

ASSET PRICES AND THE LEVEL OF BACKGROUND RISK

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1 Introduction

Although an uncountable number of papers has been written about pricing of financial assets, no asset pricing model so far has gained substantial empirical support. Instead, many so called anomalies of asset pricing have been found. The Fama-French-factors size, dividend yield and book value over market value seem to explain a substantial part of the variation in average returns of stocks [Davis, Fama and French (2000)]. A sound theoretical explanation of the role of these factors in asset pricing is not yet available. The pricing of financial derivatives has a much stronger foundation in arbitrage theory. Yet observed options prices differ in a systematic manner from the prices derived from the Black-Scholes model. Various explanations have been given for these discrepancies. Most of these explanations relate to the deviations of the exogenously given price process of the underlying from a standard geometric Brownian motion assumed by Black and Scholes.

An alternative way of pricing financial assets starts from an equilibrium model by which prices of assets are determined endogenously. If the capital market is perfect and complete, then all prices for state-contingent claims are derived endogenously. Hence all financial assets can be uniquely priced. The prices for state-contingent claims, divided by the state probability densities, reflect the risk aversion of agents. If agents are risk neutral, then all these probability-deflated prices are the same. Otherwise these prices usually are a declining function of aggregate wealth reflecting declining marginal utility. This function is called the pricing kernel. Although there is widespread agreement that the pricing kernel is declining, there is little knowledge about the precise shape of the pricing kernel. This shape is, however, critical for asset pricing, in particular for the pricing of derivatives. As has been shown by Franke/Stapleton/Subrahmanyam [Franke, Stapleton, Subrahmanyam (1999)], the Black-Scholes model of option pricing requires that the pricing kernel has constant elasticity, i.e. the elasticity of the probability-deflated price for state-contingent claims with respect to aggregate wealth must be constant. If elasticity is declining, however, then all European options on the market portfolio will be more expensive given the price of the market portfolio, i.e. the portfolio of all tradable claims on aggregate wealth.

The purpose of this paper is to investigate the shape of the pricing kernel in an economy in which agents have utility functions with hyperbolic absolute risk aversion (HARA). In the simplest case, all agents buy or sell the risk-free asset and buy a constant fraction of the market portfolio. Their sharing rules which relate their future portfolio wealth to future aggregate wealth are linear [Rubinstein (1974)]. Then the equilibrium pricing kernel is easy to derive. The equilibrium is more complicated if some agents bear background risk, i.e. risk which cannot be hedged. Background risk is created by family risks, by labor income risk etc. The question is how background risk affects the willingness of an agent to bear tradable risks. Most economists believe that background risk makes an agent more risk-averse to tradable risks. Kimball (1990) strongly makes this point. In his (1993)-paper he argues that agents are standard risk-averse. This means that the burden of a tradable

risk is aggravated by background risk; moreover, this burden is higher, the poorer the agent is. Kimball shows that standard risk-aversion is equivalent to a utility function with positive, declining absolute risk aversion and positive, declining absolute prudence. Absolute prudence is defined by the third over the negative second derivative of the utility function. Franke/Stapleton/Subrahmanyam (2000) show that a standard risk-averse agent reacts to an increase in his background risk by buying less tradable risk such that the slope of his sharing rule of aggregate wealth becomes smaller everywhere.

This paper considers HARA-utility agents who are standard risk averse. Hence, if background risk increases for many agents, the equilibrium pricing kernel for tradable risks changes. Those agents whose background risk increases want to buy a larger fraction of aggregate tradable wealth when it is low (= a “low” state) and a smaller fraction when it is high (= a “high” state) to balance their budget. Since aggregate tradable wealth is distributed exogenously, the higher demand for claims in the “low” states cannot be satisfied at the old prices for state-contingent claims. Therefore prices go up in the “low” state and down in the “high” states. Also the elasticity of the pricing kernel will be shown to increase indicating higher “relative risk aversion of the market”. As an implication, the market portfolio of all tradable claims on aggregate wealth becomes cheaper; also all European calls on the market portfolio will be shown to become cheaper while all puts become more expensive. Hence if background risk of many agents increases, then assets which tend to pay off in the “low” states become more expensive while assets which tend to pay off in the “high” states become cheaper.

It is common to price financial derivatives relative to the price of the underlyings. Hence the paper also addresses the impact of background risk on relative pricing. Suppose you observe the price of the underlying. But you do not know the pricing kernel. The pricing kernel is determined either by an equilibrium without background risk or by an equilibrium with background risk. In both cases, agents have the same HARA-utility functions. The price of the market portfolio is also the same in both cases if higher risk aversion of the agents’ utility function is offset by a higher level of background risk. It turns out the pricing kernel with background is more convex than the pricing kernel without background risk. Therefore all European options are more expensive. Any convex sharing rule on tradable aggregate wealth is more expensive relative to the market portfolio in the equilibrium with background risk. If, for example, the Black-Scholes model prices all options correctly in the equilibrium without background risk, then it underprices all options in the equilibrium with background risk.

Concerning the previous work on background risk, Nachman (1982), Kihlström et.al. (1981) and Ross (1981) were the first to discuss it. Eeckhoudt and Kimball (1992) and Meyer and Meyer (1998) discuss the impact of background risk on the demand for insurance. Eeckhoudt, Gollier and Schlesinger (1996) discuss the impact of increases in background risk on an agents absolute risk aversion as do Franke/Stapleton/Subrahmanyam (2000).

For this paper more important is the equilibrium analysis of Weil (1992). He analyses the impact of background risk on the equity premium, i.e. the difference between the expected return on the market portfolio and the risk-free rate. He shows for standard risk averse agents that background risk raises their demand for the risk-free asset and, hence, depresses the risk-free interest rate. They also reduce their demand for the market portfolio so that its price declines and the expected rate of return increases. Hence background risk raises the equity premium.

This paper does not consider the risk-free rate. Instead forward prices of assets are analysed, i.e. spot prices compounded at the risk-free rate. We concentrate on the pricing of state-contingent claims so that any financial asset can be valued.

The paper is organised as follows. In section 2 the setup of the paper and the equilibrium are derived. In section 3 the impact of an increase in background risk on the absolute prices on financial assets is analysed, in section 4 on the relative prices of financial assets. Section 5 concludes the paper.

2 Derivation of Equilibrium

2.1 The Setup

We use the same setup as in Franke/Stapleton/Subrahmanyam (1998).

We assume a two-date, pure-exchange economy, where the dates are indexed 0 and 1. There are I agents, $i = 1, 2, \dots, I$, in the economy. X is the time 1 measurable payoff on the market portfolio, i.e. the portfolio of all tradable claims, and is assumed to be continuous on R^+ . Agents have homogeneous expectations with regard to X . We assume a perfect and complete market for claims on X , so that each agent can buy state contingent claims on the portfolio. This means that an agent can buy a claim paying one unit of cash if $X > K$, and zero if $X \leq K$.¹ Hence the agent is able to choose a payoff function, which we denote as $g_i(X)$. The function relates the agent's payoff from holding state-contingent claims on the market portfolio to the aggregate payoff, X . Given the complete market for claims on X , a unique forward pricing kernel denoted $\phi = \phi(X)$ exists, with $E(\phi) = 1$. Initially, ϕ is given exogenously.

In addition to the investment in the marketable state-contingent claims, the agent also faces a non-insurable background risk. This risk has a nonpositive mean and is independent of the market portfolio payoff, X . This background risk is also a time 1 measurable random variable, denoted $e_i = \sigma_i \varepsilon_i$, where ε_i is a random variable with non-positive mean and unit variance. σ_i is a constant measuring the size of background risk. We assume that e_i is bounded from below: $e_i \geq \underline{e}_i$. The agent's total income at time 1 is,

$$y_i = g_i(X) + e_i. \quad (1)$$

¹See Nachman (1988).

The background risk is unavoidable and cannot be traded. The agent can only take this risk into account in designing an optimal portfolio of claims on X . Hence, we investigate the effect of the background risk, e_i , on the optimal payoff function $g_i(X)$.

We assume that the agent's utility function $\nu_i(\cdot)$ is of the hyperbolic absolute risk aversion (HARA) form

$$\nu_i(y_i) = \frac{1 - \gamma_i}{\gamma_i} \left[\frac{A_i + y_i}{1 - \gamma_i} \right]^{\gamma_i} \quad (2)$$

where γ_i and A_i are constants. We restrict our analysis to cases where $-\infty \leq \gamma_i < 1$, i.e., those exhibiting constant or decreasing absolute risk aversion. These cases are characterized by standard risk aversion. We also assume that any attainable payoff function yields finite expected utility for the agent. We choose the HARA-class since it is the only class of utility functions that implies linear sharing rules for all agents, in the absence of background risk. Also, we assume that it is feasible, given the agent's endowment, background risk, and the pricing kernel, to choose $g_i(X)$ so that $A_i + y_i > 0$, for all possible e_i .

Defining $x_i = g_i(X)$, and dropping the subscript i the agent solves the following maximization problem:

$$\begin{aligned} \max_x E[E_e[\nu(x + e)]] \\ \text{s.t. } E[(x - x^0)\phi] = 0 \end{aligned} \quad (3)$$

where $\nu(\cdot)$ is the utility function of the agent. In Eq.(3), $E_e(\cdot)$ is the expectation over e , and $E(\cdot)$ is the expectation over X . In the budget constraint, $x^0 = x^0(X)$ is the agent's endowment of claims on the market portfolio payoff X . Given the HARA assumption, the first order condition for a solution of (3) is

$$E_e[\nu'(x + e)] = \lambda\phi, \quad (4)$$

where λ is the Lagrangian multiplier of the budget constraint. The solution is optimal and unique.²

In order to analyze the impact of background risk on the agent's optimal demand for claims on the market payoff, it is useful to introduce Kimball's concept of the precautionary premium. Kimball (1990) defines a precautionary premium, ψ , analogous to the Arrow-Pratt risk premium, except that it applies to the *marginal* utility function itself. In the present context we define

$$E_e[\nu'(x + e)] \equiv \nu'(x - \psi) \quad (5)$$

where $\psi = \psi(x, \sigma)$. The precautionary premium is a function of the market payoff of the agent and the scale of the background risk. It is the amount of the deduction from x , which

²For details, see Franke/Stapleton/Subrahmanyam (1998, p.94).

makes the marginal utility equal to the conditional expected marginal utility of the agent in the presence of the background risk.

From equations (4) and (2) it follows that

$$\nu'(x - \psi(x, \sigma)) = \left[\frac{A + x - \psi(x, \sigma)}{1 - \gamma} \right]^{\gamma-1} = \lambda \phi \quad (6)$$

Equation (6) reveals that, given the market pricing kernel, $\phi = \phi(X)$, the payoff function $x = g(X)$ depends directly on the precautionary premium ψ . We, therefore, begin by analyzing the effect of the x and σ on the precautionary premium. The following two Lemmas have been proved in Franke/Stapleton/Subrahmanyam (1998).

Lemma 1: *In the presence of background risk, if $\nu(y)$ is of the HARA family with $-\infty < \gamma < 1$, ψ is twice differentiable and $\psi > 0$, $\partial\psi/\partial x < 0$, $\partial^2\psi/\partial x^2 > 0$, For $\gamma = -\infty$ (exponential utility), $\psi > 0$ and $\partial\psi/\partial x = \hat{0}$.*

Lemma 1 implies that, given a level of background risk, its effect, measured by the precautionary premium, declines at a decreasing rate in the income from the marketable assets. In other words, the precautionary premium is a positive, decreasing, convex function of the marketable income. The exception is the case of the exponential utility function for which the precautionary premium is independent of the marketable income. We are interested also in the effect of the scale of the non-hedgeable background risk, which is indexed by σ .

Lemma 2: *In the presence of background risk, if $\nu(y)$ is of the HARA family with $-\infty < \gamma < 1$, $\partial\psi/\partial\sigma > 0$, $\partial^2\psi/\partial\sigma\partial x < 0$, $\partial^3\psi/\partial\sigma\partial x^2 > 0$. For $\gamma = -\infty$ (exponential utility), $\partial\psi/\partial\sigma > 0$, but independent of x .*

The increase in the precautionary premium due to an increase in background risk is smaller, the higher the income x ; moreover, the convexity of the premium increases as the background risk increases. The significance of Lemma 2 is that it allows us to compare the effect of background risk on the optimal payoff functions of different agents. Other things being equal, an agent with a higher background risk (larger σ) will have a more convex precautionary premium function than one with a lower background risk (σ small). These statements are correct as long as the agent has non-exponential HARA utility.

2.2 Equilibrium

We now analyse the optimal demand of agents $i = 1, 2, \dots, I$ with different levels of background risk and derive equilibrium prices of state-contingent claims in this economy. As above, we assume a complete market for state contingent claims on the market portfolio payoff, X . Individual agents choose $x_i = g_i(X)$ claims on X . Agents have HARA utility

functions with declining absolute risk aversion $-\infty < \gamma_i < 1$ and homogenous expectations regarding the market portfolio payoff.³ In equilibrium, we require that individual demands, x_i , sum to X , the market portfolio payoff. Agents face different levels of background risks. The differing levels of background risk affect the agents' demands for shares of the market portfolio payoff.

Solving Eq. (6) for x_i , aggregating over all agents in the economy and imposing the equilibrium market clearing condition $\sum_i x_i = X$, we have

$$X = \sum_i \left[\psi_i(x_i, \sigma_i) + [\lambda_i \phi]^{\frac{1}{\gamma_i-1}} (1 - \gamma_i) - A_i \right], \quad \forall X \quad (7)$$

In principle, (7) can be solved to endogenously determine the market pricing kernel, $\phi = \phi(X)$, and then, by substituting back in the individual demand condition, (6), to determine the equilibrium optimal demand function, $x_i = g_i(X)$, for agent i . However, in general, the resulting expressions for ϕ and x_i are complex functions of the parameters γ_i , A_i , and the variables, λ_i , ψ_i for all the agents in the economy. Further insight into the portfolio behavior of agents can be gained by assuming that all the agents have the same risk aversion coefficient, γ , but face different levels of background risk, σ_i . This allows us to isolate the effect of the background risk in the portfolio behavior of the agent. If all the agents have the same γ , we can derive a simpler equation.

Define $A \equiv \sum_i A_i$, $\lambda = \sum_{i=1}^I \lambda_i^{\frac{1}{\gamma-1}}$ and $\psi(X) = \sum_{i=1}^I \psi_i(x_i)$. Then Eq. (7) yields

$$[\phi(X)]^{\frac{1}{\gamma-1}} = \frac{1}{X} \left[\frac{A + X - \psi}{1 - \gamma} \right] \quad (8)$$

In equilibrium, λ is chosen such that $E[\phi(X)] = 1$. Equation (8) defines the equilibrium pricing kernel $\phi(X)$. If no background risk exists in the economy, $\psi = 0$, then $[\phi(X)]^{1/(\gamma-1)}$ is a linear function in X . This is no longer true if background risk exists. Then $\psi > 0$ and $\phi^{\frac{1}{\gamma-1}}$ is an increasing, strictly concave function. Now, from the aggregate Eq. (8), it follows immediately that $\psi(X) = \sum_i \psi_i(X)$ is strictly convex. Background risk changes $\phi^{1/(\gamma-1)}$ from a linear function of X to a concave function⁴

3 Effects of an Increase In Background Risk on Absolute Prices of Claims

In this section, we consider the effect of a *general* increase in the background risk of individual agents in the economy. Using certain simplifying assumptions, we show the effect, first, on the pricing kernel and then on the forward price of the market portfolio and the forward price of options on the market portfolio.

³Hence we exclude the trivial case of constant absolute risk aversion, $\gamma = -\infty$, where the precautionary premium is not a function of x .

⁴This is shown in Franke/Stapleton/Subrahmanyam (1998).

Let b denote a nonnegative parameter indicating the level of background risk across agents. Assume that

$$\sigma_i(b) = \bar{\sigma}_i f_i(b) \quad ; \forall i,$$

with $f_i(b) = [>]0$ for $b = [>]0$, $df_i/db > 0$ and $\bar{\sigma}_i$ being a nonnegative constant. Hence σ_i increases with b if $\bar{\sigma}_i > 0$ and $\sigma_i(b) = 0$ if $\bar{\sigma}_i = 0$. An increase in b denotes a general increase in background risk of agents with positive background risk. The impact on the aggregate precautionary premium can be derived from

$$\psi(X) = \sum_i^I \psi_i(g_i(X), \sigma_i(b))$$

In order to characterize the effects of an increase in background risk on equilibrium prices, it is necessary to make assumptions regarding aggregation. Without such assumptions, a change in b might have complex feedback effects of price changes on initial endowments and on portfolio adjustments. Since multiple equilibria might exist, a change in b could induce a move from one equilibrium to another one making comparative statics rather meaningless. It is well known (see, for example, Arrow and Hahn [1971, p.218]) that the equilibrium is unique if only one agent or a representative agent exists. Therefore we preclude moves from one equilibrium to another one by assuming that certain aggregation properties hold for the aggregate precautionary premium. These properties necessarily hold if a representative agent exists. Otherwise it is difficult or impossible to prove them. Assuming that these properties hold is tantamount to assuming that differences among agents are small enough so that a change in b has the same impact on the aggregate precautionary premium as if a representative agent existed. Hence, we now assume that the results of Lemma 2 with respect to changes in the background risk of an individual agent also hold in aggregate, i.e. $\partial\psi/\partial b > 0$, $\partial^2\psi/\partial b\partial X < 0$, $\partial^3\psi/\partial X^2\partial b > 0$ (aggregation properties).

It is useful now to define the absolute and relative risk aversion for this economy. Since $\phi(X)$ is proportional to the ‘‘marginal utility’’ of this economy, we define the coefficient of absolute risk aversion of the pricing kernel as

$$z(X) = -\frac{\partial\phi/\partial X}{\phi(X)} = -\frac{\partial\ln\phi}{\partial X} \quad (9)$$

and the coefficient of relative risk aversion of the pricing kernel as

$$\eta(X) = -\frac{X\partial\phi/\partial X}{\phi(X)} = -\frac{\partial\ln\phi}{\partial\ln X}. \quad (10)$$

Hence the relative risk aversion of the pricing kernel equals the negative elasticity of the pricing kernel with respect to X . Differentiating (8) we find, for the absolute risk aversion in this case,

$$z(X) = (1 - \gamma) \frac{1 - \partial\psi/\partial X}{A + X - \psi(X)} \quad (11)$$

with an analogous expression holding for $r(X)$. Lemma 3 follows immediately.

Lemma 3: *In an economy composed of agents with HARA preferences and a common risk aversion parameter γ , the coefficients of absolute and relative risk aversion of the pricing kernel are increasing in background risk i.e.*

$$\frac{\partial z(X)}{\partial b} > 0, \quad \frac{\partial \eta(X)}{\partial b} > 0$$

Proof: From Lemma 2, and the assumption that changes in background risk have the same impact on the individual and the aggregate precautionary premium, it follows that $\partial\psi/\partial b > 0$, and $\partial^2\psi/\partial X\partial b < 0$. Hence, the numerator of (11) increases with b . The denominator decreases. Since $A + X - \psi(X) > 0$, $\partial z(X)/\partial b > 0$ and, by a similar argument, $\partial\eta(X)/\partial b > 0$. □

Lemma 3 is analogous to the classical risk aversion results along the lines of Pratt (1964) for HARA utility functions. The coefficient of absolute risk aversion of the pricing kernel, $z(X)$, is similar to that of the utility function. Hence, there is an analogy between the behavior of the pricing kernel and the utility function.

We can now establish the following properties of $\phi(X)$, defining $\phi_1(X)$ and $\phi_2(X)$ respectively as the pricing kernels with low and high levels of background risk:

Lemma 4: *Given that the economy satisfies the aggregation properties, the pricing kernel $\phi_2(X)$ intersects the pricing kernel $\phi_1(X)$ once from above.*

Proof: Since $E[\phi(X)] = 1$, both pricing kernels must intersect at least once. At an intersection X^* , $\phi_1(X^*)/X^* = \phi_2(X^*)/X^*$ so that $\eta_2(X^*) > \eta_1(X^*)$ implies $\partial\phi_2(X)/\partial X < \partial\phi_1(X)/\partial X$ for $X = X^*$. Hence, at X^* , $\phi_2(X)$ intersects $\phi_1(X)$ from above. Moreover, a second intersection would violate Lemma 3. □

Lemma 4 states that that the high background risk-pricing kernel intersects the low background risk-pricing kernel once from above.

We are now in a position to analyze the effect of background risk on the value of various contingent claims on the market portfolio payoff. We consider forward prices instead of spot prices. The price of a forward contract for delivery at date 1, i.e. the forward price, equals the spot price at date 0 compounded at the risk-free rate.

First, consider a forward contract to buy the market portfolio payoff, X . The forward price is the agreed price which makes the forward contract a zero-value contract. Denoting this forward price as F we have

$$0 = E[(X - F)\phi(X)] \quad (12)$$

or simply

$$F = E[X\phi(X)] \quad (13)$$

Options on the market portfolio payoff are defined in an analogous manner. The forward prices of call and put options on the market portfolio payoff at a strike price K are as follows:

$$C(K) = E[\max(X - K, 0)\phi(X)] \quad (14)$$

and

$$P(K) = E[\max(K - X, 0)\phi(X)] \quad (15)$$

We can derive the following comparative statics properties of these prices for an increase in the background risk.

Theorem 1:⁵ *Given that the economy satisfies the aggregation properties, an increase in background risk has the following effects:*

a) *The forward price of the market portfolio payoff declines, i.e.*

$$\partial F / \partial b < 0$$

b) *The forward price of a call option at strike price K declines, i.e.*

$$\partial C(K) / \partial b < 0, \forall K$$

c) *The forward price of a put option at strike price K increases, i.e.*

$$\partial P(K) / \partial b > 0, \forall K$$

Proof: Consider an increase in the background risk. It follows immediately that $\partial F / \partial b < 0$, since, from Lemma 3, risk aversion increases with the background risk.

Now consider the value of call and put options on X .

Let $\phi_1(X)$ and $\phi_2(X)$ denote the pricing kernels for low and high background risk. It follows from $E[\phi_1(X)] = E[\phi_2(X)] = 1$ and from lemma 4 that there exists a cross-over point X^* such that

$$\phi_2(X) > [=][<]\phi_1(X) \quad \text{if} \quad X < [=][>]X^*.$$

⁵The same results can also be derived from the weaker assumption of standard risk aversion instead of HARA utility.

Consider a call option at a strike price K . The forward price of this option given pricing kernel $\phi_j(X)$ ($j = 1, 2$), is $C(K|j)$.

$$C(K|j) = E[\max(X - K, 0)\phi_j(X)] ; j = 1, 2. \quad (16)$$

Suppose, first, $K \geq X^*$. Then the option has positive payoffs only if $X > K \geq X^*$. Since $\phi_1(X) > \phi_2(X)$ for $X > X^*$, the forward price of the call option must be higher under pricing kernel $\phi_1(X)$. Hence an increase in background risk lowers the forward price of the call option. Second, suppose $K < X^*$. Subtracting in every state the amount $(X^* - K)$ from the payoff of the option lowers the forward option price by $(X^* - K)$,

$$C(K|j) - (X^* - K) = E[\max(X - X^*; K - X^*)\phi_j(X)] ; j = 1, 2. \quad (17)$$

Hence it follows that

$$C(K|2) - C(K|1) = E[\max(X - X^*; K - X^*)(\phi_2(X) - \phi_1(X))]. \quad (18)$$

As $\max(X - X^*; K - X^*) < [=][>]0$ when $\phi_2(X) - \phi_1(X) > [=][<]0$, $C(K|2) - C(K|1) < 0$ follows.

Hence, the forward prices of all call options decline with a rise in background risk. A similar argument can be used to show that the prices of all put options increase with a rise in background risk.

□

The effect of an increase in background risk is to reduce the prices of claims in states with a relatively high payoff of the market portfolio and increase the prices of claims in states with a relatively low payoff of the market portfolio. Furthermore, the “average” price of claims represented by the forward price of the market portfolio also declines. It is apparent that call options with a high strike price (i.e. to the right of the crossover point X^*) get cheaper when the background risk increases. However, this is true for *all* call options regardless of the strike price. This is intuitively reasonable when one notices that the forward price of a call with a zero strike price changes by the same amount as the forward price of the market portfolio, since these two contracts differ only by a constant in their payoffs. Similarly, a put option with a low strike price becomes more expensive. This is also true of all other put options.

Theorem 1 has important testable implications. It allows us to clearly separate the effects of an increase in aggregate background risk from those of an increase in the risk of the market portfolio payoff X . An increase in the latter risk, holding the mean of X constant, should lower the forward price of the market portfolio because of risk aversion, but raise the forward prices of call options with high strike prices, since for these options the insurance value increases while the intrinsic value is zero anyway. In contrast, an increase in aggregate background risk lowers the forward prices of all call options. Therefore, a

situation in which the forward prices of all puts and of calls with high strike prices increase, indicates a rise in the risk of the market portfolio payoff X . Conversely, a situation in which the forward prices of all puts go up and those of all calls go down, indicates an increase in background risk. Hence this approach allows to distinguish empirically between increases in background risk and increases in the risk of the market portfolio payoff.

4 Effects of an Increase of Background Risk on Relative Prices of Assets

In the last section we analyzed the impact of an increase in background risk on the absolute prices of claims. Prices of financial derivatives are quoted, however, often relative to the prices of the underlyings. The Black-Scholes model derives the price of an option relative to the price of the underlying. We consider the forward price of an option relative to the forward price of the underlying market portfolio. In our model this price relative is driven by the level of risk aversion as measured by $(1 - \gamma)$, by $A (= \sum_{i=1}^I A_i)$, and by the aggregate level of background risk, b .

How can we distinguish the effects on relative option prices of an increase in background risk from those that result from a general increase in risk aversion, $1 - \gamma$, holding A constant? One way to do this is to compare the effects on option prices of an increase in background risk with those that follow from an increase in risk aversion, where both changes have the *same* effect on the forward price of the market portfolio. The result is shown in the following Theorem 2. It considers the *relative* increase in put and call option prices in the two cases. In the following theorem we consider firstly, an increase in background risk, and secondly, an increase in risk aversion where both changes have the *same* effect on the forward price of the market portfolio. We establish:

Theorem 2: *Assume an economy which satisfies the aggregation properties. Suppose initially that there is no background risk in this economy ($b = 0$). Consider now the effect of introducing background risk ($b > 0$), with no change in risk aversion, and alternatively an increase in risk aversion caused by a decline in γ from γ_2 to γ_1 , where these two changes reduce the forward price of the market portfolio by the same amount. Then, the forward prices of all call and put options are higher in the case where background risk increases than in the case where risk aversion increases, i.e.*

$$\begin{aligned} \text{If } F(b > 0, \gamma = \gamma_2) &= F(b = 0, \gamma = \gamma_1), \text{ then} \\ C(K|b > 0, \gamma = \gamma_2) &> C(K|b = 0, \gamma = \gamma_1), \forall K, \\ P(K|b > 0, \gamma = \gamma_2) &> P(K|b = 0, \gamma = \gamma_1), \forall K.^6 \end{aligned}$$

The notation $C(K|b, \gamma)$ refers to the forward price of a European call option at strike

⁶We consider only non-trivial options such that $\text{Prob}(X < K) > 0$ and $\text{Prob}(X > K) > 0$ for all

price K given that background risk is at level b and the risk aversion parameter is at level γ . Similar notation is adopted for put options and for the market portfolio.

Proof: We denote the pricing kernel in the absence of background risk ($b = 0$) as $\phi_1(X)$. With background risk, $b > 0$, the pricing kernel is denoted by $\phi_2(X)$.

Without background risk, $b = 0$, and

$$\phi_1(X) = \left[\frac{1}{\lambda_1} \frac{A + X}{1 - \gamma_1} \right]^{\gamma_1 - 1} \quad (19)$$

$$F(b = 0, \gamma = \gamma_1) = E[X \phi_1(X)] \quad (20)$$

where λ_1 is chosen so that $E[\phi_1(X)] = 1$.

Given the background risk, the pricing kernel and the forward price are given by the following equations

$$\phi_2(X) = \left[\frac{1}{\lambda_2} \frac{A + X - \psi(X)}{1 - \gamma_2} \right]^{\gamma_2 - 1} \quad (21)$$

$$F(b > 0, \gamma = \gamma_2) = E[X \phi_2(X)] \quad (22)$$

where λ_2 is chosen so that $E[\phi_2] = 1$.

It follows almost immediately that the risk aversion coefficient $1 - \gamma_2 < 1 - \gamma_1$.⁷ To see this, note that if $\gamma_1 = \gamma_2$, then by Lemma 3, $z(X|b > 0, \gamma = \gamma_2) > z(X|b = 0, \gamma = \gamma_1)$, where the $z(\cdot)$'s are the coefficients of absolute risk aversion in the two cases. This implies that $F(b > 0, \gamma = \gamma_2) < F(b = 0, \gamma = \gamma_1)$, which contradicts the assumption of the Theorem. If $\gamma_1 > \gamma_2$, the same result obtains, *a fortiori*.

We now compare the background risk pricing kernel, $\phi_2(X)$, and the kernel $\phi_1(X)$. The following Lemma states that the elasticity of $\phi_2(X)$ intersects that of $\phi_1(X)$ once from above.

Lemma 5: *Assume that the pricing kernel with background risk, $\phi_2(X)$, and the pricing kernel with no background risk, but higher risk aversion, $\phi_1(X)$, yield the same forward price of the market portfolio. Then the elasticity $\eta_2(X)$ of the pricing kernel $\phi_2(X)$ intersects the elasticity $\eta_1(X)$ of the kernel $\phi_1(X)$ once from above.*

Lemma 5 is proved in the Appendix. In Franke/Stapleton/Subrahmanyam (1999, Lemma 1) it has been shown for a constant elasticity η_1 and a declining elasticity $\eta_2(X)$ that the pricing kernels $\phi_1(X)$ and $\phi_2(X)$ intersect twice. The proof, however, does not require constant elasticity η_1 . It only requires that $\eta_2(X)/\eta_1(X)$ is monotonically declining. Hence it follows from Lemma 5 above that $\phi_1(X)$ and $\phi_2(X)$ intersect twice

options, so that there is a non-zero probability of the option finishing out-of-the-money.

⁷Weil (1992) proves this formally for the more general case of standard risk aversion.

with $\phi_2(X) > \phi_1(X)$ for very low and very high values of X . Then it follows from Franke/Stapleton/Subrahmanyam (1999, Theorem 1) that all European options on the market portfolio have a higher price under $\phi_2(X)$ than under $\phi_1(X)$. This ends the proof. \square

The surprising aspect of this result is that *all* put and call prices are higher in the background risk economy than in the economy with increased risk aversion. It mirrors the fact that the convexity of the pricing kernel is greater in this case. Therefore convex claims on the market portfolio have a higher forward price, given the same forward price of the market portfolio. The importance of the result is that it allows us to distinguish the effects of an increase in background risk on option prices relative to those of an increase in risk aversion. Suppose we observe a given reduction in the forward price of the market portfolio that could be due either to an increase in the background risk of agents or to an increase in risk aversion. Although in both cases the forward prices of puts would go up while the forward price of calls would go down, we would observe higher forward prices for puts and calls, relative to the forward price of the market portfolio, in the former case. This effect would be most pronounced for puts at very low strike prices and calls at very high strike prices. Thus, strong increases in the relative prices of these options indicate an increase in background risk.

5 CONCLUDING COMMENTS

An increase in the background risk faced by some agents in the economy changes the prices for state-contingent claims in the economy. To model the effects of an increase in background risk, all agents are assumed to have HARA-preferences with declining absolute risk aversion. Thus, they are standard risk averse. An increase in background risk makes agents more averse to tradable risks. This is reflected in higher elasticity of the pricing kernel which may be interpreted as higher relative risk aversion. Therefore the forward price of the market portfolio declines. Since the probability-deflated prices for claims in the low [high] states increase [decline], all European put [call] options become more [less] expensive. This is due to the fact that an increase in background risk imposes a relatively higher [lower] burden on the agents when their tradable income is low [high].

Usually prices of financial derivatives are quoted relative to the prices of the underlyings. The price of a European option relative to the price of the market portfolio is determined by the agents' risk aversion and by the level of background risk. If background risk increases, but risk aversion stays the same, then all European options relative to the market portfolio will be more expensive as compared to a situation where risk aversion increases, but the background risk stays the same. Thus, the model allows us to distinguish empirically between the effects of an increase in background risk and an increase in the agents' risk aversion on the pricings of assets.

Appendix
Proof Lemma 5

Given $\phi_1(X)$ and $\phi_2(X)$, define the elasticities of the two pricing kernels are $\eta_1(X)$ and $\eta_2(X)$ as defined by equation (12). We show first that

$$\frac{d}{dX} \left[\frac{\eta_2(X)}{\eta_1(X)} \right] < 0$$

To establish this, note that

$$\frac{d}{dX} [\eta_2(X)/\eta_1(X)] < 0 \quad \text{if} \quad \frac{d}{dX} \left[\ln \frac{\eta_2(X)}{X} - \ln \frac{\eta_1(X)}{X} \right] < 0$$

$$\begin{aligned} \frac{d}{dX} \left[\ln \frac{\eta_2(X)}{X} - \ln \frac{\eta_1(X)}{X} \right] &= \frac{d}{dX} [\ln(1 - \psi'(X)) - \ln(A + X - \psi(X)) + \ln(A + X)] \\ &= -\frac{\psi''(X)}{1 - \psi'(X)} - \frac{1 - \psi'(X)}{A + X - \psi(X)} + \frac{1}{A + X} \quad (23) \\ &< 0. \end{aligned}$$

This follows since, by the aggregation property, $\psi''(X) > 0$, and, by $\psi(X) > 0$ and $\psi'(X) < 0$, the second term in (23), multiplied by -1 , is greater than the third term. Hence, the elasticity of $\phi_2(X)$ can intersect that of $\phi_1(X)$ at most once. It must intersect, however. Otherwise the price of the market portfolio would always be smaller under the pricing kernel with the higher elasticity.

□

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