Effective versus Efficient Securities in Arbitrage-Free Markets with Bid-Ask Spreads: a Linear Programming Characterization

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Abstract

We consider a securities market with bid-ask spreads at any period, including liquidation. Although the minimum-cost super-replication problem is non-linear, we introduce an auxiliary problem that allows us to characterize no-arbitrage via linear programming techniques. Since no-arbitrage per se does not bound the bid-ask spread of a newly traded security, we introduce the notion of effective new security. We show that effectiveness restricts the no-arbitrage bid and ask prices of a new security to the interval defined by the minimum-cost problem. We discuss in details the cases in which the boundaries of this interval can be reached without violating no-arbitrage. We also compare effectiveness to efficiency as discussed in Jouini and Kallal (2001). We show that effectiveness is not sufficient for efficency, but is equivalent to the weaker notion of zero inefficiency cost.

Keywords: arbitrage, bid-ask prices, linear programming, effective securities, efficient trading strategies

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1 Introduction

The valuation of securities via super-replication in the presence of market frictions and its interplay with no-arbitrage constitutes one of the most active research fields in finance theory. The topic has been analyzed both in discrete-time frameworks, starting from Bensaid *et al.* (1992) and Jouini and Kallal (1995), and in continuous-time models, dating back to Cvitanic and Karatzas (1993) (see also Cvitanic, 1999, and the references therein).

The contribution of this paper, which follows the event-tree approach, is threefold. First, we show how to employ linear programming techniques to characterize no-arbitrage in markets with proportional transaction costs (i.e. bid-ask spreads). With respect to the existing literature, the novelty here is that we supply a linear programming approach that is robust to the presence of bid-ask spreads at liquidation. Second, we exploit our linear programming approach to provide a duality-based proof of the fact that no-arbitrage per se imposes only an upper bound on the bid and a lower bound on the ask price of a new security (e.g. an option). By a complementary slackness argument, we then show that the bid-ask spread is bounded if and only if the new security satisfies an additional requirement, that we name *effectiveness*. As a consequence, we highlight that no-arbitrage per se is neither necessary nor sufficient for the ask price of a new security to be bounded above by the minimum-cost to super-replicate the cash-flow from a long postion, and the bid to be bounded below by the maximum that can be borrowed against a future liability not exceeding the one from a short position. The third contribution consists in comparing our notion of effectiveness to Jouini and Kallal's (2001) notion of *efficient* trading strategies, which requires long/short positions in the new security to be optimal solutions of the portfolio problem of some Von Neumann Morgenstern investor. We show that effectiveness is necessary but not sufficient for efficiency and that effectiveness coincides instead with Jouini and Kallal's (2001) notion of *zero-inefficiency cost*.

In a seminal paper, Bensaid *et al.* (1992) incorporate bid-ask spreads in the standard binomial option pricing model. In their setting bid-ask spreads are present also at the maturity date, and the super-replication problem is approached by means of dynamic programming. In a binomial market without transaction costs at maturity, Edirisinghe *et al.* (1993) observe that the super-replication problem can be reformulated as a linear programming one. Building on this observation, Naik (1995) and Ortu (2001) analyze the general event-tree framework without transaction costs at the terminal date and use the linearized super-replication problem and its dual to provide alternative charcterizations of no-arbitrage with bid-ask spreads.

In this paper we address by linear programming the general event-tree framework with bid-ask spreads also at the final date. The presence of transaction costs at maturity arises naturally in many practical applications. Consider for example a European call option and recall that at maturity the option is typically settled either with delivery of the underlying, or by cash, or at the discretion of the short position. In a world without transaction costs at maturity, these different types of settlement are payoff-wise indistinguishable. In actual markets, however, transaction costs affect the underlying security also at maturity, so that different settlement provisions produce different payoff profiles. Bid-ask spreads at liquidation introduce however a non-linearity in the otherwise linear super-replication problem. Indeed, investors typically aggregate their long and short positions with the same broker, so that rather than the cumulative long and short positions separately, what matters at the moment of final liquidation are the net positions held in each security. With bid-ask spreads at liquidation, this makes the terminal payoff, and hence the super-replication problem, non-linear in the intertemporal trading strategies.

To deal with this non-linearity, we construct an auxiliary linear program with the same value function as the original super-replication problem, and such that any solution to the super-replication problem is a linear transformation of a solution to the auxiliary linear program. To construct the auxiliary linear program we first partition the set of feasible trading strategies according to the sign of the net positions at liquidation. Taking then one strategy for each element of this partition, we use the sum of their cashflow to super-replicate any given future payoff at the minimum cost. Equipped with this auxiliary program, we are able to extend the linear programming characterization of no-arbitrage of Naik (1995) and Ortu (2001) to the case of bid-ask spreads at liquidation. Our linear programming approach allows us to provide a dual characterization of the cases in which strict super-replication is cost-minimizing. In particular, in Theorem 3 we show that this occurs if and only if the minimum cost of super-replication strictly exceeds the value assigned to a given claim by any strictly positive linear pricing rule compatible with no-arbitrage.[1]

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[1] This result can also be interpreted as a preference-free conterpart to Remark 1 in Jouini and Kallal (2001).

Our linear programming approach constitutes the basis to discuss the interplay between no-arbitrage and the notion of effectiveness for a new security introduced in the market. We first employ linear programming duality to formally derive a fact pointed out by Jouini and Kallal (1999), namely that no-arbitrage survives the introduction of a new security if and only if the bid price does not exceeds the minimum cost to cover a short position and the ask price is greater than the maximum that can be borrowed against a liability equal to the payoff from a long position. Moreover, exploiting Theorem 3 we are able to show that the cases in which these bounds can be reached without introducing arbitrage are those in which perfect replication is cost optimal. Furthermore, we characterize the new securities with bounded bid-ask spread in terms of an additional, preference-free requirement that we name effectiveness. In words, a newly traded security is long (short)-effective when it is optimal to take a long (short) position in the new security to super-replicate some future cashflow at the minimum cost. Intuitively, our notion of effectiveness conveys the idea that the new security improves the super-hedging capabilities of the investors.

The interval bounds for the bid and ask prices of the new security highlighted in the literature[2] can then be interpreted as the outcome of the interaction of no-arbitrage and effectiveness. In particular, effectiveness forces the no-arbitrage ask price of a new security to be smaller than the minimum cost incurred to super-replicate the payoff from a long position, and the no-arbitrage bid price to be larger than the maximum that can be borrowed against a liability not exceeding the one generated by a short position. Our results identify explicitly the cases in which reaching the boundaries of the interval gives rise to arbitrage opportunities. These cases occur in fact when the initial bid-ask spread vanishes, the price of the new security collapses to the extremes of the interval and strict super-replication is cost-optimal.

Our notion of effective security can be fruitfully compared with the notion of efficient trading strategy recently introduced by Jouini and Kallal (2001). In that paper, a payoff is called *efficient* if, given an uncertain future endowment, it is the optimal choice for some non-satiated Von Neumann Morgenstern investor with some level of initial wealth. In the presence of transaction costs, optimality is defined with respect to a budget constraint identified by a sublinear pricing rule which is the value function of the super-replication problem. Using the

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[2] See in particular Jouini and Kallal (1999).

characterization of effectiveness supplied in our Proposition 3, we can readily show that under no-arbitrage the payoff from long and short positions in the new security can be efficient only if the new security is effective. However, effectiveness does not guarantee efficiency. We show this by means of a simple one-period binomial example in which a new security is effective but holding a long position in it yields an inefficient payoff. This example highlights the economic difference between effectiveness, which is based on a cost minimization principle, and efficiency, which is instead based on a utility maximization argument.

To further relate effectiveness to Jouini and Kallal's utility-based viewpoint, we adapt their definition of inefficiency cost to our framework. Basically, the *inefficiency cost* of a payoff is the difference between its price and the largest amount that investors are willing to pay to get as much utility as with the payoff itself. Using our characterizations of no-arbitrage and effectiveness together with Jouini and Kallal's characterization of zero inefficiency cost, we show that a new security is effective if and only the payoffs from long and short positions have zero inefficiency costs.

In the last decade, the literature on super-replication and arbitrage in markets with frictions has witnessed a mounting interest. Along with the papers referred to so far, other contributions related to the present work are Dermody and Rockafeller (1991), Chen (1995), Milne and Neave (1997), Charupat and Prisman (1997), Jouini (2000), Koehl *et al.* (2001), Zhang *et al.* (2002), Huang (2002), Delbaen *et al.* (2002).[3]

The rest of the paper is structured as follows. In the next section, we introduce same basic notation and definitions. In Section 3, we extend the linear programming characterization of no-arbitrage with proportional transaction costs to the case of bid-ask spreads at liquidation. In Section 4, we discuss the pure no-arbitrage bounds on the bid-ask prices of a new security, we introduce the notion of effective new security, and compare the pure no-arbitrage bounds with those that must hold for effective securities. In Section 5, we adapt Jouini and Kallal's (2001) notion of efficient trading strategy to a new security, and discuss the relations between efficiency and our notion of effectiveness. Section 6 concludes the paper. All the proofs are in the Appendix.

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[3] We refer to Cvitanic (1999) for a detailed survey of the literature on super-replication without and with frictions in continuous-time models.

2 Basic notation and definitions

We assume that J perfectly divisible securities are traded over the horizon $\mathbf{T} = \{0, 1, ..., T\}$. Investors are price-takers and allowed to short-sell securities with full use of proceeds. Trading entails however a bid-ask spread, formalized by describing each security j as a triple $\left(S_j^A, S_j^B, d_j\right) \equiv \left\{S_j^A(t), S_j^B(t), d_j(t)\right\}_{t=0}^T$ of stochastic processes,[4] where S_j^A represents the ex-dividend ask price, S_j^B the ex-dividend bid price, and d_j the dividend flow ($d_j(0) = d_j(T) = 0$ for all j without loss of generality). We assume that $S_j^A \geq S_j^B$ and in particular we allow for bid-ask spreads at liquidation. Therefore, at the terminal date T, the cost $S_j^A(T)$ of covering a unit short position may be larger than the revenue $S_j^B(T)$ from liquidating a unit long position.

We model dynamic trading in each security by means of couples $\theta_j = \{\theta_j^A(t), \theta_j^B(t)\}_{t=0}^{T-1}$ of stochastic processes, where $\theta_j^A(t)$ represents the number of units of security j bought at t, and $\theta_j^B(t)$ the number of units sold at t. Then, to be feasible, a dynamic trading strategy $\theta = \{\theta_j\}_{j=1}^J$ must be non-negative. We denote by Θ the set of all feasible dynamic trading strategies.

A dynamic trading strategy $\theta \in \Theta$ generates a *cashflow process* $x_\theta = \{x_\theta(t)\}_{t=0}^T$. To describe it, we observe that $\sum_{\tau=0}^{t-1}\left(\theta^A(\tau) - \theta^B(\tau)\right)$ represents the net position held on the J assets before trading at time t and with $\left[\sum_{\tau=0}^{T-1}\left(\theta^A(\tau) - \theta^B(\tau)\right)\right]^+$ and $\left[\sum_{\tau=0}^{T-1}\left(\theta^A(\tau) - \theta^B(\tau)\right)\right]^-$ we denote the *net long* and *short* positions held at the terminal date.[5] Therefore we have:

$$x_\theta(t) = \begin{cases} -\left[\theta^A(0)\cdot S^A(0) - \theta^B(0)\cdot S^B(0)\right], & t = 0 \\ d(t)\cdot\sum_{\tau=0}^{t-1}\left(\theta^A(\tau) - \theta^B(\tau)\right) - \left[\theta^A(t)\cdot S^A(t) - \theta^B(t)\cdot S^B(t)\right], & t = 1, ..., T-1 \\ S^B(T)\cdot\left[\sum_{\tau=0}^{T-1}\left(\theta^A(\tau) - \theta^B(\tau)\right)\right]^+ - S^A(T)\cdot\left[\sum_{\tau=0}^{T-1}\left(\theta^A(\tau) - \theta^B(\tau)\right)\right]^-, & t = T. \end{cases} \tag{1}$$

At date 0, the cashflow is just the opposite of the initial cost of θ, while at the intermediate dates $t = 1, ..., T-1$ it is the difference of the dividends earned on the net positions held in the J assets before trading and the cost to update these positions. At the terminal date T the

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[4] We assume that the underlying probability space (Ω, F, P) is finite, with $\Omega = \{\omega_1, ..., \omega_{s_T}\}$, $F = 2^\Omega$ and P strictly positive on $2^\Omega \backslash \{\emptyset\}$. In this setting, the information flow shared by all investors is an event-tree $\mathbb{P}$, with f_k^t the generic time t node, s_t the number of time t nodes, and $L = \sum_{t=0}^T s_t$ the total number of nodes. All stochastic processes introduced hereafter are adapted to $\mathbb{P}$.

[5] Given any vector $y \in \Re^n$, we denote with $[y]^+$, respectively $[y]^-$ the vectors with components $\max\{y_j, 0\}$ and $\max\{-y_j, 0\}$, $j = 1, ..., n$.

cashflow is the difference between the revenue from liquidating the net long positions at the bid prices and the cost of covering the net short positions at the ask prices. This definition of $x_\theta(T)$ assumes that long and short positions on each security are held in the same account, so that only the net positions matter at liquidation.

For expositional purposes, we recall the definition of arbitrage opportunity in the securities market with bid-ask spread introduced above.

Definition 1 *A feasible dynamic trading strategy θ such that $x_\theta(t) \geq 0$ for all t constitutes an arbitrage opportunity of:* [6]

1. *the first type if $x_\theta(t) > 0$ for some $t > 0$;*
2. *the second type if $x_\theta(0) > 0$.*

Hereafter, we say that no-arbitrage holds if the price-dividend system $\left(S^A, S^B, d\right)$ is free of both types of arbitrage opportunities.

The following condition is maintained throughout the paper.

Condition 1 *(**The Internality Condition**) There exists a feasible dynamic trading strategy θ such that $x_\theta(t) >> 0$ for all $t > 0$.*

This requirement is indeed very mild, and is satisfied whenever one of the traded assets has strictly positive bid price process, and pays non-negative intermediate dividends. Since it is readily seen that the existence of second-type arbitrage opportunities implies the existence of first-type arbitrage opportunities if the Internality Condition holds, a price-dividend system $\left(S^A, S^B, d\right)$ that satisfies the Internality Condition is arbitrage-free if and only if it is free of first-type arbitrage opportunities.

3 No-arbitrage with bid-ask spreads at liquidation

In this section, we extend the linear programming characterization of no-arbitrage with bid-ask spreads available in the literature[7] to the case in which bid-ask spreads are present also at the

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[6] Hereafter $x_\theta(t) \geq 0$ means $P(x_\theta(t) \geq 0) = 1$; $x_\theta(t) > 0$ means $x_\theta(t) \geq 0$ and $P(x_\theta(t) > 0) > 0$; $x_\theta(t) >> 0$ means $P(x_\theta(t) > 0) = 1$.

[7] See in particular Naik (1995) and Ortu (2001).

date T of final liquidation. Recall that the super-replication-at-minimum-cost problem for a given future cashflow $m = \{m(t)\}_{t=1}^{T}$ is:

$$\begin{aligned} \inf_{\theta \in \Theta} \; -x_\theta(0) = \;& \theta^A(0) \cdot S^A(0) - \theta^B(0) \cdot S^B(0) \\ & s.t. \;\; x_\theta(t) \geq m(t), \, \forall t > 0. \end{aligned} \tag{$\mathcal{P}[m]$}$$

Recall also that, since in our setting the information structure is an event-tree $\mathbb{P}$,[8] a feasible dynamic trading strategy θ is identified by a vector in $\Re_+^{2J(L-s_T)}$, the cashflow x_θ by a column vector in $\Re^L$ and the future cashflows m that parametrizes $\mathcal{P}[m]$ by a vector in $\Re^{L-1}$. The difference with the usual setting is that the presence of bid-ask spreads at the terminal date T makes the terminal payoff $x_\theta(T)$, and hence the constraint $x_\theta(T) \geq m(T)$, non-linear in θ. This implies that $\mathcal{P}[m]$ cannot be addressed directly by means of linear programming techniques. The main contribution of this section is to construct a linear program which is, in a sense to be made precise, equivalent to $\mathcal{P}[m]$. To do so, we let

$$\Theta_{\geq m} \equiv \left\{ \theta \in \Re_+^{2J(L-s_T)} \;\middle|\; x_\theta(f_k^t) \geq m(f_k^t), \, \forall t > 0 \text{ and } k \right\} \tag{2}$$

denote the feasible set of $\mathcal{P}[m]$, where f_k^t denotes the generic node of the event-tree $\mathbb{P}$ at time t. We observe that, as a consequence of the Internality Condition, $\Theta_{\geq m}$ is non-empty for any choice of m. Moreover, we denote by $\pi(m)$ the value function of $\mathcal{P}[m]$,

$$\pi(m) \equiv \inf\{-x_\theta(0) \;\mid \theta \in \Theta_{\geq m}\}. \tag{3}$$

In words, $\pi(m)$ represents the minimum cost to super-replicate m.

We are now ready to state our first result.

Proposition 1 *The following facts hold:*

1. *there exist a vector* $c \in \Re^{2J(L-s_T)}$, $K \equiv 2^{Js_{T-1}}$ *matrices* M_k *of dimension* $(L-1) \times 2J(L-s_T)$, *and* K *matrices* G_k *of dimension* $Js_{T-1} \times 2J(L-s_T)$, *such that*

$$\begin{aligned} \pi(m) = \;& \inf_{(\theta_1,..,\theta_K) \in \Re_+^{2KJ(L-s_T)}} \; \sum_{k=1}^{K} c \cdot \theta_k \\ & s.t. \; \begin{cases} \sum_{k=1}^{K} M_k \theta_k \geq m \\ G_k \theta_k \geq 0, \; k = 1, ..., K. \end{cases} \end{aligned} \tag{$\mathcal{LP}[m]$}$$

   that is, problem $\mathcal{P}[m]$ *and the linear program* $\mathcal{LP}[m]$ *have the same value function.*

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[8] Recall that L is the total number of nodes, and s_T is the number of terminal nodes.

2. *θ^* is an optimal solution to $\mathcal{P}[m]$ if and only if $\theta^* = \sum_{k=1}^{K} \theta_k^*$ for some $(\theta_1^*, ..., \theta_K^*)$ optimal solution to $\mathcal{LP}[m]$.*

3. *Denoting with $\Theta_{>m}^*$ and $\Xi_{>m}^*$ the sets of optimal solutions to $\mathcal{P}[m]$ and $\mathcal{LP}[m]$ for which the constraints are not binding, that is*

$$\Theta_{>m}^* = \{\theta \text{ solution to } \mathcal{P}[m] \mid x_\theta(t) > m(t) \text{ for some } t > 0\}$$

and

$$\Xi_{>m}^* = \left\{ (\theta_1, .., \theta_K) \text{ solution to } \mathcal{LP}[m] \;\middle|\; \sum_k M_k \theta_k > m \right\}$$

then $\Theta_{>m}^* \neq \emptyset$ if and only if $\Xi_{>m}^* \neq \emptyset$.

In words, Proposition 1 states that it is possible to construct a parametric linear program $\mathcal{LP}[m]$ with the following properties. First, the value function of $\mathcal{LP}[m]$ coincides with the value function of $\mathcal{P}[m]$. Second, the set of solutions to $\mathcal{P}[m]$ is a linear transformation of the set of solutions to $\mathcal{LP}[m]$. Third, the existence of solutions to $\mathcal{P}[m]$ for which the constraints are not binding is equivalent to the existence of solutions to $\mathcal{LP}[m]$ for which the same property holds. To understand the intuition behind Proposition 1 it is then important to understand the interpretation of $\mathcal{LP}[m]$. To construct it, the first step is to partition (modulo the boundaries) the set Θ of all feasible trading strategies into K convex cones Θ_k. Two feasible trading strategies belong to the same Θ_k, $k = 1, .., K$, if and only if at the terminal date T the sign of the net position on each asset is the same[9] on each terminal node of the event-tree $\mathbb{P}$. Since the last rebalancing of the positions in the J assets occurs at $T-1$, and since s_{T-1} denotes the number of nodes at $T-1$, it is readily seen that the number of convex cones in which we partition Θ is indeed $K \equiv 2^{Js_{T-1}}$. In the Appendix, moreover, we show that $\Theta_k = \{\theta_k \in \Theta \mid G_k\theta_k \geq 0\}$, where the matrix G_k is constructed in such a way that $G_k\theta_k$ supplies the absolute values of the net positions on each asset at each of the terminal nodes of $\mathbb{P}$. The second step to construct $\mathcal{LP}[m]$ consists in observing that for each k the set of future cashflows generated by the trading strategies in Θ_k is the covex cone $\{M_k\theta_k \mid \theta_k \in \Theta_k\}$, with the *payoff matrix* M_k once again supplied in the Appendix. Finally, we observe that the initial

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[9] For same sign we mean either weakly positive or weakly negative. This explains the "modulo the boundary" qualification when stating that $\Theta_1, ..., \Theta_K$ constitute a partition of Θ.

cost $\theta_k^A(0) \cdot S^A(0) - \theta_k^B(0) \cdot S^B(0)$ of a generic trading strategy $\theta_k \in \Theta_k$ can be represented as the inner product $c \cdot \theta_k$, with c suitably defined. Therefore, the parametric linear program $\mathcal{LP}[m]$ consists in computing the minimum cost to super-replicate m with sums of K cashflows, where the strategies that generate the K cashflows are chosen one for each of the convex cones Θ_k that partition Θ.

Proposition 1 allows us to extend all the characterizations of no-arbitrage and no-second-type-arbitrage supplied in Ortu (2001) to the case of bid ask-spreads at liquidation. We synthesize these extensions hereafter because they constitute the buildings blocks for the discussion of the effectiveness and efficiency of a newly traded security presented in following sections. First, in Theorem 1 and Corollary 1 we present the relationships between no-arbitrage, respectively no-second-type-arbitrage, and the properties of the minimum cost function $\pi(m)$ and those of the optimal solutions to the problems $\mathcal{P}[m]$ and $\mathcal{LP}[m]$.

Theorem 1 *The following statements are equivalent:*

1. *no-arbitrage holds.*
2. $\pi(m)$ *is a real-valued, strictly positive sublinear function and* $\pi(0) = 0$.
3. $\mathcal{LP}[m]$ *(equivalently,* $\mathcal{P}[m]$*), admits solutions* $\forall\ m \in \Re^{L-1}$. *Moreover,* $c \cdot \sum_k \theta_k^* = 0$ *and* $\sum_k M_k \theta_k^* = 0$ *for any* $(\theta_1^*, ..., \theta_K^*)$ *solution to* $\mathcal{LP}[0]$ *(equivalently,* $x_{\theta^*} = 0$ *for any* θ^* *solution to* $\mathcal{P}[0]$*).*
4. *There exists optimal solutions* $(\theta_1^*, ..., \theta_K^*)$ *to* $\mathcal{LP}[0]$ *(equivalently,* θ^* *to* $\mathcal{P}[0]$*) for any of which* $c \cdot \sum_k \theta_k^* = 0$ *and* $\sum_k M_k \theta_k^* = 0$ *(equivalently,* $x_{\theta^*} = 0$*).*

Corollary 1 *The following statements are equivalent:*

1. *no-second-type-arbitrage holds.*
2. $\pi(m)$ *is a real-valued, nonnegative sublinear function and* $\pi(0) = 0$.
3. *There exists optimal solution of* $\mathcal{LP}[m]$ *(equivalently,* $\mathcal{P}[m]$*) for all* $\forall\ m \in \Re^{L-1}$.
4. *There exists optimal solutions to* $\mathcal{LP}[0]$ *(equivalently,* $\mathcal{P}[0]$*).*

We now recall the definitions of underlying state-prices and semi-positive underlying state-prices for a securities market with bid-ask spreads. These objects are then employed to characterize no-arbitrage and no-second-type-arbitrage by means of duality techniques.

Definition 2 *We call underlying state-price (USP) any vector* $\psi \in \Re^{L-1}_{++}$ *such that* $(1, \psi) \cdot x_\theta \leq 0 \ \forall \theta \in \Theta$. *We call semi-positive USP any vector* $\varphi \in \Re^{L-1}_{+}$ *such that* $(1, \varphi) \cdot x_\theta \leq 0 \ \forall \theta \in \Theta$.

As usual, the USPs represent the counterpart in the case of bid-ask spreads to the Arrow-Debreu state-prices for a frictionless securities market. We denote by Ψ and Φ the set of all the USP and semi-positive USP vectors respectively. In the next result we supply an analytic characterization of these sets based on a suitable version of the theorem of the alternatives. The characterizations of Ψ and Φ exploit the matrices M_k, G_k that constitute the building blocks of the linear program $\mathcal{LP}[m]$ discussed in Proposition 1.

Proposition 2 *We have*

$$\Psi = \left\{ (1, \psi) \in R^L_{++} \ \middle| \ \psi M_k + \beta_k G_k \leq c, \ \beta_k \geq 0, \ k = 1, .., K \right\}$$

and

$$\Phi = \left\{ (1, \varphi) \in \Re^L_{+} \ \middle| \ \varphi M_k + \beta_k G_k \leq c, \ \beta_k \geq 0, \ k = 1, .., K \right\}$$

We are now ready to state the result that characterizes no-arbitrage (repectively no-second-type-arbitrage) in terms of USPs (semi-positive USPs). These characterizations are readily obtained by exploiting the relationships between $\mathcal{LP}[m]$ and $\mathcal{P}[m]$ laid out in Proposition 1, together with the fundamental theorem of linear programming.

Theorem 2 *No-arbitrage (no-second-type-arbitrage) is equivalent to the existence of USPs (semi-positive USPs). Under no-arbitrage, moreover,*

$$\pi(m) = \sup_{(1,\psi) \in \Psi} \psi \cdot m \quad \forall m \in R^{L-1}$$

while under no-second-type-arbitrage

$$\pi(m) = \max_{(1,\varphi) \in \Phi} \varphi \cdot m \quad \forall m \in R^{L-1}$$

We point out that our duality-based formulation of no-arbitrage, and the related characterization of the minimum-cost functional as the envelope of the USPs, represents an event-tree counterpart to the results of Jouini and Kallal (1995).

We conclude this section by investigating the conditions under which the supremum in Theorem 2 is actually a maximum. As we show hereafter, this fact occurs if and only if exact replication of a future cashflow is cost-optimal.[10]

Theorem 3 *If no-arbitrage holds, then* $\forall m \in R^{L-1}$ *we have* $\pi(m) = \max_{(1,\psi)\in\Psi} \psi \cdot m$ *if and only if* $\Theta^*_{>m} = \emptyset$.

To further highlight the importance of Proposition 1, we remark that our proof of Theorem 3 is based on the relationships between problems $\mathcal{LP}[m]$ and $\mathcal{P}[m]$. Since the constraints of all solutions to $\mathcal{P}[m]$ are binding (that is, $\Theta^*_{>m} = \emptyset$) if and only if the same is true for $\mathcal{LP}[m]$, the necessity part in Theorem 3 follows readily from a complementary slackness argument applied to $\mathcal{LP}[m]$. To prove the converse, we suitably modify $\mathcal{LP}[m]$ in such a way that its dual has a feasible set containing only semipositive USPs that satisfy the condition $\varphi \cdot m \geq \pi(m)$. Then, we show that the modified $\mathcal{LP}[m]$ has a strictly positive value function. This allows us to construct a USP satisfying $\psi \cdot m \geq \pi(m)$, from which the result follows.

Aside from being interesting per se, Theorem 3 constitutes the pivotal result on which we base our discussion of the arbitrage bounds for effective new securities and our comparison between effectiveness and efficiency of a trading strategy. In fact, the suitably modified version of $\mathcal{LP}[m]$ used in the proof of Theorem 3 is, in essence, the linear program that, in the market extended to a new security, is equivalent to the minimum cost super-replication problem. We address these issues in the next sections.

4 Arbitrage bounds for an effective new security

We now consider the case in which a new security, for example a derivative written on one of the original J securities, is introduced in the market. We assume that the new security trades only at time $t = 0$ and comes to maturity at the terminal date T. In other words, the only

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[10] Therefore, our next result can also be interpreted as a preference-free conterpart to Remark 1 in Jouini and Kallal (2001).

positions that the investors can take on the new security are either a long position held up to T, or a short position to be covered at T. We denote with $c^A \geq c^B$ the time 0 ask and bid prices of the new security, and with $n(t)$, $t = 1, ..., T-1$, the dividend flow. Moreover, we let $n^A(T)$ represent the cost of covering a short position on the new security at the final date T, and $n^B(T)$ the revenue from liquidating a long position, with $n^A(T) \geq n^B(T)$. Finally, we let $\zeta^A, \zeta^B \in \Re_+$ denote the units of the new security bought and shorted respectively, at time 0. Henceforth, we call *original* the securities market with securities 1 to J, and *extended market* the one in which the new security trades along with securities 1 to J. In the extended market, a trading strategy $\left(\theta, \zeta^A, \zeta^B\right)$ generates the cashflow

$$x_{\left(\theta,\zeta^A,\zeta^B\right)}(t) = \begin{cases} x_\theta\left(0\right) - \zeta^A c^A + \zeta^B c^B & t = 0 \\ x_\theta\left(t\right) + \left(\zeta^A - \zeta^B\right) n(t) & t = 1, ..., T-1 \\ x_\theta\left(T\right) + n^B(T) \cdot \left[\zeta^A - \zeta^B\right]^+ - n^A(T) \cdot \left[\zeta^A - \zeta^B\right]^- & t = T. \end{cases} \tag{4}$$

with $x_\theta\left(t\right)$ as defined in (1).

We first characterize no-arbitrage and no-second-type-arbitrage in the extended market. We base our characterizations on the comparison between the bid and ask prices of the new security and the minimum cost that, in the original market, one incurs to super-replicate the future cashflows $n^A = (n(1), .., n(T-1), n^A(T))$ and $n^B = (n(1), .., n(T-1), n^B(T))$.

Theorem 4 *If no-arbitrage holds in the original market, equivalent statements are:*

1. *no-arbitrage holds in the extended market.*

2. $c^B \leq \pi(n^A)$ *(with strict inequality if* $\Theta^*_{>n^A} \neq \emptyset$*) and* $c^A \geq -\pi(-n^B)$ *(with strict inequality if* $\Theta^*_{>-n^B} \neq \emptyset$*).*

Corollary 2 *If no-second-type-arbitrage holds in the original market, equivalent statements are:*

1. *no-second-type-arbitrage holds in the extended market.*

2. $c^B \leq \pi(n^A)$ *and* $c^A \geq -\pi(-n^B)$.

Theorem 4 characterizes no-arbitrage in the extended market by identifying the least upper bound and greatest lower bound for the bid and ask prices of the new security. Precisely, the least upper bound for c^B is the minimum cost to super-replicate n^A in the original market, while the greatest lower bound for c^A is the maximum amount that can be borrowed in the original market against a future liability not exceeding n^B. Corollary 2 shows that second-type arbitrage opportunities are absent from the extended market even if c^B reaches its least upper bound, or c^A its greatest lower bound. By Statement 2 in Theorem 4, in fact, c^B is allowed to attain its least upper bound without introducing an arbitrage opportunity if and only if perfect replication of the future cashflow n^A is cost-optimal in the original market. Likewise, no-arbitrage holds in the extended market even if c^A reaches its greatest lower bound if and only if the amount that can be borrowed in the original market against a future liability not exceeding n^B is maximized by shorting n^B itself.

As a consequence of Theorem 4 and Corollary 2, arbitrage considerations alone are not sufficient to restrict the bid-ask spread $c^A - c^B$ of the new security. To determine a condition that, together with no-arbitrage, bounds c^A and c^B both from above and from below, we need to impose an additional requirement on the new security, formalized in the next definition.

Definition 3 *For each* $m \in \Re^{L-1}$ *let* $\mathcal{P}^{ex}[m]$ *denote the super-replication problem of* m *in the extended market. Then we call the new security:*

1. *long-effective if there exist* m *and* $(\theta, \zeta^A, \zeta^B)$ *optimal for* $\mathcal{P}^{ex}[m]$ *such that* $\zeta^A > 0$.
2. *Short-effective if there exist* m *and* $(\theta, \zeta^A, \zeta^B)$ *optimal for* $\mathcal{P}^{ex}[m]$ *such that* $\zeta^B > 0$.
3. *Effective if it is both long- and short-effective.*

In words, a newly traded security is long-effective when it is *held long* in some strategy that super-replicates some future cashflow at the minimum cost. Likewise, short-effectiveness means the new security is *shorted* in some strategy that super-replicates some future cashflow at the minimum cost.[11]

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[11] We point out that our notion of long and short-effectiveness extends to a stochastic environment the notion of *attractive* long and short security introduced by Dermody and Rockafellar (1991) in a deterministic term structure framework.

The following result explores the implications of effectiveness on the ask and bid prices of the new security.

Proposition 3 *Assume that no-second-type-arbitrage holds in the original market, and denote with* $\pi^{ex}(\cdot)$ *the value function of* $\mathcal{P}^{ex}[\cdot]$. *Then the new asset is:*

1. *long-effective if* $\max\left[c^B, -\pi\left(-n^B\right)\right] \leq c^A \leq \pi\left(n^B\right)$, *equivalently* $\pi^{ex}\left(n^B\right) = c^A$.
2. *Short-effective if* $-\pi\left(-n^A\right) \leq c^B \leq \min\left[c^A, \pi\left(n^A\right)\right]$, *equivalently* $-\pi^{ex}\left(-n^A\right) = c^B$.
3. *Effective if* $-\pi\left(-n^A\right) \leq c^B \leq c^A \leq \pi\left(n^B\right)$, *equivalently* $\pi^{ex}\left(n^B\right) = c^A$, $-\pi^{ex}\left(-n^A\right) = c^B$.

Proposition 3 characterizes effectiveness in terms of both the original market and the extended one. From the standpoint of the original market, the new security is long-effective if the ask price is bounded from above by the minimum cost incurred to super-replicate n^B, and by below by the maximum amount that can be borrowed against $-n^B$, if this amount exceeds the bid price, or otherwise by the bid price itself. From the perspective of the extended market, instead, long-effectiveness is characterized by the fact that an optimal way to super-replicate the future cash-flow n^B is to buy the new security at its ask price c^A. Part *2.* of Proposition 3 characterizes short-effectiveness in a similar way, while Part *3.* Proposition 3 shows that overall effectiveness imposes $\pi\left(n^B\right) + \pi\left(-n^A\right)$ as an upper bound to the bid-ask spread $c^A - c^B$. [12]

We are now ready to state our characterization of the arbitrage bounds that must be satisfied by a newly traded effective security.

Theorem 5 *If no-arbitrage holds in the original market, equivalent statements are:*

1. *no-arbitrage holds in the extended market and the new asset is effective.*
2. $-\pi(-n^A) \leq c^B \leq c^A \leq \pi(n^B)$, *with* $c^B < \pi(n^B)$ *if* $\pi(n^B) = \pi(n^A)$ *and* $\Theta^*_{>n^A} \neq \emptyset$, *and* $c^A > -\pi(-n^A)$ *if* $\pi(-n^A) = \pi(-n^B)$ *and* $\Theta^*_{>-n^B} \neq \emptyset$.

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[12] Proposition 3 shows that effective new securities are characterized by a bid-ask price pair that is *consistent* in the sense of Jouini and Kallal (1999).To see this, one needs simply to replace in Definition 3.1 in Jouini and Kallal (1999) π' with our π^{ex}, and observe that viability coincides with no-arbitrage when the probability space is finite.

Corollary 3 *If no-second-type-arbitrage holds in the original market, equivalent statements are:*

1. *no-second-type-arbitrage holds in the extended market and the new security is effective.*
2. $-\pi\left(-n^A\right) \leq c^B \leq c^A \leq \pi\left(n^B\right).$

Therefore, the condition $-\pi(-n^A) \leq c^B \leq c^A \leq \pi(n^B)$ per se is not sufficient for no-arbitrage to be maintained after a new effective security is introduced in the market, although it guarantees that no-second-type-arbitrage will hold. From Theorem 5, in particular, it follows that a new effective security traded in an arbitrage-free market introduces (first type) arbitrage opportunities in the following two cases. The first case occurs when $c^B = \pi(n^A)$ and $\Theta^*_{>n^A} \neq \emptyset$. In this case, the effectiveness of the new security forces the bid-ask spread to vanish at the initial date, that is $c^B = c^A$. Moreover, the security can be shorted at a price equal to the minimum cost to super-replicate the future cash-flow n^A in the original market, and among the cost-minimizing strategies there is one whose future cash-flow exceeds n^A in at least one future event. In this case, therefore, an arbitrage opportunity of the first type is generated by shorting the new security and buying a strategy in $\Theta^*_{>n^A}$. The second case occurs when $c^A = -\pi(-n^B)$ and $\Theta^*_{>-n^B} \neq \emptyset$, so that effectiveness forces once again $c^B = c^A$. In this case, there exist (first type) arbitrage opportunities since it possible to finance a long position in the new security by borrowing against a future liability not exceeding n^B, and actually lower than n^B in at least one future event.[13] Summing up, our results allow us to identify explicitely the cases in which boundary bid and ask prices of a new security must be avoided in order to preserve no-arbitrage in the extended market.[14]

5 Effective securities versus efficient trading strategies

In this section we discuss the relationship between the notion of effective security and the one of efficient trading strategy as defined in Jouini and Kallal (2001), (JK(2001) henceforth). In a securities market with a finite number of states of the world, JK(2001) analyze the

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[13] A simple situation in which these cases occur is illustrated in Example 1 in the Appendix.

[14] From this standpoint, our linear programming approach details the behavior on the boundary of the bid-ask price pairs for a new security that are consistent in the sense of Jouini and Kallal (1999).

properties of the trading strategies that are optimal for some non satiated, Von Neumann-Morgenstern investor who maximizes his concave expected utility given some initial wealth and some uncertain future endowment. Imposing no-arbitrage, JK(2001) consider in fact a reduced-form framework in which the investors choose over uncertain consumption profiles that are priced by a sublinear pricing rule. The sublinearity of the pricing rule represents in reduced-form the presence of some market friction. In this setting, JK(2001) call efficient a consumption profile that, given an uncertain future endowment, is optimal for some investor with some level of initial wealth. The framework discussed in this paper constitutes the background to the analysis of JK(2001) for the case of frictions due to bid-ask spreads. The value function of the minimum-cost super-replication problem constitutes in fact the no-arbitrage sublinear pricing rule that JK(2001) take as an input.

We compare the notions of effectiveness and efficiency for a new security introduced into the original market without violating no-arbitrage. To simplify our comparison, and without any loss of generality, we restrict our attention to a single-period model with equiprobable states at date $T = 1$. In our framework a long position in the new security is an efficient trading strategy if, given an uncertain future endowment x, there exist a non-satiated and concave expected utility U and an intial wealth w such that $m = n^B$ solves $\max\{U(m + x) : \pi^{ex}(m) \leq w\}$.[15]

Comparing the notion of efficient trading strategy with the one of effective new security, it is readily seen that long effectiveness is a necessary condition for a long position in the new security to be an efficient trading strategy.[16] Effectiveness, however, is not sufficient for efficiency, as can be shown by means of a simple example whose details are worked out in Example 2 in the Appendix. Example 2 exploits the characterization of efficiency of JK(2001), Theorem 1. Such characterization implies that the terminal payoff n^B is efficient for a suitable uncertain future endowment if and only if there exists a strictly positive USP vector ψ^{ex} for the extended market such that $c^A = \psi^{ex} n^B$. Example 2 consists of a simple, arbitrage-free one-period binomial model in which a new security is added to a single pre-existing one. Both

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[15] Henceforth we compare long effectiveness to efficiency of a long positions. The comparison of short effectiveness with efficiency of a short position is specular.

[16] If a new security is not long-effective, there exists a strategy $\widetilde{\theta}$ whose cost is $\pi(n^B) < c^A$ and whose future cash-flow is at least n^B. Instead of going long in the new security, any non-satiated investor with initial wealth $w \geq c^A$ would strictly prefer to go long in $\widetilde{\theta}$ and invest $c^A - \pi(n^B)$ in a strategy θ satisfying the Internality Condition.

securities are risky and have a positive bid-ask spread at both the initial and the terminal date. We select the bid and ask prices of the two securities in such a way that the following three conditions are met. First, the introduction of the new security preserves no-arbitrage. By Theorem 2, this means that the extended market admits USPs. Second, the new security is long-effective, and by Proposition 3 we have $c^A = \pi^{ex}(n^B)$. Finally, $c^A = \pi^{ex}(n^B) > \psi^{ex} n^B$ for all USPs ψ^{ex} of the extended market. This implies that a long position in the new security is not an efficient trading strategy. Applying Theorem 3 to the extended market, it follows that in our example there exists a trading strategy θ that costs as much as a long position in the new security, and yet it produces a payoff greater than n^B in at least one terminal state.

This example highlights the economic difference bewteen effectiveness, which is based on a cost minimization principle, and efficiency, which is instead based on a utility maximization argument. Investors focusing on minimum-cost super-replication are indifferent between a long position in the new security and any strategy θ that super-replicates n^B at the same minimum cost. Investors with non-satiated expected utility would instead strictly prefer to a long position in the new security any strategy θ providing at the same cost a consumption profile that exceeds n^B in at least one terminal state. Situations like this one exhaust in fact the cases in which buying a newly traded long-effective security is not an efficient trading strategy.

Although effectiveness is not equivalent to efficiency, an equivalence can be drawn between effectiveness and the weaker notion of “zero inefficiency cost”. Given an uncertain future endowment x the inefficiency cost of a long position in the new security is the quantity $c^A - \sup_U \left\{ \min_m \left[\pi^{ex}(m) : U(m + x) \geq U(n^B + x) \right] \right\}$, with the sup taken over all non satiated and concave expected utility investors.[17] In words, the inefficiency cost is the difference between the ask price of the new security and the largest amount required by all investors with endowment x to get at least the same utility as with the payoff n^B. Suppose now that the newly traded security is long-effective, so that $c^A = \pi^{ex}(n^B)$ by Proposition 3. Applying Theorem 2 to the extended market, there exists a semi-positive USP φ^{ex} such that $\pi^{ex}(n^B) = \varphi^{ex} n^B$. By suitably choosing the uncertain future endowment $\overline{x}$ we can then invoke Theorem 3 in JK(2001)

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[17] See Definition 2 in Jouini and Kallal (2001). Specular arguments can be once again developed for short positions.

to obtain $c^A = \pi^{ex}(n^B) = \sup_U \left\{ \min_m \left[\pi^{ex}(m) : U(m + \overline{x}) \geq U(n^B + \overline{x}) \right] \right\}$. Conversely, from the definition of $\pi^{ex}(n^B)$ together with Corollary 1 in JK(2001), it follows that $c^A \geq \pi^{ex}(n^B) \geq \sup_U \left\{ \min_m \left[\pi^{ex}(m) : U(m + x) \geq U(n^B + x) \right] \right\}$ for all x, so that a newly traded security is long-effective whenever it has zero inefficiency cost for a suitably chosen future endowment $\overline{x}$. Taken together, these facts can be summarized in the form of

Proposition 4 *Suppose that a new security is traded in an original market free of second-type arbitrage opportunities. Then the new security is long (short)-effective if and only if a long (short) position in the new security has zero inefficiency cost for some suitably chosen future endowment* $\overline{x}$.

To conclude, we observe that Proposition 4 allows us to reinterpret the pricing bounds for a newly traded security discussed in the previous section from the standpoint of inefficiency cost. Indeed the bounds $-\pi\left(-n^A\right) \leq c^B \leq c^A \leq \pi\left(n^B\right)$ are necessary and sufficient for zero inefficiency cost of long/short positions in the new security and no-second-type-arbitrage in the extended market.

6 Conclusion

We have characterized no-arbitrage in a securities market with bid-ask spreads, accounting also for transaction costs at the date of final liquidation. In this case the minimum-cost super-replication problem commonly employed to characterize no-arbitrage is non-linear. Here we show how to construct an auxiliary linear programming problem that has the same value function as the minimum-cost problem, and such that the solutions of the minimum-cost problem are linear transformations of the solutions to the auxiliary problem. We use these properties to extend the linear programming characterizations of no-arbitrage without bid-ask spreads at liquidation to the case with bid-ask spreads at liquidation.

We then consider the situation in which a new security is traded in an arbitrage-free market, and we characterize the conditions under which no-arbitrage is preserved in the extended market. Observing that these conditions do not impose per se any bound on the bid-ask spread of the new security, we introduce the notion of "effective" new security. In words, a new security is effective when trading it will be a part of the optimal solution to some minimum-cost super-replication problem. Effectiveness allows us to identify the interval that *must contain* the bid and ask prices of the new security for no-arbitrage to still hold in the extended market. In particular, our analysis identifies the cases in which the bid and ask prices are allowed to reach the boundaries of the interval without perturbing no-arbitrage.

Finally, we compare our notion of effective security with the notion of efficient trading strategy introduced by Jouini and Kallal (2001). While efficiency clearly implies effectiveness, by means of an example we show that the contrary does not hold. Defining however inefficiency cost of the new security to be the difference between its ask (bid) price and the largest (smallest) amount required by all non-satiated investors to get at least the same utility as with holding (shorting) the new security, we show that effective securities are exactly those with zero inefficiency cost.

Appendix

1. Proofs of Section3

Proof of Proposition 1

Step 1. Construction of c and of the matrices G_k.

Recall that we denote with f_h^t (for $t \geq 0$ and $h = 1,..,s_t$) the generic node of the event tree $\mathbb{P}$ and with $\theta = (\theta^A(0), \theta^A(f_1^1),.., \theta^A(f_{s_{T-1}}^{T-1}), \theta^B(0), \theta^B(f_1^1),.., \theta^B(f_{s_{T-1}}^{T-1}))^T$ a generic trading strategy in $\Theta \in \Re_+^{2(L-s_T)J}$. Let $c = (S^A(0), 0,..,0, -S^B(0), 0,..,0) \in \Re^{2J(L-s_T)}$ be the vector such that the product $c\theta$ describes the cost of the strategy θ at time $t = 0$ as in (1). Let $k = 1,..,K \equiv 2^{Js_{T-1}}$ be the index associated to a generic sequence of signs of the cumulative positions on each of the J assets on f_h^{T-1}, for $h = 1,..,s_{T-1}$. The matrix G_k is given by $G_k \equiv (G_k^A, G_k^B)$, where both G_k^A and G_k^B have dimension $s_{T-1} \times J(L - s_T)$. For simplicity, suppose first that $J = 1$. The generic i-th row of the matrices G_k^A and G_k^B corresponds to a node f_h^{T-1} for some $h = 1,..,s_{T-1}$. For any $\tau < T-1$, let p_τ denote the index of the node in $\{1,..,s_\tau\}$ belonging to the history of the node f_h^{T-1}.Since k is fixed, we have a sequence of signs corresponding to the cumulative positions on the asset at each final node. Suppose first that the cumulative position on the security conditional at the node f_h^{T-1} is positive. Hence the entries of the i^{th}row of G_k^A and G_k^B referring to f_h^{T-1} are all zero, except for those corresponding to the columns 1, $1 + p_1$, $1 + s_1 + p_2$,.., $1 + s_1 + .. + s_{T-2} + h$ which are $+1$ for G_k^A and -1 for G_k^B. If the cumulative position at the node f_h^{T-1} is negative, set instead -1 at the intersection of the i^{th} row with the columns $1, 1 + p_1, 1 + s_1 + p_2, ..., 1 + s_1 + .. + h$ for G_k^A and $+1$ for G_k^B. In this way, for any strategy θ whose cumulative positions at time T correspond to the sequence of signs k the product of $G_k\theta$ is the absolute value of the cumulative position held on the stock at the liquidation date. In the case of $J > 1$ assets, the entries previously written are vectors of $\Re^J$.

Step 2. Construction of the matrices M_k.

Again, fix a generic sequence k of signs of cumulative final positions on each of the J assets. The matrix $M_k \equiv (M_k^A, M_k^B)$ is obtained by adjoining two $(L-1) \times J(L-s_T)$ matrices M_k^A and M_k^B. Again, assume first that $J=1$. The generic i^{th}row of M_k^A (and M_k^B), corresponds to a node f_h^t for some $t = 1, .., T$ and $h = 1, .., s_t$. We construct the i-th row in the following way. If $t < T$, the entries corresponding to columns $1, 1+p_1, 1+s_1+p_2, ..., 1+s_1+..+p_{t-1}$ are $d(f_h^t)$ for M_k^A and $-d(f_h^t)$ for M_k^B, respectively. The entry corresponding to the column $1+s_1+..+s_{t-1}+h$ is $-S^A(f_h^t)$ for M_k^A and $S^B(f_h^t)$ for M_k^B, respectively. All the remaining entries of the i^{th} row are 0. If $t = T$, we have to disinguish between the case of a cumulative long or short position on the asset. Recall that, since k is fixed, we have a sequence of signs corresponding to the cumulative positions on the asset at each final node. Therefore if at the node f_h^T the cumulative position on the asset is positive, the entries of columns 1, $1+p_1$, $1+s_1+p_2$,, .., $1+s_1+..+p_{T-1}+h$ are $S^B(f_h^T)$ for M_k^A, $-S^B(f_h^T)$ for M_k^B and 0 elsewhere. If the cumulative position at the node f_h^T is instead negative the entries of columns 1, $1+p_1$, $1+s_1+p_2$,..,$1+s_1+..+p_{T-1}+h$ are $-S^A(f_h^T)$ for M_k^A, $S^A(f_h^T)$ for M_k^B and 0 elsewhere. In the case of $J > 1$ assets, the entries previously written are vectors of $\Re^J$ (for example, $d(f_h^t)$ becomes $(d_1(f_h^t), ..d_J(f_h^t))$, and so on).

Step 3. We now show that the value functions of problems $\mathcal{P}[m]$ and $\mathcal{LP}[m]$ coincide for any $m \in \Re^{L-1}$. To see that $\pi(m)$ is greater or equal than the value function of $\mathcal{LP}[m]$, let $\overline{\theta}$ be feasible for problem $\mathcal{P}[m]$. Then, there exists $\overline{k} \in \{1, .., K\}$ such that $M_{\overline{k}}\overline{\theta}$ is the cashflow produced by $\overline{\theta}$ for $t > 0$ and $G_{\overline{k}}\overline{\theta}$ is the vector of cumulative positions on each stock held from $T-1$ to T. Hence, $\widehat{\theta} = (\widehat{\theta}_1, ..., \widehat{\theta}_K)$ defined by $\widehat{\theta}_k = 0$ for $k \neq \overline{k}$ and $\widehat{\theta}_{\overline{k}} = \overline{\theta}$ is feasible for $\mathcal{LP}[m]$ and $-x_{\overline{\theta}}(0) = c\sum_{k=1}^K \widehat{\theta}_k$. To prove the converse, let $\left(\overline{\theta}_1, .., \overline{\theta}_K\right)$ be feasible for problem $\mathcal{LP}[m]$ and let $\overline{\theta} = \sum_{k=1}^K \overline{\theta}_k \in \Theta$. Denoting with $x_\theta \in \Re^{L-1}$ the vector of future cashflow generated by a strategy θ,we have that $x_{\overline{\theta}} \geq \sum_{k=1}^K x_{\overline{\theta}_k} = \sum_{k=1}^K M_k\overline{\theta}_k \geq m$, so that $\overline{\theta}$ is feasible for problem $\mathcal{P}[m]$. Moreover, since $-x_{\overline{\theta}}(0) = c\overline{\theta} = c\sum_{k=1}^K \overline{\theta}_k$, the value function of $\mathcal{LP}[m]$ is greater or equal than $\pi(m)$. This concludes the proof of Part 1 of Proposition 1. Parts 2 and 3 follows immediately from Part 1.■

Throughout the rest of the Appendix we use the following notation: $\widetilde{c} \equiv (c, .., c) \in \Re^{2KJ(L-s_T)}$, $\sigma \equiv (\theta_1, .., \theta_K)^T \in \Re_+^{2KJ(L-s_T)}$,

$$\widetilde{M} \equiv \begin{pmatrix} M_1 & ... & M_K \end{pmatrix}, \; G \equiv \begin{pmatrix} G_1 & 0 & .. & 0 \\ 0 & G_2 & .. & 0 \\ .. & & .. & .. \\ 0 & 0 & .. & G_K \end{pmatrix} \text{ and } M \equiv \begin{pmatrix} -\widetilde{c} \\ \widetilde{M} \end{pmatrix}$$

so that $\widetilde{M}$ is a $(L-1) \times 2KJ(L-s_T)$ matrix and G is $KJs_{T-1} \times 2JK(L-s_T)$. The following fact follows easily applying this notation to Definition 1:

Fact: *A price-dividend system* $\left(S^A, S^B, d\right)$ *admits arbitrage opportunities if and only if the following system is feasible*

$$\begin{cases} M\sigma > 0 \\ G\sigma \geq 0 \\ \sigma \geq 0 \end{cases} \tag{5}$$

Proof of Theorem 1. The proof follows adapting the proof of Theorem 1 in Ortu (2001). To simplify the comparison, observe that $\widetilde{\mathcal{M}}\theta$ and $\mathcal{M}\theta$ in Theorem 1 in Ortu (2001) denote the vector whose entries are the cash flow $x_\theta(f_k^t)$ for all $t > 0$, $k = 1, .., s_t$ and for $t \geq 0$ respectively. Although the presence of bid-ask spreads at $t = T$ invalidates the equality $x_{\theta+\theta'}(f_k^t) = x_\theta(f_k^t) + x_{\theta'}(f_k^t)$ in the proof of Theorem 1 in Ortu (2001), the same arguments still applies on noting that $x_{\theta+\theta'}(f_k^t) \geqslant x_\theta(f_k^t) + x_{\theta'}(f_k^t)$ for all $t \geq 0$, and for all k. The link to the formulation of $\mathcal{LP}[m]$ of Proposition 1 (Part 3 and 4) is provided by Proposition 1.■

Proof of Corollary 1. Immediate consequence of Corollary 1 in Ortu (2001).■

Proof of Proposition 2. We prove the characterization for Ψ, from which the characterization of Φ follows immediately. To this end, the positivity of the USPs allows us to identify them as vectors $(1, \psi) \in \Re^L_{++}$ such that $(1, \psi)x \leq 0$ for all $x \leq M\sigma$ for some $\sigma \in \Re_+^{2KJ(L-s_T)}$. Suppose first that $\psi \in R_{++}^{L-1}$ and $\beta \in R_+^{Js_{T-1}}$ satisfy $\psi M_k + \beta G_k \leq c$ for $k = 1, .., K$, or equivalently $\psi \widetilde{M} + \gamma G \leq \widetilde{c}$, with $\gamma = (\beta, \beta, .., \beta) \in \Re_+^{KJs_{T-1}}$. We show then that $(1, \psi)$ is a USP. Indeed, for any $\sigma \in \Re_+^{2KJ(L-s_T)}$, and for any $x \leq M\sigma$, since $\gamma \geq 0$ and $G\sigma \geq 0$, we have

$$(1, \psi)x \leq (1, \psi)M\sigma = -\widetilde{c}\sigma + \psi\widetilde{M}\sigma \leq -\widetilde{c}\sigma + \psi\widetilde{M}\sigma + \gamma G\sigma \leq 0$$

where the last inequality comes from the fact that, since $\sigma \geq 0$, it preserves the inequality $(\psi\widetilde{M} + \gamma G)\sigma \leq \widetilde{c}\sigma$.

To prove the converse, let $(1, \psi)$ be a USP, and let $b \equiv -(1, \psi)M$. For any $\sigma \in \Re_+^{2KJ(L-s_T)}$ we have $b\sigma = -(1, \psi)(M\sigma) \geq 0$ (since for $x = M\sigma$ the definition of USP implies $(1, \psi)x \leq 0$). By Theorem 2.8 in Gale (1960), there exists then $\gamma \geq 0$ such that $\gamma G \leq b$, and since $M \equiv \begin{pmatrix} -\widetilde{c} \\ \widetilde{M} \end{pmatrix}$ then $\gamma G \leq -(1, \psi)M = \widetilde{c} - \psi\widetilde{M}$, which concludes the proof.■

Proof of Theorem 2 Observe first that the dual of $\mathcal{LP}[m]$ can be written in compact matrix notation as follows:

$$\begin{aligned} \pi(m) = \quad & \max_{(\varphi,\gamma)\geq 0} \varphi m \\ & s.t.\ \varphi\widetilde{M} + \gamma G \leq \widetilde{c} \end{aligned} \qquad (\mathcal{LP}'[m])$$

Hence the supremum is taken over the set $\Phi = \left\{ (1, \varphi) \in \Re^L_+ \;\middle|\; \varphi M_k + \beta_k G_k \leq c,\ \beta_k \geq 0,\ k = 1, .., K \right\}$, i.e. the semi-positive USPs. Assuming then no-arbitrage, the existence of a USP can be established by adapting the construction of a USP in the proof of Theorem 2 in Ortu (2001). Conversely, assume that there exists a USP vector. Observe that, since the Internality Condition implies $\widetilde{M}\widetilde{\theta} >> 0$ for some $\widetilde{\theta} = (0, .., \theta_k, 0, ..0)^T \in \Re^{2JK(L-s_T)}$ (with $M_k\theta_k >> 0$ for some k), the set Φ is bounded (to see this, notice that $0 \leq \varphi(\widetilde{M}\widetilde{\theta}) \leq \varphi(\widetilde{M}\widetilde{\theta}) + (\gamma G)\widetilde{\theta} \leq c\widetilde{\theta}$, for every $(1, \varphi) \in \Phi$). Absence of arbitrage follows then by adapting the corresponding part of the proof of Theorem 2 in Ortu (2001).■

Proof of Theorem 3 Under the Internality Condition and no-arbitrage, if $m = 0$, by Theorem 1 we have $x_{\theta^*} = 0$ for every θ^* solution of $\mathcal{P}[0]$, i.e. $\Theta^*_{>0} = \emptyset$. If $m \neq 0$ and $\pi(m) = \max\limits_{(1,\psi)\in\Psi} \psi m$, suppose that $\Theta^*_{>m} \neq \emptyset$ so that, by Proposition 1, $\Xi^*_{>m} \neq \emptyset$, i.e there exists σ^*_m such that $\widetilde{M}\sigma^*_m > m$. By the complementary slackness conditions, $(\widetilde{M}\sigma^*_m - m)\varphi^* = 0$, and hence for any optimal φ^* we have that $\varphi^* \notin \Re^{L-1}_{++}$, which contradicts our assumption.

Conversely, assume $\Theta^*_{>m} = \emptyset$. To prove that $\pi(m) = \max\limits_{(1,\psi)\in\Psi} \psi m$ we construct explicitely in the following four steps the (strictly positive) USP vector that attains the maximum.

Step 1. Let $K > \pi(m)$, $\widehat{M} \equiv \left[\widetilde{M}, m, -m\right]$, $\widehat{G} \equiv [G, 0, 0]$, $\widehat{\sigma} \equiv [\sigma, y_1, y_2]^T$, $\widehat{c} \equiv [\widetilde{c}, K, -\pi(m)]$ and consider the program

$$\widehat{\pi}(n) = \inf \ \widehat{c}\widehat{\sigma} \qquad \begin{cases} \widehat{M}\widehat{\sigma} \geq n \\ \widehat{G}\widehat{\sigma} \geq 0 \\ \widehat{\sigma} \geq 0 \end{cases} \qquad (\widehat{LP}(n))$$

with dual

$$\widehat{\pi}(n) = \max \ \varphi n \qquad s.t. \begin{cases} \varphi\widehat{M} + \gamma\widehat{G} \leq \widehat{c} \\ \varphi \geq 0 \\ \gamma \geq 0 \end{cases} \qquad (\widehat{LP'}(n))$$

Observe that the feasible set of $\widehat{LP'}(n)$ is the set of vectors $\varphi, \gamma \geq 0$ such that $\varphi\widetilde{M} + \gamma G \leq \widetilde{c}$, $\varphi m + \gamma 0 \leq K$ and $-\varphi m + \gamma 0 \leq -\pi(m)$. Hence, we can define

$$\widehat{\Phi} \equiv \left\{ (1, \varphi) \in \Re^L_+ \ \middle| \ \varphi\widetilde{M} + \gamma G \leq \widetilde{c}, \ \pi(m) \leq \varphi m \leq K \right\}$$

Notice that $\widehat{\Phi} \subseteq \Phi$ and is independent from n.We use this fact in Step 3.

Step 2. Here we show that $\widehat{LP}(0)$ admits optimal solutions $\widehat{\sigma}$, all of which satisfy $\widehat{c}\widehat{\sigma} = 0$ and $\widehat{M}\widehat{\sigma} = 0$. Indeed, we have

$$\begin{aligned} \inf_{\widehat{\sigma}} \widehat{c}\widehat{\sigma} &= \inf\nolimits_{(y_1,y_2)\geq 0} \left[\left(\inf_{\sigma} \left\{ \widetilde{c}\sigma \middle| \widetilde{M}\sigma \geq (y_2 - y_1)\, m, \ G\sigma \geq 0 \right\} \right) + y_1 K - \pi\,(m)\, y_2 \right] \\ &= \min\left[\inf\nolimits_{y_2 \geq y_1 \geq 0} (K - \pi\,(m)) y_1, \inf\nolimits_{y_1 \geq y_2 \geq 0} (y_1 (K + \pi\,(-m)) - y_2 (\pi\,(m) + \pi\,(-m)))\right] \\ &= 0 \end{aligned}$$

since both the infima are obtained for $y_1 = 0$. This implies that $\widehat{\pi}(0) = 0$ and any optimal solution $\widehat{\sigma} = [\sigma, y_1, y_2]^T$ is such that $y_1 = 0$. Now, if $y_2 = 0$, we have

$$0 \ = \ \widehat{\pi}(0) \ = \ \widehat{c}\widehat{\sigma} \ = \ \inf_{\sigma} \left\{ \widetilde{c}\sigma \ \middle| \ \widetilde{M}\sigma \geq 0, \ G\sigma \geq 0 \right\}$$

implying that σ is optimal for $\mathcal{LP}[0]$ and, by Theorem 1 (part 4), that $\widetilde{c}\sigma = 0$ and $0 = \widetilde{M}\sigma = \widehat{M}\widehat{\sigma}$. If instead $y_2 > 0$, then we have $0 = \widehat{\pi}(0) = \inf\limits_{\sigma} \left\{ \widetilde{c}\sigma - \pi\,(m)\, y_2 \ \middle| \ \widetilde{M}\sigma \geq y_2 m, G\sigma \geq 0 \right\}$. Hence,

$$\pi\,(m) \ = \inf_{\sigma} \ \left\{ \widetilde{c} \cdot \frac{\sigma}{y_2} \ \middle| \ \widetilde{M}\frac{\sigma}{y_2} \geq m, \ G\frac{\sigma}{y_2} \geq 0 \right\}$$

that is, $\frac{\sigma}{y_2}$ is optimal for $\mathcal{LP}[m]$. Since by assumption $\Xi^*_{>m} = \emptyset$, the minimum super-replication cost is achieved by perfect replication of m. Hence $\widetilde{M}\frac{\sigma}{y_2} = m$, which implies $\widehat{M}\widehat{\sigma} = \widetilde{M}\sigma + 0 - my_2 = 0$, concluding the proof of Step 2.

Step 3. Here we show that there exists an optimal solution to $\widehat{LP'}(n)$ for every n and $\widehat{\pi}(n) > 0$ whenever $n > 0$. Since by Step 2 $\widehat{LP}(0)$ admits optimal solutions, the set $\widehat{\Phi}$, that is independent from n, is nonnempty. Since by Step 1 $\widehat{\Phi}$ is compact, and convex, it follows that problem $\widehat{LP'}(n)$

(or equivalently $\widehat{LP}(n)$) admits solution for every n. To prove the positivity of $\widehat{\pi}(n)$, suppose by contradiction that $\widehat{\pi}(n^*) \leq 0$ for some $n^* > 0$. Denoting by $\widehat{\sigma}^*$ any optimal solution to $\widehat{LP}(n^*)$, since $\widehat{M}\widehat{\sigma}^* \geq n^* > 0$, $\widehat{G}\widehat{\sigma}^* \geq 0$ and $0 \geq \widehat{\pi}(n^*) = \widehat{c}\widehat{\sigma}^*$ then $\widehat{\sigma}^*$ is also an optimal solution to $\widehat{LP}(0)$. By Step 2 it follows that $\widehat{c}\widehat{\sigma}^* = 0$ and $\widehat{M}\widehat{\sigma}^* = 0$, which contradicts $\widehat{M}\widehat{\sigma}^* > 0$.

Step 4. Finally, we show that there exists a strictly positive USP vector $(1, \widehat{\varphi}) \in \widehat{\Phi} \subseteq \Phi$ such that $\pi(m) = \max\limits_{(1,\psi)\in\Psi} \psi m = \widehat{\varphi} m$. Let $n = I(f_h^t)$, for $t > 0$ and $h = 1, ..., s_t$.[18] From Step 3, we know that $\widehat{\pi}(n) > 0$ and that problem $\widehat{LP'}(n)$ admits an optimal solution $(1, \varphi_h^t) \in \widehat{\Phi}$ such that $\varphi_h^t I(f_h^t) = \widehat{\pi}(I(f_h^t)) > 0$. Then $\varphi = \frac{1}{L-1} \sum_{t>0} \sum_{h=1}^{s_t} \varphi_h^t \gg 0$ and $(1, \varphi) \in \widehat{\Phi}$ (since $\widehat{\Phi}$ is independent from n and convex). Hence $\pi(m) = \sup\limits_{(1,\psi)\in\Psi} \psi m \geq \varphi m$ and recalling the structure of the set $\widehat{\Phi}$, we finally have $\varphi m \geq \pi(m)$, which concludes the proof.$\blacksquare$

2. Proofs of Section 4

We first write down the auxiliary linear program $\mathcal{LP}^{ex}[m]$ for the extended market. To this end we let G^{ex} be the $(KJs_{T-1} + 2) \times (2KJ(L - s_T) + 4)$ matrix defined as follows:

$$G^{ex} = \begin{bmatrix} G & 0 \\ 0 & \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{bmatrix}$$

Moreover, we let $c^{ex} \equiv [\widetilde{c}, c^A, -c^B, c^A, -c^B]$, $\widetilde{M}^{ex} \equiv \left[\widetilde{M}, n^B, -n^B, n^A, -n^A\right]$ and $M^{ex} \equiv \begin{bmatrix} c^{ex} \\ \widetilde{M}^{ex} \end{bmatrix}$. Denoting then by $\sigma^{ex} = \left(\sigma, \zeta_1^A, \zeta_1^B, \zeta_2^A, \zeta_2^B\right)^T$ the vectors in $\Re^{2KJ(L-s_T)+4}$, the linear program $\mathcal{LP}^{ex}[m]$ is

$$\pi^{ex}(m) = \inf_{\sigma^{ex}} c^{ex}\sigma^{ex}$$
$$s.t. \begin{cases} \widetilde{M}^{ex}\sigma^{ex} \geq m \\ G^{ex}\sigma^{ex} \geq 0 \\ \sigma^{ex} \geq 0 \end{cases}$$

Applying Proposition 2 to the extended market, the set Ψ^{ex} of the USP vectors for the extended market can be written as

$$\Psi^{ex} = \left\{(1, \psi^{ex}) \in R_{++}^L \mid \psi^{ex}\widetilde{M}^{ex} + \gamma^{ex}G^{ex} \leq c^{ex},\ \gamma^{ex} = (\gamma, \gamma_1, \gamma_2) \in \Re_+^{KJs_{T-1}} \times \Re_+^2\right\}$$

or equivalently as

$$\Psi^{ex} = \left\{(1, \psi^{ex}) \in \Psi \mid c^B \leq \psi^{ex}n^B + \gamma_1 \leq c^A, c^B \leq \psi^{ex}n^A - \gamma_2 \leq c^A \text{ for } \gamma_1, \gamma_2 \geq 0 \ \right\} \tag{6}$$

Clearly, no-arbitrage in the extended market is equivalent to $\Psi^{ex} \neq \emptyset$.

Proof of Theorem 4 Since from (6) we have that $\psi^{ex}n^B \leq \psi^{ex}n^B + \gamma_1 \leq c^A$ and $\psi^{ex}n^A \geq \psi^{ex}n^A - \gamma_2 \geq c^B$ for any $(1, \psi^{ex}) \in \Psi^{ex}$, no-arbitrage in the extended market implies the existence

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[18] $\mathrm{I}(f_h^t) \in \Re^L$ is a vector of zeros except for the $(1 + s_1 + .. + s_{t-1} + h)^{th}$ entry, which is equal to 1.

of a USP $(1,\psi) \in \Psi^{ex} \subseteq \Psi$ such that $\psi n^B \leq c^A, \psi n^A \geq c^B$. The rest of the proof is in two steps. In Step 1 we show that $\psi n^B \leq c^A, \psi n^A \geq c^B$ for some $(1,\psi) \in \Psi$ implies $c^B \leq \pi(n^A)$ (with strict inequality if $\Theta^*_{>n^A} \neq \emptyset$) and $c^A \geq -\pi(-n^B)$ (with strict inequality if $\Theta^*_{>-n^B} \neq \emptyset$). In Step 2 we show that this last fact implies no-arbitrage in the extended market.

Step 1. Recall that, by Theorem 2, we have $\pi(n^A) = \sup\limits_{(1,\psi)\in\Psi} \psi n^A$. Hence, the inequality $c^B \leq \psi n^A$ for some USP implies $\pi(n^A) \geq c^B$. Similarly, we have $\pi(-n^B) = \sup\limits_{(1,\psi)\in\Psi} \psi(-n^B)$. Since $\psi(-n^B) \geq -c^A$ for some USP, we have that $\pi(-n^B) \geq -c^A$. For the strict inequalities, suppose that $\pi(n^A) = c^B$. Since $\pi(n^A) = c^B \leq \psi n^A$ for some$(1,\psi) \in \Psi$, this can be true if and only if $\pi(n^A) = \psi n^A$ i.e. $\pi(n^A) = \max\limits_{(1,\psi)\in\Psi} \psi n^A$. By Theorem 3 this is true if and only if $\Theta^*_{>n^A} = \emptyset$. A similar argument for $\pi(-n^B)$ establishes $c^A > -\pi(-n^B)$ if $\Theta^*_{>-n^B} \neq \emptyset$.

Step 2. To show that the extended market is arbitrage-free if $c^B \leq \pi(n^A)$ (with strict inequality if $\Theta^*_{>n^A} \neq \emptyset$) and $c^A \geq -\pi(-n^B)$ (with strict inequality if $\Theta^*_{>-n^B} \neq \emptyset$), we show that $\Psi^{ex} \neq \emptyset$. Since under our assumptions $-\pi(-n^B) = -\sup\limits_{(1,\psi)\in\Psi} \psi(-n^B) = \inf\limits_{(1,\psi)\in\Psi} \psi n^B \leq c^A$, there exist $\psi_1 \in \Psi$ and $\gamma_1 \geq 0$ such that

$$c^B \leq \psi_1 n^B + \gamma_1 \leq c^A \tag{7}$$

By the same token, since $\pi(n^A) = \sup\limits_{(1,\psi)\in\Psi} \psi n^A \geq c^B$, there exist $\psi_2 \in \Psi$ and $\gamma_2 \geq 0$ such that

$$c^B \leq \psi_2 n^A - \gamma_2 \leq c^A \tag{8}$$

Multiplying (7) by $\alpha \in (0,1)$, (8) by $(1-\alpha) \in (0,1)$ and summing term by term, we have

$$\begin{cases} \alpha(\psi_1 n^B + \gamma_1) + (1-\alpha)(\psi_2 n^A - \gamma_2) \leq c^A \\ \alpha(\psi_1 n^B + \gamma_1) + (1-\alpha)(\psi_2 n^A - \gamma_2) \geq c^B \end{cases}$$

Letting $\alpha \equiv \frac{\gamma_2}{\gamma_1+\gamma_2}$ and $\psi \equiv \alpha\psi_1 + (1-\alpha)\psi_2$, we find $\psi n^B \leq \alpha\psi_1 n^B + (1-\alpha)\psi n^A \leq c^A$ and $\psi n^A \geq \alpha\psi_1 n^B + (1-\alpha)\psi_2 n^A \geq c^B$, which concludes our proof.■

Proof of Proposition 3 We only prove Part 1, since Part 2 can be proved similarly to Part 1, and Part 3 follows from Part 1 and 2. Since long-effectiveness of the new security is clearly implied by $\pi^{ex}\left(n^B\right) = c^A$, we prove in Step 1 that long-effectiveness implies $\max\left[c^B, -\pi\left(-n^B\right)\right] \leq c^A \leq \pi\left(n^B\right)$, which in Step 2 is shown to imply $\pi^{ex}\left(n^B\right) = c^A$.

Step 1. Long-effectiveness of the new security implies that the feasible set $\Phi^{ex}$of the dual of $\mathcal{LP}^{ex}[m]$, where

$$\Phi^{ex}{=}\left\{(1,\varphi^{ex}) \in \Re^L_+ \mid \varphi^{ex}\widetilde{M}^{ex} + \gamma^{ex}G^{ex} \leq c^{ex},\ \gamma^{ex} = (\gamma^T, \gamma_1, \gamma_2) \in \Re_+^{KJs_{T-1}} \times \Re^2_+\right\}$$

is nonempty. Moreover, since Φ^{ex} is closed and contained in the compact set Φ, we can conclude that Φ^{ex} is compact and nonempty. This implies that the problem $\mathcal{LP}^{ex}[m]$ admits solutions for every m. This allows to construct a semi-positive USP vector for the extended market (following the same construction described in the proof of Theorem 3). Hence, second type arbitrage opportunities are banned from the extended market and, by Corollary 2, $c^A \geq \max\left[c^B, -\pi\left(-n^B\right)\right]$. To prove that $c^A \leq \pi\left(n^B\right)$, notice that, since the new asset is long-effective, the optimal strategy[19] σ^{ex} for the cash flow m is represented by the vector $\sigma^{ex} = [\sigma, \zeta^A, 0, 0, 0]^T$. By the complementary slackness condition, therefore, $\left(\varphi^{ex}\widetilde{M}^{ex} + \gamma^{ex}G^{ex} - c^{ex}\right)\sigma^{ex} = 0$. Since this is a sum of nonpositive terms, it follows that $(\varphi^{ex}n^B + \gamma_1 - c^A)\zeta^A = 0$. Moreover, the complementary slackness conditions imply also that $\varphi^{ex}(\widetilde{M}^{ex}\sigma^{ex} - m) = 0$ and $(\gamma, \gamma_1, \gamma_2)(G^{ex}\sigma^{ex} - 0) = 0$. From the last one we have that $\zeta^A\gamma_1 = 0$, which implies $\gamma_1 = 0$. Since $\varphi^{ex}n^B = c^A - \gamma_1$, it follows that $\varphi^{ex}n^B = c^A$ so that $\pi(n^B) = \max\limits_{(1,\varphi)\in\Phi} \varphi^T n^B \geq \varphi^{ex}n^B = c^A$.

Step 2: We now show that $\max\left[c^B, -\pi\left(-n^B\right)\right] \leq c^A \leq \pi\left(n^B\right)$ implies $\pi^{ex}\left(n^B\right) = c^A$. To see this, notice that since $-\pi(-n^B) \leq c^A \leq \pi(n^B)$, by continuity there exists $(1, \overline{\varphi}) \in \Phi$ such that $\overline{\varphi}n^B = c^A$. Hence by taking $\gamma_1 = 0$ and $\gamma_2 = (\overline{\varphi}n^A - c^A)^+$, we have that $(1, \overline{\varphi}) \in \Phi^{ex}$ and $\pi^{ex}(n^B) = \max\limits_{(1,\varphi)\in\Phi^{ex}} \varphi n^B = \overline{\varphi}n^B = c^A$.■

Proof of Theorem 5 Immediate consequence of Theorem 4 and Proposition 3.■

Example 1 Figures 1 and 2 illlustrate two simple 1-period examples where the introduction of an effective new security in a (originally) no-arbitrage market generates first-type arbitrage opportuinities. In both situations, the initial prices of the new security violate the strict inequalities in statement 2 of Theorem 5. Observe that the original market is arbitrage free, since the set of (strictly) positive USPs Ψ is nonempty. Moreover, the extended market is free of second-type-arbitrage, since Φ^{ex} is nonempty. However, first-type-arbitrage opportuinities are present in the extended market since the set Ψ^{ex} of strictly positive USPs for the extended market is empty. The new security is traded at the initial price $c^B = c^A$ and is affected by transaction costs at the terminal date, since $n^A(f_2^1) = n^B(f_2^1)$ and $n^A(f_1^1) > n^B(f_1^1)$. The new security is effective, since $\pi^{ex}(n^B) = c^A$ and $-\pi^{ex}(-n^A) = c^B$.

As guaranteed by Theorem 3, in Figure 1 the set $\Theta^*_{>n^A}$ is nonempty since $\pi(n^A)$ is not attained by any (strictly positive) USP. Moreover $\pi(n^A) = \pi(n^B) = c^B$ instead of $\pi(n^B) > c^B$ as required by statement 2 of Theorem 5.

Once again by Theorem 3, in Figure 2 the set $\Theta^*_{>-n^B}$ is nonempty since $-\pi(-n^B)$ is not attained by any (strictly positive) USP. Moreover $\pi(-n^B) = \pi(-n^A) = c^A$ instead of $\pi(-n^A) < c^A$ as required by statement 2 of Theorem 5.

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[19] Due to the *static* nature of the buy-and-hold strategies allowed on the new asset, it is never optimal to take both a long and a short position on the new security. More precisely, if the strategy (ϑ, ζ) is optimal for $\mathcal{P}^{ex}[m]$, then $\zeta = (\zeta^A, \zeta^B)$ is such that $\zeta^A\zeta^B = 0$.

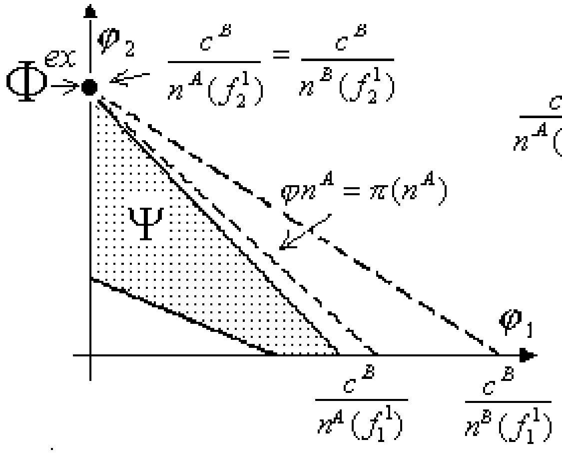

Figure 1

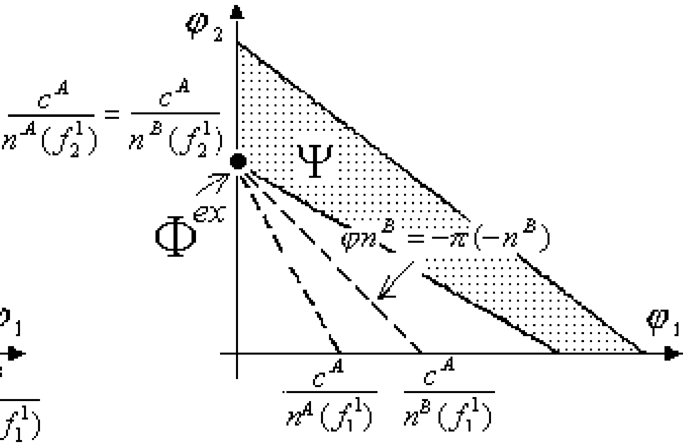

Figure 2

Proofs of Section 5

Example 2 Figure 3 illustrates an example of an effective security in which a long position is not an efficient trading strategy. The original 1-period market is constituted by a unique security S. The market is arbitrage-free, since the set of (strictly positive) USPs is nonempty. Then a new security is introduced in the market, in such a way that $\frac{S^B}{S^A(f_1^1)} < \frac{S^A}{S^B(f_1^1)} < \frac{c^B}{n^A(f_1^1)} < \frac{c^A}{n^B(f_1^1)}$ and $\frac{c^A}{n^B(f_2^1)} = \frac{S^A}{S^B(f_2^1)} > \frac{c^B}{n^A(f_2^1)} > \frac{S^B}{S^A(f_2^1)}$. Hence no-arbitrage holds in the extended market, since the set Ψ^{ex} is nonempty. Moreover, the new security is effective, since $-\pi^{ex}(-n^A) = c^B$ and $\pi^{ex}(n^B) = c^A$. However, the supremum $\pi^{ex}(n^B)$ is not attained by any $\psi^{ex} \in \Psi^{ex}$, so that by the characterization of efficiency in Jouini and Kallal (2001, Theorem 1) n^Bcannot be an efficient trading strategy.

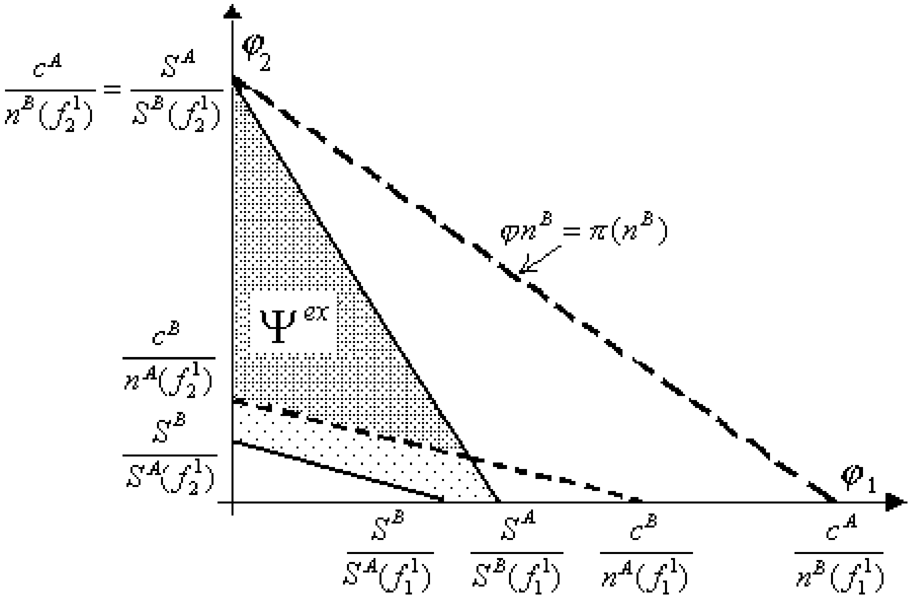

Figure 3

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