The Term Structure of Interest-Rate Futures Prices.\textsuperscript{1}

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Abstract

The Term Structure of Interest-Rate Futures Prices

We derive general properties of two-factor models of the term structure of interest rates and, in particular, the process for futures prices and rates. Then, as a special case, we derive a no-arbitrage model of the term structure in which any two futures rates act as factors. In this model, the term structure shifts and tilts as the factor rates vary. The cross-sectional properties of the model derive from the solution of a two-dimensional, autoregressive process for the short-term rate, which exhibits both mean-reversion and a lagged persistence parameter. We show that the correlation of the futures rates is restricted by the no-arbitrage conditions of the model. In addition, we investigate the determinants of the volatilities and the correlations of the futures rates of various maturities. These are shown to be related to the volatility of the short rate, the volatility of the second factor, the degree of mean-reversion and the persistence of the second factor shock. We also discuss the extension of our model to three or more factors. We obtain specific results for futures rates in the case where the logarithm of the short-term rate [e.g., the London Inter-Bank Offer Rate (LIBOR)] follows a two-dimensional process. Our results lead to empirical hypotheses that are testable using data from the liquid market for Eurocurrency interest rate futures contracts.
1 Introduction

Theoretical models of the term structure of interest rates are of interest to both practitioners and financial academics. The term structure exhibits several patterns of changes over time. In some periods, it shifts up or down, partly in response to higher expectations of future inflation. In other periods, it tilts, with short rates rising and long rates falling, perhaps in response to a tightening of monetary policy. Sometimes, its shape changes to an appreciable extent, affecting its curvature. Hence, a desirable feature of a term-structure model is that it should be able to capture shifts, tilts and changes in the curvature of the term structure.

There are several multi-factor models in the literature that capture these empirical features of the term structure. However, many of the early models, such as the long rate-spread model of Brenner and Schwartz (1979), were not presented in the “no-arbitrage” setting first proposed by Ho and Lee (1986). Today, it is recognized that a highly desirable, if not a necessary condition, for any valuation model to satisfy is the absence of arbitrage. In this paper, we develop a model of the term structure of futures rates that is consistent with the principle of no-arbitrage. Our approach yields a multi-factor framework, of which the two-factor shift-tilt model is a special case. Our framework is, therefore, similar in spirit to the previous literature, but presented in a no-arbitrage setting, based on futures rather than spot interest rates.

The no-arbitrage condition, when applied to the term structure requires the price of a long-term bond to be related to the expected value, under the equivalent martingale measure (EMM), of the future relevant short-term bond prices. This requirement links the cross-sectional properties of the term structure at each point in time to the time-series properties of bond prices and interest rates. This point has been well recognized, in a one-factor setting, since the work of Vasicek (1977). In this paper, we extend this analysis to a two-factor setting. In the context of our two-factor model, we show that, if the short rate follows a mean-reverting two-dimensional process (a process generated by two state variables), then the no-arbitrage condition implies a short rate-long rate model of the term structure of futures rates.

We show that the correlation between the long and short maturity futures
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rates is restricted by the degree of mean-reversion of the short rate and the relative volatilities of the long and short-maturity futures rates. Also, the volatility structure of futures rates of various maturities can be derived explicitly from the assumed process for the spot short rate.

We suggest a time series model in which the conditional mean of the short rate follows a two-dimensional process, similar to that proposed by Hull and White (1994). This assumption allows us to nest the popular AR(1) single-factor model as a special case. It is also general enough to produce stochastic no-arbitrage term structures with shapes that capture most of those observed empirically. A superficially similar, but distinct, model in which the conditional mean of the short rate is stochastic has been suggested by Balduzzi, Das, Foresi and Sundaram (1996), and Balduzzi, Das and Foresi (1998), and analysed by Gong and Remolona (1997).

In a recent paper, Dai and Singleton (2000) explore the structural properties and relative goodness-of-fit of affine term structure models. Within this broad category, they consider a class of models with a "stochastic central tendency," that has been considered in the academic literature, as well by practitioners for the pricing of interest rate derivatives. An example of such models is the one proposed by Balduzzi, Das, Foresi and Sundaram (1986) (BDFS). The broad class of stochastic central tendency models characterized by Dai and Singleton is somewhat similar to the general class of models discussed in this paper, with one important difference. In the stochastic central tendency models, the short rate mean-reverts to a stochastic mean, whereas in our model, the short rate mean-reverts to a deterministic mean. The class of models discussed by Dai and Singleton does not generally produce the hump-shaped term structure of volatility, that is observed empirically. For instance, in the special case of zero correlation between the errors in the processes for the short rate and the mean, the errors get reinforced in a manner so as to make the term structure of volatility monotonic, either upward or downward sloping. In contrast, as we shall see in detail below, the general class of models described in this paper, can produce this shape for all levels of the correlation between the errors. This issue is considered in detail in section 6 where the empirical properties of our model are discussed. Furthermore, the class of models discussed in this paper directly leads to a cross-sectional relationship between the futures rates for different maturities, a testable empirical prediction.

Previous work on the term structure of interest rates has concentrated
mainly on bond yields of varying maturities or, more recently, on forward rates. In contrast, this paper concentrates on futures rates, partly motivated by the relative lack of previous theoretical models of interest-rate futures prices. However, the main reason for focusing on futures rates is analytical tractability. Futures prices are simple expectations of spot prices under the EMM, whereas forward prices and spot rates involve more complex relationships. It follows that futures prices and futures rates are fairly simple to derive from the dynamics of the spot rate. In contrast, closed-form solutions for forward rates have been obtained only under rather restrictive (e.g. Gaussian) assumptions. Further, from an empirical perspective, since forward and futures rates differ only by a convexity adjustment, it is likely that most of the time series and cross-sectional properties of futures rates are shared by forward rates, to a close approximation, at least for short maturity contracts. It makes sense, therefore, to analyse these properties, even if the ultimate goal is knowledge of the term-structure behaviour of forward or spot prices. Finally, the analysis of futures rates is attractive because of the availability of data from trading on organized futures exchanges. Hence, the models derived in the paper are directly testable, using data from the liquid market for Eurocurrency interest rate futures contracts.

In section 2 of this paper we discuss the relevant literature, relate our analysis to previously proposed models and explain the incremental contribution of our work. One of the most difficult aspects of term structure modelling is notation and definition of the relevant variables and parameters. For this reason, we devote much of section 3 to a description of the set-up of the problem, the variables and our notation. In this section, we then derive some general properties that characterise two-factor models. In particular, we show that if the function of the price of a zero-coupon bond follows a two-dimensional process, then its conditional expectation is generated by a two-factor model. We also derive a two-factor, cross-sectional model for futures prices in the general case where the spot price or rate follows a two-dimensional process. In section 4, we assume that the logarithm of the London Inter-Bank Offer Rate (LIBOR) follows a two-dimensional process and derive our main result for futures contracts on the LIBOR: a two-factor cross-sectional relationship between the changes in the prices of interest-rate futures. Numerical simulations of the results for the term structure of futures rates, the futures volatility structure and the correlation of futures and spot rates are shown in section 5. The conclusions and possible applications of our model to the valuation of interest rate options and to risk manage-
ment are discussed in section 7. Here we also discuss possible extensions and, in particular, the extension of the model to three factors.

2 Term Structure Models: The Literature

The analysis in this paper can be related to several strands of research in the literature. We discuss below the main papers in this broad area and how our work relates to them. The literature on the pricing of futures contracts was pioneered by Cox, Ingersoll and Ross (1981) [CIR], who characterized the futures price of an asset as the expectation, under the risk-neutral measure, of the spot price of the asset on the expiration date. Although futures prices, in general, have been considered by many other papers in the literature, there are few that have dealt specifically with the pricing of interest rate futures contracts. This gap in the literature is striking, given that short-term interest rate futures contracts based on the LIBOR are traded in many markets and are among the most liquid futures contracts. An important exception is the paper by Sundaresan (1991) that uses the general CIR characterization to price LIBOR-based futures contracts. Sundaresan shows that, under the risk-neutral measure, the futures interest rate is the expectation of the spot interest rate in the future. This follows from the fact that the LIBOR futures contract is written on the three-month LIBOR itself, rather than on the price of a zero-coupon instrument. This fact is used and its implications are derived in Brace, Gatarek and Musiela (1997) [BGM]. In the present paper, we use this result to obtain closed-form results for the term structure of futures interest rates.

In a comprehensive paper on the term structure of futures rates that presents both theoretical and empirical results, Jegadeesh and Pennacchi (1996) [JP] provide a model of futures rates based on a two-factor extension of the Vasicek (1977) model. Similar in spirit to the Vasicek model, they assume that the (continuously-compounded) interest rate is normally distributed, and derive bond prices and LIBORs in a two-factor equilibrium model, that involves the market price of risk. They then estimate the model using futures prices of LIBOR contracts, backing out estimates of the coefficients of mean-reversion of the short rate as well as the second stochastic conditional-mean factor. Our general model is closely related to the JP paper, with the important distinction that it is embedded in an arbitrage-free, rather than
an equilibrium framework, thus eliminating the need for explicitly incorporating the market price of risk. Although our analysis is based on weaker assumptions, we are able to derive quite general, distribution-free results for futures rates. We then include, as a significant special case, a model in which the interest rate is lognormal. This is an assumption that is widely used in the modern term-structure literature. Our main result is that a cross-sectional relationship exists for futures rates, where a futures rate is log-linear in any two futures rates.

The work of Gong and Remolona (1997) is similar, in some respects, to that of JP. They also employ a two-factor model, in which the second factor is the conditional mean of the short-rate process. However, they focus on the yield of long-dated bonds rather than on the futures rates. In their model, the short-term rate is linear in the two factors. Also, in a manner similar to Vasicek (1977), they assume a market price of risk, solve for bond prices, and back-out the long-term rates and the variances of the two factors. In contrast, we work under the equivalent martingale measure and directly derive futures rates for all maturities. We are also able to compute the variances of the futures rates of different maturities and the correlations between them.

While the literature on futures rates is somewhat sparse, the same is not true for forward rates. Indeed, much of the recent literature, dating back to the work of Ho and Lee (1986), has been concerned with the evolution of forward rates. The most widely cited work in this area is by Heath, Jarrow and Morton [HJM] (1990a, 1990b, 1992). HJM provide a continuous-time limit to the Ho-Lee model and generalize their results to a forward rate which evolves as a generalized Itô-process with multiple factors. The HJM paper can be distinguished from our paper in terms of the inputs to the two frameworks. The required input to the HJM-type models is the term structure of the volatility of forward rates. In contrast, in our paper, we derive the term structure of volatility of futures rates from a more basic assumption regarding the process for the spot rate. To the extent that the futures and forward volatility structures are related, our analysis in this paper provides a link between the spot-rate models of the Vasicek type and the extended HJM-type forward rate models.

The two-factor models developed in this paper are related also to the exponential affine-class of term-structure models introduced by Duffie and Kan (1994). This class is defined as the one where the continuously-compounded
spot rate is a linear function of any $n$ factors or spot rates. In an interesting special case of our model, where the logarithm of the LIBOR evolves as a two-dimensional linear process, it is the logarithm of the futures rate that is linear in the logarithm of any two futures rates.

Our analysis is related to previous papers that have assumed a two-dimensional process for the spot interest rate, such as Hull and White (1994). Following Vasicek (1977), Hull and White investigate models where some general function of the price of a zero-coupon follows a two-dimensional process with a stochastic conditional mean. Similar, but distinct, models have been proposed in Balduzzi, Das, Foresi and Sundaran (1996), and Balduzzi, Das and Foresi (1998). In section 4 of this paper we investigate the properties of a model in which the short-term rate of interest, defined on a LIBOR basis, is lognormally distributed. Models of this type have been investigated by Miltersen, Sandmann and Sondermann (1997) and by BGM. This assumption has the advantage that the variance is dependent on the level of the rate. Thus, rates are skewed to the right in our model, which may be empirically realistic for the short term interest rate markets in several of the major currencies. Also, the Black, Derman and Toy (1990)(BDT) and Black and Karasinski (1990) models have similar assumptions. However, all these models are single-factor models. Our incremental contribution to this literature is that we analyze a particularly simple two-factor extension of the BDT model. We also provide a set of necessary and sufficient conditions for the cross-sectional two-factor model to hold in a no-arbitrage setting.

Finally, as already discussed above, the properties of a wide class of affine term-structure models are analysed in Dai and Singleton (2000). Their main conclusion is that multi-factor models in this class require negative correlation of the state variables, if they are to capture the empirical properties of the term structure. However, for the models studied in this paper, the interest rate mean-reverts to a fixed value, rather than to a stochastic central tendency. Hence, our models are distinct from the class of models analysed in Dai and Singleton. In fact, as we show, models where the rate mean-reverts to a fixed level are able to capture empirical regularities, such as a lumped volatility structure, without restrictions on the correlation of the factors.
3 Some general properties of two-factor models

In this section, we first introduce the notation that we will employ to denote zero-coupon bond prices, short-term interest rates, and futures rates. We then establish two statistical results, that hold for any two-factor process of the form that we assume for the short-term rate. These results are used to establish a general proposition, that holds for the conditional expectation of any function of the zero-coupon bond price. The conditional expectation is of key significance, since the futures price (or rate) is closely related to the conditional expectation of the future spot interest rate. These results are directly applied later in the section to establish futures prices and futures rates.

3.1 Definitions and notation

We denote $P_t$ as the time-$t$ price of a zero-coupon bond paying $\$1$ with certainty at time $t + m$, where $m$ is measured in years. The short-term interest rate is defined in relation to this $m$-year bond, where $m$ is fixed. The short-term interest rate for $m$-year money at time $t$ is denoted as $i_t$, where $i_t$ is a function of $P_t$. The conventional definition of the interest rate is the continuously compounded rate, where $i_t = -ln(P_t)/m$. In this paper, we also investigate alternative definitions of the interest rate function. The other difference between this spot rate and the interest rate in the paper of HJM is that $m$, as in BGM, is not necessarily a very short (instantaneous) period. However, as in HJM, $m$ does not vary.

We are concerned with interest rate contracts for delivery at a future date $T$. We denote the futures rate as $F_{t,T}$, the rate contracted at $t$ for delivery at $T$ of an $m$-period loan. We denote the logarithm of the futures rate as $f_{t,T} = \ln[F_{t,T}]$. Note that, under this notation, which is broadly consistent with HJM, $F_{t,t} = i_t$ and $f_{t,t} = \ln(i_t)$.

The mean and annualised standard deviation at time $t$ (of the logarithm) of the spot rate at time $T$, under the EMM, are denoted

$$
\mu(t,T,T) = E_t[f_{T,T}]
$$

$$
\sigma(t,T,T) = \left[\text{var}_t[f_{T,T}]/(T-t)\right]^{\frac{1}{2}}
$$

respectively.
In particular, at time 0, the mean and standard deviation of the log-spot rates at \( t \) and \( T \) respectively can be written as
\[
\begin{align*}
\mu(0, t, t) &= E_0[f_{t,t}] \\
\sigma(0, t, t) &= \sqrt{\text{var}_0[f_{t,t}]/t}
\end{align*}
\]
and
\[
\begin{align*}
\mu(0, T, T) &= E_0[f_{T,T}] \\
\sigma(0, T, T) &= \sqrt{\text{var}_0[f_{T,T}]/T}
\end{align*}
\]

In the case of futures rates, we define the mean and standard deviation at time-0 of the log-futures at time-\( t \) for delivery at time-\( T \) as
\[
\begin{align*}
\mu(0, t, T) &= E_0[f_{t,T}] \\
\sigma(0, t, T) &= \sqrt{\text{var}_0[f_{t,T}]/t}
\end{align*}
\]

Table 1 summarizes the notation used in the paper.

Note that the mean and variance of the spot rate are statistics of a time-\( t \) or a time-\( T \) measurable random variable. In the case of the futures rates, these statistics relate to a time-\( t \) measurable random variable.

### 3.2 General properties of two-factor models

We now establish that, if a variable follows a two-dimensional process, the conditional expectation of the variable is necessarily governed by a two-factor cross-sectional model.\(^1\) We begin by proving this result quite generally, and then apply it to the special case of the logarithm of interest rates.

We first assume that some function of the zero-coupon bond price, \( P_t \), follows a two-dimensional process as follows:

\[
g(P_t) = E_0[g(P_t)] + (1 - c) \{ g(P_{t-1}) - E_0[g(P_{t-1})] \} + y_{t-1} + \epsilon_t, \quad (1)
\]

\(^1\)Hull and White (1994), for example, assume a similar two-dimensional process for short-term interest rates. As discussed below, this model is distinct from the class of stochastic central tendency models used by several authors and synthesized by Dai and Singleton (2000).
where
\[ y_t = (1 - \alpha)y_{t-1} + \nu_t, \]
and
\[ E_{t-1}(\epsilon_t) = 0, E_{t-1}(\nu_t) = 0, \]

where \( g(P_t) \) is the function and \( E_0[\cdot] \) is its expectation at time 0. Note that this specification allows for a general relationship between interest rates and bond prices, covering all standard alternative definitions, including continuous or discrete compounding. Further, no restrictions are placed on the correlation between \( \epsilon_t \) and \( \nu_t \).

We now state and prove the following proposition:

**Proposition 1** (General Cross-Sectional Relationship for the Change in Expected Values)

A function of the price of an \( m \)-year zero-coupon bond \( P_t \) follows the two-dimensional process defined by equation (1) if and only if the conditional expectation, \( E_t[g(P_{t+k})] \), is given by
\[
E_t[g(P_{t+k})] - E_0[g(P_{t+k})] = a_k [g(P_t) - E_0[g(P_t)]] + b_k [E_t[g(P_{t+1})] - E_0[g(P_{t+1})]]
\]

where
\[
b_k = \sum_{r=1}^{k} (1 - c)^{k-r} (1 - \alpha)^{r-1} \tag{2}
\]
and
\[
a_k = (1 - c)^{k} - (1 - c)b_k. \tag{3}
\]

**Proof.** See Appendix 1.

Proposition 1 states the implications of a two-dimensional, stochastic conditional mean, process for an arbitrary function of the zero-coupon bond price. The function could be a rate of interest, such as the continuously compounded rate (as in HJM) or the LIBOR (as in BGM), or it could be the price of the zero-coupon bond itself. The proposition restricts the cross-sectional properties of the conditional expectation. As we will see in the
next section, these properties are directly relevant for the investigation of future prices and rates. The intuition behind Proposition 1 is that the two influences on the function of the zero-bond prices, one of which is lagged, yield a cross-sectional structure with two factors. This contrasts with the single factor case, where there is a simple correspondence between what is driving the time series process and the cross-sectional factor structure.

We first examine an important special case of the Proposition 1, where the logarithm of the short-term interest rate follows the two-factor process. This is also a special case of Hull and White (1994). In Hull and White, the function \( g(P_t) = \ln(i_t) \), where \( i_t \) is the continuously compounded short rate, follows the continuous-time stochastic process

\[
d\ln(i_t) = \left[ \theta_t - \lambda \ln(i_t) + \gamma \right] dt + \sigma_1 dz_1
\]

where \( \theta_t \) is a parameter chosen to match the model parameters to the initial term structure, \( y \) is a stochastic conditional mean, and \( \lambda \) is the mean-reversion coefficient. \( \sigma_1 \) is the instantaneous standard deviation of the Wiener process, \( dz_1 \). The variable \( y \) itself follows the stochastic process

\[
dy = -\beta y dt + \sigma_2 dz_2
\]

where \( \beta \) is the mean-reversion of \( y \) and \( \sigma_2 \) is the instantaneous standard deviation of the Wiener process, \( dz_2 \). The two Wiener processes have an instantaneous correlation of \( \rho \).

Given the notation introduced earlier and assuming there are \((1/n)\) periods in a year, the above model, in discrete form, leads to:\(^2\)

\[
f_{t,t} - \mu(0,t,t) = \left[ f_{t-1,t-1} - \mu(0,t-1,t-1) \right] (1 - c) + y_{t-1} + \epsilon_t, \quad \forall t,
\]

where

\[
y_t = (1 - \alpha)y_{t-1} + \nu_t
\]

where

\[
\alpha = 1 - e^{-\beta n},
\]

\(^2\)To see this, take the unconditional expectation of the discrete version of equation (2) and the expression for the stochastic mean factor, and substitute for the means of the log-interest rate and the stochastic mean factors. \( n \), the length of the time period from \( t \) to \( t + 1 \), appears in the formulae for \( \alpha \) and \( c \) since these parameters represent mean-reversion on a periodic basis. The expressions for \( \alpha \) and \( c \) follow by summing the progressions and taking limits. We thank the referee for suggesting this.
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\[ c = 1 - e^{-\lambda n} \]

and

\[ E_{t-1}(\epsilon_t) = 0, \quad E_{t-1}(\nu_t) = 0, \]

In this case, we have, as a corollary of Proposition 1:

**Corollary 1** (Cross-Sectional Relationship for the Change in Expected Values for the Case of the Logarithmic Spot Rates)

The logarithm of the spot rate follows the process

\[ f_{t,t} - \mu(0, t, t) = [f_{t-1,t-1} - \mu(0, t-1, t-1)](1 - c) + y_{t-1} + \epsilon_t, \quad \forall t \]

where

\[ y_t = y_{t-1}(1 - \alpha) + \nu_t \]

if and only if the expectation of the logarithm of the interest rate \( i_{t+k} \) at time \( t \) is

\[
\begin{align*}
\mu(t, t + k, t + k) - \mu(0, t + k, t + k) &= a_k[f_{t,t} - \mu(0, t, t)] \\
&\quad + b_k[\mu(t, t + 1, t + 1) - \mu(0, t + 1, t + 1)]
\end{align*}
\]

**Proof.** The corollary follows directly from Proposition 1, where \( g(P_t) = \ln(i_t) \) and \( i_t \) is any interest rate function of \( P_t \). \( \square \)

In this case, the spot rate follows a logarithmic, mean-reverting process with a stochastic conditional mean. The implication is that the conditional expectation of the logarithmic rate for maturity \( t + k \) is generated by a two-factor cross-sectional model. The corollary has direct implications for the behaviour of futures rates in a logarithmic short-rate model. These are explored in section 4, where we assume that the interest rate function is the \( m \)-year LIBOR, rather than the continuously compounded rate.

The model should be contrasted with the similar, but distinct class of stochastic central tendency models analysed by Dai and Singleton (2000). In those models, the short rate \( r_t \) mean-reverts to a variable \( \theta_t \), which is itself stochastic. An example of this type of model, can be written as follows:

\[
\begin{align*}
    dr_t &= k_1(\theta_t - r_t)dt + \sigma_1dz_1 \\
    d\theta_t &= k_2(\overline{\theta} - \theta_t)dt + \sigma_2dz_2.
\end{align*}
\]
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Dai and Singleton (2000) (in the discussion following equation (20) in their paper) argue that models of this type, with the additional restriction that the innovations in \( r_t \) and \( \theta_t \) are uncorrelated, fail to capture the empirical term structure of volatility. Humped-shaped volatility term structures are not possible in the above model and the volatility structure is monotonic.\(^3\) In contrast, in our model the short rate mean-reverts to fixed parameters. Consequently, our model, which is outside the Dai-Singleton class, can produce a humped-shape volatility term structure even when the innovations are uncorrelated.\(^4\) The continuous-time version of our model, recast in the Dai-Singleton notation is of the form:

\[
\begin{align*}
    dr_t &= k_1(\theta_t - r_t)dt + y_t + \sigma_1 dz_1 \\
    dy_t &= k_2(\gamma_t - y_t)dt + \sigma_2 dz_2.
\end{align*}
\]

Models of this type are, therefore, outside the Dai-Singleton class.

3.3 Futures prices and rates in a no-arbitrage economy

In this sub-section, we apply the results in the previous sub-section 3.2 to derive futures prices and futures interest rates in a no-arbitrage setting. We assume here that the two-dimensional process, specified in equation 1 for prices or rates defined above, holds under the EMM. The EMM is the measure under which all zero-coupon bond prices, normalised by the money market account, follow martingales.

Cox, Ingersoll and Ross (1981) and Jarrow and Oldfield (1981) established the proposition that the futures price, of any asset, is the expected value of the future spot price, where the expected value is taken with respect to the equivalent martingale measure. We can now apply this result to determine, for example, the behaviour of the futures prices of zero-coupon bonds, assuming that the bond prices are generated by the two-dimensional

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\(^3\)Dai and Singleton conclude that "across a wide variety of parameterizations of ATSMs, the data consistently called for negative conditional correlations among the state variables." They also find that "a key role of the factor correlations is in explaining the shape of the term structure of volatility..." (Dai and Singleton, p. 1970).

\(^4\)The humped-shaped volatility term structure results more generally. This is shown, for example, by Hull and White (1994), Figure 3, but with positive correlation between the errors. The simulations in Section 5 below illustrate this point in some detail for a range of correlation coefficients.
process analysed in sub-section 3.2. Since there is a one-to-one relationship between zero-coupon bond prices and short-term interest rates, defined in a particular way, we can then proceed to derive a model for futures interest rates.

Initially, we make no specific distributional assumptions. We assume only a) the existence of a no-arbitrage economy in which the EMM exists, b) that a function of the time \( t \) price of an \( m \)-year zero-coupon bond, \( g(P_t) \), follows a two-dimensional process of the general form assumed in Proposition 1, and c) that a market exists for trading futures contracts on \( g(P_t) \), where the contracts are marked-to-market at the same frequency as the definition of the discrete time-period from \( t \) to \( t+1 \). We first establish a general result, for any function \( g(P_t) \), and then illustrate it with some familiar examples. We denote the futures price, at \( t \), for delivery of \( g(P_{t+k}) \), at \( t+k \), as \( g(P_{t,t+k}) \). We now have:

**Proposition 2** (General Cross-Sectional Relationship for Futures Prices)

Assume that equation (2) holds for \( g(P_t) \) under the EMM, then

\[
g(P_{t,t+k}) - g(P_{0,t+k}) = a_k [g(P_t) - g(P_{0,t})] + b_k [g(P_{t,t+1}) - g(P_{0,t+1})]
\]

where \( a_k \) and \( b_k \) are given by (3) and (2) respectively.

**Proof.** From CIR (1981), proposition 2 and Pliska (1997), the futures price of any payoff is its expected value, under the EMM. Applying this result to \( g(P_{t+k}) \), yields

\[
g(P_{t,t+k}) = E_t[g(P_{t+k})]
\]

Substituting equation (6) in Proposition 1 yields Proposition 2. □

The rather general result in Proposition 2 is of interest because of two special cases. The first is the case where the futures contract is on the zero-coupon bond itself. The second is the case of a futures contract on an interest rate, which is a function of the zero-coupon bond price. We consider these cases in the corollaries below.

We first have, as an implication of Proposition 2:
Corollary 2 (Cross-Sectional Relationship for Futures Prices for the Case of a Linear Process for the Zero-Bond Price)

The price of an $m$-year zero-coupon bond $P_t$ follows a two-dimensional process under the equivalent martingale measure (EMM):

$$P_t = E_0(P_t) + (1 - c)[P_{t-1} - E_0(P_{t-1})] + y_{t-1} + \epsilon_t$$

where

$$y_t = (1 - \alpha)y_{t-1} + \nu_t$$

and

$$E_{t-1}(\epsilon_t) = 0, \quad E_{t-1}(\nu_t) = 0,$$

if and only if the $k$th futures price $P_{t,t+k}$ is given by

$$P_{t,t+k} - P_{0,t+k} = a_k[P_t - P_{0,t}] + b_k[P_{t,t+1} - P_{0,t+1}]$$

where $a_k$ and $b_k$ are given by (3) and (2) respectively.

Proof. This follows as a special case of Proposition 2 with $g(P_t) = P_t$, and $g(P_{t,t+k}) = P_{t,t+k}$. □

The above results show that futures prices at time $t$ are generated by a linear two-factor model if and only if the zero-bond price follows a process of the Hull-White type. Note that the two factors generating the $k$th futures price are the spot price of the bond and the first futures price, i.e., the futures with maturity equal to $t + 1$. Similarly, the variance of the $k$th futures price is determined by the variance of the spot bond price, the variance of the conditional mean and the mean-reversion coefficients.

Corollary 2 is helpful in understanding the conditions under which the term structure follows a two-factor process. Essentially, if futures prices of long-dated futures contracts are given by the cross-sectional model in Proposition 2, then forward prices, and also futures and forward rates will follow two-factor models. The relationship for interest rates, however, is, in general, complex, since the function $i_t(P_t)$ is, in general, non-linear.

We next illustrate the use of Proposition 2 in the case of interest rate (as opposed to bond-price) futures. Instead of assuming that the price of a
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zero-coupon bond follows a two-dimensional, linear process, we now assume that the interest rate, defined as any function of the zero-coupon bond price, follows a two-dimensional, linear process. We have the following corollary of Proposition 2:

**Corollary 3 (Cross-Sectional Relationship for Futures Prices for the Case of a Linear Process for the Short Term Interest Rate)**

In a no-arbitrage economy, the short-term rate of interest follows a process of the form

\[ F_{t,t} = E_0[F_{t,t}] + (1 - c)[F_{t-1,t-1} - E_0(F_{t-1,t-1})] + \gamma_{t-1} + \epsilon_t \]

where

\[ \gamma_t = (1 - \alpha)\gamma_{t-1} + \nu_t \]

and

\[ E_{t-1}(\epsilon_t) = 0, \quad E_{t-1}(\nu_t) = 0, \]

if and only if the term structure of futures rates at time \( t \) is generated by a two-factor model, where the \( k \)th futures rate is given by

\[ F_{t,t+k} - F_{0,t+k} = a_k[F_{t,t} - F_{0,t}] + b_k[F_{t,t+1} - F_{0,t+1}] \quad (7) \]

where \( a_k \) and \( b_k \) are given by (3) and (2) respectively.

**Proof.** The proof of the corollary again follows as a special case of Proposition 2, where \( g(P_t) = i_t \). \( \square \)

The corollary illustrates the simple two-factor structure of futures rates that is implied by the two-dimensional process for the spot rate. Note that the mean-reversion coefficients are embedded in the cross-sectional coefficients, \( a_k \) and \( b_k \). Also, it follows from (7), given the linear structure, that the futures rates will be normally distributed, if the spot rate and the first futures rate are normally distributed. Hence, the corollary could be helpful in building a Gaussian model of the term structure of futures rates.\(^5\)

\(^5\) In a two-factor extension of the Vasicek (1977) framework, Jegadeesh and Pennacchi
4 LIBOR futures prices in a lognormal short-rate model

In the previous section, we showed that if either the price of a zero-coupon bond, or a short-term interest rate, evolves as a two-dimensional mean-reverting process under the risk-neutral measure, then a simple cross-sectional relationship exists between futures prices (or rates) of various maturities. In principle, these models could be applied to predict relationships between the prices of Eurocurrency futures contracts, based on LIBOR or some other similar reference rate, which are the most important short-term interest rate futures contract traded on the markets. However, in the case of LIBOR, the consensus in the academic literature and in market practice is that changes in interest rates are dependent on the level of interest rates. In particular, a lognormal distribution for short-term interest rates is commonly assumed.\(^6\)

When the logarithm of the short-term interest rate follows a linear process, the results of the analysis of futures prices in section 3 cannot be used, since the market does not trade futures on the logarithm of the LIBOR. However, if it is assumed that the LIBOR follows a lognormal process, standard results relating the mean of the lognormal variable to its logarithmic mean can be used to derive results for futures prices in this case, using Corollary 1, from section 3.

The standard Eurodollar futures contract is defined on the LIBOR. We now assume that the the function \(g(P_t)\) in Proposition 1 gives us the logarithm of the LIBOR. Since the LIBOR, \(i_t\), is defined on an “add-on basis”, it is related to the zero-coupon bond price, \(P_t\), by the relation

\[
P_t = 1/(1 + i_m),
\]

(1996) estimate a two-factor term structure model that is superficially similar to that in equation (7) under the assumption of normally distributed interest rates. (As discussed before, there is an important difference between the Jegadeesh and Pennachi model and the model discussed later on in this paper; their model is a special case of the “stochastic central tendency models” analyzed by Dai and Singleton (2000), while our models are not.) They show that their model fits the level of Eurodollar short-term interest rates contracts rather well for maturities of up to two years, and changes in the rates for longer-dated contracts. It is possible that this is because of ignoring the skewness effect (due to the normality assumption), which becomes significant for longer-dated contracts.

\(^6\)This is borne out by the empirical research of Chan et al. (1992) and more recently of Eom (1994) and Bliss and Smith (1998). There continues to be debate over the elasticity parameter of the changes in interest rates with respect to the level.
where $m$ is the proportion of a year. The logarithm of $i_t$ is, therefore, given by

$$f(t, t) = g(P_t) = \ln\left[\frac{1}{P_t} - 1\right]/m.$$  

We assume now that the logarithm of the LIBOR follows a two-dimensional lognormal process, under the equivalent martingale measure. We make use of the following lemma:

**Lemma 1 (Lognormal Futures Rates)**

In a no-arbitrage economy, if the LIBOR rate follows a lognormal process under the equivalent martingale measure, then

**a)** the $k$-period LIBOR futures rate at time $t$ is

$$F_{t,t+k} = \exp[\mu(t, t+k, t+k) + \frac{kn}{2}\sigma^2(t, t+k, t+k)]$$

where $n$ is the length, in years, of the period $t$ to $t+1$.

**b)** Also, the $k$-period LIBOR futures rate at time $t$, $F_{t,t+k}$ is lognormal, with logarithmic mean

$$\mu(0, t, t+k) = \mu(0, t+k, t+k) + \frac{kn}{2}\sigma^2(t, t+k, t+k)$$

We also have:

$$\mu(t, t+k, t+k) = f_{t,t+k} - \frac{kn}{2}\sigma^2(t, t+k, t+k),$$

$$\mu(0, t+k, t+k) = f_{0,t+k} - \frac{kn}{2}\sigma^2(0, t+k, t+k),$$

**Proof.** See Appendix 2.

Lemma 1 establishes first that the lognormality of the spot LIBOR implies lognormality of the LIBOR futures. This is important for our analysis of the

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7In the LIBOR contract, $m$ has to be adjusted for the day-count convention. Hence, $m$ becomes the actual number of days of the loan contract divided by the day-count basis (usually 360 or 365 days). Note that for quotation purposes, the day-count is always taken to be 360, to get a round money figure for the tick-size of the contract.
behaviour of the futures rate, in this section. This property follows from the CIR (1981) result that the futures price is the expectation, under the EMM, of the spot price. Second, the lemma establishes a useful relationship between the logarithmic mean of the futures rate and that of the corresponding spot rate.

We use this relationship, which itself follows from the lognormality of the futures and spot rates, in the proof of the following proposition. We then have:

**Proposition 3 (Cross-Sectional Relationship between Futures Rates)**

Consider a no-arbitrage economy, in which the LIBOR rate follows a two-dimensional lognormal process, under the equivalent martingale measure, of the form

\[ f_{t,t} = \mu(0, t, t) + [f_{t-1,t-1} - \mu(0, t-1, t-1)](1 - \epsilon) + \eta + \epsilon_t \]

where

\[ \eta = (1 - \alpha)\eta_{t-1} + \nu_t, \]

and where \( \epsilon_t \) and \( \nu_t \) are independent, normally distributed variables, i.e., the term structure of futures rates at time \( t \) is generated by a two-factor model.

Then, the \( k \)th futures rate is given by

\[ f_{t,t+k} - f_{0,t+k} = M_k + a_k[f_{t,t} - f_{0,t}] + b_k[f_{t,t+1} - f_{0,t+1}] \]

where \( a_k \) and \( b_k \) are given by (3) and (2) respectively, and \( M_k \) is defined as follows:

\[ M_k = \frac{kn}{2} \sigma^2(t, t + k, t + k) - \frac{(t + k)n}{2} \sigma^2(0, t + k, t + k) \]
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\[
+ a_k\left[\frac{tn}{2}\sigma^2(0, t, t)\right] \\
+ b_k\left[-\frac{n}{2}\sigma^2(t, t + 1, t + 1) + \frac{(t + 1)n}{2}\sigma^2(0, t + 1, t + 1)\right].
\]

Also, using Lemma 1 b), we can write the drift term in terms of the futures volatilities:

\[
M_k = -\frac{nt}{2}\sigma^2(0, t, t + k) \\
+ a_k\left[\frac{tn}{2}\sigma^2(0, t, t)\right] \\
+ b_k\left[\frac{nt}{2}\sigma^2(0, t, t + 1)\right]
\]

Proof. From Corollary 1,

\[
\mu(t, t + k, t + k) - \mu(0, t + k, t + k) = a_k[\mu(t, t - k) - \mu(0, t, t)] \\
+ b_k[\mu(t, t + 1, t + 1) - \mu(0, t + 1, t + 1)]
\]

is a necessary and sufficient condition. Substituting the results of Lemma 1 then yields the statement in the proposition. □

Proposition 3 is the main result of this paper. The Proposition shows the conditions under which a simple log-linear relationship exists for futures rates of various maturities. In this cross-sectional model, futures rates are related to the spot LIBOR and the first LIBOR futures. The result extends to the lognormal LIBOR case the prior results on the term structure of Duffie and Kan (1993). Proposition 3 relates the kth futures rate, (i.e., the one expiring in k periods) to the spot rate $f_{t,t}$ and the first futures rate, $f_{t+1,t+1}$. For example, this means that the kth three-month futures rate is related to the spot three-month rate and the one-period, three-month futures rate. However, following Duffie and Kan (1993), if the model is linear in two such rates, it can always be expressed in terms of any two futures rates. In the present context, therefore, the kth futures rate can be expressed as a function of the spot rate and the Nth futures rate. We have the following implication of Proposition 3:
Corollary 4 Suppose any two futures rates are chosen as factors, where $N_1$ and $N_2$ are the maturities of the factors, then the following linear model holds for the $k$th futures rate:

$$f_{t,t+k} = \mu(0,t,t+k) + A_k(N_1,N_2)[f_{t,t+N_1} - \mu(0,t,t+N_1)]$$

$$+ B_k(N_1,N_2)[f_{t,t+N_2} - \mu(0,t,t+N_2)]$$

(8)

where

$$B_k(N_1,N_2) = (a_k b_{N_1} - b_k a_{N_1})/(a_{N_2} b_{N_1} - b_{N_2} a_{N_1}),$$

$$A_k(N_1,N_2) = (-a_k b_{N_1} + b_k a_{N_1})/(a_{N_2} b_{N_1} - b_{N_2} a_{N_1}),$$

and where $a_k$ and $b_k$ are defined as before.

Proof. Corollary 4 follows by solving equation (7) for $k = N_1$, and $k = N_2$ and then substituting back into equation (7). $\square$

The $k$th futures rate is log-linear in any two futures rates. The meaning of the result is illustrated by the following special case, where there is no mean-reversion in the short rate, i.e., the logarithm of the LIBOR follows a random walk.

Corollary 5 The Random Walk Case

Suppose that $c = 0$, i.e., the logarithm of the LIBOR follows a random walk. In this case, the $k$th futures LIBOR is

$$f_{t,t+k} = \mu(0,t,t+k) + \left(1 - \frac{k}{N}ight)[f_{t,t+k} - \mu(0,t,t)] + \left(\frac{k}{N}\right)[f_{t,t+N} - \mu(0,t,t+N)].$$

(9)
Proof. Corollary 5 follows directly from Corollary 4 with

$$b_{k,N} = \frac{k}{N},$$

and hence,

$$a_{k,n} = \frac{N - k}{N}.\]

\(\Box\)

Here, the \(k\)th futures is affected by changes in the \(N\)th futures according to how close \(k\) is to \(N\). Equation (9) is a simple two-factor “duration-type” model, in which the term structure of futures rates shifts and tilts. This and other special cases are illustrated, using numerical examples, in the next section.

5 LIBOR Futures, Volatilities and Correlation

The model developed in the previous section relates the futures, with maturity \(k\), to the spot LIBOR. The implication of Proposition 3 is that the variance of the \(k\)-th futures and the correlation of the \(k\)-th futures with the spot are also determined by the model. In this section, we derive the explicit implications of the two-dimensional lognormal spot-rate process for the volatility and correlation structure of the futures rates. We first derive expressions for the volatility and correlation of the futures rates, given the correlation of the two stochastic factors in the model. We then compute the volatilities and correlations for a variety of parameter values by simulating our two-factor model. Finally, we analyse empirical estimates of the volatilities and the correlations empirical estimates using LIBOR futures data for the period 1990-9.

It follows directly from the result in Proposition 3 that the conditional (log) variance of the \(k\)-th futures is

$$\text{var}_{t-1}(f_{t,t+k}) = a^2_{t-1}(f_{t,t}) + b^2_{t-1}(f_{t,t+1})$$

$$+ 2a_{t} b_{t} \text{cov}_{t-1}(f_{t,t}, f_{t,t+1})$$

(10)
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where

\[ \text{cov}_{t-1}(f_{tt}, f_{t,t+1}) = (1 - c)\text{var}_{t-1}(f_{tt}) + \text{cov}_{t-1}(\epsilon_t, \nu_t). \]  \(11\)

It now follows that:

**Proposition 4 (The Volatilities and Correlations of Futures Rates)**

Suppose the logarithm of the spot rate follows the process in equation (5). The variance of the \(k\)-th futures rate is given by

\[
\sigma^2(t - 1, t, t + k) = a_k^2\sigma^2(t - 1, t, t) + b_k^2\sigma^2(t - 1, t, t + 1) + 2a_kb_k[(1 - c)\sigma^2(t - 1, t, t) + 2\rho\sigma(t - 1, t, t)\sigma_\pi]
\]

and the correlation of the spot and futures rates is therefore given by:

\[
\rho(t - 1, t, t + k) = \frac{(1 - c)^k\sigma^2(t - 1, t, t) + b_k\rho\sigma(t - 1, t, t)\sigma_\pi}{\sigma(t - 1, t, t)\sigma(t - 1, t, k)}
\]

where

\[
\sigma^2(t - 1, t, t + 1) = (1 - c)^2\sigma^2(t - 1, t, t) + \sigma_\pi^2 + 2(1 - c)\rho\sigma_\pi\sigma(t - 1, t, t)
\]

and where \(\rho\) is the correlation between the errors \(\epsilon_t\) and \(\nu_t\). \(\sigma_\pi\) is the volatility of \(\pi\).

**Proof.** See Appendix 3.

In Proposition 4, the volatility of the \(k\)-th futures rate is determined by the mean-reversion of the short rate, the variance of the short rate, the mean-reversion and variance of the stochastic mean factor, and the correlation \(\rho\) of the innovations in the two factors.

The correlation between the futures rates and the spot rate is important for two reasons. First, the correlation between any two futures rates, which may be taken as factors in the above model, cannot be determined independently of the mean-reversion of the short rate, \(c\), and the persistence of the conditional mean shock factor, \(\alpha\). Second, the correlation is an important determinant of the value of certain derivatives, whose payoff depends on the difference between various rates of interest in the term structure.

This expression for the correlation of the short rate and the \(k\)-th futures rate illustrates an important implication of the no-arbitrage model. Given the
volatilities of the spot and futures rates and the correlation of the errors, we cannot independently choose both the correlation and the degree of mean-reversion. The no-arbitrage model restricts the correlation between the two factors to be a function of the degree of mean-reversion of the short rate.\footnote{This would imply that one cannot arbitrarily specify a two-factor model such as Brennan and Schwartz (1979) or Bühler et al., (1999) without restricting the correlation between the short and long rates.} Further, because the futures volatility depends, in addition, on the degree of persistence of the premium factor shock, all the parameters affect the correlation.

5.1 The Cross-sectional properties of LIBOR futures rates

In this section, we present some examples that illustrate, within the context of the model, the effect of parameter values on the LIBOR futures rates, the volatilities of the futures rates, and the coefficients of correlation between them. In these examples, we use different values of the mean-reversion coefficients of the two factors. We refer to the mean-reversion of the second factor (the stochastic conditional mean) as persistence. This is because \((1 - \alpha)\) determines the persistence or memory effect of a conditional mean shock. Also, referring to \((1 - \alpha)\) as persistence serves to distinguish it clearly from the mean-reversion of the short rate \((c)\), which we refer to simply as mean-reversion.

What types of term structures are the possible results of the two-factor model derived in Proposition 3? This question is best answered by considering numerical examples, where we choose different parameter values. According to equation (3), the mean-reversion coefficient \((c)\), the persistence parameter \((\alpha)\), and the correlation coefficient \((\rho)\), together with the changes in the short rate and the conditional mean factor should determine the cross-sectional shape of the term structure. We present a series of examples, for different parameter values, assuming an initial term structure where all futures rates are 5%. In all cases the short rate shifts up or down by 1%. Figures 1-5 illustrate the effects of 1) persistence \((1 - \alpha)\), 2) mean-reversion coefficient of the short rate \((c)\), and 3) the shock to the conditional mean \((y)\). We use two values each for the persistence \((1 - \alpha)\) (0.25 and 0.75) and mean-reversion coefficient \((c)\) (0 and 0.05). The conditional mean factor moves up or down by 0.3% in the same direction in Figures 1-4 and
in the opposite direction in Figure 5. Figures 1 and 2 illustrate the case of no mean-reversion with different degrees of persistence. Figures 3 and 4 illustrate cases of mean-reversion.

The effects of a shock to the conditional mean are as follows:

1. **The persistence factor**

   In Figures 1 and 2, we compare a one-factor model, where the short rate moves by 1%, up or down, with a two-factor model, where the first futures rate moves by a further 0.3%. The one-factor model, with mean-reversion, \( c = 0 \), yields a parallel shift in the term structure of futures rates. In the two-factor model, high persistence of the premium shock (low \( \alpha \)) implies relatively large changes in the long-maturity futures rates. Figure 2 is similar to Figure 1 except that the persistence is low (high \( \alpha \)). With higher persistence, as in Figure 1, the shock to the stochastic mean lasts longer, resulting in a large effect for longer maturity rates. In contrast, in Figure 2, the shock dies out more quickly, resulting in a smaller effect for longer maturity rates.

2. **Mean-reversion of the LIBOR**

   In Figures 3 and 4 we introduce mean-reversion, with \( c = 0.05 \). In the one-factor case, the effect is to produce continuously downward or upward sloping term structures according to whether the initial shock to the LIBOR is positive or negative. In the two-factor case, the result depends also on the degree of persistence. If persistence is relatively high, as in Figure 3, the net effect can be a rising and then falling futures curve. When persistence is low, as in Figure 4, the effect of mean-reversion dominates and the futures curve is downward sloping over most of its range.

3. **The stochastic mean factor**

   The difference between the two-factor and one-factor models in Figures 1-4 depends upon the size and direction of the shock to the first futures premium. In these cases, the shock is in the same direction as the shock to the LIBOR. However these shocks could have opposite signs, a case considered in Figure 5. Here, the one-factor model yields a continuously rising or falling curve, while the two-factor model produces rates that can have positive or negative changes at different points of the yield curve. In this case, the term structure of futures
rates can tilt as well as shift. For the parameter values in Figure 5, there is no change in the fifth futures rate, where the effects of the two factor changes just happen to cancel out.

5.2 The volatility of LIBOR futures rates

Proposition 4 (which is implied by Proposition 3) allows us to investigate the properties of the volatility of futures rates. Again, the shape of the volatility structure of futures LIBORs depends on the persistence parameter (α) and the mean-reversion parameter (c). In addition, it also depends on the relative volatilities of the short rate and the stochastic mean factor, defined as σ₁ and σ₂ respectively, for simplicity, as well as the correlation coefficient (ρ) between the two errors. Figures 6-9 correspond to the cases in Figures 1-4 respectively. In each of these figures, there are four graphs. The solid line indicates the futures volatility term structure for the one-factor model, and the dashed line indicates the futures volatility term structure for the two-factor model in the case where the correlation coefficient is 0. The line with crosses is the two-factor case where the correlation coefficient is +0.5. The line with circles is the two-factor case where the correlation coefficient is -0.5. In Figures 6-9, we compare a one-factor model (the solid line), where the short rate exhibits a volatility of 10%, with a two-factor model, where there is, in addition, a stochastic mean factor with a volatility of 6%, for the case of zero correlation of errors (the dashed line).

1. The persistence factor

The one-factor model, with no mean-reversion, yields a flat volatility structure, which mirrors the parallel shift in the term structure of futures rates, shown in Figure 1. In contrast, the two-factor model produces an increasing volatility curve. The persistence factor also affects the shape of the term structure of volatility. High persistence of the stochastic mean shock (low α) implies relatively large volatilities for long-maturity futures rates. Comparing Figure 6 with Figure 7, we see that where persistence is low (high α). Note that when persistence of the stochastic mean shock is low (Figure 7), the two-factor model has a volatility structure close to that of a one-factor model with higher volatility.

2. Mean-reversion of the LIBOR
In Figures 8 and 9, we introduce mean-reversion, with \( c = 0.05 \). In the one-factor case, the effect is to produce a continuously downward sloping volatility structure. This illustrates the well-known effect of mean-reversion of the short rate. In the two-factor case, the result again depends also on the degree of persistence. If persistence is relatively high, as in Figure 8, then the net effect can be a humped futures volatility curve. When persistence is low, as in Figure 9, the effect of mean-reversion tends to dominate and the futures volatility curve is downward sloping over most of its range. In any case, given the degree of mean-reversion, the point at which the volatility curve starts declining, for a given volatility of the second factor, depends on the degree of persistence of the stochastic mean factor.

3. The (relative) volatility of the stochastic mean

The difference between the two-factor and one-factor models in Figures 6-9 depends upon the volatility of conditional mean factor, \( \sigma_2 \), in relation to the persistence and mean-reversion parameters. This is illustrated by the humped-shape of the term structure of volatility in Figures 8 and 9. A different ratio of the volatilities of the two factors would change both the slope of the hump and its location in the term structure of volatility. It also depends on the interaction between this effect, the mean-reversion of the short rate, and the persistence of the shock. For instance, in Figure 8, the volatility peaks at period 6, as the persistence is high, whereas in Figure 9 it peaks earlier at period 2, since the shock decays more quickly when the persistence is low.

4. The correlation between the errors

Overall, the correlation between the errors accentuates the shape of the volatility term structure. For negative correlation, the volatility impact over time gets dampened downwards, reducing the slope of the graph; for positive correlation, the volatility impact over time gets magnified, increasing the slope. Although there is an interaction between the correlation coefficient and the mean-reversion and the persistence, there are some general effects of correlation. In both Figures 6 and 7, with low persistence, the volatility actually dips the initial volatility in period 2 and then goes up slowly. With a positive correlation, the volatility impact gets magnified so that it goes up steadily. The degree of persistence determines how rapidly the volatility goes up over time; when the persistence is low, it goes up faster. In Figures
8 and 9, mean-reversion causes the volatility to decline over time, dipping below not just the initial level, but even the level in the 1-factor case.

5.3 The correlation of LIBOR spot and futures rates

A further implication of Proposition 4 concerns the correlation of spot and futures rates. The degree of correlation of the logarithmic rates depends on the relative importance of the second factor. In a one-factor model, all futures rates are perfectly correlated with each other and with the spot rate. In our two-factor model, the correlation structure depends on the mean-reversion and persistence parameters, in addition to the ratio of the volatilities of the two factors. Specifically, given the volatility of the first factor, we need to examine the effect of mean-reversion ($c$), persistence ($1 - \alpha$), the volatility of the second factor, $\sigma_2$, and the correlation coefficient ($\rho$) between the two errors, on the correlation of the spot LIBOR with the $k$th futures rate. Again, this is best analysed with the help of numerical examples. We now look at the effect of $c$, $\alpha$, $\sigma_2$ and ($\rho$) on the correlation of the spot LIBOR with the $k$th futures rate, using numerical examples similar to those illustrated in Figures 1-9 above. This is illustrated in Figures 10-13. In Figure 10, we compare a two-factor model, where the short rate moves have a volatility of 10% and the second, stochastic mean factor has a volatility of 6%, with a two-factor model, where the short rate moves have a volatility of 10% and the second factor has a volatility of 3%. In each case, the mean reversion, $c$, is zero, and the persistence is relatively high ($\alpha$ is low), $\alpha = 0.25$. In Figures 12 and 13 we introduce mean-reversion, with $c = 0.05$.

1. The persistence factor

As expected, the correlation declines with the maturity of the futures contract, in both cases. In the 6% and the 3% volatility cases respectively, the correlation falls to about $\rho = 0.4$ and $\rho = 0.65$ for the longest maturity futures contract. Figure 11 has the comparable graphs for the low persistence case ($\alpha$ is high). Comparing the correlation structure in Figures 10 and 11, we notice the influence of the persistence parameter, $\alpha$. When persistence is low ($\alpha$ high) as in Figure 11, the size of the premium shock is far lower in later periods,
and consequently, the correlation of the longer maturity futures rates
with the spot rate is higher, only falling to about $\rho = 0.8$ and $\rho = 0.9$
for the 6% and the 3% volatility cases respectively for long maturity
futures contracts.

2. Mean-reversion of the LIBOR

In Figures 12 and 13 we introduce mean-reversion, with $c = 0.05$.
Comparing Figures 10 and 12, we see that the correlation function
falls more steeply with positive mean-reversion. It also falls to a sig-
nificantly lower level. In the case of low persistence, comparing Figures
11 and 13, the effect of mean-reversion is somewhat marginal. This
again highlights the important role of the persistence parameter in
two-factor models.

3. The correlation between the errors

Overall, the correlation between errors accentuates the shape of the
correlation term structure. With negative correlation, the correlation
term structure moves towards zero very sharply. In the case of pos-
tive correlation, the correlation term structure moves in the other
direction, towards one. In Figures 10 and 11, positive correlation ac-
centuates the correlation between the spot and futures rates. With
high persistence, however, the correlation stays high. Figures 12 and
13 illustrate the effect of mean-reversion. In the zero correlation case,
the mean-reversion causes the correlation to decline more sharply over
time. Again, the correlation between the errors dampens the effects
in the negative correlation case and magnifies it in the positive corre-
lation case, with the degree of persistence determining how slowly the
correlation declines.

5.4 Empirical Analysis of the Correlation between Futures
Interest Rates

We construct a time series of futures rates from historical data on Eurodollar
futures prices from January 1, 1990 to December 31, 1999, for all contracts.
(The futures rate is defined as 100 minus the futures price.) During the
first half of the sample period, the first 15 contracts were reasonably liquid,
while in the second half, the first 20 were liquid. Since the futures maturity dates are fixed, we interpolate between two adjacent contracts to construct a time-series of futures prices and rates for fixed maturity periods. For instance, if on a particular trading day, the first contract matures in 63 days and the next one in 154 days, we obtain the futures rate for a maturity of 3 months (91 days), by linearly interpolating between the two futures rates for maturities on either side of 91 days. In this manner, we obtained the futures rates for maturities ranging from 3 months to 42 months.

The annualized volatilities of the changes in the futures rates of different maturities are computed from this data and plotted in Figures 14 and 15, for the two sub-periods, 1990-94 and 1995-99, as well as the overall period, 1990-99. Figure 14 illustrates the term structure of volatility based on observations with a monthly frequency, while Figure 15 presents similar data based on observations with a quarterly frequency. In all cases, the figures show a hump-shaped volatility term structure, that is very similar to that obtained in the simulations of our model discussed earlier, particularly in Figure 9, where $c = 0.05$, $\sigma = 0.75$ and $\sigma_2 = 6\%$.

Using the same data as for the volatilities, we construct the correlogram between the logarithm of the 1st futures rate (maturity of 3 months) and the logarithms of the next 13 futures rates (maturities of 6,9,...,42 months). This is done for each sub-period, 1990-94 and 1995-99 as well as the overall period, 1990-99. We present the correlation matrices, based on monthly and quarterly frequencies, in Tables 2 and 3 respectively. The correlation coefficients between the 3-month futures rate (the 1st futures rate) and the next 13 futures rates for the two sub-periods are plotted in Figures 16 and 17 for the monthly and quarterly frequencies of observations. Figures 16 and 17 show how the correlation coefficients changed during the two sub-periods.

---

9 We also analyzed the data for the 3-month LIBOR spot rates over this sample period. We found that these data were noisy due to problems of discreteness (since the quotations are usually in 1/16's) and non-synchronicity (as the spot data are based on the London fixing, a few hours before the Chicago futures markets close). The correlations were consequently lower.

10 More frequent samples of the data, such as weekly and daily, proved to be too noisy, especially for the near-term futures contracts, and resulted in lower correlations. The noise is probably due to trading frictions, including discreteness, non-synchronicity of observations, as well as errors introduced by the interpolation procedure to obtain constant maturity period futures prices and rates. Only one panel, representing the correlogram for the whole period, 1990-99, is presented in Table 3, since the other numbers are fairly similar.
in comparison with the overall period.

The correlograms shown in Tables 2 and 3, and illustrated in Figures 16 and 17 indicate a pattern that is fairly similar to those shown in the prior simulations of our model. For instance, comparing Figure 17 to Figure 12, based on our model, it is clear that the correlations are higher for contracts that are closer to each other in maturity. Also, the model with \( c = 0.05 \), \( \alpha = 0.75 \), \( \sigma_2 = 6\% \) and a zero correlation of the error terms produces similar correlations to the empirical estimates. It goes without saying that these comparisons are only suggestive and anticipate a formal empirical study, which we leave to future research.

6 Conclusions and Extensions

There is a close relationship between the cross-sectional characteristics of term structure models and the time-series process followed by the short-term interest rate. This paper has explored this relationship in the context of futures prices and futures interest rates. If we assume that the price of a zero-coupon bond (or, indeed, any function of the price) follows a two-dimensional process, then no-arbitrage restrictions imply that the term structure of futures prices or rates can be represented by a two-factor cross-sectional model. In an important special case, if we assume that the logarithm of the LIBOR interest rate follows a two-dimensional, mean-reverting process under the equivalent martingale measure, the term structure of futures rates can be written as a log-linear function of any two rates. The coefficients of this two-factor model are determined by the rates of mean-reversion of the two factors generating the time-series process of the LIBOR.

Perhaps the most important theoretical implications of the paper concern the relationship between HJM-type forward rate models and Vasicek-Hull and White-type models of the spot rate process. We have shown in particular that the degree of persistence of the second, conditional mean, factor shock is a critical determinant of the futures-volatility structure. Given the close relationship of futures and forward rates, it must also be an important determinant of the forward-volatility structure, which is an input to the HJM-type models. The well known humped volatility structure has been reproduced in our two-factor model with mean-reversion of the short rate
Interest Rate Futures

and persistence of the conditional-mean factor shock, even in the case where correlations between the innovations in the two factors are zero.

The results in the paper also have some interesting empirical implications. Mean-reversion of short term interest rates is a crucial determinant of the pricing of interest rate contingent claims, in general, and interest rate caps, floors and swaptions, in particular. It is well-known that it is extremely difficult to estimate the coefficient of mean-reversion of short term interest rates from historical data, due to low power. Our model provides an alternative method of estimating the mean-reversion and persistence factors using futures rather than spot data, and using both cross-sectional and time-series data rather than time-series data alone. This derives from the fact that the mean-reversion coefficient, together with the volatility and persistence of the second factor, determines the shape of the futures volatility curve. Hence, observation of the futures volatility curve could lead to improved estimation of mean-reversion. In addition, the model provides the inputs required to judge when a two-factor model may substantially change the pricing and hedging of interest rate contingent claims, and when a one-factor model is sufficient. Empirical analysis of futures rate volatilities and correlations, using data for the period 1990-99, provides some initial support for the two-factor model.

The two-factor model has the characteristic that any futures rate can be written as a log-linear function of any other two futures rates. The restriction to two stochastic variables and the assumption of lognormal LIBORs are important. The restriction to two factors can be relaxed, with some cost of increased complexity of the model. In Appendix 4, we show that if the second factor $y_i$ is itself generated by a two-factor model, then a three-factor cross-sectional relationship results. Such a three-factor model nests the two-factor model analysed here and provides a natural generalization. Another possible generalization would consider non-lognormal processes with, for example, stochastic volatility, generated perhaps with a GARCH process. Recent empirical findings in Brenner, et al. (1996) suggest that the short-term interest rate follows a stochastic volatility process. We leave such extensions for subsequent research.
Appendix 1: Properties of the conditional mean for two-dimensional time-series processes (Proof of Proposition 1)

Define

\[ x_t = g(P_t) - E_0[g(P_t)] \]

Equation (1) can then be written as

\[ x_t = (1 - c)x_{t-1} + y_{t-1} + \epsilon_t \]

**Sufficiency**

Successive substitution \( x_1, x_2, \ldots, x_{t+k} \) and taking the conditional expectation yields

\[ E_t(x_{t+k}) = x_t(1 - c)^k + \sum_{\tau=0}^{t-1} \nu_{t-\tau} (1 - \alpha)^\tau \cdot \sum_{\tau=1}^{k} (1 - c)^{k-\tau} (1 - \alpha)^{\tau-1} \tag{12} \]

Substituting the corresponding expression for \( E_t(x_{t+1}) \):

\[ E_t(x_{t+1}) = x_t(1 - c) + \sum_{\tau=0}^{t-1} \nu_{t-\tau} (1 - \alpha)^\tau \]

yields

\[ E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1}). \tag{13} \]

Hence, we have

\[ E_t[g(P_{t+k})] - E_0[g(P_{t+k})] = a_k [g(P_t) - E_0[g(P_t)]] \tag{14} \]
\[ + b_k [E_t[g(P_{t+1})] - E_0[g(P_{t+1})]] \tag{15} \]

where

\[ b_k = \sum_{\tau=1}^{k} (1 - c)^{k-\tau} (1 - \alpha)^{\tau-1} \]
and 

\[ a_k = (1 - c)^k - (1 - c)b_k. \]

**Necessity**

Assume that

\[
E_t[g(P_{t+k})] - E_0[g(P_{t+k})] = a_k [g(P_t) - E_0[g(P_t)]] + b_k [E_t[g(P_{t+1})] - E_0[g(P_{t+1})]]
\]

(16)

or

\[
E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1})
\]

(17)

where \( a_k \) and \( b_k \) are defined by (13) above, and \( x_t \) and \( E_t(x_{t+1}) \) are not perfectly correlated. Consider the orthogonal component \( z_t \) from

\[
E_t(x_{t+1}) = \gamma x_t + z_t
\]

(19)

Then

\[
E_t(x_{t+1}) = (a_1 + b_1 \gamma)x_t + b_1 z_t
\]

and hence, since \( a_1 = 0 \) and \( b_1 = 1 \)

\[
x_{t+1} = \gamma x_t + z_t + \epsilon_{t+1}.
\]

(20)

where \( E_t(\epsilon_{t+1}) = 0. \)

Hence \( x_t \) follows a two-dimensional process with innovations \( z_t, \epsilon_{t+1}. \)

We now show that \( \gamma = (1 - c) \) and also that \( z_t \) follows a mean-reversion process with mean-reversion \( \alpha. \) Suppose by way of contradiction, that \( \gamma = (1 - c'). \) Also, suppose there is a shock such that \( x_t \) changes while the difference, \( E_t(x_{t+1}) - x_t, \) is constant; then, \( E_t(x_{t+k}) \) will not be given by equation (13), since \( c \neq c'. \) It follows that we must have \( \gamma = (1 - c). \) Second, suppose that \( \gamma = (1 - c), \) but \( z_t \) mean-reverts at a rate different from \( \alpha. \) Then, if the difference, \( E_t(x_{t+1}) - x_t, \) changes, while \( x_t \) is constant, then again \( E_t(x_{t+k}) \) will not be given by equation (13). Hence, a necessary condition is that \( z_t \) mean-reverts at a rate \( \alpha. \)

In other words,
\[ g(P_{t+1}) = E_0[g(P_{t+1})] + (1 - c)[g(P_{t+1}) - E_0[g(P_{t+1})]] + y_t + \epsilon_{t+1} \quad (21) \]

or one period earlier,

\[ g(P_t) = E_0[g(P_t)] + (1 - c)[g(P_{t-1}) - E_0[g(P_{t-1})]] + y_t + \epsilon_t \quad (22) \]
Appendix 2: Properties of Lognormal LIBOR Rates
(Proof of Lemma 1)

From CIR (1981), the futures rate is equal to the expectation of the LIBOR under the equivalent martingale measure, \( F_{t,t+k} = E_t(i_{t+k}) \). Since by assumption \( i_{t+k} \) is lognormal, under the EMM, with a conditional logarithmic mean and annualised volatility of \( \mu(t, t + k, t + k) \) and \( \sigma(t, t + k, t + k) \), we have

\[
F_{t,t+k} = E_t(i_{t+k}) = \exp \left[ \mu(t, t + k, t + k) + \frac{kn}{2} \sigma^2(t, t + k, t + k) \right]
\]

Now since

\[
F(t, t + k) = E_t(i_{t+k})
\]

the expectation of the futures rate is

\[
E_0[F(t, t + k)] = E_0(i_{t+k}), \quad (23)
\]

by the law of iterated expectations.

Taking the logarithm of equation (23) and using the relationship of the mean and variance of lognormal variables, we have

\[
\mu(0, t, t+k) + \frac{tn}{2} \sigma^2(0, t, t+k) = \mu(0, t+k, t+k) + \frac{(t+k)n}{2} \sigma^2(0, t+k, t+k). \quad (24)
\]

From the lognormality of \( i_{t+k} \),

\[
(t+k)n \sigma^2(0, t+k, t+k) = \text{var}_0[\mu(0, t+k, t+k)] + kn \sigma^2(t, t+k, t+k). \quad (25)
\]

Moreover,
\[ F_{t,t+k} = \exp[\mu(t, t + k, t + k) + \frac{kn}{2} \sigma^2(t, t + k, t + k)] \]

\[ \text{var}_0[\mu(t, t + k, t + k)] = n\sigma^2(0, t, t + k). \quad (26) \]

Substituting equations (26) into (25), and then (25) into (24), yields

\[ \mu(0, t, t + k) = \mu(0, t + k, t + k) + \frac{kn}{2} \sigma^2(t, t + k, t + k). \]

From the definition of the futures interest rate as the expectation of the future spot interest rate under the risk-neutral measure, it follows that:

\[ \mu(t, t + k, t + k) = f_{t,t+k} - \frac{kn}{2} \sigma^2(t, t + k, t + k), \]

\[ \mu(0, t + k, t + k) = f_{0,t+k} - \frac{kn}{2} \sigma^2(0, t + k, t + k), \]

\[ \square \]
Appendix 3: The Volatilities and Correlations of Futures Rates (Proof of Proposition 4)

From Proposition 3, we have

$$\text{var}_{t-1}(f_{t,t+k}) = a_k^2 \text{var}_{t-1}(f_{t,t}) + b_k^2 \text{var}_{t-1}(f_{t,t+1})$$

$$+ 2a_k b_k \text{cov}_{t-1}(f_{t,t}, f_{t,t+1})$$

(27)

and

$$\text{cov}_{t-1}(f_{t,t}, f_{t,t+k}) = a_k \text{var}_{t-1}(f_{t,t}) + b_k \text{cov}_{t-1}(f_{t,t}, f_{t,t+1}).$$

(28)

First, we evaluate \( \text{var}_{t-1}(f_{t,t+1}) \) and \( \text{cov}_{t-1}(f_{t,t}, f_{t,t+1}) \). From equation (5) the time \( t+1 \) spot rate is

$$f_{t+1,t+1} = \mu(0, t + 1, t + 1) + [f_{t,t} - \mu(0, t, t)](1 - c) + y_t + \epsilon_{t+1}.$$ 

Hence, the conditional expectation at time \( t \) is

$$\mu(t, t + 1, t + 1) = \mu(0, t + 1, t + 1) + [f_{t,t} - \mu(0, t, t)](1 - c) + y_t.$$ 

But, from Lemma 1, we have

$$\mu(t, t + 1, t + 1) = f_{t,t+1} - \frac{n}{2} \sigma^2(t, t + 1, t + 1).$$

Hence, we can write

$$f_{t,t+1} = \mu(0, t + 1, t + 1) + \frac{n}{2} \sigma^2(t, t + 1, t + 1) + [f_{t,t} - \mu(0, t, t)](1 - c) + y_t.$$ 

It then follows directly that

$$\text{cov}_{t-1}(f_{t,t}, f_{t,t+1}) = (1 - c) \text{var}_{t-1}(f_{t,t}) + \text{cov}_{t-1}(y_t, f_{t,t}),$$

where \( \text{cov}_{t-1}(y_t, f_{t,t}) = \text{cov}_{t-1}(\nu_t, \epsilon_t) \). Hence,

$$\text{cov}_{t-1}(f_{t,t}, f_{t,t+1}) = (1 - c) \text{var}_{t-1}(f_{t,t}) + \text{cov}_{t-1}(\nu_t, \epsilon_t).$$

(29)

Similarly

$$\text{var}_{t-1}(f_{t,t+1}) = (1-c)^2 \text{var}_{t-1}(f_{t,t}) + 2(1-c)\text{cov}_{t-1}(\nu_t, \epsilon_t) + \text{var}_{t-1}(\nu_t).$$

(30)
Substitution of (29) and (30) in (27) and (28) then yields

\[
\text{var}_{t-1}(f_{t,t+k}) = a_k^2 \text{var}_{t-1}(f_{t,t}) \\
+ b_k^2 \left[(1-c)^2 \text{var}_{t-1}(f_{t,t}) + 2(1-c)\text{cov}_{t-1}(\nu_t, \epsilon_t) + \text{var}_{t-1}(\nu_t)\right] \\
+ 2a_kb_k[(1-c)\text{var}_{t-1}(f_{t,t}) + \text{cov}_{t-1}(\nu_t, \epsilon_t)]
\]

and

\[
\text{cov}_{t-1}(f_{t,t}, f_{t,t+k}) = [(1-c)^k] \text{var}_{t-1}(f_{t,t}) + b_k \text{cov}_{t-1}(\nu_t, \epsilon_t).
\]

Finally, using the annualised volatility definitions then yields:

\[
\sigma^2(t - 1, t, t + k) = a_k^2 \sigma^2(t - 1, t, t) + b_k^2 \sigma^2(t - 1, t, t + 1) \\
+ 2a_kb_k[(1-c)^k \sigma^2(t - 1, t, t) + 2\rho \sigma(t - 1, t, t) \sigma_{\pi}]
\]

and the correlation of the spot and futures rates is therefore given by:

\[
\rho(t - 1, t, t + k) = \frac{(1-c)^k \sigma^2(t - 1, t, t) + b_k \rho \sigma(t - 1, t, t) \sigma_{\pi}}{\sigma(t - 1, t, t) \sigma(t - 1, t, k)}
\]

where

\[
\sigma^2(t - 1, t, t + 1) = (1-c)^2 \sigma^2(t - 1, t, t) + \sigma_{\pi}^2 + 2(1-c)\rho \sigma_{\pi} \sigma(t - 1, t, t)
\]

and where \(\rho\) is the correlation between the errors \(\epsilon_t\) and \(\nu_t\). \(\sigma_{\pi}\) is the volatility of \(\pi\). □
Appendix 4: Generalisation of the Model to Three Factors

In this appendix, we show that our model can be generalised to three-factors. We present the following proposition, based on three variables, $x$, $y$ and $z$:

**Proposition 5** (General Cross-Sectional Relationship for the Change in Expected Values for the Three-Factor Case.)

If the variable $x_t$ follows the time series process

$$x_t = (1 - c)x_{t-1} + y_{t-1} + \epsilon_t$$

where

$$y_t = (1 - \alpha)y_{t-1} + z_{t-1} + \nu_t$$

and

$$z_t = (1 - \beta)z_{t-1} + \eta_t$$

the conditional expectation of $x_{t+k}$ is of the form

$$E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1}) + c_k E_t(x_{t+2})$$

where

$$a_k = a'_k - (1 - c)b'_k - (1 - c)(1 - \alpha)c'_k$$

$$b_k = b'_k - (1 - c)c'_k - (1 - \alpha)c'_k$$

$$c_k = c'_k$$

where

$$a'_k = (1 - c)^k$$

$$b'_k = \sum_{\tau=1}^{k} (1 - c)^{k-\tau}(1 - \alpha)^{\tau-1}$$

$$c'_k = \sum_{\tau=2}^{k} (1 - c)^{k-\tau}b'_{\tau-1}$$
where

\[ b'_k = \sum_{\tau=1}^{k} (1 - \alpha)^{k-\tau} (1 - \beta)^{\tau-1} \]

Proof.

Successive substitution and taking expectations yields for the \( z_t \) variable:

\[ E_t(z_{t+k}) = (1 - \beta)^k z_t, \quad (31) \]

Similarly, for the \( y_t \) variable, using (31):

\[ E_t(y_{t+k}) = (1 - \alpha)^k y_t + \sum_{\tau=1}^{k} (1 - \alpha)^{k-\tau} (1 - \beta)^{\tau-1} z_t, \quad (32) \]

Successive substitution for \( x_t \), using (32) then yields

\[
E_t(x_{t+k}) = (1 - c)^k x_t + \sum_{\tau=1}^{k} (1 - c)^{k-\tau} (1 - \alpha)^{\tau-1} y_t \\
+ \sum_{\tau=2}^{k} (1 - c)^{k-\tau} \left[ \sum_{s=1}^{\tau-1} (1 - \alpha)^{\tau-1-s} (1 - \beta)^{s-1} \right] z_t, \quad (33) \]

The proposition then follows by substitution of equations (31) and (32) in (33).

\[ \square \]
References


### Table 1
Notation for the Mean and Volatility of Spot and Futures Rates

<table>
<thead>
<tr>
<th>Time Period</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Spot prices and interest rates for m-year money</strong></td>
<td>( \mu(0, t, t) )</td>
<td>( P_t )</td>
<td>( P_T )</td>
</tr>
<tr>
<td>( \sigma(0, t, t) )</td>
<td>Unconditional logarithmic mean of ( i_t )</td>
<td>Zero bond price at ( t ) for delivery of $1 at ( (t + m) )</td>
<td>Zero bond price at time ( T ) for delivery of $1 at time ( T + m )</td>
</tr>
<tr>
<td>( i_t = F_{t,t} )</td>
<td>( m )-year interest rate at time ( t )</td>
<td>( i_T = F_{T,T} )</td>
<td>( m )-year interest rate at time ( T )</td>
</tr>
</tbody>
</table>

| Futures interest rates for bonds maturing at time \( T + m \) | \( \mu(0, t, T) \) | \( F_{t,T} \) | \( f_{t,T} \) |
| \( \sigma(0, t, T) \) | Mean of \( f_{t,T} \) | \( F_{t,T} \) futures interest rate at \( t \) for delivery at \( T \) \( (m \)-year money) \( f_{t,T} \) Logarithm of \( F_{t,T} \) |
| Conditional mean of \( f_{T,T} \) | \( \mu(t, T, T) \) | Conditional \( (\text{annualised}) \) volatility of \( F_{T,T} \) | \( \sigma(t, T, T) \) | \( \sigma(t, T, T) \) \( (\text{annualised}) \) volatility of \( F_{T,T} \) |
**Table 1: Correlation Coefficients between Monthly Changes in the Logarithms of Futures Rates for Different Maturities**

This table presents the coefficients of correlation between changes in the logarithms of futures rates of different maturities for Eurodollar futures contracts. The data used are for the period January 1, 1990 to December 31, 1999, sampled on a monthly basis. The data from adjacent futures contracts, interpolated to produce a time-series of futures prices and rates for fixed maturity periods: 3, 6, 9, ... months. Panel A represents the correlation matrix for the period 1990-94, Panel B for 1995-99, and Panel C for the whole period, 1990-99.

<table>
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<th>3</th>
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<th>9</th>
<th>12</th>
<th>15</th>
<th>18</th>
<th>21</th>
<th>24</th>
<th>27</th>
<th>30</th>
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<td>0.98</td>
<td>0.97</td>
<td>0.97</td>
<td>0.95</td>
<td>0.93</td>
<td>0.91</td>
<td>0.91</td>
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<tr>
<td>21</td>
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<td>0.92</td>
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**Panel A: January 1, 1990 to December 31, 1994**

**Panel B: January 1, 1995 to December 31, 1999**
### Interest Rate Futures

#### Panel C: January 1, 1990 to December 31, 1999

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Table 2: Correlation Coefficients between Quarterly Changes in the Logarithms of Futures Rates for Different Maturities

This table presents the coefficients of correlation between changes in the logarithms of futures rates of different maturities for Eurodollar futures contracts. The data used are for the period January 1, 1990 to December 31, 1999, sampled on a quarterly basis. The data from adjacent futures contracts, interpolated to produce a time-series of futures prices and rates for fixed maturity periods: 3, 6, 9, ... months.

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Figure 1: The Futures Term Structures: Mean Reversion, \( c = 0 \); Persistence, \( \alpha = 0.25 \).

The figure shows the term structures of futures rates for the base case, where the rates are 5% for all maturities, and the cases where the changes in the logarithms of the short rate and the premium factors are the same in the shifts up and down. In the one-factor case, the logarithm of the short rate shifts up by 1%. The shift down is similar in logarithmic terms. In the two-factor case, the premium factor also moves up by 0.3% from 1, in the same direction. The shift down is similar in logarithmic terms. The dotted line indicates the base case, the solid lines the term structures with the one-factor model, and the dashed lines the term structures with the two-factor model.
Figure 2: The Futures Term Structures: Mean Reversion, $c = 0$; Persistence, $\alpha = 0.75$.

The figure shows the term structures of futures rates for the base case, where the rates are 5% for all maturities, and the cases where the changes in the logarithms of the short rate and the premium factors are the same in the shifts up and down. In the one-factor case, the logarithm of the short rate shifts up by 1%. The shift down is similar in logarithmic terms. In the two-factor case, the premium factor also moves up by 0.3% from 1, in the same direction. The shift down is similar in logarithmic terms. The dotted line indicates the base case, the solid lines the term structures with the one-factor model, and the dashed lines the term structures with the two-factor model.
Figure 3: The Futures Term Structures: Mean Reversion, \( c = 0.05 \); Persistence, \( \alpha = 0.25 \).

The figure shows the term structures of futures rates for the base case, where the rates are 5% for all maturities, and the cases where the changes in the logarithms of the short rate and the premium factors are the same in the shifts up and down. In the one-factor case, the logarithm of the short rate shifts up by 1%. The shift down is similar in logarithmic terms. In the two-factor case, the premium factor also moves up by 0.3% from 1, in the same direction. The shift down is similar in logarithmic terms. The dotted line indicates the base case, the solid lines the term structures with the one-factor model, and the dashed lines the term structures with the two-factor model.
Figure 4: The Futures Term Structures: Mean Reversion, $c = 0.05$; Persistence, $\alpha = 0.75$.

The figure shows the term structures of futures rates for the base case, where the rates are 5% for all maturities, and the cases where the changes in the logarithms of the short rate and the premium factors are the same in the shifts up and down. In the one-factor case, the logarithm of the short rate shifts up by 1%. The shift down is similar in logarithmic terms. In the two-factor case, the premium factor also moves up by 0.3% from 1, in the same direction. The shift down is similar in logarithmic terms. The dotted line indicates the base case, the solid lines the term structures with the one-factor model, and the dashed lines the term structures with the two-factor model.
Figure 5: Futures Term Structures: Mean Reversion, $c = 0$; Persistence, $\alpha = 0.25$.

The figure shows the term structures of futures rates for the base case, where the rates are 5% for all maturities, and the cases where the changes in the logarithms of the short rate and the premium factors are the same in the shifts up and down. In the one-factor case, the logarithm of the short rate shifts up by 1%. The shift down is similar in logarithmic terms. In the two-factor case, the premium factor also moves up by 0.3% from 1, in the opposite direction. The shift down is similar in logarithmic terms. The dotted line indicates the base case, the solid lines the term structures with the one-factor model, and the dashed lines the term structures with the two-factor model.
Figure 6: The Futures Volatility Structure: Mean Reversion, $c = 0$; Persistence, $\alpha = 0.25$.

The figure shows the term structure of volatilities for the one-factor base case, where the volatilities are 10% for all maturities, and the two-factor cases where the volatilities take into account the effects of mean-reversion and persistence in the short term interest rate (using Proposition (3)). In the one-factor case, the volatility of the premium factor is zero. In all the two-factor cases, the volatility of the premium factor is 6%. In the two-factor cases, the correlation coefficient between the errors is -0.5, 0 or +0.5. The solid line indicates the futures volatility term structure for the one-factor model, and the dashed line indicates the futures volatility term structure for the two-factor model in the case where the correlation coefficient is 0. The line with crosses is the two-factor case where the correlation coefficient is +0.5. The line with circles is the two-factor case where the correlation coefficient is -0.5.
Figure 7: The Futures Volatility Structure: Mean Reversion, $c = 0$; Persistence, $\alpha = 0.75$.

The figure shows the term structure of volatilities for the one-factor base case, where the volatilities are 10% for all maturities, and the two-factor cases where the volatilities take into account the effects of mean-reversion and persistence in the short term interest rate (using Proposition (3)). In the one-factor case, the volatility of the premium factor is zero. In all the two-factor cases, the volatility of the premium factor is 6%. In the two-factor cases, the correlation coefficient between the errors is -0.5 0 or +0.5. The solid line indicates the futures volatility term structure for the one-factor model, and the dashed line indicates the futures volatility term structure for the two-factor model in the case where the correlation coefficient is 0. The line with crosses is the two-factor case where the correlation coefficient is +0.5. The line with circles is the two-factor case where the correlation coefficient is -0.5.
Figure 8: The Futures Volatility Structure: Mean Reversion, $c = 0.05$; Persistence, $\alpha = 0.25$.

The figure shows the term structure of volatilities for the one-factor base case, where the volatilities are 10% for all maturities, and the two-factor cases where the volatilities take into account the effects of mean-reversion and persistence in the short term interest rate (using Proposition (3)). In the one-factor case, the volatility of the premium factor is zero. In all the two-factor cases, the volatility of the premium factor is 6%. In the two-factor cases, the correlation coefficient between the errors is -0.5 0 or +0.5. The solid line indicates the futures volatility term structure for the one-factor model, and the dashed line indicates the futures volatility term structure for the two-factor model in the case where the correlation coefficient is 0. The line with crosses is the two-factor case where the correlation coefficient is +0.5. The line with circles is the two-factor case where the correlation coefficient is -0.5.
Figure 9: The Futures Volatility Structure: Mean Reversion, \( c = 0.05 \); Persistence, \( \alpha = 0.75 \).

The figure shows the term structure of volatilities for the one-factor base case, where the volatilities are 10% for all maturities, and the two-factor cases where the volatilities take into account the effects of mean-reversion and persistence in the short term interest rate (using Proposition (3)). In the one-factor case, the volatility of the premium factor is zero. In all the two-factor cases, the volatility of the premium factor is 6%. In the two-factor cases, the correlation coefficient between the errors is -0.5, 0 or +0.5. The solid line indicates the futures volatility term structure for the one-factor model, and the dashed line indicates the futures volatility term structure for the two-factor model in the case where the correlation coefficient is 0. The line with crosses is the two-factor case where the correlation coefficient is +0.5. The line with circles is the two-factor case where the correlation coefficient is -0.5.
Figure 10: The Futures Correlation Structure: Mean Reversion, $c = 0$; Persistence, $\alpha = 0.25$.

The figure shows the term structures of correlations between the spot rate and the $k^{th}$ futures rate for the two-factor case, for different values of the correlation coefficient between the errors. The correlations take into account the effects of mean-reversion and persistence in the short term interest rate, using equation (10). The dashed line and the solid line indicate the futures correlation term structure when the volatility of the premium factor are 3% and 6% respectively, in the case where the correlation coefficient is 0. The line with crosses is the case where the correlation coefficient is +0.5. The line with circles is the case where the correlation coefficient is -0.5.
Figure 11: The Futures Correlation Structure: Mean Reversion, \( c = 0 \); Persistence, \( \alpha = 0.75 \).

The figure shows the term structures of correlations between the spot rate and the \( k^{th} \) futures rate for the two-factor case, for different values of the correlation coefficient between the errors. The correlations take into account the effects of mean-reversion and persistence in the short term interest rate, using equation (10). The dashed line and the solid line indicate the futures correlation term structure when the volatility of the premium factor are 3% and 6% respectively, in the case where the correlation coefficient is 0. The line with crosses is the case where the correlation coefficient is +0.5. The line with circles is the case where the correlation coefficient is -0.5.
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Figure 12: The Futures Correlation Structure: Mean Reversion, \( c = 0.05 \); Persistence, \( \alpha = 0.25 \).

The figure shows the term structures of correlations between the spot rate and the \( k^{th} \) futures rate for the two-factor case, for different values of the correlation coefficient between the errors. The correlations take into account the effects of mean-reversion and persistence in the short term interest rate, using equation (10). The dashed line and the solid line indicate the futures correlation term structure when the volatility of the premium factor are 3\% and 6\% respectively, in the case where the correlation coefficient is 0. The line with crosses is the case where the correlation coefficient is +0.5. The line with circles is the case where the correlation coefficient is -0.5.
Figure 13: The Futures Correlation Structure: Mean Reversion, $c = 0.05$; Persistence, $\alpha = 0.75$.

The figure shows the term structures of correlations between the spot rate and the $k^{th}$ futures rate for the two-factor case, for different values of the correlation coefficient between the errors. The correlations take into account the effects of mean-reversion and persistence in the short term interest rate, using equation (10). The dashed line and the solid line indicate the futures correlation term structure when the volatility of the premium factor are 3% and 6% respectively, in the case where the correlation coefficient is 0. The line with crosses is the case where the correlation coefficient is +0.5. The line with circles is the case where the correlation coefficient is -0.5.
Figure 14: The Futures Volatility Term Structure: Empirical Evidence based on Monthly Changes in Futures Rates

The figure shows the term structures of volatilities of the logarithm of the futures interest rate for various maturities based on monthly (interpolated) Eurodollar futures data for the period January 1, 1990-December 31, 1999. The solid line represents the data for the period 1990-94. The line with crosses represents the data for the period 1995-99 and the line with circles the data for the whole period 1990-99.
Figure 15: The Futures Volatility Term Structure: Empirical Evidence based on Quarterly Changes in Futures Rates

The figure shows the term structures of volatilities of the logarithm of the futures interest rate for various maturities based on quarterly (interpolated) Eurodollar futures data for the period January 1, 1990-December 31, 1999. The solid line represents the data for the period 1990-94. The line with crosses represents the data for the period 1995-99 and the line with circles the data for the whole period 1990-99.
Figure 16: The Futures Correlation Term Structure: Empirical Evidence based on Monthly Changes in Futures Rates

The figure shows the term structures of correlations between the logarithm of the 3-month futures interest rate (based on the 1st futures contract) and the logarithm of the futures rate of the relevant maturity based on monthly (interpolated) Eurodollar futures data for the period January 1, 1990-December 31, 1999. The solid line represents the data for the period 1990-94. The line with crosses represents the data for the period 1995-99 and the line with circles the data for the whole period 1990-99.
Figure 17: The Futures Correlation Term Structure: Empirical Evidence based on Quarterly Changes in Futures Rates

The figure shows the term structures of correlations between the logarithm of the 3-month futures interest rate (based on the 1st futures contract) and the logarithm of the futures rate of the relevant maturity based on quarterly (interpolated) Eurodollar futures data for the period January 1, 1990-December 31, 1999. The solid line represents the data for the period 1990-94. The line with crosses represents the data for the period 1995-99 and the line with circles the data for the whole period 1990-99.