The Term Structure of Interest Rates: Alternative Approaches and Their Implications for the Valuation of Contingent Claims[1]

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One of the most active areas of research within the literature on financial economics, in recent years, has been the modeling of the term structure of interest rates and its relationship to the pricing of interest rate contingent claims. The interest in this area stems both from the conceptual richness of the problems posed, as well as the practical issues that arise in the implementation of the models developed. It goes without saying that a considerable impetus for this research, particularly on questions of implementation, has come from the world of practice, which has experienced a virtual explosion in the variety and quantity of interest rate derivative products.

A casual survey of the area suggests that researchers have addressed a variety of issues in the modeling of the term structure, often in the context of particular models. These issues span the spectrum from the equilibrium determination of interest rates to alternative specifications of interest rate processes, and from arbitrage-free pricing models for bonds and interest rate contingent claims to the implementation of these models. Often, the papers in the area are concerned with both the modeling of such claims with a focus on pricing issues from a financial economist's perspective, as well as a solution to problems of implementation from a financial engineering point of view. These diverse perspectives sometimes make it difficult to place the work in the area in a broader framework.

Given the vast array of issues in the area, as well as the variety of perspectives, ranging from a theoretical financial economist's to that of a practitioner in the field, there is an amazing quantity of detail that confronts anyone attempting to get an overall perspective of the field. Not surprisingly, perhaps, there have been a number of surveys of the literature in the area. One problem with these surveys is that while many of them catalog the various papers in the area, sometimes in a fair amount of detail, it is difficult to get a grip on what the central issues are, what questions have been answered satisfactorily and where the gaps are in our understanding. In addition, the distinction is not often made between the general requirements for such models, such as an arbitrage-free term structure, that have to be satisfied by *any* model that is proposed, and issues of detail, such as the precise specification of the stochastic processes, which have to be resolved ultimately by empirical examination.

This paper attempts to provide such an analysis but from an overall perspective, rather than on questions of detail. Two issues of general interest in the area of term structure modeling and the valuation of interest rate contingent claims will not be discussed here in much detail: problems of implementation of the models and the empirical evidence. Each of these issues requires a detailed examination in its own right and it would be overly ambitious to survey them in the context of the present general overview of the literature. However, we present here the main conclusions of the empirical evidence as well as the highlights of the implementation issues.

Section 1 discusses the main issues that have been posed in the broad area of term structure modeling and places them in context. Section 2 provides an overview of five seminal models in the area that provide the foundation for further work. A brief review of the empirical issues in the area and a summary of the main conclusions is given in section 3. In the next section, section 4, an attempt is made to synthesize the work by defining issues where there appears to be general agreement. Section 5 concludes by discussing the issues that remain unresolved and possible directions for further work in the area.

1 Issues in modeling the term structure of interest rates and the pricing of interest rate contingent claims

At a general level, financial economists are concerned with the equilibrium foundations of the model they develop. Although most financial models are pure-exchange models and are not concerned with production economies, it is important to ensure that there is no inconsistency between the asset return processes that are assumed and the underlying production economy. In particular, the question is how the state variables in the underlying production economy, when placed in equilibrium setting, yield the commonly-used specifications of the stochastic processes generating asset returns.

The issue of consistency between the underlying production economy and the stochastic process generating asset returns is virtually impossible to answer at a completely general level. Since the equilibrium prices of assets are determined by the interaction of the preferences of consumer-investors and the production processes that are available, it would be necessary to specify both assumptions in some detail in order to get an explicit analytical solution for the stochastic process for asset prices.[1]

Since most financial models deal only with pure exchange equilibria, they are partial equilib-

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[1] Recently, there have been a few papers that claim that an equilibrium model can be developed to support virtually any kind of interest rate process. See Boldrin and Montrucchio (1986), for example. However, even if one accepts such a claim, it is necessary to develop an explicit model for several applications.

rium models viewed from the above perspective. However, even within such partial equilibrium models, the issue of consistency arises. In many cases, this amounts to a requirement that assets be priced consistently with respect to each other. This requirement can be stated as an arbitrage condition within the model, for instance that pricing of different assets should imply the same market price of risk, i.e., the model has to be internally arbitrage-free. Again, it is difficult to derive such specific pricing results for arbitrary stochastic processes. Restrictions have to be placed on the form of the stochastic process in order to derive closed-form solutions for bond prices and the prices of contingent claims. Unfortunately, some processes that have the virtue of analytic tractability, may have undesirable economic (or even econometric) properties. For instance, Gaussian interest rate processes yield closed-form solutions for the prices of bonds and contingent claims, but allow for negative nominal interest rates, which would be inconsistent with basic arbitrage requirements. From an econometric perspective, the choice of a process with no known unconditional distribution makes the use of estimation methods other than the generalized method of moments (GMM) hard to employ.

A basic requirement of contingent-claim models since the work of Black-Scholes (1973) is that the price of the underlying asset be taken as given, and also that its current market price be consistent with the price dynamics implied by the model. For instance, in the context of the original Black-Scholes model, this would suggest that the price of the stock be taken as given, and its price be determined as the present value of the expected stock price in the future under the risk-neutral distribution. However, in the context of term structure models, there is more than one underlying asset involved, since the prices of interest contingent claims depend, in principle, on the current prices of bonds of all maturities along the term structure. Furthermore, these bonds themselves have to be priced relative to the future possible movements in an arbitrage-free manner. Since there are several bonds involved, this translates into two types of no-arbitrage conditions. Firstly, each bond has to be priced correctly with respect to its future price evolution. Secondly, the price movements across two or more bonds should not imply arbitrage opportunities. The latter feature is indeed the key distinguishing feature of term-structure models. To illustrate, the pricing of the stock options involves a single-step exercise, given the stochastic process for stock prices. In contrast, the pricing of bond options is , in principle, a two-step effort requiring first the derivation of the bond price process from the interest rate process, and then the pricing of bond options as a second step.

An additional issue that arises in specifying the current term structure (the underlying assets) is whether to specify it in terms of spot interest rates or forward interest rates. Although at a broad conceptual level, the two specifications would be equivalent, the input data and the details of the models may be quite different, depending on the stochastic processes assumed. As an example, the derivation of results on the equivalent martingale measure may be easier with the forward rate specification. Also, as we shall discuss below, it may be in line with no-arbitrage restrictions to assume that the volatility of forward rates is constant, while this may be difficult to do with spot interest rates. Thus, for particular purposes, one representation may turn out to be more convenient than the other.

The specification of the interest rate process, either directly assumed or derived from the underlying state variables, has to be in line with certain stylized facts. Although some of these stylized facts can be discussed at a broad conceptual level, others can only be related to the empirical evidence. The basic issue that separates models is the number of factors used to describe the term structure movements over time. The simplest structure would involve only one factor so that all bond prices are instantaneously perfectly correlated, with the single factor usually being taken as the short term interest rate. Generalization to two or more factors requires care on two grounds. Firstly, from a conceptual perspective, the relationship between the factors has to be explicitly taken into account, in order to be consistent with the no-arbitrage requirement. For example, if the two factors are a short-term and a long-term interest rate, it should be borne in mind that the long-term bond is related to a series of short-term bonds in a risk-neutral world and hence, the movements have to be tied together in some fashion. As a practical matter, due to problems of computational complexity, the number of factors is usually restricted to a maximum of two.[2]

From an empirical viewpoint, there is considerable latitude in the choice of the two factors. Examples include the short term interest rate and the long term interest rate [e.g., Brennan and Schwartz (1979)], the short term interest rate and a spread between the long-term and short term interest rates [Schaefer and Schwartz (1984)] the short term interest rate and the rate of inflation [Cox, Ingersoll and Ross (1985b)] and the short term interest rate and its volatility [Longstaff and Schwartz (1992)], the short rate and duration [Schaefer and Schwartz

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[2] It is often convenient to depict term structure movements in terms of (1) parallel shifts, (2) twists, and (3) changes in curvature, which effectively implies a three factor model. There are some three factor models available such as Chen and Scott (1993), although the implementation of such models for general bond options would be extremely complex.

(1984)] and the short rate and its mean [Das and Foresi (1996)]. At a more detailed level, there is the question of what the best proxies are for each of these variables. For instance, should the short-term interest rate be a three-month rate or a four-year rate?[3] The empirical analysis could be at two levels, either to see how various alternative specifications fit the term structure data, or, to examine how the contingent claims models implied by these structures fit the prices of interest rate options such as caps, floors and swap options.

The other aspects of the term structure specification are the distributional assumptions built in. Three issues are relevant for discussion here. Firstly, is the process a pure diffusion or does it allow for jumps? Secondly, in the case of diffusion process, are there any restrictions placed on the drift term, such as mean-reversion or some type of time-dependence, for example? Thirdly, what is the specification of the diffusion term and what implication does this have for the distribution of interest rates?

Turning to the first issue of jumps, most option pricing models assume diffusion processes for the price of the underlying asset. This is natural, given the dynamic hedging arguments on which contingent claim pricing models are based. However, empirical work has pointed to the existence of jumps in the interest rate processes. Discontinuities in the price paths of the underlying asset can be accommodated in these models only with the imposition of risk-neutrality or the assumption of a state variable that correlates perfectly with the jump component.

The second issue of mean-reversion is an aspect of the process governing movements in the short-term interest rates that has often been postulated. Although the empirical evidence in support of the existence of mean-reversion is somewhat weak, both academics and practitioners feel that this is an important stylized fact to be built into interest rate option pricing models, particularly when applied to high frequency data.

The third issue of the choice of the specification of the diffusion term is dictated by certain stylized facts tempered by analytical tractability. At a conceptual level, the main constraints imposed by these stylized facts would be the restriction that *nominal* interest rates stay non-negative, with the mean-reversion term being taken into account. At an empirical level, the Gaussian and the lognormal distributions for interest rates are taken to be the polar cases, with

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[3] See Nelson and Schaefer (1983) and Elton, Gruber and Michaely (1990), for example, for a detailed discussion of the considerations involved in making the choice about the number and type of factors and the empirical evidence from the U.K. and U.S. respectively.

other cases in between being characterized by the elasticity of the volatility in the diffusion term. Although the empirical evidence is so far inconclusive, it appears that the elasticity lies between one-half and unity (if stochastic volatility is not assumed). If the criterion of analytical tractability is used to obtain closed-form solutions for bond and contingent claim prices the candidates for the interest rate processes are three special cases: Gaussian and the square root processes (for which the elasticity parameter is one-half) and the lognormal case under some special circumstances.

2 An Overview of the Major Approaches to Term Structure Modeling

Although the literature in the area is vast, it turns out that five papers capture most of the considerations discussed in the previous section. Much of the other work in the area may be considered to be extensions or elaborations of these five papers. It would be useful, in a survey of this nature, to isolate the main features of these five approaches and the strengths and weaknesses based on the specific aspects of each of the various approaches, in order to provide a synthesis of the area and point out the lacunae in the present paradigm.

2.1 The Black-Scholes-Merton Model

The Black-Scholes (1973) and Merton (1973b) papers establish the basic paradigm for the pricing of contingent claims. Although these papers primarily discuss the pricing of options on stocks, the application of the approach to the case of bond options is also discussed by Merton. The Merton modification of the basic Black-Scholes model views the stochastic process for the price of a long-term bond as being similar to that of a stock, i.e. as a lognormal diffusion process

$$dB/B = \mu dt + \sigma dz \tag{1}$$

where B is the discount bond price, μ is the drift term or instantaneous expected return, σ is the volatility and z is the standard Brownian motion.

The price of any contingent claim based on such a bond can be determined by constructing a dynamic hedge based on the underlying bond and the (short-term) riskless asset. The price of the contingent claim, determined by this replication argument, depends on the payoffs on this claim and can be determined by a variety of methods. The original method proposed by Black-Scholes involves solving the fundamental partial differential equation for the price of a contingent claim, subject to the boundary conditions defined by the payoffs of the claim. The alternative martingale approach, which is now more commonly used, involves the evaluation of the discounted expected values of the payoff of the option. For example, for a European call option on a bond, with maturity T, the modified Black-Scholes model proposed by Merton, implies that the price would be given by

$$C = BN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{2}$$

where

$$d_1 = \frac{\ln(B/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

where C is the price of the European call option, K is its exercise price, r is the instantaneous risk-free rate (assumed constant), σ is the volatility of the instantaneous change in the (log) bond price and $T-t$ is the time to expiration. $N(\cdot)$ is the cumulative normal density function.

The above model is a fairly straight-forward extension of the Black-Scholes model to the case of bond options and is used widely for relatively short-dated options on long-term bonds. The main virtue of this basic model for pricing bond options is its simplicity and intuitive appeal, given the wide-spread use of the Black-Scholes model. However, at a conceptual level, the model has some fairly serious shortcomings. The basic insight of the Black-Scholes approach is the risk-neutral valuation argument that the instantaneous drift term of the return on the underlying asset (which is preference-dependent and not observed) can be replaced by the short-term interest rate, r, which is observable. This does not raise any conceptual problems in the case of options on stocks, since it is possible, in principle, for stock prices to be volatile, while the risk-free rate is deterministic and, in particular, constant. However, when applied to bond options, this argument is problematic. It is hard to imagine how the short-term interest rate can be non-stochastic while the long-term bond prices are volatile. More fundamentally, the no-arbitrage conditions between bonds of different maturities are not explored. Furthermore, the information contained the current term structure which would be relevant to the characterization of the price of the underlying asset, the long-term bond is not taken into account.

A second issue, that is probably more important for the valuation of bond options is the specification of the volatility parameter. For the case of options on stocks, which are analyzed in the basic Black-Scholes framework, it is not unreasonable to suppose that the volatility of the return on the underlying asset is constant through time, since the asset is infinite-lived. Thus, for stocks, it makes sense to assume that the range of possible stock prices changes gets larger as the expiration date of the option is moved further out in time. For the case of bonds, however, this is not a sensible assumption, since the price of a (default-free) bond on its maturity date is known with certainty. Hence, unlike the prices of stocks, bond prices must have an increasing range of values for a period, but thereafter, have a decreasing range, converging to a fixed price on the maturity date of the bond.[4]

In order to address these concerns, some researchers model the behavior of the volatility of the bond price over time, in an exogenous fashion, while preserving the basic Black-Scholes structure. These approaches include the Brownian-bridge process proposed by Ball and Torous (1983), for which the bond prices at the current date and the maturity date are "tied down", leaving the process in-between to fluctuate, similar to a suspension bridge. It is necessary, therefore to specify the process followed by the bond price between the current date and the maturity date. Particular specifications of this process include the Macaulay duration-based specification of the volatility parameter suggested by Schaefer and Schwartz (1984). The intuition behind this approach is that the duration of the bond is highest at the current date, declining to zero on the maturity date of the bond, so that the volatility parameter exhibits a similar behavior. Schaefer and Schwartz suggest that a time-varying parameter could then be built into the Black-Scholes framework to obtain at least a partial resolution to the volatility problem. However, in these *ad hoc* approaches, basic arbitrage conditions may be violated and hence the solutions may be unsatisfactory.

A third problem with applying the Black-Scholes model to bond options is the implication for interest rates. Since the price of the bond follows a lognormal diffusion process, the instantaneous returns would be normally distributed. In a one-factor model, this would imply that the short term interest rate is also normally distributed, which would mean that interest rates could become negative. An alternative to preserve the property of positive interest rates is to assume that interest rates are lognormally distributed. The Black-Scholes model can be modified for options on forward contracts to yield the Black (1976) model, which can be written

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[4] However, it may be sensible to model bond options based on the *forward* price of bonds, using the Black-Scholes-Merton approach. This approach is widely used in practice to price European-style option products, such as interest rate caps and floors, as discussed below.

as follows:

$$\begin{aligned} C_f &= e^{-r(T-t)}[FN(d_1) - KN(d_2)] \\ d_1 &= \frac{\ln(F/K) + \sigma^2/2(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

where C_f is the price of a call option on a forward contract on the underlying asset, F is the forward price, and the other variables are as defined earlier. The d_1 and d_2 terms do not have the interest rate in them since it is already reflected in the forward price. The Black model is widely used to value interest rate caplets and floorlets, which can be regarded roughly as options on the forward interest rate multiplied by a face value. In other words, the forward price and the strike price are replaced by the forward interest rate and the strike rate of the caplet/floorlet. Again, this solution may be practically quite useful, but is not properly grounded in an arbitrage-free specification.

2.2 The Vasicek Model

Soon after the development of the Black-Scholes model, Vasicek (1977) developed a partial equilibrium model of the term structure of interest rates. Although the model does not fully use the dynamic hedging arguments of Black-Scholes, since it does not take the current term structure as given, it applies the continuous-time methodology and the concept of arbitrage that is central to the original Black-Scholes formulation.

At a general level, the model postulates that the short-term interest rate, the only state-variable in the model, follows a diffusion process

$$dr = \alpha(r,t)dt + \sigma_r(r,t)dz \tag{3}$$

where $\alpha(r,t)$ and $\sigma_r(r,t)$ are the instantaneous drift and standard deviation respectively of the process for $r(t)$ and z is a Brownian motion. Since r is the only state variable in the model (a key assumption of the model), the price of any discount bond, whose price depends on the current date, the maturity date and the state variable, can be written by Ito's Lemma, as a function of these variables

$$dB = B\mu(t)dt - B\sigma(t)dz \tag{4}$$

Since there is only one stochastic variable driving all bond prices, it is possible to set up a hedge portfolio, consisting of bonds of two different maturities, that is instantaneously riskless. Note that this explicitly rules out, by assumption, other possible influences on the bond price process.

The arbitrage mechanism across bonds, which is similar in spirit to the Black-Scholes argument, yields the condition that the ratio of excess return to risk, the measure of the market price of risk, has to be equated across bonds

$$\lambda(t,r) = \frac{\mu(t,T_1) - r(t)}{\sigma(t,T_1)} = \frac{\mu(t,T_2) - r(t)}{\sigma(t,T_2)} \tag{5}$$

where T_1 and T_2 are the maturity dates of the two discount bonds and λ is the market price of risk. The arbitrage condition also yields a partial differential equation similar to the one derived by Black-Scholes, that can be solved subject to the boundary condition that the bond price at maturity has to be equal to its face value of unity:

$$\frac{\partial B}{\partial t} + (\mu + \lambda\sigma)\frac{\partial B}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 B}{\partial r^2} - rB = 0 \tag{6}$$

There is one important difference between the above equation and its Black-Scholes counterpart: the presence of the market price of risk, λ. Since this parameter would depend on the risk attitudes of investors, the model is not preference-free unlike the Black-Scholes formulation. The reason this is so is not hard to seek: the Black-Scholes model takes the stock price, which reflects the market price of risk, as given. In contrast, the Vasicek model solves for all bond prices relative to each other. The only way to tie down the prices is by invoking the exogenous parameter, the market price of risk.

After deriving the general model, which cannot yield bond prices in closed form, Vasicek provides an explicit solution for the case of the Ornstein-Uhlenbeck (O-U) process for the case of a constant λ:

$$dr = \kappa(\ell - r)dt + \sigma dz \tag{7}$$

where $\kappa > 0$ is the "elasticity" of the mean-reversion term and ℓ is the long term mean towards which the short rate is pulled. The attractive feature of this process is that the process converges to a stationary distribution, unlike a diffusion process without the mean-reversion term. Furthermore, the process is in line with empirical work on interest rate, which suggests mean-reversion in the short rate, although the evidence in support is somewhat mixed. Since the conditional expectation and variance of the process can be computed, it is possible to derive closed-form expressions for bond prices and interest rates, either spot or forward. However, all these expressions are dependent on the exogenous parameter, the market price of risk, λ, which is assumed to be constant. The latter assumption obviously implies strong restrictions on the preferences of investors.

The strength of the Vasicek formulation is that it focuses attention on the notion of arbitrage across bonds, in the spirit of Black-Scholes and Merton [(1971), (1973a) and (1973b)]. It proposes a special case of a mean-reverting Gaussian interest rate process which has the twin virtues of intuitive appeal and analytical tractability. The intuitive appeal rests on the concept of mean-reversion which is reflected in this process. The closed form expressions that the model yields allow for empirical testing, without too much computational effort.

The main weaknesses of the model are the assumptions of an exogenous market price of risk, which means that the model is not preference-free. Worse, particular choices of the parameter may imply arbitrage opportunities, causing the model to be internally inconsistent. Also, the model does not incorporate the observed current term structure, which means that the model prices do not reflect current market information (although this could be achieved with a choice of the market price of risk, λ, that varies with maturity). In turn, this means that a pricing of contingent claims would be subject to two possible sources of error: the error in explaining the current term structure and the errors in the model for pricing contingent claims, given the model term structure. The specific mean-reverting Gaussian model assumes a constant market price of risk, which may be inconsistent with the absence of arbitrage. Also, given that interest rates are normally distributed, there is a positive probability of negative interest rates, which is not reasonable.

The Vasicek model has been extended in three directions in subsequent research. Firstly, alternative processes have been proposed for the stochastic process for the short term interest rate (1978). The work of Dothan (1978), Courtadon (1982), Cox, Ingersoll and Ross (1985b) [CIR] and Stapleton and Subrahmanyam (1993) can also be placed in this category, although the CIR paper has several other additional aspects of interest, that will be discussed below. A general formulation which allows for non- Gaussian interest rates would be:

$$dr = \kappa(\ell - r)dt + \sigma_r r^{\beta} dz \tag{8}$$

where β is between 0 and 1. $\beta = 1$ would be the lognormal case, while $\beta = 1/2$ permits an analytical solution as shown by CIR. Secondly, the Vasicek model has been extended to include multiple factors, in particular two factors, as in the work of Brennan and Schwartz (1979) and Longstaff and Schwartz (1992), among others. There is considerable latitude in picking the second factor, for instance, the long-term interest rate, the spread between the short rate and the long-term rate, inflation and volatility of the short rate itself. One common structure proposed by Brennan and Schwartz (1979) includes a short rate which is mean-reverting to the long rate, ℓ, the second factor:

$$\begin{aligned} dr &= \kappa(\ell - r)dt + \sigma_1 r dz_1 \\ d\ell &= \sigma_2 \ell dz_2 \\ dz_1 dz_2 &= \rho dt \end{aligned}$$

where σ_1 and σ_2 are standard deviations of the short and long rate process respectively and ρ is the correlation between them. In this model, the long rate is taken to be the yield on a perpetual bond, which can be observed (approximately). However, the market price of risk, is still required to be specified in the pricing of all bonds, since the short rate process is not hedgeable. The attractive feature of a two-factor model is that it can explain a greater variety of term structure movements over time. The disadvantages are the reliance on an exogenous parameter, the market price of risk, which is not observable and the cumbersome numerical implementation of the model, since closed-form solutions are often difficult to obtain. Also, even such a two-factor model is unable to accommodate all shapes of the term structure, in particular, the inverted hump shape.

A third direction in which the Vasicek model has been extended is to derive the prices of contingent claims. Since all contingent claims have to satisfy the Vasicek partial differential equation, but with different boundary conditions, one can solve for prices of interest rate options. However, the solution will involve the market price of risk parameter. If one uses the bond price equation as well, in principle, one can eliminate λ from the pricing equation for contingent claims. This is essentially the Gaussian interest rate model of Jamshidian (1989), which leads to closed-form solutions for options on discount bond prices, with the important difference that the λ parameter is not eliminated.

2.3 The Cox-Ingersoll-Ross Model

The CIR model [(1985a) and (1985b)] is the first general equilibrium model of asset prices which can also be applied to model the term structure of interest rates. Although there is a considerable amount of notation and technical detail in the general version of the model, CIR (1985a), the basic economic arguments can be summarized in an intuitive manner. The model assumes a single-good, continuous-time economy with n linear production activities, whose expected return vector and variance-covariance matrix are specified. There is also a vector of state variables whose expected return and variance-covariance matrix are given. There are many identical consumer-investors who maximize a time-additive utility function by choosing their consumption and portfolio weights for each activity, as well as their holdings of contingent claims based on these activities. In equilibrium, the wealth is fully invested in all the production activities. The solution yields the optimal decision of the consumer-investor and the equilibrium interest rate as a function of the investor preferences, wealth and the attributes of the state variables. In addition, the model also yields the partial differential equation to be satisfied by all contingent claims.

The general specification of the model does not permit an explicit solution of the interest rate process in the pricing model for bond prices or contingent claims. The particular specification used by CIR to yield such results involves two explicit assumptions : the preferences of investors are logarithmic and the single state variable describing the economy follows a "square-root" process. The motivation behind the first assumption is that portfolio choices are independent of the investors' wealth, i.e. investors' decisions are myopic. This considerably simplifies the computation of the equilibrium, since it separates the present outcome of wealth from the future portfolio decisions. The second assumption permits the process for the short term interest rate to be written as

$$dr = \kappa(\ell - r)dt + \sigma\sqrt{r}dz \tag{9}$$

which is a particular case of the model in equation (8), with $\beta = 1/2$.

The CIR model has the attractive feature of being firmly grounded in a general equilibrium framework and hence, is completely internally consistent. It makes explicit the variables on which the equilibrium bond prices depend and also derives the bond price dynamics. In contrast, partial equilibrium models make implicit assumptions about these variables. Apart from this general benefit, it avoids the problems associated with *ad hoc* specifications, for instance for the market price of risk. CIR show, for instance, that assuming a simple linear structure for λ, the market price of risk, to be a function of r, could lead to internal inconsistency, where arbitrage is possible. Thus, there is a distinct advantage to determining the market price of risk endogenously. Turning to the specification of the short rate process, it has several advantages, apart from analytical tractability and the existence of a steady state distribution for the interest rate. The interest rates implied are in line both with intuition and the empirical findings. Under the square root process, at least for a range of parameter values, interest rates can never become negative or become absorbed at zero.[5] Furthermore, the process has the property that the variance of the interest rate increases with its level, a feature which accords with the stylized facts from the empirical work.

The general version of the CIR model provides a complete solution to the problem of equilibrium interest rates and bond prices. This model has no explicit solution, unless one specifies the model further. However, the problem with the specific version of the CIR model, which is implementable, is that it rests on the fairly specific assumptions of a square-root technology and logarithmic utility. The latter assumption implies that portfolio decisions in succeeding periods are independent of wealth at the end of the current period. Hence, investor preferences have a myopic property, which probably restricts the movements in the term structure over time. More general preferences, which allow dependence between the investors decisions over time, may allow for a wider range of possible term structure movements. However, this may prove to be difficult to solve analytically. In addition, as in the case of the Vasicek model, the CIR model does not take into account the current term structure observed in the market. Hence, since the model would not explain the present term structure perfectly, there would be arbitrage possibilities between the term structure implied by the model and that observed in the market. This mispricing by the model would also be reflected in the pricing of contingent claims, with the result that basic arbitrage relationships between the model prices for such claims and the observed term structure, such as put-call parity would fail. This would severely restrict the applicability of the model to practical problems of valuation. In addition, the model would predict movements in the term structure which would systematically deviate from ob-

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[5] The problem with absorbing and reflecting barriers at zero is that they admit arbitrage opportunities, since in both cases, the future direction of interest rates is known with certainty when the interest rate is at the barrier. Gaussian models do not have this problem, although they have the other problem of negative interest rates.

served data. These drawbacks of the CIR approach, which would apply to any model that prices both bonds and interest rate contingent claims, are mainly due to the ambitious nature of the task undertaken. Despite these problems, the model remains the only fully consistent equilibrium model of the term structure.

The one-factor version of the CIR model has been extended in several different ways. One extension is to explicitly incorporate the effect of expected inflation into the model. By interpreting the equilibrium result in the basic model in the context of a single-good economy, this approach views the square-root process as a more appropriate model for real rather than nominal interest rates. Gibbons and Ramaswamy (1993) and Brown and Schaefer (1994) use this interpretation in their empirical work and achieve some success in explaining real bond prices. To price nominal bonds, however, is necessary to specify the inflation process, or more generally, the effect of inflation on the real economy. In their specific model, CIR(1985b) proposed a partial equilibrium approach to accommodate the uncertainty of inflation within bond pricing models by assuming that the expected inflation rates govern both the drift and the diffusion of the price level process. Another assumption in CIR (1985b) is that the state variable governing the production is not correlated with innovations to inflation. Thus, the treatment of inflation in CIR requires only the exogenous determination of the rate of inflation in the economy, but also the assumption of neutrality of money.

Pearson and Sun (1994) extend the original CIR specification for nominal bond pricing model by adding a constant term to the inflation and real interest rates. Their specification, which is the most general version of "affine" yield model, assumes that the variance of the state variable is linear in the level of the state variable rather than strictly proportional to it. As discussed previously, the non-negativity of the square-root process is appealing for nominal interest rates but may be a rather severe restriction for real interest rates. Pearson and Sun (1994) take this into account by introducing a constant term that changes the level of the boundary for the process, so that negative real interest rates can be accommodated. In order to test the assumption of neutrality of money, Sun (1992) allows for correlation between the state variable in the consumption growth and the innovation in inflation. In Pennacchi (1991), the expected inflation is modeled to govern production process by affecting the correlation as well as the expected return of real investments. The empirical results of these papers indicate that it may be unrealistic to assume that inflation is independent of the real economy. Given that the inflation process is exogenously specified in these models, another extension is to admit money into a more general equilibrium setting where the inflation is endogenously determined. For example, in Giovannini and Labadie (1991), the term structure of indexed bonds and nominal bonds is analyzed in the cash-in-advance model of Svensson (1985).

Longstaff and Schwartz (1992) and Chen and Scott (1992) extend one-factor CIR model by including additional factors which follow the square-root process. These models, as well as the one by Duffie and Kan (1993), can be classified as multi-factor affine models since these models are mathematically isomorphic, and share the property that the short rate, bond yields and forward rates are affine in the underlying state variables. A remaining issue in these multi-factor models is how to identify the observable factors in the implementation of the models. In fact, one of the biggest advantages of one-factor models is that the short rate provides sufficient information to the state of economy, so that the factor can be observed without any errors. In the Duffie and Kan (1993) two-factor model, factors can be identified by any two bonds yields. In Ho, Stapleton and Subrahmanyam (1995), two factors are identified by the short rate and a forward rate. Longstaff and Schwartz (1992) observe that the conditional variance of the short term interest rate is also a linear function of state variables, so that state variables can be identified by the short rate and the conditional volatility. Because conditional volatility can be measured with satisfactory precision due to the progress in econometric technique related to the GARCH type model, the Longstaff and Schwartz model can be implemented without losing too much of the information embedded in the yield curves. Given that there is no consensus regarding the choice of identifying state variables in multi-factor models, the only alternative is to compare the pricing errors for different choices of the two factors. Although there is some work in this direction, more research remains to be done using alternative data sets and econometric methodologies.

2.4 The Heath-Jarrow-Morton model

Although not an equilibrium model, the Heath-Jarrow-Morton (1992) model is a complete model of the term structure specified in an arbitrage-free manner in the spirit of Harrison and Kreps (1979) and Harrison and Pliska (1981). The essential feature of the approach is its characterization of the sufficient conditions under which a unique equivalent martingale measure exists. Given such conditions, markets are complete and the valuation of contingent claims is well-defined at a conceptual level, although issues of implementation remain.[6] At one level, the HJM approach can be viewed as a multi-factor extension of the basic Black-Scholes framework, if one could imagine a multi-asset economy where forward prices of assets follow general diffusion processes. Under certain conditions for the drift terms and volatilities, an equivalent martingale measure exists, and hence, all claims, defined in a completely general manner, based on the multiple assets can be valued by taking the expectation of their payoffs under the equivalent martingale measure. As in the case of the Black-Scholes framework, the current prices of the underlying assets, in this case, the entire term structure today, have to fit the model perfectly. The only additional feature of the term structure problem, as compared to the general problem of asset pricing, is that there are restrictions on the pricing of bonds of different maturities. Since the bonds have different maturities, unlike assets in general, one has to be careful to specify the price process. Specifically, there should be no arbitrage possibilities across bonds, in the sense that the instantaneous return on one bond should not stochastically dominate that of another. Thus, in a binomial model, for example, the return in the first period on a two-period bond should straddle the return over the same period from a one-period bond: in one state in the future, it would be higher, and in the other it would be lower.

The basic set-up of the model is similar in spirit to the Vasicek model with the important difference that we are dealing with forward rather than spot interest rates. The process for the forward interest rates is written as

$$df(t,T) = a(t,T)dt + b(t,T)dz \tag{10}$$

where $a(t,T)$ and $b(t,T)$ are the drift and diffusion terms of the forward rate process, t is the current date, T is the maturity date, and z is a Brownian motion. $f(t,T)$ is the instantaneous forward interest rate at time t for delivery at date T. In order to keep the notation simple, we consider only one source of uncertainty. The spot interest rate process can be written as a special case of the above as

$$dr(t,t) = a(t,t)dt + b(t,t)dz \tag{11}$$

where the second arguments of $a(\cdot)$ and $b(\cdot)$ vary over time. Note that the specification of spot

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[6] A parallel paper by Artzner and Delbaen (1989) also proposes a martingale approach to modeling the term structure. However, this approach suggests a two-step procedure to price contingent claims, first to price zero-coupon bonds and then to price contingent claims. The HJM procedure, in contrast, avoids this two-step procedure by taking the bond prices and forward price processes as given.

rate process here is the same as in the general version of the Vasicek model, except that in the latter case, the forward rate process is not similarly defined. Thus, the Ornstein-Uhlenbeck version of the Vasicek model as well as the CIR model are special cases of the above formulation, with restrictions being placed on $a(t,t)$ and $b(t,t)$.

The discount bond price can be defined by integrating over the entire structure of forward rates. Also, since the forward rates can be written as

$$f(t,\tau) = f(0,\tau) + \int_0^t a(s,\tau)ds + \int_0^t b(s,\tau)dz_s, \tag{12}$$

the bond price process can be written as the definitional relationship

$$\begin{aligned} \ln B(t,T) &= -\int_t^T f(t,\tau)d\tau \\ &= -\int_t^T f(0,\tau)d\tau - \int_t^T \int_0^t a(s,\tau)dsd\tau - \int_t^T \int_0^t b(s,\tau)dz_s d\tau \end{aligned} \tag{13}$$

Superficially, this step resembles the analysis in the Vasicek model, where the bond price is written by integrating over the path of spot interest rates. The important difference is that in the Vasicek case, the analysis is driven by the *assumption* of a one-factor model, where the factor is the short rate. In other words, there is no guarantee that some other state variable has not been left out. In the HJM case, the analogous condition for the bond price is in terms of forward rates, and hence, follows definitionally.

Another important difference between modeling the spot rate process versus the forward rate process stems from the structure of the instantaneous volatility parameter. Since the forward interest rate process can be assumed to be trend-stationary, it is possible to integrate over the $a(\cdot)$ and $b(\cdot)$ and write the process for the bond price as

$$dB(t,T) = B(t,T)[\mu(t,T)dt + \sigma(t,T)dz] \tag{14}$$

This is the same as equation (14) in the Vasicek model, except that equation (14) is derived from the definition of bond price as a function of the forward interest rates, rather than by assuming a single-state variable model as Vasicek does.

At this stage, it is useful to point out the essential difference between the martingale approach of Harrison and Kreps (1979) and Harrison and Pliska (1981) and the Black-Scholes approach to the pricing of contingent claims. The Black-Scholes model starts with a specific structure and derives the necessary condition for the absence of arbitrage, which happens to yield a risk-neutral valuation relationship. In contrast, the martingale approach aims to derive the sufficient conditions under which the market price of risk parameters can be eliminated and contingent claims can be valued under the equivalent martingale measure.

The key result of the HJM is to show the equivalence between three alternative conditions that lead to preference-free pricing:

1. the market price of risk is independent of the vector of horizon dates.
2. a unique equivalent martingale measure exists.
3. there is a particular relationship between the drift and volatility terms of the forward rate process and the market price of risk.

Operationally, the last condition is the one that can be applied directly to eliminate the market price of risk.

The main insight of the HJM method can be illustrated for the special case of a single factor model. In this case, the last condition implies

$$a(t,T) = b(t,T)[\lambda(t) + \int_t^T b(t,\tau)d\tau] \tag{15}$$

If we assume further that $b(t,\tau) = b > 0$, i.e. that the forward rate volatility is constant, equation (15) can be written as

$$a(t,T) = b\lambda(t) + b^2(T-t) \tag{16}$$

which, upon substituting into the forward rate process, and using the transformation from the original Brownian motion to its risk-neutral counterpart

$$\hat{z} = z + \int_0^t \lambda(s)ds \tag{17}$$

yields

$$f(t,T) = f(0,T) + b^2 t(T - t/2) + b\hat{z} \tag{18}$$

where the drift term has been eliminated and the Brownian motion is replaced by the risk-neutral counterpart $\hat{z}$.

From the above expression for the forward rate process, it is easy to write down the spot rate process as a special case. Also, the bond price can be written down by integration over the forward rate between t and T, in terms of the risk-neutral process. It is straightforward now to value any contingent claim by computing its payoff, evaluating its expectation under the risk-neutral distribution and discount it to present using the integral over the instantaneous forward rates over the life of the contingent claim. It should also be emphasized that all preference-dependent parameters have been eliminated from the valuation model.

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2.5 The Ho-Lee model

The Ho-Lee model deserves mention as a special case of the HJM model, because it is the first approach to the implementation of arbitrage-free interest rate processes. In continuous-form, it is essentially the single-factor example analyzed in the previous section. Indeed, without mean reversion, it is simply a one-factor version of the HJM model where the forward rates follow arithmetic Brownian motions with constant volatility coefficients on the diffusion term. However, since this model was originally formulated as a discrete-time model, it is also useful to think of it as an extension of the Cox-Ross-Rubinstein (1979)[CRR] model. The basic structure of the model can also be stated as a special case of the Vasicek model without mean-reversion

$$dr = \theta(t)dt + \sigma(t)dz \tag{19}$$

The distinguishing feature of the model is its practical appeal due to its emphasis on implementation, although the HJM model provides the conceptual underpinnings of the Ho-Lee formulation. Also, the Ho-Lee model has the advantage of being an intuitively appealing special case of the HJM model.

The Ho-Lee approach involves fitting a binomial lattice for discount bond prices, taking the current term structure as given and imposing the restriction that the bond price is "pulled to par" at maturity. Since the model assumes only one factor, the movements in the entire discount function over time can be explained by just one state variable. The evolution of this state variable is captured by what Ho-Lee call a perturbation function, which simply represents the up- and down- movements in the binomial tree, which is time-dependent, unlike the basic CRR formulation. These perturbation functions are specified as deviations from the implied forward price function. Hence the risk-neutral probability has to be specified such that the expected value across the two states adds up to unity, since the analysis is in terms of forward prices. This is strictly analogous to the approach of CRR with three minor differences. Firstly, the analysis is in terms of forward prices rather than spot prices. Secondly, the up- and down-movements are time-dependent. Lastly, in the Ho-Lee model, the whole term structure is shifted up or down, rather than a single asset price.

The key attribute of the Ho-Lee model is that, by construction, there is no arbitrage allowed between the pricing along the lattice that is constructed and the current market interest rates. However, since no equilibrium model is specified, there is no question of explaining the current term structure, either. Further, once the lattice is constructed and the rates change, a gap will open up between the actual rates observed and the pricing along the lattice, one time-step later. This problem of "time-inconsistency" is faced by all current arbitrage-based models of the term structure.

Several extensions and modifications of the Ho-Lee approach have been proposed, prominent among them being the models of Black, Derman and Toy (1990) and Hull and White (1990). In these alternative models, there are differences in the specification of the interest rate process in terms of variables such as mean-reversion, the time-dependence of the volatility and drift terms, and the elasticity of the diffusion. In addition, there are details relating to implementation, such as whether the lattice is binomial or trinomial, and whether it is constructed for spot or forward interest rates.

3 Empirical Issues

The empirical examination of term structure models probably calls for a separate survey, focusing on the econometric issues. However, it would be useful to review briefly the main issues that have been addressed in the empirical work to date and what the findings are, so far. Models of the term structure can be tested either directly, or in terms of their implications for the pricing of interest rate contingent claims. The direct test of the term structure models can be conducted either cross-sectionally, across bonds of different maturities, or with time-series observations.

The major advantage of cross-sectional tests is that they explicitly identify the market price of risk, which is fundamental to the modeling of the term structure, from a theoretical perspective. The drawback, however, is that it is difficult to test a whole class of models in a nested fashion, so that models can be rejected or not, at various levels. In contrast, time-series tests cannot directly impose the restriction on the market price of risk across bonds without extremely cumbersome computation, but can test the models in a nested fashion. Thus, the time series tests are tests of "goodness of fit" to the stochastic process for interest rates, while the cross-sectional tests are tests of specific bond pricing models and they nest a test of the process, by construction.

Brown and Dybvig (1986) were the first to test the CIR model with cross-sectional data. Their broad conclusion is that, the estimates of the long-term mean, to which the short rate reverts, as well as the instantaneous volatility parameter, are unstable over time. Thus, the special form of the basic CIR model may be mis-specified, although an extended version, along the lines discussed above, may better explain the bond price dynamics. However, this extended specification may have problems of under-identification of the parameters. Gibbons and Ramaswamy (1993) use generalized method of moments (GMM) procedures to test the unconditional distribution of the steady-state interest rate process and find that the CIR model cannot be rejected in cross-sectional data. However, this failure to reject may stem from the weak power of their test. In contrast, Pearson and Sun (1994) used the more powerful maximum likelihood estimate (MLE) procedure, and were able to reject the CIR model in nested tests.

The time-series tests of term structure models infer the parameters from short term interest rates (typically one-month Treasury bill rates) in tests of the underlying stochastic process. Apart from the difficulty in identifying the market price of risk, the tests require much longer data series, since cross-sectional data is not used. Also, since the data are obtained from discrete sampling, it is necessary to use the Euler approximation to discretize the continuous-time process. On the other hand, for data of frequencies up to monthly, there is little difference between using an exact estimator and the Euler approximation.[7]

Marsh and Rosenfeld (1983) and Chan, Karolyi, Longstaff and Sanders (1992) [CKLS], use more general processes to nest the CIR model. They both reject the CIR model, and in the case of CKLS, they find that the elasticity term is about 1.5! This would imply that the short term interest rate process is non-stationary, which is out of line with any theory of the term structure. Recent work by Eom (1995) indicates that conclusions of these tests may be due to inefficient estimation methods.

A test of the contingent claims pricing implications of alternative term structure models was first conducted in a comprehensive manner by Amin and Morton(1994). Their results indicate

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[7] This further validates similar results Bergstrom (1972) demonstrated long ago.

that the implied volatilities from the six alternative special cases of the HJM formulation are unstable. Although the goodness of fit improves with two-parameter specifications, the instability gets worse. The empirical evidence to date suggests that one-factor models that are interesting special cases of HJM may be mis-specified because the parameter estimates are unstable. However, Amin and Morton show that estimates from a one-parameter volatility model outperform those from a two-parameter volatility model, and that the Gaussian models do better than lognormal ones! The one-parameter model does better possibly because it is more robust over time, and is less susceptible to the time inconsistency issue.

There are two possible directions for new empirical research on the term structure. The first would be to look at multi-factor models more closely, along the lines of Longstaff and Schwartz(1992). The second would be to specify the trade-off between fitting the current term-structure perfectly in cross-sectional data versus forecasting the future term structure with time-series data. Of course fitting better cross-sectionally means that time inconsistency may be exacerbated. Furthermore, it is necessary to apply the models to data on important interest rate derivatives such as caps, floors and swaptions.

4 A Brief Synthesis of Models of the Term Structure

We are now in a position to put together a broad picture of the field emphasizing the main areas where we have fairly good answers to the issues raised. In doing so, we use as a basis the papers surveyed in Section 2, but supplemented by some others that focus on issues of implementation.

At a general level, following CIR, we have a formal understanding of connection between technology, preferences and asset prices in an equilibrium setting, with particular reference to interest rates. At a general equilibrium level, we are aware of a set of sufficient conditions that yield closed-form solutions for the short-term interest rate and for bond prices: square root technology and logarithmic utility functions. Two other combinations of these variables can also be solved in a similar fashion - lognormal diffusions and constant proportional risk aversion (CPRA) or normal diffusion and constant absolute risk aversion (CARA). However, the bond prices and interest rate processes they yield have undesirable properties such as negative interest rates or even bond prices.

Although it is important to ensure that the general equilibrium foundations of the models in the area are well established, it is not necessary to go back to them in every model that is developed. At the next level, we could work with the pure-exchange partial equilibrium models of the type developed by Merton (1973a) and several others. Special cases of these models, such as the Gaussian interest rate model of Vasicek, in a partial equilibrium setting, are well-understood. This approach provides a necessary condition for the absence of arbitrage which essentially amounts to equating the market price of risk implied across bonds. This restriction leads to a partial differential equation that has to be satisfied by all bonds and contingent claims, which can solved subject to their respective boundary conditions. Both the general and partial equilibrium approaches have the disadvantage of not perfectly fitting the current term structure; consequently, standard arbitrage relationships such as put-call parity would fail with respect to current market prices. Further, these errors could get further compounded in the pricing of contingent claims in these models.

The next step of establishing a sufficient condition for arbitrage-free pricing is taken in the approach of HJM. The important feature of this approach is that the market price of risk parameters are eliminated by working with the equivalent martingale measures. Furthermore, these models are fully consistent with the current observed term structure. The major difference between the equilibrium and the martingale approaches is that in the former, we are forced to derive the market price of risk explicitly or specify it exogenously. In the latter approach, we take the forward rate processes as given, and hence, are able to eliminate the market price of risk parameter altogether. The problem is that in the equilibrium approach, the market price of risk can be derived explicitly only in special cases; on the other hand, specifying it exogenously may lead to problems of internal consistency. The martingale approach is perfectly adequate and further, practically implementable, if one is only concerned with deriving prices of bonds and interest rate contingent claims, rather than specifying the equilibrium supporting these prices.

Special cases of the HJM set up have been derived, which are implementable. Most of these use a single-factor structure in the spirit of Ho-Lee. Implementation of binomial lattices in various cases including normal and lognormal interest rates, different structures for mean-reversion, time-dependent volatility, etc. has led to a fairly detailed understanding of the biases from the various alternative specification. In the special case of Gaussian model, closed-form solutions can be derived as shown by Jamshidian (1989). Although the HJM approach is a multi-factor approach, implementation of even a two-factor model has been quite cumbersome computationally, although there has been some recent work in the area proposing improvements in efficiency.

5 Conclusions and Future Directions

It is fair to conclude, based on the above survey, that we have progressed considerably in our understanding of term structure modeling and the pricing of interest rate contingent claims. In particular, we are aware of the importance of building models that are arbitrage-free, in order to be internally consistent.

It is clear, however, that for practical purposes, it is simpler to use models such as Black-Scholes or Black which can be stated in closed-form.[8] These models may work quite adequately even though they are not theoretically well-grounded. For example, the Black-Scholes model is often employed for the valuation of short-term options on long-term bonds with satisfactory results. Similarly, for the valuation of interest rate caps and floors, the widespread use of the Black model may be justified on practical grounds. However, for many other applications such as the valuation of path-dependent options on interest rates, or even American-style options, e.g. mortgage-backed securities, it is important to model the interest rate process in an arbitrage-free manner. The methods discussed in the previous sections become very relevant for such purposes.

However, several gaps remain in our understanding. Although, the HJM model is well-specified from an arbitrage perspective, it suffers from the problem of time-inconsistency. In other words, the term-structure has to be re-calibrated every period to fit the market prices of bonds. Future re-calibration is not taken into account in fitting the tree on any given date. It is not clear how this general problem can be solved, although it appears that some sort of Bayesian procedure is necessary.

At a different level, the equilibrium underpinnings of the HJM model are unclear. Without attempting to go to the full-blown general equilibrium level of CIR, which is intractable in its general form, the issue is whether it is possible to characterize the equilibrium upon which the HJM specification rests. Specifically, what restrictions can be placed on the key parameter, the market price of risk, based on stylized facts and how does this accord with the HJM model?

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[8] A variation of these models, particularly for the pricing of interest rate caps and floors, is the alternative approach suggested by Stapleton and Subrahmanyam (1993).

Some work along these lines, for example by Ritchken and Sankarasubramanian (1995b), is under way. However, more research is required in this area.

Extensions of the basic model to include more general structure such as stochastic volatility and jumps in the interest rate process have been undertaken, for example, the work of Ahn and Thompson (1988), Longstaff and Schwartz (1992) and Das[(1994) and (1995)] and Das and Foresi (1996). However, these have to be related more closely to the general HJM framework.

At the level of implementation, single-factor versions of the HJM model with alternative specifications to incorporate stylized facts such as mean-reversion, time-dependent drift and volatility terms, as well as alternative processes for the diffusion term, are now fairly well-understood. However, it is necessary to explore further multi-factor versions of the HJM model. In particular, since multi-factor models are difficult to work with from a computational standpoint, more work is required to simplify the structure to make such models Markovian and hence implementable.[9]

At an empirical level, we need to gain a better understanding of the number of factors that would be required to explain term structure movements adequately. In addition, more work is required on assessing the importance of mean-reversion and the elasticity terms for the short rate process, as well as the role of jumps in interest rates.

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[9] Some work in this direction has been done by Ritchken and Sankarasubramanian [(1995a) and (199b)] and Pennacchi, Ritchken and Sankarasubramanian (1995).

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