A Two-factor Lognormal Model of the Term Structure and the Valuation of American-Style Options on Bonds

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Abstract

We build a no-arbitrage model of the term structure, using two stochastic factors, the short-term interest rate and the premium of the forward rate over the short-term interest rate. The model can be regarded as an extension to two factors of the lognormal interest rate model of Black-Karasinski. It allows for mean reversion in the short rate and in the forward premium. The method is computationally efficient for several reasons. First, interest rates are defined on a bankers' discount basis, as linear functions of zero-coupon bond prices, enabling the use of the no-arbitrage condition to compute bond prices without resorting to iterative methods. Second, the multivariate binomial methodology of Ho-Stapleton-Subrahmanyan is extended so that a multiperiod tree of rates with the no-arbitrage property can be constructed using analytical methods. The method uses a recombining two-dimensional binomial lattice of interest rates that minimizes the number of states and term structures over time. Third, the problem of computing a large number of term structures is simplified by using a limited number of 'bucket rates' in each term structure scenario. In addition to these computational advantages, a key feature of the model is that it is consistent with the observed term structure of volatilities implied by the prices of interest rate caps and floors. We illustrate the use of the model by pricing American-style and Bermudan-style options on bonds. Option prices for realistic examples using forty time periods are shown to be computable in seconds.
1 Introduction

Perhaps the most important and difficult problem facing practitioners in the field of interest rate derivatives in recent years has been to build inter-temporal models of the term structure of interest rates that are both analytically sound and computationally efficient. These models are required both to help in the pricing and in the overall risk management of a book of interest rate derivatives. Although many alternative models have been suggested in the literature and implemented in practice, there are serious disadvantages with most of them. For example, Gaussian models of interest rates, which have the advantage of analytical tractability, have the drawback of permitting negative interest rates, as well as failing to take into account the possibility of skewness in the distribution of interest rates. Also, many of the term-structure models used in practice are restricted to one stochastic factor.

Since the work of Ho and Lee (1986), it has been widely recognized that term-structure models must possess the no-arbitrage property. In this context, a no-arbitrage model is one in which the forward price of a bond is the expected value of the one-period-ahead spot bond price under the risk-neutral measure. Building models that possess this property has been a major pre-occupation of both academics and practitioners in recent years. One model that achieves this objective in a one-factor context is the model proposed by Black, Derman and Toy (1990)(BDT), and extended by Black and Karasinski (1991)(BK). In essence, the model which we build in this paper can be thought of as a two-factor extension of this type of model. In our model, interest rates are lognormal and are generated by two stochastic factors. The general approach we take is similar to that of Hull and White (1990)(HW), where the conditional mean of the short rate depends on the short rate and an additional stochastic factor, which can be interpreted as the forward premium. In contrast to HW, and in line with BK, we build a model where the conditional variance of the short rate is a function of time. It follows that the model can be calibrated to the observed term structure of interest rate volatilities implied by interest rate caps/floors. Essentially the aim here is to build a term structure model which can be applied to value American-style contingent claims on interest rates, which is consistent with the observed market prices of European-style contingent claims.

Our approach to building a no-arbitrage term structure for pricing interest rate derivatives is consistent with the general framework proposed by Heath, Jarrow and Morton (1992)(HJM). In the HJM formulation, assumptions are made about the volatility of the forward interest rates. Since the forward rates are related in a no-arbitrage model to the future spot interest rates, there is a fairly close relationship between this approach and the one we are taking. In fact, the HJM forward-rate volatilities can be thought of as the outputs of our model.
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If the parameters of the HJM model are known, this represents a satisfactory alternative approach. However, the BDT-HW approach has the advantage of requiring as inputs the volatilities of the short rate and of longer bond yields which are more directly observable from market data on the pricing of caps, floors and swaptions.

We would like any model of the stochastic term structure to have a number of desirable properties. Apart from satisfying the no-arbitrage property, we want the model inputs to be consistent with the observed conditional volatilities of the variables and the mean-reversion of the short rate and the premium factor. We also require that the short term interest rates be lognormal, so that they are bounded from below by zero and skewed to the right. From a computational perspective, we require the state space to be non-explosive, i.e. recombining, so that a reasonably large span of time-periods can be covered. The complexities caused by these model requirements are discussed in section 2.

We introduce a number of new aspects into our model that allow us to solve these requirements. The most important simplification arises from modeling the bond's discount interest rate, where the interest rate is a linear function of the price of a short-term bond. We then extend and adapt the recombining binomial methodology of Nelson and Ramaswamy (1990) and Ho, Stapleton and Subrahmanyam (1996)(HSS) to model lognormal rates rather than prices. These computational techniques are discussed in detail in section 3. Section 4 presents the basic two-factor model, and discusses some of its principal characteristics. In section 5, we explain how the multiperiod tree of rates is built using a modification of the HSS methodology. In section 6, we present some numerical examples of the output of our model, apply the model to the pricing of American-style and Bermudan-style options and discuss the computational efficiency of our methodology. Section 7 presents our conclusions.

2 Requirements of the model

There are several desirable features of any multi-factor model of the term structure of interest rates. Some of these features are requirements of theoretical consistency and others are necessary for tractability in implementation. Keeping in mind the latter requirements, it is important to recognize that the principal purpose of building a model of the evolution of the term structure is to price interest rate options generally and, in particular, those with path-dependent payoffs. The simplest examples of options that need to be valued using such a model are American-style and Bermudan-style options on an interest rate.

First, since we wish to be able to price any term-structure dependent claim, it is important that the model output is a probability distribution of the term structure of interest rates.
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at each point in time. A realistic model should be able to project the term structure for
ten or twenty years, at least on a quarterly basis. With the order of forty or eighty sub-
periods, the computational task is substantial and complex. If we do not compute each
term structure point at each node of the tree, then we need to be able to interpolate, where
necessary, to obtain required interest rates or bond prices. As in the no-arbitrage models of
HL, HJM, HW, BDT, and BK, we first build the risk-neutral or martingale distribution of
the short-term interest rate, since other maturity rates and bond prices can be computed
from the short-term rate.

The second and crucial requirement is that the interest rate process be arbitrage-free. In
the context of term structure models, the no-arbitrage requirement, in effect, means that
the one-period forward price of a bond of any maturity is the expected value, under the risk-
neutral measure, of the one-period-ahead spot price of the bond. Since HL, this requirement
has been well-recognized, within the context of single-factor models, and is a property
satisfied by the HW, BDT, and BK models. However, the requirement is more demanding
in the two-factor setting as shown by HJM, Duffie and Kan (1993). In a two-factor model
in which the factors are themselves interest rates, the no-arbitrage condition restricts the
behavior of the factors themselves as well as the behavior of bond prices. However, the no-
arbitrage property is also an advantage in a computational sense, allowing the computation
of bond prices at a node by taking simple expectations of subsequent bond prices under the
risk-neutral measure.

The third requirement is that the term structure model should be consistent with the
current term structure and with the term structure of volatilities implicit in the prices of
European-style interest rate caps and floors. Models that are consistent with the current
term structure have been common in the literature since the work of Ho and Lee (1986)(HL).
For example, the HJM, HW, BDT and BK models are all of this type. The second part of
the requirement is rather more difficult to accommodate, since if volatilities are not constant
over time, the tree of rates may be non-recombining, as in some implementations of the HJM
model, leading to an explosion in the number of states. The HW implementation of a two-
factor model, in Hull and White (1994), specifically excludes time-dependent volatility. BK
on the other hand, accommodate both time varying volatility and mean reversion of the
short rate within a one-factor model by varying the size of the time steps in the model.
This procedure is difficult to extend to a two-factor case.

At a computational level, it is necessary that the state space of the model does not explode,
producing so many states that the computations become infeasible. Even in a single-factor
model, this fourth requirement means that the tree of interest rates or bond prices must
recombine. This is a property of the binomial models of HL, BDT, and BK, and also of the trinomial model of HW. A number of computational methods have been suggested.
in guarantee this property, including the use of different time steps and state-dependent probabilities. In the context of a two-factor model, the requirement is even more important. In our bivariate-binomial model we require that the number of states is no more than \((n+1)^2\), after a time step. This is the bivariate generalization of the "simple" recombining one-variable binomial tree of Cox, Ross and Rubinstein (1979).

Based on the empirical evidence as well as on theoretical considerations, the fifth requirement for our two-factor model is that the interest rates are lognormally distributed. Based on the work of Vasicek (1977), it is well known that the class of Gaussian models, where the interest rate is normally distributed, are analytically tractable, allowing closed-form solutions for bond prices. However, apart from admitting the possibility of negative interest rates, these models are not consistent with the right skewness that is considered important, at least for some currencies. Furthermore, such a model would be inconsistent with the widely-used Black model in value interest rate options, which assumes that the short-term rate is lognormally distributed. Of the models in the literature, the HJM and HK models explicitly assume that the short rate is lognormal. The HJM and HW multi-factor models are general enough to allow for lognormal rates, but at the expense of computational complexity.

In addition to these five requirements, there is an overall necessity that the model be computable for realistic scenarios, efficiently and with reasonable speed. In this context, we aim to compute option prices in a matter of seconds rather than minutes. To achieve this we need a number of modelling innovations, compared to the techniques used in prior models. These methodological innovations are discussed in the next section.

3 Features of the methodology

3.1 The stochastic process for interest rates

The dynamics of the term structure of interest rates can be modeled in terms of one of three alternative variables: zero-coupon bond prices, spot interest rates, or forward interest rates. If the objective of the exercise is to price contingent claims on interest rates, it is sufficient to model forward rates, as demonstrated by HJM. However, there are some problems with adopting this approach in a multi-factor setting. First, from a computational perspective, for general forward rate processes the tree may be non-recombining, which implies that a large number of time steps becomes practically infeasible. Second, it is difficult to estimate the volatility inputs for the model directly from market data. Usually, the implied volatility
data are obtained from the market prices of options on Euro-currency futures, caps, floors and swaptions, which cannot be easily transformed into the volatility inputs required to build the forward rates in the HJM model.

We choose to model interest rates rather than prices because existing methodologies, introduced by BBS, can be employed to approximate a process with a log-binomial process. One major problem arises in modelling rates rather than prices, however, and that concerns the no-arbitrage property. Under the risk-neutral measure, forward bond prices are related to one-period-ahead spot prices, but the relationship for interest rates is more complex, as shown by HJM. This non-linearity makes the implementation of the binomial lattice much more cumbersome. We overcome this problem by modelling interest rates defined on a bankers’ discount basis as suggested in Stapleton and Subrahmanyam (1983). The short-term interest rate defined on a bankers’ discount basis, is a linear function of the price of a semi-annual bond of the same maturity. Further, we assume that the three-month rate, defined on a bankers’ discount basis, is lognormally distributed. There is a small difference between assuming that the short-term interest rate defined in the usual manner and on a bankers’ discount basis are lognormally distributed. Simulations show that for short maturities and low levels of the mean and volatility, the two assumptions are not statistically distinguishable from each other. The main advantage of the bankers’ discount is that the linear relationship of the interest rate and the bond price allows the no-arbitrage condition to be enforced in a straightforward manner.

We choose here to model interest rates, and then derive bond prices, forward prices and forward rates as required, from the spot rate process. We prefer to directly model the short rate, which we interpret here as the three-month interest rate, since it is used to determine the payoffs on many contracts such as interest rate caps, floors and swaptions. One advantage of doing so is that implied volatilities from caps/floors/section floor prices may be used to determine the volatility of the short-rate process fairly directly.

As in HW, we model the short rate under the risk-neutral measure as a two-dimensional AR process. In particular, we assume that the logarithm of the short rate, follows such a process where the second factor is an independent shock to the forward premium. The short rate itself and the premium factor each mean revert, at different rates, allowing for quite general shifts and tilts in the term structure. Also, in contrast to HW, we assume time-dependent volatility functions for both the short rate and the premium factor. In this

Note that we only assume that the short-term interest rate is lognormal. If the short-term interest rate is represented by the three-month rate, we would expect a price in the region of 0.95±0.01. It follows that the probability of a negative rate is negligible.

Taking conditional expectations of the process, it can be shown that the term structure of futures rates is given by a log-linear model in any two futures rates. The proof relies on the fact that the futures rate is the
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model, the bond prices and forward rates for all maturities can be computed by backward
induction, using the no-arbitrage property. Since the interest rate process is defined under
the risk-neutral measure, forward bond prices and interest rates (defined on a bankers'
discount or linear functions of zero-bond prices) are expectations of one-period-ahead bond
prices under this measure. This property permits the rapid computation of bond prices of
all maturities by backward induction, at each point in time.

3.2 Building a multivariate tree for interest rates

The principal computational problem is to build a tree of interest rates, which has the prop-
erty that the conditional expectation of the rate at any point depends on the rate itself and
the premium factor. A methodology available in the literature, which allows the building
of a multivariate tree, approximating a lognormal process with non-stationary variances
and covariances, is described in Ho, Stapleton and Subrahmanyam (1990). The HSS
methodology is itself a generalization to two or more variables of the method advocated
by Nelson and Ramaswamy (1990), who devised a method of building a 'simple' or re-
combining binomial tree for a single variable. In HSS, the expectation of a variable depends
on its current value, but not on the value of a second stochastic variable. However, as we
show in section 8, the methodology is easily extended to this more general case. Essentially,
the HSS method relies on fixing the conditional probabilities on the tree to accommodate
the mean reversion of the interest rates, the changing volatilities of the variables and the
covariances of the variables. In the case of interest rates, it is crucial to model changes in
the short rate as to reflect the second, premium factor. This is the key; in a two-factor
model, to maintaining the no-arbitrage property, while avoiding an explosion in the number
of states. Using our extension of the HSS methodology allows us to model the bivariate
distribution of short rate and the premium factor, with \( n + 1 \) states for each variable after
\( n \) time steps, and a total of \( (n + 1)^2 \) term structures after \( n \) time steps. This is achieved by
allowing the probabilities to vary in such a manner as to guarantee that the no-arbitrage
property is satisfied and the tree is consistent with the given volatilities and mean rever-
ence of the process.

One problem with extending the typical interest rate tree building methods of HW, BDT,
and BK to two or more factors arises from the forward induction methodology normally
employed in these models. The tree is built around the current term structure and the
calculation proceeds by moving forward period-by-period. This is expensive in computing
time, and could become prohibitively so, in the case of multiple factors. To avoid this
expectation, under the risk-neutral measure, of the future spot rate, given the linear relationship between
the interest rate and the price.
problem, we devise a new dynamic method of implementation of the HSS tree, which allows us to compute the multivariate tree in a matter of seconds for up to eighty periods. This method uses the feature of HSS which allows a variation in the density of the tree over any given time step. A forward, dynamic procedure is used whereby a two-period tree with changing density is converted into the required multi-period tree. For example, when the eighty-step time step is computed, the program computes a two-period tree with a density over the first period of seventy-nine and a density over the second period of one. This allows us to compute the tree nodes and the conditional probabilities analytically, and without recourse to iterative methods.

3.3 Improving computational efficiency

In practice much of the skill in building realistic models rests on deciding exactly what to compute. Potentially, in a tree covering eighty time steps, we could compute bond prices for between one and eighty maturities at each node of the tree. Not only is this a vast number of bond prices, but also, most of the bond prices will not be required for the solution of any given option valuation problem. We assume here that it will be sufficient to compute bond prices for maturities one year apart from each other. Intermediate maturity prices, if required, can always be computed by interpolation. Hence, in our eighty-time step example, where each time step is a quarter of a year, we compute at most twenty bond prices. This saving reduces the number of calculations by almost seventy-five percent. For large numbers of time steps, it can turn an almost infeasible computational task into one that can be accomplished within a reasonable time frame. For example, with three hundred time steps, the number of bond price calculations can be reduced from approximately twenty-seven million to approximately one million.

In spite of the computational savings that are made by having a recombining tree methodology and reducing the number of bond prices that need to be calculated, it may still be the case that the computation time is excessive for a given problem. For example, for a Bermudan-style bond option that is exercisable every year for the first six years of the underlying bond’s twenty year life, we only require bond prices at the end of each of the first six years. One computational advantage of the HSS methodology, is that the binomial density can be altered so that this problem is reduced to a seven-period problem with differential density (numbers of time steps). The binomial density ensures sufficient accuracy in the computations, while the number of bond and option price calculations is minimized.
4 The Two-factor Model

4.1 No-arbitrage properties of the model

We assume that, under the risk-neutral measure, the logarithm of the short-term interest rate, for loans of maturity \( m \), follows the process

\[
d \ln(r) = (\theta_r(t) - \alpha \ln(r) + \ln(x_1) + \sigma_r(t))dt + \sigma_r(t)dz_1
\]

where

\[
d \ln(x) = (\theta_x(t) - \beta \ln(x) + \ln(x_2) + \sigma_x(t))dt + \sigma_x(t)dz_2
\]

In the above equations \( d \ln(r) \) is the change in the logarithm of the short rate, \( \theta_r(t) \) is a time-dependent constant term that allows the model to be calibrated to the current term structure, \( \alpha \) is the speed at which the short rate mean reverts, \( \pi \) is a shock to the conditional mean of the process. We refer to \( \pi \) as the forward premium factor, since it is crucial in determining the forward rates in the model. \( \sigma_r(t) \) is the instantaneous volatility of the short rate. The forward premium factor itself follows a diffusion process with mean \( \theta_x \), mean reversion \( b \) and instantaneous volatility \( \sigma_x(t) \). Note that, in this notation \( \sigma_x \) and \( \sigma_r \) refer to the unconditional logarithmic standard deviation of the variable, over the period \( 0 - t \), on a non-annualized basis. Although this structure is broadly similar to the model of Hull and White (1984), note that we do not restrict the volatilities \( \sigma_r(t) \) and \( \sigma_x(t) \) to be constant over time, although they are assumed to be non-stochastic. Also, we assume that the short-term interest rate is defined on a bankers' discount basis, i.e. \( r_t = (1 - B_{t+1}/m) / m \), where \( m \) is a fixed maturity of the short rate and \( B_{t+1} \) is the price of a \( 1+1 \) year zero-coupon bond at time \( t \). \( dz_1 \) and \( dz_2 \) are standard Brownian motions.

In discrete form, equation (1) can be written as

\[
\ln(x_{t+1}) - \ln(x_t) = \theta_r(t) - \alpha \ln(x_{t+1}) + \ln(x_1) + \sigma_r(t) + \epsilon_r(t) + \epsilon_r(t+1).
\]

where

\[
\ln(x_{t+1}) - \ln(x_t) = \theta_x(t) - \beta \ln(x_{t+1}) + \ln(x_2) + \sigma_x(t) + \epsilon_x(t) + \epsilon_x(t+1).
\]

and where \( \epsilon_r \) and \( \epsilon_x \) are independently distributed, normal variables, and the mean and unconditional standard deviation of the logarithm of \( x_1 \) and \( x_2 \) are \( \mu_x \), \( \sigma_x \), and \( \mu_r \), \( \sigma_r \), respectively.
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First, we solve the model to determine the constants $\theta_\tau(t)$ and $\theta_\sigma(t)$. We have, for any lognormal process of the form of equation (2):

Lemma 1. Suppose that the short-term interest rate follows the process in equation (2). Then

$$\ln(r_t) - \mu_r = [\ln(r_{t-1}) - \mu_{r-1}] (1 - a) + \ln(\tau_t) - \mu_{\tau_{t-1}} + \epsilon_t$$

where

$$\ln(\tau_t) - \mu_{\tau_{t-1}} = [\ln(\tau_{t-1}) - \mu_{\tau_{t-1}}] (1 - b) + \nu_t$$

with

$$\mu_{\tau_{t-1}} = \ln[E_0(\tau_{t-1})] - \sigma_{\tau_{t-1}}^2/2,$$

and

$$\mu_{\tau_{t-1}} = \ln[E_0(\tau_{t-1})] - \sigma_{\tau_{t-1}}^2/2.$$  

Proof
Taking the unconditional expectation of equation (2)

$$\mu_r - \mu_{\tau_{t-1}} = \theta_r(t) - a \mu_{\tau_{t-1}} + \mu_{\tau_{t-1}};$$

$$\mu_{\tau_{t-1}} = \theta_\tau(t) - a \mu_{\tau_{t-1}}$$

Then, substituting for $\theta_r(t)$ and $\theta_\tau(t)$ in (2) yields (3).

Also since $r_t$ and $\tau_t$ are lognormal, it follows from the moment generating function of the normal distribution that

$$E_0(r_t) = \exp(\mu_r + \sigma_r^2/2)$$

$$E_0(\tau_t) = \exp(\mu_{\tau_{t-1}} + \sigma_{\tau_{t-1}}^2/2).$$

Hence, taking logarithms yields the statement in the lemma. □

Lemma 1 is essential in the calibration of the model. We have to choose the means $\mu_r$ of the short rate to be consistent with the absence of arbitrage and with the term structure of interest rates at $t = 0$. We achieve this by assuming, as given, a set of futures rates $[f_{tT}]$. These are futures rates for $n$-month money defined on a bankers' discount basis.

Assuming for the moment that these futures rates are given, we build a model of $r_t$ in which the expected value under the risk-neutral measure is the futures rate, $[f_{tT}]$, $\forall t$, $0 < t < T$.

Working back from the terminal data $T$, we then compute zero bond prices assuming that
the forward price under the risk-neutral measure is the expected value of the one-period-ahead, spot price. This ensures that the no-arbitrage property of the model is guaranteed in every state, and at each date. The no-arbitrage properties of the model are established in the following proposition. First, we define the no-arbitrage property, precisely. We have, from Pliska (1987), p. 218.

**Definition (The No-Arbitrage Property)**

A term-structure model satisfies the no-arbitrage property, if and only if the zero-coupon bond prices satisfy

\[ B_{s,t} = E_s(B_{s,t+1}), \quad 0 \leq s < t \leq T. \]

We now show that, if the model is suitably calibrated, then the model is arbitrage-free. We first state without proof's result from Cox, Ingersoll and Ross (1981) which we require in the proof of the no-arbitrage property that follows:

**Lemma 2.** The futures price \( F_{0,t} \) of an asset with spot price \( S_t \) at time \( t \) is the value of an asset which pays \( S_{t}/B_{0,t}B_{1,t}...B_{i,t} \) at time \( t \).

**Proof**


**Proposition 1. (The No-Arbitrage Property)** Suppose that the short rate, \( r_s \), defined on a bankers' discount basis, follows the process:

\[ \ln(r_s) - \ln(r_{s-1}) = \theta(r) - \alpha \ln(r_{s-1}) + \ln(r_{s-1}) + \sigma_{s-1}, \]

where

\[ \ln(r_s) - \ln(r_{s-1}) = \theta(r) - \ln(r_{s-1}) + \nu_s. \]

under the risk-neutral measure, with \( E(r_s) = 1, \forall t \), where \( r_s \) and \( \nu_s \) are independently distributed, normal variables. Then, if the model is calibrated so that

\[ \theta(r) = \ln(f_{0,t}) - \frac{\sigma^2}{2}, \forall t, \]

where \( f_{0,t} \) is the futures price at time 0, for delivery at time \( t \) and if the forward price at \( t \), of a \( t + k + 1 \) maturity zero-coupon bond, for delivery at \( t + 1 \) is given by

\[ B_{s,t+k+1,t+k+1} = E_s(B_{s,t+k+1,t+k+1}), \forall t, k \]

then the no-arbitrage property holds.
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Proof

From lemma 1,

\[ \ln(r_k) - \mu_{r_k} = [\ln(r_{k-1}) - \mu_{r_{k-1}}](1 - a) + \ln(r_{k-1}) - \mu_{r_{k-1}} + \sigma_r^2 \]

where

\[ \ln(r_{k-1}) - \mu_{r_{k-1}} = [\ln(r_{k-2}) - \mu_{r_{k-2}}](1 - b) + \sigma_r^2 

and where

\[ \mu_{r_k} = \ln[E_0(r_k)] - \sigma_r^2/2 \]

and

\[ \mu_{r_{k-1}} = \ln[E_0(r_{k-1})] - \sigma_r^2/2 \]

From the calibration of the model,

\[ \mu_{r_k} = \ln(f_{a+1}) - \sigma_r^2/2 \]

and hence it follows that \( f_{a+1} = E_0(r_k) \). Since \( r_k \) and \( f_{a+1} \) are defined on a bank's discount basis, \( r_k = (1 - B_{k+1})/m \) and \( f_{a+1} = (1 - B_{a+1})/m \) and it follows that the futures price of the zero-coupon bond is

\[ \tilde{B}_{a+1} = 1 - mf_{a+1} = 1 - mE_0(r_k) = E_0(B_{k+1}) \].

Using the property of expectations, we can write

\[ \tilde{B}_{a+1} = E_0(E_1(\ldots E_k(E_{k+1}))) \]

and

\[ \tilde{B}_{a+1} = B_{a+1}E_0(E_1(\ldots E_k(E_{k+1}))) \left( \frac{B_{k+1}}{B_{a+1}E_{k+1}} \right). \]

From lemma 2, this equation gives the value of an asset paying \( \left( \frac{B_{k+1}}{B_{a+1}E_{k+1}} \right) \) at time \( t \). Hence, the value of the bond itself must be given by

\[ \tilde{B}_{a+1} = B_{a+1}E_0(E_1(\ldots E_k(E_{k+1}))) \]

From spot-forward parity, and the condition of the proposition

\[ \frac{B_{k+1}}{B_{a+1}} = \frac{B_{k+1}}{B_{a+1}} \]

or

\[ B_{k+1} = E_0(B_{k+1}) \]
Successive substitution of the condition (5) in (4) then yields

$$E_{0,t} = E_{0,t}E_{0}(B_{1,t}), \quad 0 < t < T.$$  

Hence, the calibration of the model ensures that the no-arbitrage equation holds for $\sigma = 0$. Finally, (6) guarantees that it holds for $\sigma > 0$. \(\square\)

Proposition 1 guarantees that the model is arbitrage free, if it is calibrated correctly to futures rates. The novel and important feature of the model is that the short-term interest rate, defined on a bankers' discount basis, follows a lognormal process. The implication, in proposition 1, is that the futures rate is a martingale, under the risk-neutral measure. Hence we can easily calibrate the model to the given term structure of futures rates, and thereby guarantee that the no-arbitrage property holds.

### 4.2 Regression Properties of the Model

The two-factor model of the term structure, described above, has the Hull-White characteristic that the conditional mean of the short rate is stochastic. Since the forward rate directly depends on the conditional mean, this induces an imperfect correlation between the short rate and the forward rate. In this section we establish the regression properties of the model, using the covariances of the short rate and premium processes. These properties are required as inputs for the building of a binomial approximation model of the term structure.

In the following lemma, we define the covariance of the logarithm of the short rate and the premium factor as $\sigma_{\text{r},\text{p}}$. The process assumed in Lemma 1 has the following properties:

**Lemma 3** Assume that

$$\ln(r_t) - \mu_r = [\ln(r_{t+1}) - \mu_{r_{t+1}}](1 - \alpha) + \ln(r_{t+1}) - \mu_{r_{t+1}} + \epsilon_t$$

where

$$\ln(r_t) - \mu_r = [\ln(r_{t+1}) - \mu_{r_{t+1}}](1 - \beta) + \nu_t$$

with $E_0(r_t) = 1$, $\forall t$.

Then,

1. the multiple regression

$$\ln\left[\frac{r_t}{E_0(r_t)}\right] = \alpha_r + \beta_r \ln\left[\frac{r_{t+1}}{E_0(r_{t+1})}\right] + \gamma_r \ln(r_{t+1}) + \epsilon_t \quad (8)$$
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has coefficients

\[ \alpha_n = \left( a \sigma^2_n \gamma_n \sigma^2_{n+1} + \gamma_n \sigma^2_{n+1} \right)^{1/2} \]
\[ \beta_n = (1 - a) \]
\[ \gamma_n = 1 \]

2. the regression

\[ \ln(r_n) = \alpha_n + \beta_n \ln(r_{n-1}) + e_n \]

has coefficients

\[ \alpha_n = \left( a \sigma^2_n + \sigma^2_{n+1} (1 - b) \right)^{1/2} \]
\[ \beta_n = (1 - b) \]

3. the conditional variance of \( \ln(r_n) \) is given by

\[ \sigma^2_{n-1, r_n} = a^2 \sigma^2_n + \sigma^2_{n+1} - \sigma^2_{n+1} - (1 - a) \sigma^2_{n+1, n+1} \]

where \( \sigma^2_{n+1} \) denotes the annualized covariance of the logarithm of the short rate and the premium factor.

Proof

First, we derive the following covariances from equation (3)

\[ \sigma_{n+1, n+1} = (1 - a) \sigma^2_n + \sigma^2_{n+1} \]
\[ \sigma_{n+1, n} = (1 - b) \sigma^2_{n+1} + (1 - a)(1 - b) \sigma^2_{n+1, n+1} \]

and

\[ \sigma_{n+1, n+1} = (1 - b) \sigma^2_n \]
\[ \sigma_{n+1, n+1} = (1 - a) \sigma^2_{n+1} + \sigma^2_{n+1} \]

Now from the multiple regression

\[ \ln \left( \frac{r_n}{E(r_n)} \right) = \alpha_n + \beta_n \ln \left( \frac{r_{n-1}}{E(r_{n-1})} \right) + e_n \]

the regression coefficients are

\[ \beta_n = \frac{\sigma_{n+1, n+1} \sigma^2_{n+1} - \sigma_{n+1, n} \sigma_{n+1, n+1}}{\sigma^2_{n+1} - (\sigma_{n+1, n+1})^2} \]
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\[ \gamma_n = \frac{\sigma_{\delta n+1, \sigma_{n, n-1}}^2}{\sigma_{n+1, n-1}^2 - (\sigma_{n, n-1})^2} \]

Substituting the covariances, and simplifying yields

\[ \beta_n = (1 - a) \]  \hspace{1cm} (9)

and

\[ \gamma_n = 1 \]  \hspace{1cm} (10)

From the lognormality of \( \tau_k \) and \( x_k \) we can write equation (3) as

\[ \ln(\tau_k) - \ln(E_0(\tau_k)) + \sigma_{\delta n}^2/2 = \{ \ln(\tau_k + 1) - \ln(E_0(\tau_k + 1)) + \sigma_{\delta n}^2/2 \}(1 - a) \]

\[ + \ln(\tau_k + 1) - \{ \ln(E_0(\tau_k + 1)) + \sigma_{\delta n}^2/2 \} + \epsilon_k \]

Re-arranging yields

\[ \ln \left[ \frac{\tau_k}{E_0(\tau_k)} \right] = \{ -\sigma_{\delta n}^2 + \sigma_{\delta n}(1 - a) + \sigma_{\delta n}^2/2 \}

\[ + \ln \left[ \frac{\tau_k + 1}{E_0(\tau_k + 1)} \right] (1 - a) + \ln(\tau_k + 1) + \epsilon_k \]

Given (8), (9), and (10), we have \( \alpha_n \) as stated in the lemma. Similarly, with \( E_0(\tau_k) = 1 \), we have

\[ \ln(\tau_k) = \alpha_n + \ln(\tau_k + 1)(1 - b) + \epsilon_k \]

and

\[ \alpha_n = E_0[\ln(\tau_k)] - (1 - b)E_0[\ln(\tau_k + 1)] \]

\[ \alpha_n = [-\sigma_{\delta n}^2 + (1 - b)\sigma_{\delta n}^2]/2 \]

Finally, the variance of \( \epsilon_k \), given \( \tau_k \), is

\[ \text{var}_\tau \epsilon_k(\tau_k) = \text{var}_\tau \left\{ \ln \left[ \frac{\tau_k}{E_0(\tau_k)} \right] \right\} - \beta_n \text{var}_\tau \left\{ \ln \left[ \frac{\tau_k + 1}{E_0(\tau_k + 1)} \right] \right\} \]

\[ - \gamma_n \text{var}_\tau[\ln(\tau_k + 1)] - \beta_n \gamma_n \text{cov}[\ln(\tau_k + 1), \ln(\tau_k)] \]
\[
\text{var}_t (x_t) = \sigma_n^2 - (1 - \alpha)^2 \sigma_{n-1}^2 - \sigma_{n-1}^2 \cdot (1 - \alpha) \sigma_{n-1, n-1}.
\]

Note that we require the multiple regression coefficients \( \alpha, \beta_n, \text{ and } \gamma \) in order to build the binomial approximation of the multi-asset process, using the method of Ho, Stapleton, and Subrahmanyan (1996). From part 1 of the lemma, the \( \beta_n \) coefficients simply reflect the mean reversion of the short rate. The \( \gamma \) coefficients are all unity, reflecting the one-to-one relationship between \( \tau \), the forward premium factor, and the expected spot rate. The \( \alpha \) coefficients reflect the drift of the lognormal distribution, which depends on the variances of the variables. Part 2 of the lemma shows that the regression relation for \( \tau \) is a simple regression, where the \( \beta_n \) coefficients reflect the constant mean reversion of the premium factor. Lastly, part 3 of the lemma gives an expression for the conditional variance of the logarithm of the short rate.

4.3 Determining the Volatility Inputs of the Model

In order to build the model outlined above, we need the parameters of the premium process, as well as those for the short rate process itself. The result in Lemma 3, part 3 gives the relationship of the conditional volatility of the short rate to the unconditional volatilities of the short rate, the volatility of the premium factor, and mean reversion of the short rate. We assume that the unconditional volatilities of the short rate are available, perhaps from capital/financial volatilities and that the mean reversion \( \gamma \) is given. The premium process, \( \tau_n \) on the other hand, determines the extent to which the first forward rate differs from the spot rate in the model. Since the premium factor is not directly observable, we need to be able to estimate the mean and volatility of the premium factor from the behavior of forward or futures rates. In order to discuss this, we first establish the following general results:

**Lemma 4.** Assume that

\[
\ln (\tau_t) - \mu_{\tau_t} = [\ln (\tau_{t-1}) - \mu_{\tau_{t-1}}] (1 - \alpha) + \ln (x_t) - \mu_{x_t} + \sigma^2_t
\]

where

\[
\ln (\tau_t) - \mu_{\tau_t} = [\ln (\tau_{t-1}) - \mu_{\tau_{t-1}}] (1 - \beta) + \nu_t
\]

with \( E_0(\tau_t) = 1, \forall t \), then the conditional volatility of \( \tau_t \) is given by

\[
\sigma^2_t (\tau) = [\sigma^2_t (\tau) - (1 - \alpha)^2 \sigma^2_{(\tau)}]^{\text{er}}
\]
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where \( s = E \theta(r_{t+1}) \) and \( \text{var} \ s[\ln(s_t)] = \sigma_s^2(t) \).

Proof

Taking the conditional expectation of equation (3) at \( t \)

\[
E \theta[\ln(r_{t+1})] - \mu_{w_t} = [\ln(r_t) - \mu_{r_t}](1-a) + \ln(s_t) - \mu_s.
\]

Given \( s_t = E \theta(r_{t+1}) \) and using the lognormal property of \( r_{t+1} \),

\[
\ln(s_t) = E \theta[\ln(r_{t+1})] + \sigma_s^2(t+1)/2 = \sigma_s^2(t+1)/2 + \mu_{w_t+1} + [\ln(r_t) - \mu_{r_t}](1-a) + \ln(s_t) - \mu_s + \sigma_s^2(t)(1-b) + \eta_t
\]

Hence

\[
\sigma_s^2(t) = \text{var} \ s[\ln(s_t)] = (1-a)^2 \text{var} \ s[\ln(r_t)] + \text{var} \ s[\eta_t]
\]

or

\[
\sigma_s^2(t) = \sigma_s^2(t) + (1-a)^2 \sigma_r^2(t)
\]

and the statement in the lemma follows \( \Box \)

Lemma 4 relates the volatility of the premium factor to the volatility of the conditional expectation of the short rate. To apply the result in the current context, we first assume that the short rate follows the process assumed in the lemma under the risk-neutral process. We then use the fact that the unconditional expectation of the 8th rate is \( f_{k,t+1} = E \theta(\tau_{t+1}) \), i.e. the first futures (or forward) rate is the expected value of the next period spot rate. This implication of no-arbitrage leads to

\[
\ln(f_{k,t}) = \mu_{\tau_{t+1}} + [\ln(r_t) - \mu_{r_t}](1-a) + \ln(s_t) - \mu_s + \sigma_s^2(t)/2.
\]  

It follows that the conditional logarithmic variance of the first futures rate is given by the relation

\[
\sigma_f^2(t) = (1-a)^2 \sigma_s^2(t) + \sigma_s^2(t).
\]

Hence, the volatility of the premium factor is potentially observable from the volatility of the first futures rate. This in turn could be estimated empirically or implied from options on the futures rate.
4.4 Estimating the Mean from Futures Rates

One important requirement of any financial model is that the parameters should be observable, or at least capable of being estimated, from market data. The use of the bank's discount definition of the rate means that the expected value of the short-term interest rate, under the risk-neutral measure, is directly observable from market prices of futures contracts. However, traded futures contracts are for LIBOR rates rather than for rates defined on a bank's discount basis. In the analysis that follows, we assume that the futures prices of zero-coupon bonds for the relevant maturity dates are either directly observable from the market for traded futures contracts, or estimable from the prices of bonds.\footnote{For the major currencies, Eurocurrency (based on LIBOR rates) futures rates are observable from the market. For US$, futures rates for the Eurodollar contract can be observed for up to ten years, with substantial liquidity. However, for most currencies, some estimation will be required, using market forward prices of bonds.}

The following argument can be used to back out estimates of the mean parameter, \( \mu_{eq} \), from market prices of futures contracts. First, we know, from market data, the LIBOR futures rate, for delivery of an \( m \)-maturity loan at date \( t \). We denote this rate as \( f_{0, m} \). The corresponding futures price of an \( m \)-period zero-coupon bond is denoted \( F_{0, m} \) and is given by

\[
F_{0, m} = \frac{1}{1 + f_{0, m}}
\]

where \( m \) is the loan maturity, adjusted for the Libor day-count convention. We now define the futures rate, on a bank's discount basis, as

\[
f_{0, m} = \frac{(1 - F_{0, m})}{m}.
\]

From now on, for simplicity, we refer to this rate as the futures rate.
5 The Multivariate-Binomial Approximation of the Process

5.1 The HSS approximation

The general approach we take to building a bivariate-binomial lattice, representing a discrete approximation of the process in equation (1), is to construct separate recombining binomial trees for the short-term interest rate and the forward-premium factor. The no-arbitrage property and the covariance characteristics of the model are then captured by choosing the conditional probabilities at each node of the tree.

We now outline our method for approximating the two-factor process interest rate process described above. We use three types of inputs: the unconditional means $E_0(r_t)$, $t = 1, \ldots, T$; the short-term rate, the volatilities of $r_t$ (i.e., the conditional volatility of the short rate, given the previous short rate and the one-period forward premium factor, denoted $\sigma_r(t)$) and the conditional volatilities of the premium, denoted $\sigma_\nu(t)$, estimates of the mean reversion, $a$, of the short rate, and the mean reversion, $b$, of the premium factor. The process in (3) is then approximated using an adaptation of the methodology described in Ho, Stapleton and Subesh Manram (1985) (HSS). HSS show how to construct a multiperiod multivariate-binomial approximation to a joint-lognormal distribution of $M$ variables with a recombining binomial lattice. However, in the present case, we need to modify the procedure, allowing the expected value of the interest rate variable to depend upon the premium factor. That is, we need to model the two variables $r_t$ and $\nu_t$, where $r_t$ depends upon $\nu_t$. Furthermore, in the present context, we need to implement a multi-period process for the evolution of the interest rate, whereas HSS only implement a two-period example of their method. In this section, these modifications and the multiperiod algorithm are presented in detail.

We divide the total time period into $T$ periods of equal length of $m$ years, where $m$ is the maturity period, in years, of the short-term interest rate. Over each of the periods from $t$ to $t + 1$, we denote the number of binomial time steps, termed the binomial density, by $m$. Note that, in the HSS method, $m$ can vary with $t$ allowing the binomial tree to have a finer density; if required for accurate pricing, over a specified period. This might be required, for example if the option exercise price changes between two dates, increasing the likelihood of the option being exercised, or for pricing barrier options.

We use the following result, adapted from HSS:

Lemma 5 Suppose that $X_t$ follows a lognormal process, where $E_0(X_t) = 1, \forall t$. Let the conditional logarithmic standard deviation of $X_t$ be $\sigma_\nu(t)$. If $X_t$ is approximated by a log-binomial distribution with binomial density $N_t = N_t(1 + m)$ and if the proportionate up and
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down movements, $v_k$ and $d_k$ are given by

$$d_k = \frac{2}{1 + \exp(2\sigma_2 \sqrt{2/\mu_2})}$$

$$v_k = 2 - d_k$$

and the conditional probability of an up-move at node $v$ of the lattice is given by

$$q_{uv} = \frac{E_u s(X_u) + (N_v - 1)\ln(v_k) - (v_k + v)\ln(d_k)}{v_k\ln(v_k) - \ln(d_k)}$$

then the unconditional mean and conditional volatility of the approximated process approach their true values, i.e., $E_u(X_u) \to 1$ and $\sigma_{uv} \to \sigma_v$ as $n \to \infty$.

**Proof** If $E_u(X_u) = 1$, then we obtain the result as a special case of HSS (1996), Theorem 1. $\square$

5.2 Computing the nodal values

In this section, we first describe how the vectors of the short-term rates and the premium factor are computed. We approximate the process for the short-term interest rate, $v$, with a binomial process, which moves up or down from its expected value, by the multiplicative factors $d_v$ and $v_v$. Following HSS, equation (7), these are given by

$$d_v = \frac{2}{1 + \exp(2\sigma_2 \sqrt{2/\mu_2})}$$

$$v_v = 2 - d_v$$

We then build a separate tree of the forward premium factor $x$. The up-factors and down-factors in this case are given by

$$d_x = \frac{2}{1 + \exp(2\sigma_2 \sqrt{2/\mu_2})}$$

$$v_x = 2 - d_x$$
At node $j$ at time $t$, the interest rates $r_k$ and premium factors $x_k$ are calculated from the equations

$$
\begin{align*}
  r_{k,j} & = \nu_{k_j}^{N_k} \nu_{p_{k_j}}^{x_{k_j}} (r_k), \\
  x_{k,j} & = \nu_{x_{k_j}}^{N_k} \nu_{p_{x_{k_j}}^{x_{k_j}}} (x_k), \\
  j & = 0, 1, \ldots, N_k
\end{align*}
$$

where $N_k = \sum_{k} u_k$. In general, there are $N_k + 1$ nodes, i.e., states of $r_k$ and $x_k$, since both binomial trees are recombining. Hence, there are $(N_k + 1)^2$ states after $t$ time steps.

### 5.3 Computing the conditional probabilities

As in HW, in general, the covariance of the two variables may be reflected by varying the conditional probabilities in the binomial process. Since the tree of the rates and the forward premium are both recombining, the time-series properties of each variable must also be captured by adjusting the conditional probabilities of moving up or down the tree, as in HSS and in Nelson and Ramaswamy (1991). Since increments in the premium variable are independent of $r_k$, this is the simplest variable to deal with. Using the result of lemma 3, we compute the conditional probability using HSS, equation (10). In this case the probability of a up-move, given that $x_{k-1}$ is at node $j$, is

$$
\begin{align*}
  p_{u_{x_{k}}} & = \frac{\omega_{x_{k}} + \beta_{x_{k}} \ln \pi_{x_{k+1}} - (N_k - j) \ln \pi_{x_{k}} - (j + 1) \ln \pi_{x_{k}+1}}{\ln \pi_{x_{k}} - \ln \pi_{x_{k}+1}}
\end{align*}
$$

where

$$
\begin{align*}
  \beta_{x_{k}} & = (1 - h) \\
  \omega_{x_{k}} & = (-\sigma_{x_{k}}^{2} + \beta_{x_{k}} \sigma_{x_{k-1}}^{2})/2
\end{align*}
$$

and where $h$ is the coefficient of mean reversion of $x$, and $\sigma_{x_{k}}^{2}$ is the unconditional logarithmic variance of $x$ over the period $(0 - t)$.

The key step in the computation is to fix the conditional probability of an up-movement in the rate $r_k$ given the outcome of $x_{k-1}$, the mean reversion of $r$, and the value of the premium factor $x_{k-1}$. In discussing the multi-period, multi-factor case, HSS present the formula for the conditional probability when a variable $x_k$ depends upon $x_{k-1}$ and a contemporaneous
variable, \( y_t \). Again, using the regression properties derived in lemma 3, and adjusting HSS, equation (12) in the present case, we compute the probability

\[
q_n = \alpha_n + \beta_n \ln \left( \frac{\pi_n(1)}{E(\pi_n)} \right) + \gamma_n \ln \pi_{n+1} - (N_t - j) \ln \pi_n - (j + n) \ln \pi_{n+1}
\]

(16)

where

\[
\beta_n = (1 - a)
\]

\[
\gamma_n = 1
\]

\[
\alpha_n = [\gamma_n \sigma^2_n + \beta_n \sigma^2_{n-1} + N_t \sigma^2_{n+1}] / 2
\]

Then, by lemma 3, the process converges to a process with the given mean and variance inputs.

5.4. The multi-period algorithm

HSS (1986) provides the equations for the computation of the modal values of the variables, and the associated conditional probabilities in the case of two periods \( t \) and \( t + 1 \). Efficient implementation requires the following procedure for the building of the \( T \) period tree. The method is based on forward induction. First, compute the tree for the case where \( t = 1 \). This gives the modal values of the variables and the conditional probabilities, for the first two periods. Then, treat the first two periods as one new period, but with a binomial density equal to the sum of the first two binomial densities. The computations are then made for period three modal values and conditional probabilities. Note that the equations for the up and down movements of the variables always require the conditional volatilities of the variables to compute the vectors of modal values. The following steps are followed:

1. Using equation (13), compute the \([n_3 \times 1]\) vectors of the modal outcomes of \( r_4, x_4 \) using inputs \( \sigma_4(1), E(\pi_4), \sigma_4(1), E(\pi_4) \) and binomial density \( \pi_4 \). Also compute the \([N_4 + n_3] \times 1\) vectors \( \pi_5, x_5 \) using inputs \( \sigma_5(2), E(\pi_5), \sigma_5(2) \), \( E(\pi_4) \) and binomial density \( \pi_5 \). Assume the probability of an up-move in \( r_4 \) is 0.5 and then compute the conditional probabilities \( q_{t+1} \) using equation (14) with \( t = 1 \). Then compute the conditional probabilities \( q_{t+1}, q_{t+1} \), using equations (14) and (16), with \( t = 2 \).

2. Using equation (13), compute the \([(N_4 + n_3) \times 1]\) vectors \( \pi_6, x_6 \) using inputs \( \sigma_6(3), E(\pi_6), \sigma_6(3), E(\pi_6) \) and binomial density \( \pi_6 \). Then compute the conditional probabilities \( q_{t+1}, q_{t+1} \), using equations (14) and (16) with \( t = 3 \).
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3. Continue the procedure until the final period $T$.

In implementing the above procedure, we first complete step 1, using $t = 1$ and $t = 2$, and with the given binomial densities $m_1$ and $m_2$. To effect step 2, we then redefine the period from $t = 0$ to $t = 2$ as period 1 and the period 3 as period 2 and re-run the procedure with a binomial densities $n_1^3 = n_1^4 + m_3$ and $n_2^3 = m_3$. This algorithm allows the multiperiod lattice to be built by repeated application of equations (13), (14) and (15).

5.5 A summary of the approximation method

We will summarize the methodology by using a two-period and a three-period example. Figure 1 shows the recombining nodes for the two-factor process in the two-period case. The interest rate goes up to $r_{1,0}$ or down to $r_{1,1}$ at $t = 1$. The forward premium factor goes up to $r_{1,0}$ or down to $r_{1,1}$ at $t = 1$, with probability $q_0$. In the second period, there are just 3 nodes of the interest rate tree, together with 3 possible premium factor values. There are 9 possible states, and the probability of an $r_2$ value materialising in $q_0$. Note that this probability depends on the level of the premium factor and of the interest rate at time $t = 1$. The recombining property of the lattice, which is crucial for the computability of the lattice, is emphasised in Figure 2, where we show the process for the interest rate over periods $t = 2$ and $t = 3$. After two periods there are 3 interest rate states and 9 states representing all the possible combinations of the interest rate and premium factor. The interest rate then goes to 4 possible states at time $t = 3$ and there are 18 states representing all the possible combinations of rates and premium factor. Note that the probability of reaching an interest rate state at $t = 3$, depends on both the interest rate and the premium factor at $t = 2$. It is these probabilities that allow the no-arbitrage property of the model to be fulfilled. In the model, the term structure at time $t$ is determined by the two factors, one representing the short rate and the premium factor. Thus, with a binomial density of $n = 1$, there are $(t + 1)^3$ term structures generated by the binomial approximation, at time $t$.

6 Model Validation and Examples of Inputs and Outputs

This section shows the results from several numerical examples and examines the two-factor term structure model described in previous sections in detail. Firstly, we show an example of how well the binomial approximation converges to the mean and unconditional volatility inputs. Secondly, we show that a two-factor term structure model can be run in a speedy
and efficient manner. Thirdly, we discuss the input and output for an eight-period example, showing illustrative output of zero-coupon bond prices, and conditional volatilities. Finally we show the output from running a forty-eight quarter model, including the pricing of European and Bermudan style options on coupon bonds.

4.1 Convergence of Model Statistics to Exogenous Data Inputs

The first test of the two-factor model is how quickly the mean and variance of the short rate generated converge to the exogenous data. Table 1 shows an example of a twenty-period model, where the input mean of the spot rate is 5%, with a 10% conditional volatility. There is no mean reversion and the premium has a volatility of 1%. Note that as the binomial density of 1, the accuracy of the binomial approximation deteriorates for later periods. This is due to the premium factor increasing with maturity and the difficulty of coping with the increased premium, by adjusting the conditional probabilities.

One way to increase the accuracy of the approximation is to increase the binomial density. In the last three columns of the table we show the effect of increasing the binomial density to 2, 3, and 4 respectively. By comparing different binomial densities in a given row of the table we observe the convergence of the binomial approximation to the exogenous inputs as the density increases. Even for the 20 period case, high accuracy is achieved by increasing the binomial density to 4.

Table 1 here

4.2 Computing Time

The most important feature of the two-factor model proposed in this paper is the computation time. With two stochastic factors rather than one, the computation time can easily increase dramatically. In table 2, we illustrate the efficiency of our model by showing the time taken to compute the zero-coupon bond prices and option prices. With a binomial density of 1, the 48-period model takes 123 seconds. Doubling the number of periods increases the computer time by a factor of six. There is a trade-off between the number of periods, the binomial density of each period, and the computation time for the model. This is illustrated by the second line in the table, showing the effect of using a binomial density of 2. Again the computation time increases more than proportionately as the density in-
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The time taken for the 28-period model, when the binomial density is 2, is roughly the same as that for the 48-period model with a density of 1.

Table 2 here

4.3 Numerical Example: An Eight-Period Bond

This subsection shows a numerical example of the input and output of the two-factor term structure model, in a simplified eight-quarter example. It illustrates the large amount of data produced by the model, even in this small scale case. The input is shown first in Table 3. We assume a rising curve of future rates, starting at 6% and rising to 6%. These are used to fix the means of the short rate for the various periods. The second row shows the conditional volatilities assumed for the short rate. These start at 14% and fall through time to 12%. We then assume a constant mean reversion of the short rate of 10%, and constant conditional volatilities and mean reversion of the premium factor, of 2% and 40% respectively.

Table 3 here

Tables 4 and 5 show a selection of the basic output of the model. For a binomial density of one, there are 4 states at time 1, 9 states at time 2, 16 states at time 3, and so on. In each state the model computes the whole term structure of zero-bond prices. In Table 4, we show just the longest bond price, paying one unit at period 8. These are shown for the 4 states at time 1, in the first block of the table. The subsequent blocks show the 9 prices at time 2, the 16 prices at time 3, and so on.

Table 4 here

One of the most important features of the methodology is the way that the no-arbitrage property is preserved, by adjusting the conditional probabilities at each node in the tree of rates. In Table 5, we show the probability of an up move in interest rate given a state, where the state is defined by the short rate and the premium factor. In the first block of the table is the set of probabilities conditional on being in one of four possible states at time 1. The second block shows the conditional probabilities at time 2, in the 9 possible states, and so on.
6.4 An Example of an Option on a Coupon-Bond

An important application of the model is to price and hedge contingent claims such as options with American and path-dependent features. A good example is a Bermudan option on a coupon bond. We illustrate our methodology by pricing a six-year option on a twelve-year coupon bond. The option has the Bermudan feature that it is exercisable, at par, at the end of each year up to the option maturity in year six. We first build the tree of rates, assuming the data detailed in the notes to Table 6. Note that the model uses 48 quarterly time periods, to cover the twelve-year life of the coupon bond.

Table 6 here

Table 6 shows that the Bermudan option on the coupon bond is worth considerably more than the European option. Also there is a small positive effect of valuing the options with the binomial tree with a density of two. Illustrative output of the price of the underlying coupon bond is shown in Table 7.

Table 7 here

7 Conclusions

In this paper we have presented a model of the term structure of interest rates which can be regarded as a two-factor extension of the Black-Karasinski lognormal-rate model. However, in this model, we assume that the short-term interest rate, defined on a bankers' discount basis follows a lognormal process. We have shown that, by calibrating to the current term structure of futures rates, the model is arbitrage-free in the sense of Ho and Lee (1986) and Pliska (1987).

The model was implemented by using a multivariate-binomial tree approach. By extending previous work of Nelson and Ramaswamy (1990) and Ho, Stapleton and Subrahmanyan (1993), we have developed a recombining bivariate-binomial tree, which has a non-explosive number of nodes.
A two-factor model of term structure

We have applied the model to the valuation of American-style and Bermudan-style options on coupon bonds. We have shown that prices are computable in seconds for examples with a realistic number of periods. Also, prices in the two-factor model exceed those in the one-factor model, calibrated to the same data.

A number of related research issues remain to be resolved in future work. First, the relationship between the Hull-White, Black-Karasinski type model developed here and the Heath-Jarrow-Morton multifactor model needs to be explored further. Specifically, we intend to generate forward-rate volatilities as outputs of our model in an extension of the analysis in the paper. Second, it is not clear to what extent a two-factor model, as opposed to a one-factor model, is necessary for the accurate valuation of specific interest-rate related contingent claims. We intend to investigate the properties of the model, in a subsequent paper, by applying it to a range of American-style, Bermudan-style and exotic options on bonds and interest rates. Third, the application of models of this type to derive risk management measures for interest-rate dependent claims should be studied further. We hope that the framework presented in this paper can be used to address these issues in further research.
References


Figure 1: A Recombining Two-factor Process for the Short-term Interest Rate (Two-period case)

$t = 1: 4$ states

$t = 2: 9$ states

[1] The probability of moving, for example, to $x_{2,0}$ given $(x_{1,0}, x_{1,0})$ is $g_{00}$, defined in Equation (16).

[2] The probability of moving, for example, in $(x_{1,0}, x_{1,0})$ given $x_{1,0}$ and $x_{1,0}$ is $g_{01}$, defined in Equation (14).
Figure 2: A Recombining Two-factor Process for the Short-term Interest Rate (Three-period case)

$t = 2$: 9 states

$t = 3$: 16 states

[1] The probability of moving for example, to \( r_{2,0} \) given \((r_{2,0}, r_{2,0})\) is \( \sigma_{r_{2,0}} \), defined in Equation (15).

[2] The probability of moving, for example, to \((r_{3,0}, r_{3,1})\) given \( r_{3,1} \) and \( r_{2,0} \) is \( \sigma_{r_{3,0}} \), defined in Equation (14).
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Table 1: Convergence of Term Structure Model

<table>
<thead>
<tr>
<th>Period</th>
<th>Binomial Density</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, 1]</td>
<td>mean</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
</tr>
<tr>
<td></td>
<td>volatility</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>[0, 2]</td>
<td>mean</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>volatility</td>
<td>0.07</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>[0, 3]</td>
<td>mean</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>volatility</td>
<td>0.05</td>
<td>0.07</td>
<td>0.07</td>
<td>0.07</td>
</tr>
<tr>
<td>[0, 4]</td>
<td>mean</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>volatility</td>
<td>0.02</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>[0, 5]</td>
<td>mean</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>volatility</td>
<td>0.02</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>[0, 10]</td>
<td>mean</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>volatility</td>
<td>0.02</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>[0, 20]</td>
<td>mean</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>volatility</td>
<td>0.02</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
</tr>
</tbody>
</table>

The numbers in the table are the computed means and volatilities, in percent, for the short rate over periods 1, 2, 3, 4, 5, 10, and 20, using the output of the two-factor model. The means are calculated using the possible outcomes and the nodal probabilities. The volatilities are the annualized standard deviations of the logarithm of the short rate. The binomial density refers to the density of the binomial tree of the short rate and the premium factor, over each sub interval. The input parameters in this case are a constant mean of 5%, and conditional volatility of 10% with no mean reversion of the short rate. The premium factor has a volatility of 1%, a mean of 1, and no mean reversion.
A two-factor model of term structure.........................................................32

Table 2: Computing Time for Bond and Option Pricing (seconds)

<table>
<thead>
<tr>
<th>Number of Periods</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binomial Density 1</td>
<td>0</td>
<td>0.3</td>
<td>2.5</td>
<td>12.3</td>
</tr>
</tbody>
</table>
| Binomial Density 2| 0.3| 1  | 12.7|-

The table shows the time taken to compute all the zero-bond prices, coupon bond prices and option prices, given the tree of rates. The computer speed is 266 MHz, and the processor is Pentium.
A two-factor model of term structure

Table 3: 8-period Example Input

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Futures rate</td>
<td>5.0</td>
<td>5.2</td>
<td>5.4</td>
<td>5.6</td>
<td>5.7</td>
<td>5.8</td>
<td>5.9</td>
<td>6.0</td>
</tr>
<tr>
<td>Conditional volatility (r)</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td>1.3</td>
<td>1.3</td>
<td>1.25</td>
<td>1.25</td>
<td>1.20</td>
</tr>
<tr>
<td>Mean reversion (r)</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>Conditional volatility (x)</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>Mean reversion (x)</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
</tr>
</tbody>
</table>

All numbers are in percent. The table shows the exogenous data input for an 8-period example. The short rate is the quarterly rate, so the period length is quarter of one year. Input data relating to the short rate appears in the first three rows; data relating to the premium in the last two rows.
Table 4: Illustrative Output of Zero-Coupon Bond Prices

\[
\begin{array}{cccccc}
0.9002111 & 0.9004773 \\
0.9110627 & 0.9114909 \\
0.9009610 & 0.9001703 & 0.9123043 \\
0.9163451 & 0.9192411 & 0.9220527 \\
0.9236451 & 0.9282168 & 0.9307135 \\
0.9148397 & 0.9173619 & 0.9198222 & 0.9222217 \\
0.9238716 & 0.9262163 & 0.9286166 & 0.9306271 \\
0.9321501 & 0.9341906 & 0.9361978 & 0.9381456 \\
0.9391666 & 0.9413137 & 0.9431125 & 0.9448636 \\
0.9528570 & 0.9528722 & 0.9530141 & 0.9532155 & 0.9538211 \\
0.9541906 & 0.9560650 & 0.9576500 & 0.9591859 & 0.9606925 \\
0.951610 & 0.9527034 & 0.9541116 & 0.9555023 & 0.9566622 \\
0.9473504 & 0.9486192 & 0.9499236 & 0.9511710 & 0.9521006 \\
0.9528166 & 0.9539860 & 0.9551371 & 0.9562645 & 0.9573702 \\
0.9416127 & 0.9426355 & 0.9436678 & 0.9446880 & 0.9456387 & 0.9466177 \\
0.9476198 & 0.9486547 & 0.9496386 & 0.9506641 & 0.9516251 & 0.9521436 \\
0.9530188 & 0.9536514 & 0.9546810 & 0.9557460 & 0.9563968 & 0.9571062 \\
0.9579691 & 0.9586239 & 0.9593701 & 0.9601090 & 0.9606899 & 0.9615607 \\
0.9622218 & 0.9629064 & 0.9635504 & 0.9642179 & 0.9649062 & 0.9655575 \\
0.9661353 & 0.9667350 & 0.9673558 & 0.9679650 & 0.9685555 & 0.9691435 \\
\end{array}
\]

All prices are for a zero-coupon bond paying one unit of currency at the end of period 5. The first set of 4 numbers are the time 1 bond prices \( B_{1,s} \), the second set of 9 numbers are the time 2 bond prices \( B_{2,s} \), through to the time 5 set of 36 prices \( B_{5,s} \). The 19 prices, \( B_{3,s} \), and 64 prices, \( B_{4,s} \), are not shown for reasons of space.
Table 5: Illustrative Output of Conditional Probabilities

\[
\begin{align*}
0.5530794671 & \times 1.0087419001 \\
0.5906241619 & \times 0.418190776 \\
0.6622172125 & \times 0.5086479060 \\
0.6516626419 & \times 0.4976668530 \\
0.6106951673 & \times 0.4871225154 \\
0.7780765869 & \times 0.6126741059 \\
0.7546745612 & \times 0.5804720682 \\
0.7316725635 & \times 0.5662700705 \\
0.7084704158 & \times 0.5430686025 \\
0.8091566018 & \times 0.6577849469 \\
0.8216723262 & \times 0.6355006704 \\
0.8105805066 & \times 0.6092163017 \\
0.8363087719 & \times 0.6814321191 \\
0.8772019498 & \times 0.7006475435 \\
1.0000000000 & \times 0.8611991841
\end{align*}
\]

All the probabilities are conditional probabilities of an up-move in the interest rate, given the short rate and the premium factor. The first set of 9 numbers are the probabilities at time 1, the second set of 9 numbers are the conditional probabilities at time 2, through to the time 5 set of 36 conditional probabilities. The 48 probabilities at time 6, and the 64 probabilities at time 7, are not shown for reasons of space. In each case the columns show the probabilities for different (increasing to the right) values of the short rate. The rows show the values (increasing downwards) for different values of the premium factor.
Table 6: 48 period Option Price ($)

<table>
<thead>
<tr>
<th>Binomial Density</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bermudan Call</td>
<td>0.867</td>
<td>0.874</td>
</tr>
<tr>
<td>European Call</td>
<td>0.505</td>
<td>0.509</td>
</tr>
</tbody>
</table>

The above table shows the value of a 6-year call option on a coupon bond that pays $100 in 12 years time. The annual coupon rate of the bond is 5%. The interest rate tree is built assuming a future rate of 5% for each maturity, a volatility of the short rate of 10%, a coefficient of mean reversion of the short rate of 10%, and volatility of the premium at 1% with 30% coefficient of mean reversion.
Table 7: Illustrative Output of Coupon Bond Prices

<table>
<thead>
<tr>
<th>Bond Price</th>
<th>0.9341104</th>
<th>0.9418602</th>
<th>0.9495429</th>
<th>0.9571560</th>
<th>0.9647215</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.9541104</td>
<td>0.9619713</td>
<td>0.9698134</td>
<td>0.9776517</td>
<td>0.9851703</td>
</tr>
<tr>
<td></td>
<td>0.9740350</td>
<td>0.9813479</td>
<td>0.9886932</td>
<td>0.9957732</td>
<td>1.0029000</td>
</tr>
<tr>
<td></td>
<td>0.9959119</td>
<td>1.0000000</td>
<td>1.0070360</td>
<td>1.0140930</td>
<td>1.0210150</td>
</tr>
<tr>
<td></td>
<td>1.0210800</td>
<td>1.0279730</td>
<td>1.0354350</td>
<td>1.0428210</td>
<td>1.0502390</td>
</tr>
</tbody>
</table>

The coupon bond prices are computed at each of 25 nodes at the end of year 1, shown in the first block in the table. 91 prices are computed at the end of year 2, shown in the second block of the table. The coupon bond has an annual coupon of 5% and a par value of 1 unit. Prices are also computed at the end of year 3, year 4, year 5, and year 6, but are not shown here for reasons of space.