and the objective function (3) is replaced by $E_0 \sum_{n=0}^{\infty} e^{-\rho nh}c^{\gamma}(nh)$. For this problem the optimal investment policy is known to be simple, as argued above. One can proceed by deriving a difference equation analogous to the differential equation (8) and obtain a general solution that is of the same general form as eq. (11). The whole discussion may be couched in this framework. With the appropriate choice of the parameters $a = a(h)$ and $p = p(h)$, the stochastic process of $P_1(t)$ tends to either the diffusion process (1b) or a Poisson process. For an illustration of this approach in option pricing, see Cox, Ross, and Rubinstein (1979).
For \( y/x \leq \lambda \), the assumed investment policy is to sell \((1 + k)v\) units of the riskless asset and purchase \(v\) units of the risky asset so that the investment proportions equal \(\lambda\); that is, \((y + v)/(x - (1 + k)v) = \lambda\). Solving for \(v\), we obtain \(v = (\lambda x - y)/(1 + (1 + k)\lambda)\). Hence \(J(x, y) = J(x - (1 + k)v, y + v) = J(x - (1 + k)(\lambda x - y)/(1 + (1 + k)\lambda), y + (\lambda x - y)/(1 + (1 + k)\lambda)]\). It is easily shown that \(J\) is homogeneous of degree one in its arguments. Invoking homogeneity, we can simplify the expression above into \(J(x, y) = [x + (1 + k)y]/[1 + (1 + k)\lambda], \lambda/[1 + (1 + k)\lambda]\), which satisfies the boundary condition

\[
(1 + k)J_x = J_y, \quad \frac{y}{x} \leq \lambda. \tag{9a}
\]

Essentially, the marginal rate of substitution of the riskless asset for the risky asset equals \(1 + k\) because in the range \(y/x \leq \lambda\) the optimal simple policy is to sell \((1 + k)v\) units of the riskless asset and use the proceeds to purchase \(v\) units of the risky asset.

On the other side of the region of no transactions the boundary condition is

\[
(1 - k)J_x = J_y, \quad \frac{y}{x} \geq \bar{\lambda}, \tag{9b}
\]

stating that the marginal rate of substitution of the riskless asset for the risky asset equals \(1 - k\).

I invoke continuity of the first derivatives of the \(J\) function and impose the conditions that the solution to equation (8) satisfies equation (9a) at \(y/x = \lambda\) and equation (9b) at \(y/x = \bar{\lambda}\).

We can substitute \(c = \beta x\) in equation (8) and obtain

\[
\left(\frac{Bx}{\gamma}\right)^{\gamma} + (r - \beta)xJ_x + \mu yJ_y + \frac{\sigma^2}{2} y^2 J_{yy} - \rho J = 0, \quad \lambda \leq \frac{y}{x} \leq \bar{\lambda}. \tag{10}
\]

We can easily show that \(J\) is homogeneous of degree \(\gamma\) in \(x\) and \(y\). If \(x\) and \(y\) are both positive, the general solution to equation (10) is

---

5 The problem defined by eqns. (8) and (9) and without the restriction that the consumption policy be simple is a two-security version of the multisection problem formulated in Magill and Constantinides (1976). Unable to solve the partial differential equation, they solve a closely related “\(\epsilon\) problem” and obtain the control parameters \(\lambda\), \(\bar{\lambda}\), by taking the limit as \(\epsilon \to 0^+\). In a private communication, Duffie (1983) points out that the claimed solution to the \(\epsilon\) problem does not satisfy the partial differential equation in the continuation region because certain \(x, y\) functions are mistakenly treated as constants. The quantitative results in Magill and Constantinides (1976) are invalid and are corrected in the present paper in the special case of two securities and a simple consumption policy.

6 If \(x\) is positive but \(y\) is negative, the general solution to eq. (10) is given by eq. (11), where \(y\) is replaced by \(-y\). If \(y\) is positive but \(x\) is negative, the general solution is still given by eq. (11), where \(x\) and \(\beta^y\) are replaced by \(-x\) and \((-\beta)^y\), respectively. In either case, the parameters \(s_1\) and \(s_2\) are the roots of the quadratic equation (12).
\[ J(x, y; \beta, \lambda, \tilde{\lambda}) = \frac{\beta^\gamma}{\rho - \gamma(r - \beta)} \left( \frac{x^\gamma}{\gamma} + A_1 x^{\gamma-s_1} y^{s_1} + A_2 x^{\gamma-s_2} y^{s_2} \right), \]  

(11)

where \( A_1 \) and \( A_2 \) are free parameters and \( s_1 \) and \( s_2 \) are the roots of the quadratic equation

\[ \frac{\sigma^2}{2} s^2 + \left( \mu - \frac{\sigma^2}{2} - r + \beta \right) s - [\rho - \gamma(r - \beta)] = 0. \]  

(12)

If we substitute the solution (11) in the boundary conditions (9) and divide by \( x^\gamma \), we obtain the following pair of linear equations in \( A_1 \) and \( A_2 \):

\[
(1 + k)[1 + A_1(\gamma - s_1)\lambda^{s_1} + A_2(\gamma - s_2)\lambda^{s_2}] = A_1 s_1 \lambda^{s_1-1} + A_2 s_2 \lambda^{s_2-1}.
\]  

(13)

and

\[
(1 - k)[1 + A_1(\gamma - s_1)\tilde{\lambda}^{s_1} + A_2(\gamma - s_2)\tilde{\lambda}^{s_2}] = A_1 s_1 \tilde{\lambda}^{s_1-1} + A_2 s_2 \tilde{\lambda}^{s_2-1}.
\]  

(14)

Assuming that the corresponding matrix of coefficients is nonsingular, we see that these equations uniquely determine \( A_1 = A_1(\beta, \lambda, \tilde{\lambda}) \) and \( A_2 = A_2(\beta, \lambda, \tilde{\lambda}) \) in terms of the controls \( \beta, \lambda, \) and \( \tilde{\lambda} \).

Before we assert that

\[ J^*(x, y) = \max_{\beta, \lambda, \tilde{\lambda}} J(x, y; \beta, \lambda, \tilde{\lambda}), \quad \lambda \leq \frac{y}{x} \leq \tilde{\lambda}, \]  

(15)

determine the optimal controls, we need to demonstrate that the maximizing triplet \((\beta, \lambda, \tilde{\lambda})\) is independent of \(x\) and \(y\). Since (13) is not an explicit function of \( \tilde{\lambda} \), differentiating this equation with respect to \( \tilde{\lambda} \), we obtain the result that \( \partial A_1/\partial \tilde{\lambda} = 0 \) implies \( \partial A_2/\partial \tilde{\lambda} = 0 \), and vice versa. Likewise, from equation (14) we see that \( \partial A_1/\partial \lambda = 0 \) implies \( \partial A_2/\partial \lambda = 0 \). We conclude that the same pair \((\lambda, \tilde{\lambda})\) that satisfies the necessary conditions of optimality of \( A_1 \) also satisfies the necessary conditions of optimality of \( A_2 \) and, more to the point, of \( J(x, y; \beta, \lambda, \tilde{\lambda}) \) irrespective of \(x, y\).

We cannot prove that the control \( \beta \) maximizing \( J(x, y; \beta, \lambda, \tilde{\lambda}) \) is independent of \(x, y\). This is to be expected because of the constraint imposed on \( c(t)/x(t) \) to be independent of \( x(t) \) and \( y(t) \).

In the numerical solution we can proceed as follows. For a given triplet \((\beta, \lambda, \tilde{\lambda})\) we obtain \( A_1(\beta, \lambda, \tilde{\lambda}) \) from equations (13) and (14). We maximize \( A_1(\beta, \lambda, \tilde{\lambda})/(\rho - \gamma(r - \beta)) \) with respect to \((\lambda, \tilde{\lambda})\). By the earlier argument, this pair also maximizes \( J(x, y; \beta, \lambda, \tilde{\lambda}) \). Defining

\[ J(x, y, \beta) = \max_{(\lambda, \tilde{\lambda})} J(x, y; \beta, \lambda, \tilde{\lambda}), \]  

(16)
we then maximize $J(x, y, \beta)$ with respect to $\beta$ at the value of $y/x$ corresponding to the optimal portfolio proportions in the absence of transaction costs. From equation (5b), this value of $y/x$ is

$$
\lambda^* = \left[ \frac{\mu - r}{(1 - \gamma)\sigma^2} \right] \left[ 1 - \frac{\mu - r}{(1 - \gamma)\sigma^2} \right]^{-1}.
$$

In tables 1, 2, and 3 I report the maximizing triplet $(\beta, \lambda, \tilde{\lambda})$ for various values of the model parameters. The properties of the region of no transactions and the optimal consumption rate are summarized below.

A. Transaction costs broaden the region of no transactions. In table 1 and figure 1 I report the lower ($\lambda$) and upper ($\tilde{\lambda}$) bounds of the risky to the riskless asset ratio in the region of no transactions for different transaction cost rates and for parameter values $\gamma = -1, \rho = .10/\text{year}, r = .10/\text{year}, \mu = .15/\text{year},$ and $\sigma^2 = .04/\text{year}$. The bounds of the region of no transactions are sensitive to a change in transaction costs, especially for low levels of the transaction cost rate.

B. Transaction costs shift the region of no transactions toward the riskless asset. Table 1 and figure 1 illustrate that the lower bound $\lambda$ is decreasing in $k$ faster than the upper bound is increasing in $k$. As expected, the average demand for the risky asset is decreasing in the transaction cost rate.

The same conclusion obtains if the control limits are expressed in terms of the ratio of the risky asset to total wealth, $y/(x + y)$, instead of the ratio $y/x$. With the same parameter values as in table 1, the optimal ratio $y/(x + y)$ in the absence of transaction costs is .625 and the control limits with $k = .20$ are .670 and .394, respectively, illustrating that the lower bound is decreasing in $k$ faster than the rate at which the upper bound is increasing in $k$.

C. The width of the region of no transactions is insensitive to risk aversion. Table 2 presents the bounds of the region of no transactions for different levels of risk aversion for $k = .01$ or .10, $\rho = .10/\text{year}, r = .10/\text{year}, \mu = .15/\text{year},$ and $\sigma^2 = .04/\text{year}$. Whereas the region of no transactions narrows as measured by $\tilde{\lambda} - \lambda$, the terms $\lambda/\lambda^*$ and $\tilde{\lambda}/\lambda^*$ are insensitive to the level of risk aversion. Also the width of the region of no transactions, as measured by the difference in the ratio $y/(x + y)$ at the boundaries, is insensitive to risk aversion. For $k = .01$ and $\gamma = -1$ this width is $.64 - .58 = .06$, and for $k = .01$ and $\gamma = -5$ it is $.22 - .17 = .05$.

D. An increase in risk aversion shifts the region of no transactions toward the riskless asset. This is illustrated in table 2. Also the average demand for the risky asset is decreasing in risk aversion.

E. The width of the region of no transactions is insensitive to the variance of the asset’s rate of return. Table 3 presents the bounds of
TABLE 1

Optimal Policy Parameters and Liquidity Premia for Different Values of the Transaction Cost Rate

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>.005</th>
<th>.01</th>
<th>.02</th>
<th>.03</th>
<th>.04</th>
<th>.05</th>
<th>.10</th>
<th>.15</th>
<th>.20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>1.667</td>
<td>1.450</td>
<td>1.377</td>
<td>1.277</td>
<td>1.202</td>
<td>1.140</td>
<td>1.087</td>
<td>.891</td>
<td>.754</td>
<td>.650</td>
</tr>
<tr>
<td>$\bar{\lambda}$</td>
<td>1.667</td>
<td>1.767</td>
<td>1.784</td>
<td>1.803</td>
<td>1.818</td>
<td>1.832</td>
<td>1.844</td>
<td>1.905</td>
<td>1.965</td>
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</tr>
<tr>
<td>$\beta$</td>
<td>.2875</td>
<td>.2869</td>
<td>.2869</td>
<td>.2864</td>
<td>.2859</td>
<td>.2852</td>
<td>.2845</td>
<td>.2803</td>
<td>.2754</td>
<td>.2700</td>
</tr>
<tr>
<td>$\delta(k)/\text{year}$</td>
<td>0</td>
<td>.0008</td>
<td>.0014</td>
<td>.0025</td>
<td>.0037</td>
<td>.0049</td>
<td>.0061</td>
<td>.0130</td>
<td>.0216</td>
<td>.0347</td>
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<tr>
<td>$\delta(k)/k$</td>
<td>...</td>
<td>.16</td>
<td>.14</td>
<td>.13</td>
<td>.12</td>
<td>.12</td>
<td>.12</td>
<td>.13</td>
<td>.13</td>
<td>.17</td>
</tr>
<tr>
<td>$\delta(k)/\text{year}$</td>
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<td>.0013</td>
<td>.0024</td>
<td>.0046</td>
<td>.0068</td>
<td>.0091</td>
<td>.0114</td>
<td>.0250</td>
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<tr>
<td>$\delta(k)/k$</td>
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<td>.24</td>
<td>.23</td>
<td>.23</td>
<td>.23</td>
<td>.23</td>
<td>.25</td>
<td>N.A.</td>
<td>N.A.</td>
</tr>
</tbody>
</table>

Note.—The table displays the lower ($\lambda$) and upper ($\bar{\lambda}$) bounds of the risky to the riskless asset ratio in the region of no transactions, the optimal consumption rate ($\beta$), and the liquidity premia ($\delta, \bar{\delta}$) on the risky asset, for different values of the transaction cost rate ($k$). The assumed parameter values are $\gamma = -1$, $\rho = .10/\text{year}$, $r = .10/\text{year}$, $\mu = .15/\text{year}$, and $\sigma^2 = .04/\text{year}$. 
the region of no transactions for different levels of variance for \( k = .01 \) or .10, \( \gamma = -1, \rho = .10/\text{year}, r = .10/\text{year}, \) and \( \mu = .15/\text{year}. \) Whereas the region of no transactions widens as measured by \( \bar{\lambda} - \lambda, \) the width of the region of no transactions, as measured by the difference in the ratio \( y/(\kappa + y) \) at the boundaries, is insensitive to risk aversion. For \( k = .01 \) and \( \sigma^2 = .2^2/\text{year}, \) the width is \( .64 - .58 = .06, \) and for \( k = .01 \) and \( \sigma^2 = .5^2/\text{year} \) the width is \( .13 - .06 = .07. \)

F. An increase in the variance of the risky asset's rate of return shifts the region of no transactions toward the riskless asset.

G. Transaction costs decrease the consumption rate, but the effect is weak. The consumption rate (\( \beta \)) as a function of the transaction cost rate is reported in table 1. Transaction costs affect the consumption rate in two ways. First, through an income effect, transaction costs decrease the consumption rate. Second, through a substitution effect, they make current consumption less costly (in terms of transaction costs) than future consumption and shift consumption to the earlier
TABLE 2
Optimal Policy Parameters and Liquidity Premia for Different Values of the Risk Aversion Parameter

<table>
<thead>
<tr>
<th>γ</th>
<th>-1</th>
<th>-2</th>
<th>-3</th>
<th>-4</th>
<th>-5</th>
</tr>
</thead>
<tbody>
<tr>
<td>k = .01</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>λ</td>
<td>1.377</td>
<td>.576</td>
<td>.364</td>
<td>.266</td>
<td>.209</td>
</tr>
<tr>
<td>λ/λ*</td>
<td>.83</td>
<td>.81</td>
<td>.80</td>
<td>.80</td>
<td>.80</td>
</tr>
<tr>
<td>λ*</td>
<td>1.667</td>
<td>.714</td>
<td>.455</td>
<td>.333</td>
<td>.263</td>
</tr>
<tr>
<td>λ</td>
<td>1.784</td>
<td>.767</td>
<td>.489</td>
<td>.360</td>
<td>.284</td>
</tr>
<tr>
<td>λ/λ*</td>
<td>1.07</td>
<td>1.07</td>
<td>1.08</td>
<td>1.08</td>
<td>1.08</td>
</tr>
<tr>
<td>β (k = 0)</td>
<td>.2875</td>
<td>.1833</td>
<td>.1540</td>
<td>.1400</td>
<td>.1318</td>
</tr>
<tr>
<td>β (k = .01)</td>
<td>.2868</td>
<td>.1814</td>
<td>.1524</td>
<td>.1387</td>
<td>.1307</td>
</tr>
<tr>
<td>δ/year</td>
<td>.0014</td>
<td>.0016</td>
<td>.0017</td>
<td>.0018</td>
<td>.0019</td>
</tr>
<tr>
<td>δ/year</td>
<td>.0024</td>
<td>.0026</td>
<td>.0027</td>
<td>.0028</td>
<td>.0028</td>
</tr>
</tbody>
</table>

k = .10

| λ   | .892  | .374  | .236  | .172  | .135 |
| λ/λ* | .54   | .52   | .52   | .52   | .52  |
| λ*  | 1.667 | .714  | .455  | .333  | .263 |
| λ   | 1.905 | .825  | .528  | .388  | .306 |
| λ/λ* | 1.14  | 1.16  | 1.16  | 1.16  | 1.16 |
| β (k = .1) | .2803 | .1783 | .1501 | .1368 | .1291 |
| δ/year | .0130 | .0133 | .0134 | .0136 | .0136 |
| δ/year | .0250 | .0236 | .0230 | .0226 | .0224 |

Note.—The table displays the lower (λ) and upper (λ*) bounds of the risky to the riskless asset ratio in the region of no transactions, the target portfolio ratio (λ*), the optimal consumption rate (β), and the liquidity premia (δ) on the risky asset for different values of the risk aversion parameter (γ) and the transaction cost rate (κ). The assumed parameter values are p = .10/year, r = .10/year, μ = .15/year, and σ² = .04/year.

periods. Whereas it is not a priori obvious which effect should dominate, table 1 illustrates that the consumption rate is decreasing in the transaction cost rate. This point is discussed further in the Appendix.

Finally, tables 2 and 3 illustrate that an increase in risk aversion or in the risky asset’s variance of return increases the consumption rate. These properties are the same as in the absence of transaction costs.

IV. The Liquidity Premium

Consider two assets with perfectly correlated rates of return and equal variance of their rates of return. Trading in the first asset is subject to proportional transaction costs κ. Trading in the second asset is exempt from transaction costs. I refer to the second asset as the liquid counterpart of the first one. If both assets are held in equilib-
Table 3
Optimal Policy Parameters and Liquidity Premia for Different Values of the Variance of the Risky Asset’s Rate of Return

<table>
<thead>
<tr>
<th>$\sigma^2$/Year</th>
<th>$\lambda$</th>
<th>$\lambda/\lambda^*$</th>
<th>$\lambda^*$</th>
<th>$\bar{\lambda}$</th>
<th>$\bar{\lambda}/\lambda^*$</th>
<th>$\beta(k = 0)$</th>
<th>$\beta(k = .01)$</th>
<th>$\delta$/year</th>
<th>$\hat{\delta}$/year</th>
</tr>
</thead>
<tbody>
<tr>
<td>.2</td>
<td>1.377</td>
<td>.506</td>
<td>.274</td>
<td>.172</td>
<td>.118</td>
<td>.086</td>
<td>.064</td>
<td>.58</td>
<td></td>
</tr>
<tr>
<td>.25</td>
<td>.83</td>
<td>.76</td>
<td>.71</td>
<td>.67</td>
<td>.64</td>
<td>.61</td>
<td>.111</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.3</td>
<td>1.667</td>
<td>.667</td>
<td>.385</td>
<td>.256</td>
<td>.185</td>
<td>.141</td>
<td>.111</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.35</td>
<td>1.784</td>
<td>.739</td>
<td>.442</td>
<td>.305</td>
<td>.228</td>
<td>.178</td>
<td>.144</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.4</td>
<td>1.07</td>
<td>1.11</td>
<td>1.15</td>
<td>1.19</td>
<td>1.23</td>
<td>1.26</td>
<td>1.30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.45</td>
<td>.2875</td>
<td>.1750</td>
<td>.1433</td>
<td>.1289</td>
<td>.1208</td>
<td>.1159</td>
<td>.1125</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.5</td>
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<td>.1727</td>
<td>.1412</td>
<td>.1270</td>
<td>.1192</td>
<td>.1144</td>
<td>.1112</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\lambda/\lambda^*$</th>
<th>$\lambda^*$</th>
<th>$\bar{\lambda}$</th>
<th>$\bar{\lambda}/\lambda^*$</th>
<th>$\beta(k = .1)$</th>
<th>$\beta$/year</th>
<th>$\delta$/year</th>
<th>$\hat{\delta}$/year</th>
</tr>
</thead>
<tbody>
<tr>
<td>.2</td>
<td>.891</td>
<td>.294</td>
<td>.140</td>
<td>.078</td>
<td>.047</td>
<td>.030</td>
<td>.020</td>
<td></td>
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<tr>
<td>.25</td>
<td>.53</td>
<td>.44</td>
<td>.36</td>
<td>.30</td>
<td>.25</td>
<td>.21</td>
<td>.18</td>
<td></td>
</tr>
<tr>
<td>.3</td>
<td>1.667</td>
<td>.667</td>
<td>.385</td>
<td>.256</td>
<td>.185</td>
<td>.141</td>
<td>.111</td>
<td></td>
</tr>
<tr>
<td>.35</td>
<td>1.905</td>
<td>.805</td>
<td>.488</td>
<td>.341</td>
<td>.259</td>
<td>.205</td>
<td>.168</td>
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<tr>
<td>.4</td>
<td>1.14</td>
<td>1.21</td>
<td>1.27</td>
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<td>1.40</td>
<td>1.45</td>
<td>1.51</td>
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<td>.45</td>
<td>.2803</td>
<td>.1692</td>
<td>.1379</td>
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<td>.1163</td>
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<td>.1086</td>
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</tr>
<tr>
<td>.5</td>
<td>.0130</td>
<td>.0137</td>
<td>.0150</td>
<td>.0166</td>
<td>.0185</td>
<td>.0205</td>
<td>.0227</td>
<td></td>
</tr>
<tr>
<td>$\hat{\delta}$/year</td>
<td>.0250</td>
<td>.0225</td>
<td>.0217</td>
<td>.0219</td>
<td>.0225</td>
<td>.0235</td>
<td>.0247</td>
<td></td>
</tr>
</tbody>
</table>

Note.—The lower ($\lambda$) and upper ($\bar{\lambda}$) bounds of the risky to the riskless asset ratio in the region of no transactions, the target portfolio ratio ($\lambda^*$), the optimal consumption rate ($\beta$), and the liquidity premia ($\delta$, $\hat{\delta}$) for different values of the variance of the risky asset’s rate of return ($\sigma^2$) and the transaction cost rate ($k$). The assumed parameter values are $\gamma = -1$, $\rho = .10$/year, $r = .10$/year, and $\mu = .15$/year.

The expected rate of return on the first asset must exceed that of its liquid counterpart by some liquidity premium $\delta(k)$.

I assume that the investor follows an optimal simple consumption and investment policy. I define the liquidity premium, $\delta(k)$, on the risky asset in the presence of proportional transaction costs ($k$) as the decrease in the risky asset’s mean return ($\mu$), which, combined with the elimination of transaction costs, leaves unchanged the investor’s expected utility at $y/x = \lambda^*$.

From equation (11) the maximized expected utility in the presence of transaction costs is the left-hand side of the following equation:

$$
\frac{\beta^\gamma}{\rho - \gamma(r - \beta)} \left( \frac{x^\gamma}{\gamma} + A_1 x^{\gamma - \mu y} + A_2 x^{\gamma - \delta(k)} \right) = \left( \frac{1}{1 - \gamma} \right) \left( \rho - \gamma r - \left[ \frac{\mu - \delta(k) - r^2}{2(1 - \gamma)\sigma^2} \right] \right)^{-1} \frac{(x + y)^\gamma}{\gamma}. 
$$

(18)
The right-hand side is the maximized expected utility in the absence of transaction costs and with the expected rate of return on the risky asset replaced by $\mu - \delta(k)$. If we divide both sides of equation (18) by $x^\gamma$ and substitute $\lambda^*$ for $y/x$, we obtain

$$\frac{\beta^\gamma}{(1 + \lambda^*)^\gamma[(\rho - \gamma(r - \beta)]} \left[ A_1(\lambda^*)^\gamma + A_2(\lambda^*)^\gamma \right]$$

$$= \left( \frac{1}{1 - \gamma} \left[ (\rho - \gamma) - \frac{[\mu - \delta(k) - r1^2\gamma]}{2(1 - \gamma)\sigma^2} \right] \right)^{\gamma^{-1}} / \gamma.$$

This definition of the liquidity premium does not reflect the transaction costs of initially adjusting the portfolio proportions to the ratio $\lambda^*$. An alternative definition that accounts for the initial transaction costs is provided shortly.

The liquidity premium is a decreasing function of the left-hand side of equation (19). This left-hand side is the maximized expected utility under the constraint that the policy is simple and, therefore, underestimates the unconstrained maximized expected utility. Thus the liquidity premium, as defined by equation (19), is an upper bound to the liquidity premium when the investor is not constrained to follow a simple policy.

Table 1 presents the liquidity premium $\delta(k)$ and the ratio $\delta(k)/k$ for parameter values $\gamma = -1$, $\rho = .10$/year, $r = .10$/year, $\mu = .15$/year, and $\sigma^2 = .04$/year. The liquidity premium is one order of magnitude smaller than the one-way proportional transaction cost rate; that is, $\delta(k)/k \approx .15$ for $k = .005$ to .20. As the investor becomes more risk averse, the liquidity premium rises but only slightly. Table 2 shows that, for $k = .01$, the liquidity premium rises from $\delta = .0014$ to .0019 as $\gamma$ decreases from $-1$ to $-5$; and for $k = .10$, the liquidity premium rises only from $\delta = .0130$ to .0136 as $\gamma$ decreases from $-1$ to $-5$.

An increase in the risky asset’s variance of return increases the frequency of transactions, increases the transaction costs, and raises the liquidity premium. Table 3 shows that, for $k = .01$, the liquidity premium triples as the variance increases from $\sigma^2 = .04$/year to .25/year: for $k = .10$, the liquidity premium doubles as the variance increases from $\sigma^2 = .04$/year to .25/year.

Banz (1981) documents a negative association between average returns to stocks and the market value of the stocks after controlling for risk. Classifying all New York Stock Exchange–listed stocks into 10 groups by the market value of the stock, he finds that the difference in the annual mean rate of return between the smallest and largest deciles is about .15. I readily dismiss transaction costs as an explanation for this anomaly. If small firms are traded with $.10$ one-way proportional transaction costs and if $\sigma^2 = .25$/year, table 3 indicates
that the liquidity premium is .023/year. The liquidity premium is one order of magnitude smaller than the observed anomaly even with the assumed large transaction costs and variance of return.

The second definition of a liquidity premium, \( \hat{\delta}(k) \), incorporates the transaction costs both in setting up the portfolio and in maintaining the portfolio proportions within the region of no transactions. The investor’s endowment is \( x_1, y_1 \), where \( y_1 = 0 \); that is, the endowment is in the riskless asset alone. The investor optimally purchases \( y_2 = \lambda x_1/[1 + (1 + k)\lambda] \) units of the risky asset. Net of transaction costs, the investment in the riskless asset becomes

\[
x_2 = x_1 - \frac{(1 + k)\lambda x_1}{1 + (1 + k)\lambda} = \frac{x_1}{1 + (1 + k)\lambda}.
\]

(20)

The ratio \( y_2/x_2 \) is \( \lambda \); that is, the investor optimally transacts to the lower boundary of the region of no transactions.

The definition of the liquidity premium \( \hat{\delta}(k) \) is given by

\[
\frac{\beta^\gamma}{\rho - \gamma(r - \beta)} \left( \frac{x_2^\gamma}{\gamma} + A_1 x_2^{-\gamma^1} y_2^{\gamma^1} + A_2 x_2^{-\gamma^2} y_2^{\gamma^2} \right)
\]

\[
= \left( \left( \frac{1}{1 - \gamma} \right) [\rho - \gamma r - \frac{[\mu - \hat{\delta}(k) - r]^{\gamma^2}}{2(1 - \gamma)\sigma^2}] \right)^{\gamma - 1} / \gamma. \tag{21}
\]

Dividing both sides by \( x_2^\gamma \), we obtain

\[
\frac{\beta^\gamma}{[1 + (1 + k)\lambda]^{\gamma}} \left[ \rho - \gamma(r - \beta) \right]
\]

\[
= \left( \left( \frac{1}{1 - \gamma} \right) [\rho - \gamma r - \frac{[\mu - \hat{\delta}(k) - r]^{\gamma^2}}{2(1 - \gamma)\sigma^2}] \right)^{\gamma - 1} / \gamma. \tag{22}
\]

Tables 1, 2, and 3 report the liquidity premium \( \hat{\delta}(k) \) and illustrate that it is almost double the liquidity premium \( \delta(k) \). Recall that \( \delta(k) \) is the liquidity premium that compensates for the transaction costs in maintaining the portfolio proportions within the region of no transactions. I interpret the difference \( \hat{\delta}(k) - \delta(k) \) as the liquidity premium in setting up the portfolio if the endowment is in the form of the riskless asset. The difference \( \hat{\delta}(k) - \delta(k) \) is small compared with the transaction cost rate because the transaction costs of setting up the portfolio are amortized over the investor’s infinite lifetime.

A major difficulty in incorporating transaction costs in the intertemporal consumption and investment problem is that the (stopping) times at which the investor optimally trades are endogenously determined. In the particular problem examined here, trades occur when the ratio of the risky to the riskless asset falls outside the region of no transactions.
By contrast, in single-period models trades may occur only at the
two exogenously determined times, the beginning and the end of the
period. The problem is reduced to the determination of the optimal
size of a trade at the beginning of the period. The simplification
associated with single-period models comes at a cost. Since the
assumed length of the time period is arbitrary, the length of time over
which the transaction costs are amortized is also arbitrary. The liquidity
premium (to be defined) implied by a single-period model is a
function of the arbitrary length of the time period. The problem may
not be resolved by simply choosing the "correct" length of the time
period. The following example illustrates that the correct length of the
time period is asset specific.

An investor is endowed with $W_0$ units of the consumption good.
The investor buys $y_0$ units of the risky asset and $W_0 - y_0$ units of the
riskless asset without incurring transaction costs. No trading is al-
lowed between now and date $T$. The asset dynamics are given by
equation (1). At date $T$ the risky asset becomes $y(T) = y_0 \exp \int_0^T [\mu \, dt + \sigma dw(t)]$, gross of transaction costs; it is converted into $(1 - k)y_0 \exp \int_0^T [\mu \, dt + \sigma dw(t)]$ units of the riskless asset and is consumed along with the $(W_0 - y_0) \exp(\sigma T)$ units of the riskless asset. The corresponding
definition of the liquidity premium, $\delta(k)$, is

\[
(1 - k)y_0 \exp \int_0^T [\mu \, dt + \sigma dw(t)] = y_0 \exp \int_0^T ([\mu - \delta(k)] \, dt + \sigma dw(t)),
\]

which implies

\[
\delta(k) = - \frac{1}{T} \ln(1 - k).
\]

The single-period model states that the liquidity premium is indepen-
dent of the variance of the asset's rate of return unlike the optimal
trading model, which states that the liquidity premium is strongly
positively related to the variance of the asset's rate of return. Thus the
single-period model either underestimates the liquidity premium of
high-variance stocks or overestimates the liquidity premium of low-
variance stocks.\(^\text{7}\) The single-period model also states that the liquidity
premium is independent of the risk aversion of the investor unlike

\(^\text{7}\) Single-period models of capital market equilibrium with fixed transaction costs
have been presented in Levy (1978) and Mayshar (1979, 1981). The argument may be
extended to the case of fixed transaction costs to show that single-period models imply a
substantially different liquidity premium than the optimal trading model. See the dis-
tussion in Constantinides (1985).
the optimal trading model, which states that the liquidity premium is weakly positively related to the risk aversion of the investor.

With parameter values $\gamma = -1$, $\rho = .10/\text{year}$, $r = .10/\text{year}$, $\mu = .15/\text{year}$, $\sigma^2 = .04/\text{year}$, and $k = .01$, the optimal trading model states that the liquidity premium is $\delta(0.01) = .0014$ (see table 3). If the single-period model is to give the same liquidity premium in equation (24), the length of the period must be 7.2 years. If, instead, the variance is $\sigma^2 = .25/\text{year}$, the optimal trading model gives a liquidity premium $\delta(0.01) = .0043$, and the length of the period in the single-period model must shrink to 2.3 years.

Despite their shortcomings, single-period models capture the realistic possibility that the investor may be forced to liquidate the portfolio at some future time. The infinite-horizon model ignores the possibility of forced liquidation. A simple modification of the basic model accommodates forced liquidations.

We can model forced liquidations as Poisson arrivals uncorrelated with the stock return realization. We can add one term to the right-hand side of equation (8), which is the product of the force of the Poisson process and the utility of consumption, net of transaction costs, in the event of a forced liquidation. The problem can be solved by the numerical methods discussed earlier. If the expected time of arrival of the forced liquidation is short, the model resembles the single-period model, while if it is long, it resembles the infinite-horizon model.

V. Extensions and Concluding Remarks

There are several directions in which the model presented in this paper can be extended. First, we can allow for more than one risky asset. In principle this extension is straightforward. The computational requirements, however, are enormous. I conjecture that, as the number of risky assets is increased, each with the same variance of their rate of return and correlation less than one, the liquidity premium drops. Indirect supporting evidence is provided in Mayshar (1979) and Kovenock and Rothschild (1985).

Second, we can introduce fixed transaction costs. Single-period models with fixed transaction costs are discussed in Leland (1974), Mukherjee and Zabel (1974), Brennan (1975), Goldsmith (1976), Levy (1978), and Mayshar (1979, 1981). In multiperiod extensions of these models the optimal investment policy is complex because the derived utility function, $J(x, y)$, is no longer homogeneous of some degree in $x$ and $y$. Kandel and Ross (1983) introduce quasi-fixed transaction costs in a multiperiod model. They capture some aspects of fixed transaction costs yet maintain the homogeneity of the derived
utility function in its arguments. Constantinides (1985) computes the liquidity premium with quasi-fixed transaction costs and confirms the earlier conclusion that transaction costs have only a second-order effect on equilibrium asset returns.

Third, we can model the process by which firms supply their shares, endogenize share prices, and study the serial correlation of stock returns. Fourth, we can model the way in which the release of information by firms affects prices and trading volume around the event date. Finally, we can study the process by which market makers facilitate trade and endogenize prices and transaction costs.

In this paper I abstract from many realistic features of the market in order to highlight the implications of the endogenously chosen times to trade in the presence of transaction costs. My first conclusion is that investors accommodate transaction costs by drastically reducing the frequency and volume of trade. Second, an investor's expected utility is insensitive to deviations from the optimal portfolio proportions. Hence the liquidity premium due to transaction costs is small. Third, the higher the variance of an asset's return, the more frequent the trade in this particular asset and the higher the liquidity premium. Fourth, even large transaction costs may not explain the empirical anomaly that small firms have mean returns that are substantially larger than the mean returns of large firms. Finally, the implications of the model with endogenous tradings are fundamentally different from those of a single-period model.

Appendix

A Generalized Consumption Policy

In the text I confine my attention to simple consumption policies, defined by the property that the consumption rate is a constant fraction of the holding in the riskless asset; that is, \( c(t)/x(t) \) is a constant \( \beta \) in the region \( \bar{\lambda} \leq y(t)/x(t) \leq \bar{\lambda} \). As a first step in exploring the sensitivity of the calculated liquidity premium to the assumed simple policy, I derive the liquidity premium under the assumption that the consumption policy is of the following generalized form: The ratio \( c(t)/x(t) \) is a constant \( \bar{\beta} \) in the region \( \lambda \leq y(t)/x(t) \leq \lambda^* \) and is a possibly different constant \( \bar{\beta} \) in the region \( \lambda^* \leq y(t)/x(t) \leq \bar{\lambda} \). The constant \( \lambda^* \) is defined by equation (17), and \( \lambda, \bar{\lambda} \) are the parameters of the simple investment policy in the presence of proportional transaction costs.

I proceed as in Section IIII to find the optimal parameter values \( \beta, \bar{\beta}, \lambda, \) and \( \bar{\lambda} \) in the presence of proportional transaction costs. I define \( J(x, y; \beta, \bar{\beta}, \lambda, \bar{\lambda}) \) to be the expected utility of consumption given \( x \) and \( y \) and given policy parameters \( \beta, \bar{\beta}, \lambda, \) and \( \bar{\lambda} \). The function \( J \) satisfies equation (8) and the boundary conditions (9).

In the region \( \lambda \leq y(t)/x(t) \leq \lambda^* \), I set \( c = \beta x \) and obtain equation (10) with solution given by (11), where \( \beta \) is replaced by \( \beta \) and the free parameters \( A_1 \) and \( A_2 \) are replaced by \( A_1 \) and \( A_2 \). The boundary condition (9a) imposes a
TABLE A1
Comparisons of the Optimal Policy Parameters and Liquidity Premia under Different Constraints on the Consumption Policy

<table>
<thead>
<tr>
<th>Panel 1*</th>
<th>Panel 2†</th>
<th>Panel 3‡</th>
<th>Panel 4§</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = \bar{\beta}$</td>
<td>$\beta \leq \bar{\beta}$</td>
<td>$\beta = \bar{\beta}$</td>
<td>$\beta \leq \bar{\beta}$</td>
</tr>
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<td>$\lambda^*$</td>
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<td>0.776</td>
<td>1.377</td>
</tr>
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<td>1.905</td>
<td>1.892</td>
<td>1.784</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.2803</td>
<td>0.2589</td>
<td>0.2869</td>
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<tr>
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<td>0.2869</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.0130</td>
<td>0.0127</td>
<td>0.0014</td>
</tr>
</tbody>
</table>

**Note.**—The lower ($\lambda$) and the upper ($\bar{\lambda}$) bounds of the risky to the riskless asset ratio in the region of no transactions, the optimal consumption rates ($\bar{\beta}, \bar{\beta}$) in the regions ($\lambda^*, \lambda^*$) and ($\bar{\lambda}^*, \bar{\lambda}^*$), and the liquidity premium ($\bar{\delta}$) for different values of the risk aversion parameter ($\gamma$), variance of the risky asset's rate of return ($\sigma^2$), and transaction cost rate ($\rho$). The assumed parameter values are $\rho = .10$/year, $r = .10$/year, and $\mu = .15$/year.

* $\gamma = -1, \sigma^2 = .04$/year, $k = .1$.
† $\gamma = -1, \sigma^2 = .04$/year, $k = .01$.
‡ $\gamma = -4, \sigma^2 = .04$/year, $k = .1$.
§ $\gamma = -1, \sigma^2 = .16$/year, $k = .1$.

The restriction equivalent to equation (13):

\[
(1 + k)[1 + A_1(\gamma - s_1)\lambda^{s_1} + A_2(\gamma - s_2)\lambda^{s_2}] = A_1 s_1 \lambda^{s_1 - 1} + A_2 s_2 \lambda^{s_2 - 1}.
\]

(A1)

In the region $\lambda^* \leq y(t)/x(t) \leq \bar{\lambda}$, I set $\epsilon$ and $\bar{\beta}x$ and proceed as above. The boundary condition (9b) imposes a restriction equivalent to (14):

\[
(1 - k)[1 + \bar{A}_1(\gamma - s_1)\bar{\lambda}^{s_1} + \bar{A}_2(\gamma - s_2)\bar{\lambda}^{s_2}] = \bar{A}_1 s_1 \bar{\lambda}^{s_1 - 1} + \bar{A}_2 s_2 \bar{\lambda}^{s_2 - 1}.
\]

(A2)

Continuity of $\partial J/\partial x$ at $y/x = \lambda^*$ imposes the restriction

\[
\frac{\bar{\beta}^y}{\rho - \gamma(r - \bar{\beta})} [1 + A_1(\gamma - s_1)(\lambda^*)^{s_1} + A_2(\gamma - s_2)(\lambda^*)^{s_2}] = \frac{\bar{\beta}^y}{\rho - \gamma(r - \bar{\beta})} [1 + \bar{A}_1(\gamma - s_1)(\lambda^*)^{s_1} + \bar{A}_2(\gamma - s_2)(\lambda^*)^{s_2}].
\]

(A3)

Continuity of $\partial J/\partial \gamma$ at $y/x = \lambda^*$ imposes the restriction

\[
\frac{\bar{\beta}^y}{\rho - \gamma(r - \bar{\beta})} [A_1 s_1 (\lambda^*)^{s_1 - 1} + A_2 s_2 (\lambda^*)^{s_2 - 1}] = \frac{\bar{\beta}^y}{\rho - \gamma(r - \bar{\beta})} [\bar{A}_1 s_1 (\lambda^*)^{s_1 - 1} + \bar{A}_2 s_2 (\lambda^*)^{s_2 - 1}].
\]

(A4)

The four equations (A1)–(A4) are linear in $A_1$, $A_2$, $\bar{A}_1$, and $\bar{A}_2$ and uniquely determine these parameters in terms of the policy parameters $\bar{\beta}, \bar{\beta}, \lambda$, and $\bar{\lambda}$, provided the corresponding matrix of coefficients is nonsingular. Finally I maximize $J(x, y; \bar{\beta}, \bar{\beta}, \lambda, \bar{\lambda})$ at $y/x = \lambda^*$ with respect to the policy parameters.

In table A1 I compare the optimal simple investment policy and liquidity
premium in the case that the optimal consumption policy is constrained to be simple and in the case that it is generalized as above. Throughout the table I set \( \rho = .10/\text{year}, r = .10/\text{year}, \) and \( \mu = .15/\text{year}. \) In panel 1 I set \( \gamma = -1, \) \( \sigma^2 = .04/\text{year}, \) and \( k = .1. \) If the optimal consumption policy is constrained to be simple, the liquidity premium is \( \delta = .0130/\text{year}. \) If the optimal consumption policy is of the generalized form, the investor has greater flexibility in accommodating the adverse effects of transaction costs and the liquidity premium is reduced to \( .0127/\text{year}. \) The striking conclusion from panel 1 as well as from panels 2, 3, and 4 is that the liquidity premium with the generalized consumption policy is only slightly lower than the liquidity premium with the simple policy. I also conclude that the optimal simple investment policy and consumption rate are robust to the assumption that the consumption policy is simple.

In principle I can further generalize the set of consumption policies by dividing the region of no transactions into \( N \) subregions with the ratio \( c/dx \) in the \( nth \) region given by the parameter \( \beta_n. \) I can then proceed as above to find the optimal parameter values \( \beta_n, n = 1, 2, \ldots, N. \) However, table A1 provides convincing evidence that the primary results of the paper remain essentially unchanged as the set of consumption policies is generalized.

References


