THE VALUATION OF STOCK INDEX OPTIONS

by

Menachem Brenner*, Georges Courtadon**, and Marti Subrahmanyan***

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* New York University and Hebrew University, Jerusalem
** Shearson Lehman Brothers
*** New York University
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I. Introduction

The most recent innovations in the equity options market are options on stock market indices, both on the spot and on futures contracts. The most successful of these options are the options on the Standard and Poor's 100 (SP 100) index, which account for a large proportion of the volume of trading on the Chicago Board Options Exchange (CBOE). Other examples include options on the New York Stock Exchange (NYSE) index, options on the Standard and Poor's 500 (SP500) index, options on the Major Market Index (MMI) and options on the Institutional Index (XII). While most of these options, such as the NYSE and the MMI options, are of the American type, the SP500 and the XII options are of the European type.

Although the stock index options markets have been very successful, there has been very little research dealing directly with the valuation of these instruments, either at a theoretical, or an empirical, level. An examination of this limited literature indicates that extending the standard option pricing framework to the pricing of stock index options is not a simple matter. Brenner, Courtadon and Subrahmanyam (1985), and Ramaswamy and Sundaresan (1985) have dealt with the valuation of options on futures and options on spot, in general, with some reference to stock index options. They have not analyzed, however, certain important issues that are unique to index options. Whaley (1986a)
examines empirically options on stock index futures, while Ball and Torous (1986) study options on commodity futures. Evnine and Rudd (1985), Figlewski (1985) and Eytan and Harpaz (1986) apply essentially the Black-Scholes model to index options and investigate the deviations from the model. As we shall see below, there are major differences between options on individual stocks and options on stock indices. Thus, the deviations between model prices and actual prices reported in previous studies may be due to the simplifying assumptions underlying their valuation models.

There are three important issues related to the pricing of index options that need to be taken into account. Their influences are exacerbated, in many cases, for American options. The first problem has to do with the irregular pattern of dividends on the index. Figlewski (1985) and Evnine and Rudd (1985) assume a continuous dividend stream, which may, or may not, be a good approximation for the actual dividend payments. Not only could there be a difference in valuation due to differences in the present value of the dividends, but the timing and size of the dividends may affect the timing of early exercise, and hence, affect the valuation of American options.

The second issue concerns the effect of interest rate volatility on option valuation. This is a familiar issue that

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1 Evnine and Rudd (1985) also use a binomial model to value index options.
relates to options on any asset: stocks, bonds, currencies, commodities, etc. The strong link between interest rates, and in particular, interest rate volatility, and the stock market is evident even on the basis of casual empiricism. It is reasonable to expect that, in turn, this would affect option valuation. Courtadon [(1982a) and (1983)] and Ramaswamy and Sundaresan (1985) examine the effect of interest rate uncertainty on the probability of early exercise and on valuation. The negligible effect that these authors find is due to the fact that their model does not take into account the effect of interest rate uncertainty on the value of the stock index, either directly, or indirectly via its effect on the stock market volatility. This effect could be important for both American as well as European options. For American options, we have the added dimension that interest rate uncertainty may have a significant effect on early exercise.

The third issue arises from the fact that the underlying asset, the stock index, is not directly traded - only the component stocks are. This creates a hedging problem for options written on large indices, like the NYSE index. The costs involved in buying and selling the component securities make it practically impossible to adjust the hedge continuously. Hence, there is no arbitrage relationship that will guarantee a particular link between the options and the index. Of course, in cases where there is a futures contract on the index, it can be used to hedge the options more effectively. Since, however, the options are American, the efficiency of the hedge hinges upon the relationship between the
futures contract and the index. In fact, the deviation of the futures price from its theoretical level is substantial at times, and could be either positive or negative. The stochastic nature of this deviation is due to the fact that the spot index is not traded, as stated earlier. This issue is further complicated due to the method of delivery, cash settlement, which exposes the writer of stock index options to the risk of early exercise. This risk cannot be hedged even if the investor holds all the components of the index.

In this paper, we analyze stock index option valuation in the context of the three issues discussed above. Since each of these issues is rather complex, we consider them one at a time. In Section II, we compare the effect on option valuation and exercise policy of the actual dividend stream on stock indices versus a continuous dividend stream. The actual dividend streams for two different indices, NYSE and MMI, are used for these comparisons. Section III deals with the effect of interest rate volatility on valuation. We model the direct effect of interest rate volatility on the value and volatility of the stock index and contrast this with previous models in the literature. In Section IV, we model the uncertainty of the basis and its effect on the valuation of index options. Finally, in Section V, we offer some conclusions and conjectures regarding basis risk and cash settlement.

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The issue of cash settlement is discussed in more detail in Brenner, Courtadon and Subrahmanyam (1986).
II. Dividend Payments and the Valuation of Stock Index Options

An important adjustment to be made in the valuation of put and call options on stock indices is for the effect of dividend payments. Firstly, the payment of dividends affects the cash payoffs to the options, even if held to expiration. Secondly, as in the case of options on individual stocks, the payment of dividends on the underlying asset may affect the probability of early exercise of the options, and hence, their valuation. However, there is an important qualitative difference between options on individual stocks and options on stock indices, namely the pattern of dividend payments. In the case of options on individual stocks, there are, typically, four dividend payments every year. In the case of stock indices, by contrast, the pattern of dividend payments on the component stocks is more continuous. This distinction may have an impact on the early exercise decision, and hence, on the valuation of stock index options compared with options on individual stocks.

Suppose the stock index (the spot) follows a geometric Wiener process:

\[ dS = \mu S dt + \sigma S dz \]

where \( S \) is the value of the spot index at time \( t \), \( dS \) is the

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3 It is assumed here that the index is averaged arithmetically using market value weights and is updated continuously. The SP500 and the NYSE indices fit this description. See Eytan and Harpaz (1986) for an analysis of options on a geometrically-weighted index, the Value Line Index.
instantaneous change in its value over a small time interval \( dt \), \( \mu \) is the mean of the instantaneous rate of return on the index, \( \sigma \) is the standard deviation of the instantaneous rate of return on the index and \( dz \) is a Brownian motion. In the absence of interim distributions, such as dividend payments on the underlying asset, the Black-Scholes model applies to the valuation of call options on the spot index. Since there is no incentive to exercise American call options early, American and European call options are equally valuable.

The standard approach to deriving the valuation relationship is by solving the following partial differential equation for \( C(S, t) \), the value of the call option, as a function of the spot index \( S \), and time \( t \):

\[
\frac{1}{2} \sigma^2 S^2 C_{SS} + rSC_S - rC + C_t = 0
\]

(2)

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4 It should be pointed out that the same argument applies to options on forward contracts on the index. Since, unlike futures contracts, there is no marking-to-the-market on forward contracts, there is no immediate benefit from exercising American call options on forward contracts early and hence, no incentive to do so, in the absence of dividend payments. See Courtadon (1983) and Jarrow and Oldfield (1986) for details.
This valuation equation is solved subject to the boundary condition for a European or American call option:

\[ C(S, T) = \text{Max} (S - X, 0) \quad (3) \]

where \( X \) is the exercise price of the call option, and \( T \) is the expiration date of the option, to obtain the standard Black-Scholes formula. For a European put option, we impose the boundary condition:

\[ P(S, T) = \text{Max} (X - S, 0) \quad (4) \]

where \( X \) is the exercise price of the put option. (This yields the same result as the one from the put-call parity condition).

In contrast to the case of American call options on a non-dividend paying asset, American put options may be exercised early. Since no closed-form solution exists to take this into account, it is necessary to use numerical methods such as those in Schwartz (1977), or Courtadon (1982c).

Interim distributions such as the payment of dividends on

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5
See Brennan and Schwartz (1977), Parkinson (1977), and Geske and Johnson (1984), for alternative solution techniques.

6
Details of the boundary conditions are provided in Brenner, Courtadon and Subrahmanyam (1985).
the component stocks of the index change the above analysis substantially. In the case of European call options, we simply adjust the Black-Scholes model by subtracting the present value of the dividend payments from the index value. The price of a European put option may be similarly adjusted to take into account the benefit from an effective reduction in the spot index value through payment of dividends.

For American call options, there is an additional effect. The payment of dividends on the underlying asset, may trigger early exercise of options. There is an incentive to exercise an American call option early only when a dividend payment is imminent. Between dividend payment dates, on the other hand, there is no reason to exercise the option, by the same argument as for American call options on non-dividend paying assets.

For American put options, the payment of dividends

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7 The dividends are translated into index-equivalent values. The NYSE index uses market value weights while the MMI uses prices as weights. Also, we ignore the timing differences between the ex-dividend dates and the dividend payment dates.

8 In this adjustment, it is presumed that the stochastic process in equation (1) refers to the cum-dividend stock index value.

9 See Merton (1973).
diminishes the motivation to exercise the option early. The optimal decision depends on the trade-off between the interest foregone on the exercise price on the one hand, and the premium over parity and the exercise price on the other.

Of course, these arguments apply to American options on individual stocks as well. The difference, however, is that there are many more interim payments in the case of a stock index than on an individual stock, since each stock may pay dividends on a different date. Although there is a certain amount of "clustering" of dividend payment dates across stocks, the dividend payment stream for a stock index is more continuous than for the component stocks. Hence, there are many more dates when early exercise of call options may be optimal.

To study the importance of the dividend assumption we examine two stock indices: the New York Stock Exchange (NYSE) index and the Major Market Index (MMI). The NYSE index is a market value index of all stocks traded on the NYSE and is continuously updated. In contrast, the MMI is an equally weighted price index comprised of twenty large corporations, sixteen of which are also in the Dow Jones Industrial Average. This index also is continuously updated. There is a futures

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10 For details see Brenner, Courtadon and Subrahmanyan (1985).
contract on the NYSE index traded on the New York Futures Exchange (NYFE) and one on the MMI traded at the Chicago Board of Trade (CBOT).

The options on the NYSE index (the spot options) are traded on the same floor as the index futures. The options have a monthly cycle while the futures contracts have a three-month cycle of expiration dates. At the end of each quarter, the futures and the options expire together. The options on the MMI are traded on the American Stock Exchange. The options on the MMI and the futures contract based on it are both on a monthly cycle of expiration dates. Both the MMI and NYSE options are American and have the same features as any stock option except for the method of delivery which is by cash settlement. This institutional difference may affect option values on the index substantially as we will argue later on in the paper.

The main reason for analyzing these two indices is the difference in the costs involved in trading the components of the index. While it is prohibitively costly to trade continuously all the stocks in the NYSE index, it is relatively inexpensive to do so with the components of the MMI. This difference should show up in the futures-spot relationship and thereby affect options prices as we will see in Section IV. The secondary reason is that the pattern of the dividend for the NYSE index is, to a large extent, smooth and continuous over the course of the year. For the MMI, in contrast, the pattern is quite different, since the dividends appear at a few discrete points during a quarter. Thus, the pattern of dividend
payments for the MMI is somewhere between the pattern for an individual stock and that of the NYSE index. The difference in the dividend patterns of the two indices should show up in differences in the values of the respective options.

We now turn to the valuation of options on the two indices, taking into account these differences in the dividend payout patterns, using the numerical methods described earlier. For illustrative purposes, we present results for the theoretical values of American options on the two indices with one month (30 days) to expiration. The calculations are performed based on three alternative assumptions regarding the dividend stream using the actual dividends paid in 1984. The first is that the dividends are paid every day, (i.e. almost continuously) so that the total amount paid during the year equals the total amount actually paid in 1984. In other words, the dividend paid each day is set equal to the sum of the actual dividend payments divided by the number of days. The two other patterns analyzed are based on the actual timing of the dividends paid. The distinction is that the values are for two "extreme" months, the month with the maximum and minimum dividends paid over the annual cycle.

Tables 1A and 1B provide the theoretical values for American call and put options, respectively, on the NYSE Index.

Our 30 days maturity cycle starts in the middle of every month. For the NYSE index, the minimum dividend period is June 16 to July 15, 1984 and the maximum dividend month is July 16 to August 15, 1984. For the MMI, the minimum dividend month is June 16 to July 15, 1984 while the maximum dividend month is November 15 to December 15, 1984.
while Tables 2A and 2B provide the same information for American call and put options, respectively, on the Major Market Index, MMI. All calculations are based on the actual dividend payments made in 1984. The option values are compared using the three alternative dividend streams for in-the-money, at-the-money and out-of-the-money options, for two interest rates (r = .06, r = .12) and for three volatility measures (σ = .15, σ = .20, σ = .25).

The differences between option values with a continuous dividend and option values that incorporate the actual payments and their timing could be sizeable. The differences are dependent on the depth-in-the-money volatility, interest rates, option type and index composition. The differences for options on the narrow index, the MMI, are two to three times as large as they are for options on the broad, NYSE, index. For the MMI, that has 3 to 4 ex-dividend days a month, the assumption of a constant daily dividend may be a serious misspecification. For example, in the case of in-the-money calls, the average difference is about 2 percent for the NYSE options vs. 5.4 percent for the MMI options. The difference is even larger for out-of-the-money options. While out-of-the-money NYSE calls show a maximum difference of 6.25 percent (for σ = .15 and r = .06), the out-of-the-money MMI calls show a difference of 15.2 percent.

The difference is also affected by the distance of the index value from the strike price. The more the option is out-of-the-money, the larger is the proportional difference
between the value of an option with a constant dividend stream and one with a discrete dividend stream. In Table 1A, for $\sigma = .15$, and $r = .06$, for example, the difference for in-the-money, at-the-money and out-of-the-money calls is 2.3, 3.9 and 6.25 percent respectively. This is due to the fact that the difference in values is a function of the difference in hedge ratios and the dividend value. The difference in hedge ratios declines very slowly compared with the decline in the call value that is used to compute the proportional difference. The same pattern is observed, Tables 1B, 2A, 2B, for other combinations of volatility and interest rates and for both puts and calls. There is, however, a difference between calls and puts. For any given $\sigma$, $r$ and $S/K$ the differences observed for puts are smaller than the ones observed for calls. This is especially true for in-the-money options. For example, the difference for MMI in-the-money calls, Table 2A, is 6.4 percent ($\sigma = .15$, $r = .06$) while the difference for the comparable put options, Table 2B, is just 3.6 percent. The reason is that for put options, unlike call options, the effects of early exercise and dividend payments work in opposite directions, as argued earlier.

12 The differences should decline with volatility but increase with interest rates.
Figures 1A and 2A show the optimal early exercise boundaries for NYSE and MMI calls respectively for a $\sigma$ of 0.15 and $r$ of 0.06. The same information for puts is given in Figures 1B and 2B. As can be observed in Figures 1A and 2A, when dividends are spread out over the entire year there is no incentive, even for rather deep in-the-money calls to be exercised early. However, when the actual timing of dividends is considered, we find several instances of early exercise triggered by large dividends, paid in certain months. Again, such instances are more pronounced for the MMI options.

Figures 1B and 2B show the optimal exercise boundaries for index put options. In the case of a constant dividend stream, the early exercise boundary is smooth and continuous while the actual dividend stream may trigger exercise on some days and not on other days. Here too the difference between the policies is larger for MMI options than with NYSE options.

In short, the assumption of a continuous dividend may provide grossly biased option values. This bias is larger for call options than for put options and it is especially important for narrowly-based indices like the MMI.
Table 1A

Values of American Call Options on the NYSE Index, with 30 days to Maturity.*

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* Cc    Call values using a constant dividend
Cm    Call values for a month with maximum dividend payments
Cn    Call values for a month with minimum dividend payments

Values are given as a proportion of the exercise price
Table 1B

Values of American Put Options on the NYSE Index, with 30 Days to Maturity.*

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* $P_c$ Call values using a constant dividend

$P_m$ Call values for a month with maximum dividend payments

$P_n$ Call values for a month with minimum dividend payments

Values are given as a proportion of the exercise price
Table 2A

Values of American Call Options on the MMT, with 30 days to Maturity.*

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* Cc  Call values using a constant dividend
Cm  Call values for a month with maximum dividend payments
Cn  Call values for a month with minimum dividend payments

Values are given as a proportion of the exercise price
Table 2B

Values of American Put Options on the MMI, with 30 days to Maturity.*

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<td>r</td>
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<td>.06</td>
</tr>
<tr>
<td></td>
<td>.15</td>
<td>.20</td>
</tr>
<tr>
<td>Pc</td>
<td>.0165</td>
<td>.0223</td>
</tr>
<tr>
<td>Pm</td>
<td>.0178</td>
<td>.0235</td>
</tr>
<tr>
<td>Pn</td>
<td>.0152</td>
<td>.0209</td>
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</tbody>
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<table>
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<tr>
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<th>X = 250</th>
<th>S = 240</th>
</tr>
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<tr>
<td>r</td>
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<td>.06</td>
</tr>
<tr>
<td></td>
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<td>.20</td>
</tr>
<tr>
<td>Pc</td>
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<td>.0470</td>
</tr>
<tr>
<td>Pm</td>
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<td>.0489</td>
</tr>
<tr>
<td>Pn</td>
<td>.0416</td>
<td>.0454</td>
</tr>
</tbody>
</table>

* Pc  Call values using a constant dividend
Pm  Call values for a month with maximum dividend payments
Pn  Call values for a month with minimum dividend payments

Values are given as a proportion of the exercise price
III. Interest Rate Uncertainty and the Valuation of Stock Index Options

A crucial assumption of most stock option valuation models that use the Black and Scholes (1973) framework is that the riskless rate of interest is non-stochastic, and, in particular, constant. The only uncertainty in these models arises from the volatility of the price of the underlying stock, measured by the standard deviation of its rate of return. By assumption, this volatility is non-stochastic, and hence, unaffected by the level of the riskless rate of interest. In reality, of course, interest rates are stochastic. One index of the stochastic nature of interest rates is that, since 1979, the price volatility implied by options on Treasury bond futures using the Black (1976) model has been in the range of 10% to 25%. It is reasonable to expect that interest rate volatility may have an indirect effect on stock option prices. Several authors have recognized the need to incorporate interest rate uncertainty in option valuation and provide analyses of the issues involved.

The most recent work in this area, by Ramaswamy and Sundaresan (1985), examines the effect of interest rate uncertainty on the valuation of stock index options, both on the

---

13 See for example, Brennan and Schwartz (1980), Courtadon [(1982a) and (1983)] and Ramaswamy and Sundaresan (1985).
spot and on the futures. Their model results in option values
that are relatively unaffected by interest rate uncertainty.

The reason why interest rate uncertainty has such a
marginal effect on option valuation in previous research is due
to the modelling of this uncertainty, where interest rate
volatility, \( \sigma_r \), affects the risk free rate, \( r \), and not, more
directly, the stock price. Since in "standard" option
valuation models changes in the interest rate have a negligible
effect on option values, modelling \( \sigma_r \) to affect \( r \) will also
result in a marginal effect. These "standard" models do not
consider the direct impact of interest rates on stock price
volatility. Stock price volatility, \( \sigma_S \), may be affected by
interest rate volatility, \( \sigma_r \) and by the level of the interest
rate, \( r \). If the stock price volatility is also interest rate
induced, a model that incorporates \( \sigma_r \) in this manner should

\[\text{Ramaswamy and Sundaresan (1985) conclude that their model provides option values that are significantly different from those derived from the constant interest rate model. They state, however, that changes in the interest rate volatility parameter result in changes in option values that differ "only marginally" from their base case. However, in their model the location parameter of the interest rate "relative to its long-run mean value rather than the volatility of interest rates ..... makes a sizeable difference in (option) values." Thus, the difference in option values that they report are due to the changes in the risk free rate through the time value effect rather than interest rate uncertainty per se.}\]

\[\text{There is a small additional effect through the correlation between the two "diffusions." Empirically, however, it is difficult to justify a sizeable correlation coefficient.}\]
produce more significant changes in option values than in the "standard" model used by Courtadon [(1982a)] and Ramaswamy and Sundaresan (1985).

We consider first a "standard" model of interest rate uncertainty where the impact on option valuation is through the correlation in the stochastic terms of the diffusion processes. In this case, there is no impact of interest rate uncertainty on either the level, or volatility, of the stock index.

\[
\begin{align*}
\text{d}S &= (\alpha S - D)\text{d}t + \sigma_S S\text{d}z_S \\
\text{d}r &= k(\mu - r)\text{d}t + \sigma_r r\text{d}z_r \\
\text{d}z_S \text{d}z_r &= \rho \text{d}t
\end{align*}
\]

(5)

where $S$ is the stock index value, $D$ is the dividend, $\sigma_S^2$ is the instantaneous variance of the rate of change in $S$, $\alpha$ is the instantaneous expected rate of change, $r$ is the rate of interest, $\mu$ is the long-run mean of the rate of interest, $k$ is the speed of adjustment of $r$ to $\mu$, $\sigma_r^2$ is the variance of the rate of change in $r$, $\text{d}z_S$ and $\text{d}z_r$ are standard Wiener processes, and $\rho$ is the correlation between $\text{d}z_S$ and $\text{d}z_r$.

The two state variables in this model are the index value and the interest rate. In this model, the effect of $\sigma_r$ on the

16 Theory and empirical evidence can better support a relationship between stock market volatility and interest rate volatility for the market as a whole than for individual stocks. See Copeland and Stapleton (1985) for a discussion of the theory.

17 This "standard" model is essentially the same model as in Brennan and Schwartz (1980) and Courtadon [(1982a)]. It is similar in spirit to the approach in Ramaswamy and Sundaresan (1985), except that they use a "square root" process for interest rates.
option price is transmitted through $r$ - the volatility affects the change in $r$ and hence the level of $r$. However, since $r$ has a relatively small effect on option values, in most models, changes in $r$ caused by $\sigma_r$ will have a marginal effect on the option price. This can be seen in Tables 3A and 3B, where values of calls and puts derived from the above "standard model" are presented. These values are generated using the numerical procedure described in Courtadon (1982a). For a given set of parameter values, we compare the option values using a $\sigma_r$ of 10 percent and 15 percent respectively. There is almost no impact on the values of call and put options as a result of this shift in interest rate volatility. This observation applies to European options as well as American options. If we would like to concentrate on the effect of $\sigma_r$ on valuation, apart from the effect on early exercise, we could compare the values of out-of-the-money options, where the options values approach European values. This lack of effect of $\sigma_r$ on option values is true for a wide range of deviations of $r$ from $\mu$ as is evident from Tables 3A and 3B.

Since this model shows almost no effect on the option values, it is inconceivable that we would find any effect on early exercise policy. Indeed, as can be observed in Tables 4A and 4B, the early exercise boundaries for $\sigma_r = 10\%$ and $\sigma_r = 15\%$ are almost identical. The boundaries are monotonic in the rate of interest, but are virtually unaffected by the shift in $\sigma_r$.

\[ \text{For the case of } \mu = r \text{ and } \sigma_r = 0 \text{ this model reduces to the Black-Scholes model.} \]
It is obvious from the above discussion that the previous models have missed an essential aspect of interest rate uncertainty - its effect on the stock volatility parameter. Following our earlier discussion about the effect of interest rate volatility on stock market volatility, we propose the following model:

\[
dS = (\alpha S - D)dt + (a_1 + a_2 \sigma_r)Sdz_S \\
dr = (\mu - r)dt + \sigma_r rdz_r \\
dz_S dz_r = \rho dt
\]  

(6)

In the above model, the stock volatility has two components; the stock specific component \(a_1\), the volatility that is not affected by \(\sigma_r\), and the interest rate volatility component which affects \(\sigma_S\) through the multiplier \(a_2\). This formulation is consistent with two possible scenarios. First, for a given level of the interest rate, the higher the \(\sigma_r\), the higher is \(\sigma_S\). Since all financial markets are related, high volatility in one market must affect the volatility in other markets due to a "substitution effect." Observe, for example, foreign exchange markets and the bond market, or the bond market and the stock market. Second, for a given \(\sigma_r\), an increase in \(r\) should increase \(\sigma_S\). A plausible explanation may be that the increased interest rate reflects increased inflation expectations. In turn, this increases the uncertainty of inflation, due to problems in nominal contracting, which gets reflected in the uncertainty of nominal interest rates.
Another way to explain these relationships is to use a duration-type model for stock prices. In a manner similar to bonds, the "duration" measures the sensitivity of the relative change in the stock index to the relative changes in the level of interest rates. Thus, the uncertainty in the interest rate directly affects stock price volatility.

Based on the above specification in an arbitrage-free market, the valuation equation for a call option is given by:

\[
\frac{1}{2} \sigma_S^2 S^2 C_{SS} + \frac{1}{2} \sigma_r^2 \sigma^2 C_{rr} + C_S \rho \sigma_S \sigma_r \\
+ C_r \kappa (\mu - r) + C_S (\alpha S - D) - r C + C_t = 0
\]  

(7)

where

\[
\sigma_S^2 = (\sigma_1 + \sigma_2 \sigma_r)^2
\]

In the case of an American call option, the boundary conditions to be imposed on (7) are:

\[
C(S, r, t) = \text{Max} \ [0, S-X] \\
C(0, r, t) = 0 \\
C_S(\infty, r, t) = 1 \\
C(S, \infty, t) = S \\
C(S, r, t) = \text{Max} \ [C(S, r, t), S-X]
\]  

(8)
In the case of an American Put the boundary conditions to be imposed on (7) are:

\[ C(S, r, t) = \text{Max} \left[ 0, X - S \right] \]

\[ C(0, r, t) = X \]

\[ C(\infty, r, t) = 0 \]

\[ C(S, \infty, t) = \text{Max} \left[ 0, X - S \right] \]

\[ C(S, r, t) = \text{Max} \left[ C(S, r, t), X - S \right] \]

(9)

We now consider the significance of the effect of interest rate uncertainty in our new model. The following parameter values were used in our computations: \( a_1 = 0.10; a_2 = 4; \mu = 0.08; \) \( k = 0.5, d = 0.05. \) Again, using the numerical procedure described in Courtadon (1982a) we obtained the values for American call and put options for different values of the \( \sigma_r \) parameter. These values are tabulated in tables 5A and 5B. The differences are rather sizeable.

For example, for an out-of-the-money call with 6 months to maturity, when \( r = .10, \) the difference in values between the case where \( \sigma_r = .10 \) and \( \sigma_r = .15 \) is about 38 percent (\$1.34 vs. \$ .97). The magnitude of the difference gets smaller as the option gets...

\[ \text{-----------} \]

\[ \text{-----------} \]

\[ \text{-----------} \]

\[ \text{-----------} \]

The values of \( a_1 \) and \( a_2 \) were set to be approximately consistent with a stock index volatility of 10--15 percent, in the absence of interest rate uncertainty.
more into-the-money, since the effect of volatility gets smaller. For at-the-money call options, the difference is still about 11 percent while for in-the-money call options, the difference is about 3 percent. For put options the differences are even larger. For out-of-the-money puts, it is as large as 83 percent, while for at-the-money puts it is not less than 10 percent.

For out-of-the-money and in-the-money options, the difference in values increases with \( r \) up to a certain value and then the effect of \( r \) becomes so dominant that it swamps the effect of \( \sigma_r \). If we compare the effect of \( \sigma_r \) on a near-maturity option versus a long maturity option, we find that the near-maturity, at-the-money options show a bigger impact of \( \sigma_r \) than long maturity at-the-money options. The reason is that the closer they get to maturity, the more sensitive are the options to changes in volatility.

Tables 6A and 6B show the differences in exercise policies for calls and puts based on the model developed here. For calls, we see that early exercise is triggered at a lower \( r \) when \( \sigma_r \) is 10 percent compared with a \( \sigma_r \) of 15 percent. This is consistent with the fact that a higher \( \sigma_r \) means higher \( \sigma_S \), which, in turn, means a smaller probability of exercise. For puts, however, a higher \( r \) means a higher probability of exercise and, therefore, for lower \( \sigma_r \), early exercise is triggered at a higher price than in the case of a higher \( \sigma_r \). However, at a higher \( \sigma_r \), we observe that the exercise policy can become a decreasing function of \( r \). This is due to the fact that the component of
stock market volatility due to the interest rate uncertainty increases as a function of $r$ and $\sigma_r$ and swamps the pure time value effect of interest rates. This reversal is a function of time to maturity - the closer is the maturity date, the less likely it would be to find such a reversal.

To summarize, in a "standard" model, as in (5), where interest rate uncertainty does not affect stock market volatility directly, the effect of this uncertainty on option values is practically zero. In contrast, in equation (6) where interest rate uncertainty is related to stock market uncertainty, the effect on option values is strong and may result in sizeable differences in option values. It should be emphasized, however, that our model is, by no means, the only way to model the direct impact of interest rate uncertainty on stock market volatility. This is just one plausible case where the impact of interest rate uncertainty on option values is more substantial than was previously supposed. This issue is of sufficient importance, in the valuation of options in general, to warrant greater attention in further research.
Table 3A

Values of American Call Options Using the "Standard" Interest Rate Volatility Model

<table>
<thead>
<tr>
<th>$t=0.5$</th>
<th>$a_1 = 0.15$</th>
<th>$a_2 = 0.0$</th>
<th>$\mu = 0.08$</th>
<th>$k = 0.5$</th>
<th>$d = 0.05$</th>
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</thead>
<tbody>
<tr>
<td>$r$</td>
<td>.05</td>
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<td>.08</td>
<td>.10</td>
</tr>
<tr>
<td>$S/X$</td>
<td>$a_r$</td>
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<td>.15</td>
<td>.10</td>
<td>.15</td>
</tr>
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<td>.0101</td>
<td>.0101</td>
<td>.0117</td>
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<td>.0414</td>
<td>.0480</td>
<td>.0480</td>
<td>.0527</td>
</tr>
<tr>
<td>1.10</td>
<td>.1117</td>
<td>.1117</td>
<td>.1224</td>
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<td>.1297</td>
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</table>

Table 3B

Values of American Put Options Using the "Standard" Interest Rate Volatility Model

<table>
<thead>
<tr>
<th>$t=0.5$</th>
<th>$a_1 = 0.15$</th>
<th>$a_2 = 0.0$</th>
<th>$\mu = 0.08$</th>
<th>$k = 0.5$</th>
<th>$d = 0.05$</th>
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</thead>
<tbody>
<tr>
<td>$r$</td>
<td>.05</td>
<td>.05</td>
<td>.08</td>
<td>.08</td>
<td>.10</td>
</tr>
<tr>
<td>$S/X$</td>
<td>$a_r$</td>
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<td>.15</td>
<td>.10</td>
<td>.15</td>
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<td>.0345</td>
<td>.0316</td>
</tr>
<tr>
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<td>.0100</td>
<td>.0100</td>
<td>.0079</td>
<td>.0079</td>
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Table 4A

Optimal Exercise Boundaries for Call Options Using the "Standard" Interest Rate Volatility Model

<table>
<thead>
<tr>
<th>t</th>
<th>a₁</th>
<th>a₂</th>
<th>μ</th>
<th>k</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.5</td>
<td>.15</td>
<td>.0</td>
<td>.0</td>
<td>.05</td>
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</table>

<table>
<thead>
<tr>
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<th>.045</th>
<th>.05</th>
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</thead>
<tbody>
<tr>
<td>σ_₀ₙₑ₀ₜ</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>.10</td>
<td>1.15</td>
<td>1.20</td>
<td>1.25</td>
<td>1.35</td>
<td>N.E.</td>
</tr>
<tr>
<td>.15</td>
<td>1.15</td>
<td>1.20</td>
<td>1.25</td>
<td>1.40</td>
<td>N.E.</td>
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</tbody>
</table>

Table 4B

Optimal Exercise Boundaries for Put Options Using the "Standard" Interest Rate Volatility Model

<table>
<thead>
<tr>
<th>t</th>
<th>a₁</th>
<th>a₂</th>
<th>μ</th>
<th>k</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.5</td>
<td>.15</td>
<td>.0</td>
<td>.0</td>
<td>.05</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>r</th>
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<th>.05</th>
<th>.055</th>
<th>.065</th>
<th>.10</th>
<th>.175</th>
<th>.020</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ_₀ₙₑ₀ₜ</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>.10</td>
<td>.00</td>
<td>.75</td>
<td>.80</td>
<td>.85</td>
<td>.90</td>
<td>.95</td>
<td>1.00</td>
</tr>
<tr>
<td>.15</td>
<td>.00</td>
<td>.70</td>
<td>.80</td>
<td>.85</td>
<td>.90</td>
<td>.95</td>
<td>1.00</td>
</tr>
</tbody>
</table>

33
### Table 5A

**Values of American Call Options Based on a New Interest Rate Volatility Model**

\[
\begin{array}{cccccccc}
  t=0.5 & a_1 = 0.10 & a_2 = 4.0 & \mu = 0.08 & k = 0.5 & d = 0.05 \\
  r & .05 & .05 & .08 & .08 & .10 & .10 & .15 & .15 \\
  S/X & \sigma_r & .10 & .15 & .10 & .15 & .10 & .15 & .15 \\
  0.90 & .0040 & .0053 & .0071 & .0097 & .0097 & .0134 & .0181 & .0245 \\
  1.00 & .0331 & .0362 & .0429 & .0475 & .0497 & .0551 & .0673 & .0747 \\
  1.10 & .1068 & .1085 & .1197 & .1221 & .1282 & .1311 & .1491 & .1532 \\
\end{array}
\]

### Table 5B

**Values of American Put Options Based on a New Interest Rate Volatility Model**

\[
\begin{array}{cccccccc}
  t=0.5 & a_1 = 0.10 & a_2 = 4.0 & \mu = 0.08 & k = 0.5 & d = 0.05 \\
  r & .05 & .05 & .08 & .08 & .10 & .10 & .15 & .15 \\
  S/X & \sigma_r & .10 & .15 & .10 & .15 & .10 & .15 & .15 \\
  0.90 & .1031 & .1043 & .1000 & .1008 & .1000 & .1000 & .1000 & .1000 \\
  1.00 & .0315 & .0347 & .0294 & .0340 & .0285 & .0340 & .0271 & .0349 \\
  1.10 & .0052 & .0069 & .0051 & .0076 & .0051 & .0081 & .0053 & .0097 \\
\end{array}
\]
Table 6A
Optimal Exercise Boundaries for Call Options Using a New Interest Rate Volatility Model

\[ t = 0.5 \quad a_1 = 0.10 \quad a_2 = 4 \quad \mu = 0.08 \quad k = 0.5 \quad d = 0.05 \]

\[
\begin{array}{ccccccc}
  r & 0.01 & 0.025 & 0.03 & 0.04 & 0.045 & 0.05 \\
  \sigma_r & 0.10 & 1.10 & 1.10 & 1.15 & 1.20 & 1.25 & \text{N.E.} \\
  & 0.15 & 1.10 & 1.15 & 1.15 & 1.20 & 1.30 & \text{N.E.} \\
\end{array}
\]

Table 6B
Optimal Exercise Boundaries for Put Options Using A New Interest Rate Volatility Model

\[ t = 0.5 \quad a_1 = 0.10 \quad a_2 = 4 \quad \mu = 0.08 \quad k = 0.5 \quad d = 0.05 \]

\[
\begin{array}{ccccccc}
  r & 0.04 & 0.05 & 0.06 & 0.08 & 0.10 & 0.40 \\
  \sigma_r & 0.10 & 0.00 & 0.80 & 0.85 & 0.85 & 0.90 & 0.90 \\
  & 0.15 & 0.00 & 0.75 & 0.85 & 0.90 & 0.90 & 0.85 \\
\end{array}
\]
IV. **Basis Risk and the Valuation of Stock Index Options**

In a market where there are no taxes, transactions costs (including bid-ask spreads) or restrictions on short sales, the price of a futures (or a forward) contract can be determined by arbitrage considerations. If the risk-free interest rate is non-stochastic, and, in particular, constant, the price of the futures contract equals the price of a forward contract. If there are no interim distributions, such as dividends paid to holders of the spot instrument, the price of the futures (and the forward) contract, $F$, can be written as the compounded value of the spot price $S$:

$$F = S e^{r(T-t)} \quad (10)$$

where $T$ is the maturity date of the futures contract, $t$ is the current calendar time and $r$ is the risk-free interest rate, which is assumed to be constant. This follows by arbitrage arguments - buying the spot instrument and selling the futures contract initially, and reversing the position by delivering the spot instrument into the futures contract at maturity should earn the risk-free rate of interest over the remaining maturity of the contract.

---


21 Alternatively, the argument can be stated in terms of first order stochastic dominance.
If there are interim distributions such as dividends, they must be taken into account as part of the return to holding the spot. Assume, as in Section II, that there are $N$ dividend payments between $t$ and $T$, $D_i$ is the dollar amount of the $i$th dividend and $T_i$ is the payment date, which, for convenience, is the same as the ex-dividend date. In this case, by the same arbitrage argument, the futures price is lower than in the previous case, to the extent of the distributions to holders of the spot that are unavailable to the buyers of the futures contract. Thus:

$$F = Se^{r(T-t)} - \sum_{i=1}^{N} D_i e^{r(T-T_i)}$$ (11)

In practice, however, the theoretical relationship between the stock index futures price and the spot index prices is violated. The difference between the spot and futures prices, also known as the basis, is not simply due to the force of interest over the remaining maturity of the futures contract and the dividends. There seem to be considerable deviations of the actual basis from the theoretical basis described in equations (10) and (11). Basis risk, or the risk of such deviations, seems to be particularly high for commodities, where the cost of storage, as well as the convenience of owning the spot commodity are large\(^{22}\). Among financial assets, which have no convenience yield or cost of storage,

\(^{22}\) See Brennan (1986) for an empirical analysis of convenience yields for commodities.
one would expect basis risk to be small. This is indeed the case for highly liquid instruments such as Treasury bills. However, relatively large departures have been noted between the stock index spot and futures prices relative to the theoretical relationship. See Figures 3A and 3B for a plot of the deviation between the futures and the forward price (the theoretical price) for the SP500 index and MMI respectively. Not surprisingly, the deviations are much larger for the broader based SP500 index.24

Given the general efficiency of stock markets, the observed departures are too large to be accounted for merely by market "inefficiencies." Several alternative explanations have been proposed to account for the deviations from the theoretical relationship:

1. The importance of interest rate risk and marking-to-the-market.
2. The difficulties involved in simultaneously buying or selling the large number of component stocks in an index.
3. The costs and margin requirements associated with short sales.
4. Non-trading effects in one or more of the constituent stocks. The contribution of an individual stock to the current level of the spot index is based on its last

23 However, even in this case, systematic, although numerically small deviations have been noted. See, for example, Elton, Gruber and Rentzler (1984) for such evidence.

24 In 1984, the percentage premiums, (futures-implied forward)/implied forward, were as large as 2.5 percent for the SP500 index, but only about .1 percent for the MMI.
traded price. If the stock has not traded for several minutes and the market has moved in the interim, the spot index is likely to be an erroneous indicator of the (unobserved) true spot index.

In principle, marking-to-the-market may cause a divergence between the forward price and the futures price in a world of interest rate uncertainty. The sign of the difference depends on the covariance between interest rates and the spot index. Since the theoretical relationship in equation (10) assumes no interest rate risk, the observed discrepancies could be due to marking-to-the-market. However, evidence from other markets suggests that this risk is not very significant. Even in the Treasury-bill market, where the departures from the theoretical relationship are relatively small, Elton, Gruber and Renzler (1984) find that this risk is insufficient to explain the observed irregularities in prices. Given the relatively large magnitude of the deviation in the case of stock indices, interest rate risk and its impact on marking-to-the-market are unlikely to explain the observed discrepancies.

The second factor, the logistical problem of buying and selling a large number of stocks simultaneously, may be important, especially for the broader market indices such as the Standard and Poor's 500 (SP500) Index or the New York Stock Exchange (NYSE) Index. It is less likely to be a problem for a narrowly defined index such as the Major Market Index (XMI).

See, for example, Cox, Ingersoll and Ross (1981) for a discussion.
The third factor, the difficulties and costs associated with short sales, may cause the spot index to be "too high" in relation to the futures and yet not be arbitrated away. Furthermore, according to New York Stock Exchange rules, short sales are permitted only on an "up-tick," making a continuously hedged position difficult to achieve. Again, this is likely to present more of a problem for the broader market indices. If this would have been the dominant factor we should find more cases of discounts than premiums which is not consistent with observed data.

The fourth reason, non-trading effects, could be significant in the context of volatile markets when the market has moved substantially in a short interval of time, and an individual stock (or stocks) has not traded in the interim. However, this argument is not sufficient to explain the persistent departures, sometimes lasting days, that have been observed in the stock index spot-futures markets. The last and residual factor is market "inefficiency." This is a "catch-all" explanation and may include the effects of transactions costs, taxes and constraints on trading which have been ignored in the discussion here.

The divergence of the basis from its theoretical value has an impact on the pricing of stock index options. If the divergence is large and persistent, options on the spot may be priced off the futures value rather than the contemporaneous spot index. This is valid for two reasons. The first is that the futures price may be a better indication of where the spot would have been in the absence of the frictions discussed above. The
second is that, since it is impossible to trade the spot index directly, it is far less cumbersome and costly to hedge an options position with stock index futures contracts than with the component stocks of the index. Since most option pricing models are crucially dependent on the construction and maintenance of a riskless hedge, it may be argued that the relevant variable in the valuation of stock index options is the contemporaneous futures price rather than the level of the spot index.

This distinction is unimportant if there were a simple functional relationship between the spot index and the futures price at each point in time. An obvious example is a case where the deviation from the theoretical relationship follows some deterministic path. Even a casual examination of the data suggests that this is not a reasonable approximation. [See Figures 3A and 3B].

We now proceed to value contingent claims based on the assumption that the basis is not at its theoretical level, i.e.\[ B = F - S = S[e^{r(T-t)} - 1] \] \hspace{1cm} (12a)

Alternatively, the "modified basis" is defined as follows:

\[ b = F - S e^{r(T-t)} \neq 0 \] \hspace{1cm} (12b)

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26 See Brenner, Courtadon and Subrahmanyam (1985) for an analysis of the case where the basis is at its theoretical level. European call and put options on the futures contract are worth the same as similar options on the spot, if the options expire when the futures contract matures. This is true whether or not the basis is at the theoretical level, since the basis goes to zero at maturity. In the case of American options, this argument is no longer valid, since the optimal exercise decision for options on the futures contract is, in general, different from that for options on the spot index.
Since the futures price is the relevant variable from a hedger's perspective, and hence, for option valuation, we start by making assumptions about the stochastic process driving the futures price:

\[ dF = \mu F dt + \sigma F dz \]  

(13a)

In addition, we assume that the "modified basis" is not at its theoretical (deterministic) level, but follows a Brownian bridge process:

\[ db = \frac{b}{T-t} dt + \sigma^* dz^* \]  

(13b)

\[ dz^* dz = \rho dt \]

where \( b \) is the modified basis, \( db \) is the change in the modified basis, \( \sigma^* \) is the volatility parameter of the modified basis, \( dz^* \) is a Wiener process and \( \rho \) is the coefficient of correlation between the two Wiener processes.

Note that this process for the "modified basis" allows for departures between the forward price and the futures price at all points in time except at maturity. At maturity, the process forces an equality between the forward (and hence the spot) and the futures prices. This process is derived by imposing a restriction on the Wiener process formulation for the "modified basis." The restriction is that the modified basis is identically zero at maturity, when \( t = T \).

The restriction imposed by the boundary condition, implies that the drift term in the Brownian bridge process is dependent

See Ball and Torous (1983) for a discussion on the Brownian bridge process in the context of European options on default-free bonds.
on how far the forward prices deviates from the futures price. In other words, the process exerts a "pull" on the "modified basis" to bring it back in line with the boundary condition. The greater the discrepancy, the greater the drift term exerting the "pull." The shorter the time to maturity the greater the "pull."

Notice, however, that the deviation from the boundary condition has no implications for the instantaneous standard deviation of the process which is state independent.

If the futures contract and the "modified basis" were both traded, option prices would not be affected by the terminal constraint imposed by the Brownian bridge process once the volatility parameter, \( \sigma^* \), is specified. The reason is that if a continuous hedge were maintained, the value of a contingent claim would depend only on the instantaneous standard deviations, \( \sigma \) and \( \sigma^* \), and the correlation coefficient, \( \rho \). Since, in this case, the volatility parameter is

28

See equation (7) in Ball and Torous (1983). Notice that while the drift terms are dependent on time and the price of the underlying asset, the volatility parameter is constant.
state independent, option values would be unaffected.

The imposition of the terminal constraint through the Brownian bridge process may affect the valuation of contingent claims in approaches that do not rely exclusively on arbitrage principles. In particular, the degree of risk aversion or equivalently, the market price of basis risk may play a role in the valuation model in a preference-based approach.  

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The same argument applies to the Ball and Torous (1983) model of pricing of default-free bonds. The imposition of the constraint that the price of a default-free bond equals unity, modifies the deterministic term in the Brownian bridge process, characterizing the log of the bond price. However, the stochastic term remains state independent. As a result, bond option values are no different from a case where this constraint is not imposed. This is so because, in an arbitrage-based model, the drift term plays no role in option pricing. On the other hand, the parameter defining the stochastic term, which remains state independent, is the only one characterizing the bond price process which is relevant to the arbitrage-based pricing of contingent claims, such as options on bonds. (It should be noted, however, the drift term affects the valuation of the underlying asset i.e. the bond). Option values are unaffected by the terminal boundary condition imposed by the Brownian bridge process in the Ball and Torous (1983) model.

This problem is analogous to the problem in the Brennan and Schwartz [(1979) and (1982)] model of bond (and contingent claims) valuation in a market with two state variables, the short term interest rate and the long term interest rate. In their model, it is assumed that it is possible to hedge perfectly against changes in the long term interest rate, but not against changes in the short term interest rate. This leads to a valuation model where it is necessary to estimate a market price of risk term.
The first step in developing a valuation model is to derive the stochastic process for the spot index which is assumed not to be directly tradeable. This "synthetic spot" index, which is tradeable via the futures contract, represents the value of the spot if it had been traded in a frictionless market.

\[ f = F e^{r(T-t)} \]  \hspace{1cm} (14)

Applying Ito's Lemma to equations (13a) and (14), the stochastic process for the "synthetic spot" can be written as:

\[ df = (\mu+r)fdt + \sigma fdz \]
\[ = \mu^{*}fdt + \sigma fdz \]  \hspace{1cm} (15)

The value of any contingent claim on the "synthetic spot" can be written by applying Ito's Lemma to equations (13b) and (15). The valuation equation for the claim \( C(f, b) \) can be written as:

\[ \frac{dC}{C} = \mu_{C}dt + \frac{\sigma f}{C} C_{f}dz + \frac{\sigma^{*}}{C} C_{b}dz^{*} \]  \hspace{1cm} (16)

where

\[ \mu_{C} = \frac{1}{C} [ \frac{1}{2} \sigma^{2} f^{2} C_{ff} + \frac{1}{2} \sigma^{2} b^{2} C_{bb} + \rho \sigma^{*} \sigma f C_{fb} + \mu^{*} f C_{f} + (\frac{b}{T-t}) C_{b} + C_{t} ] \]  \hspace{1cm} (17)

Since the instantaneous rates of return on all claims in this market are linear combinations of the two stochastic increments, \( dz \) and \( dz^{*} \), in order to avoid arbitrage, we must have:

\[ \mu_{C} - r = \frac{\lambda_{1}}{C} \sigma f + \frac{\lambda_{2}}{C} \sigma^{*} \]  \hspace{1cm} (18)

\[ 31 \]

The argument is similar to the one used by Brennan and Schwartz [(1979) and (1982)], in the pricing of bonds.
where $\lambda_1$ and $\lambda_2$ are, respectively the market prices of the spot index and basis risk respectively, and are the same for all claims, including contingent claims in this market.

In particular, the "synthetic spot" which is tradeable via the futures contract, satisfies the same relationship except that it is not affected by basis risk.

Thus,

$$\mu^* - r = \lambda_1/f . 1.\sigma_f$$

(19)

In other words:

$$\lambda_1 = (\mu^*-r)/\sigma = \mu/\sigma$$

(20)

given that $\mu^* = \mu + r$. If we now substitute equation (20) in equation (18), we have:

$$\mu_C - r = \mu/C . C_f f + \lambda_2/C C_b \sigma^*$$

(21)

Finally, by replacing the $\mu_C$ by its value in equation (17) we obtain the valuation equation from (21) for the index option:

$$\frac{1}{2} \sigma^2 f^2 C_{ff} + \frac{1}{2} \sigma^2 \sigma^2 C_{bb} + \rho \sigma^2 \sigma f C_{fb}$$

$$+ \left[ (b/T-t) - \lambda_2 \sigma^* \right] C_b + rfC_f - rC + C_t = 0$$

(22)
The valuation equation can also be rewritten as a function of the futures price given equation (14). If \( U(F, b, t) = C(f, b, t) \), then the value of any contingent claim and, in particular, the index option as a function of the futures price, the basis and time will satisfy:

\[
\frac{1}{2} \sigma_F^2 U'_{FF} + \frac{1}{2} \sigma^2 U'_{bb} + \rho \sigma \sigma U'_{Fb} \\
+ [(b/T-t) - \lambda \sigma^2] U_b - ru + UT = 0
\]  

(23)

The value of any option as a function of the future price and the basis may be determined by solving this valuation equation subject to the appropriate boundary conditions. In particular, it must be recognized that a negative "modified basis" (forward price greater than the futures), will push the holder of a call option to exercise the option earlier to the extent that the early exercise will allow him to "capture" the difference between the forward and the future. The argument is reversed for a put option. In the case of the American call option, we have:

\[
U(F, b, t) = \text{Max} [U^*(F, b, t), (F-b)e^{-r(T-t)} - X]
\]  

(24)

where \( U^* (F, b, t) \) is the value of the American call option if kept "alive" until the next instant of time and \( X \) is the exercise price. Similarly, if \( U(F,b,t) \) represents the value of an

32 The spot index value is equal to \((F-b)e^{-r(T-t)}\), given our definition of the "modified basis" in equation (12b).
American put option, it will satisfy the previous valuation operation subject to the additional time boundary condition:

\[ U(F, b, t) = \max \left[ U^*(F, b, t), X - (F-b)e^{-r(T-t)} \right] \]  \hspace{1cm} (25)

where \( U^*(F,b,t) \) is the value of the American put, option if kept "alive". In other words, the investor will have a higher incentive to exercise the put option early if the futures contract is worth more than the forward, or equivalently, if the "modified basis" is positive.

The solution technique necessary to solve the valuation problem in equation (23) subject to the appropriate boundary conditions, for American puts and calls would be similar to that used in the previous section of the paper in the case of interest rate uncertainty. We will not undertake this numerical solution here and will leave the actual computations of the valuation model to future research.
S&P FUTURES PREMIUM/DISCOUNT

FIGURE 3A
V. Concluding Remarks

There are several aspects of stock index options that distinguish them from options on individual stocks. These differences may have implications for the valuation of stock index options that require modifications of the basic Black-Scholes (1973) framework. The first issue deals with how the dividend stream on the stock index departs from the discrete, quarterly dividend payment pattern for individual stocks, on the one hand, and the limiting case of a continuous dividend stream, on the other. Our analysis shows that the dividend stream on a broad-based index like the NYSE index is more or less continuous. Even the small departures we detect do not seem to affect option values which are virtually identical to those obtained by assuming a continuous stream of dividends. For a narrow index such as the MMI, the dividend stream is discontinuous and does have an impact on option values, relative to the case of continuous dividends.

The second issue is the impact of interest rate uncertainty on the stochastic process driving the level of the stock index. We first examine previous models in the literature where interest rate uncertainty has only an indirect and hence, small, effect on option values through a change in the level of interest rates. We contrast this with a model where interest rate volatility affects stock index volatility directly and, therefore, has a substantial impact on option values.

The final issue we deal with is the effect of basis risk in a market where the stock index futures contract used to hedge the
stock index option. If the relationship between the stock index futures price and the spot is not at the theoretical level, and, in particular, stochastic, an additional risk is introduced into the problem. We model basis risk on the assumption that it cannot be hedged away and develop a valuation equation with a preference-based parameter. This valuation equation with appropriate boundary conditions would yield values for put and call options.

A specific aspect of the institutional arrangements for trading stock index options is the issue of cash settlement. Unlike options on an individual stock that are settled with the underlying asset, stock index options are settled with cash. Hence, the sellers of stock index options assume the risk of early cash settlement. For example, if a writer of a deep-in-the-money call attempts to hedge himself with the component stocks of the index, he runs the risk of paying the cash value of the index, when exercised, and is then exposed to the risk of the market moving against him. This problem is particularly acute, with delays in exercise notification, when a big market movement takes place in the interim. Thus, the problem of basis risk and its implications for option valuation are further accentuated by the cash settlement procedure. We leave a more detailed examination of this question to future research.
REFERENCES


