# The Accounting Rate of Return as a Framework for Analysis

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September 1999

This paper has gone through a number of iterations. Along the way, Ken Peasnell, participants at seminars at New York University and London School of Economics and several anonymous reviewers made helpful comments.

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Research on the accounting rate of return (ARR) began with Harcourt (1965), Solomon (1966) and Vatter (1966), and continued with work by Kay (1976, 1978), Peasnell (1982), Franks and Hodges (1984), Kay and Edwards (1986), Edwards, Kay and Mayer (1987), Brief and Lawson (1992), Brief (1996) and others. The main objective was either to study the relationship between the ARR and the internal rate of return (IRR) or to use the ARR in the valuation process. In both cases, the basic framework was provided by an economic model and the accounting model was fit into the economic framework.

In contrast, the goal of this paper is to separate the accounting process from the economic model, first, by emphasizing the analytical properties of the accounting model that defines the ARR as a constant and, second, by showing how these properties can provide insights into the accounting process which cannot be gleaned from the more traditional approach which focuses on the economic model from the start.

Twenty-six properties of the accounting model are discussed. The first 14 deal with properties related to the constant ARR and the smoothed time series derived from the constant ARR. Property 15 defines two economic models as special cases of the accounting model. The remaining properties concern the relationship between the IRR and ARR, the residual income valuation model and the direct comparison of accounting model and the economic models. Some of these properties have been derived by others and some are discussed for the first time. Taken together, and viewed as a framework for analysis, these properties can be used to study a diverse set of questions relating to the accounting process.

## Constant ARR

The time horizon is assumed to begin in period $t = 0$, the end of period 0 (start of period 1), and to terminate $T$ periods later at the end of period $T$. The time slice, $(0, T)$, may be the entire life of the firm or any window of time during the firm's life. There are $T(T + 1)/2$ time segments of various lengths in this time span, i.e. $T$ one-period segments, $T - 1$ two period segments, etc. The main concern is with the ARR defined over the T one-period time segments and the ARR defined over the interval, $(0, T)$. We also consider the steady-state situation that begins at the end of period T and continues indefinitely, i.e., the time period $(T, \infty)$.

The accounting data might relate to the past or to the future. If the time perspective is ex ante, the future is assumed to be known with certainty. Unless stated otherwise, all variables are defined from the viewpoint of the firm's equity interests. The perspective could be switched to a firm's total assets, to a single investment project or to a collection of projects.

Let

$A_t$ = accounting book value at the end of period $t$

$C_t$ = net dividends paid at the end of period $t$.

$I_t$ = comprehensive income in period $t$

where $I_t = C_t + A_t - A_{t-1}$. The accounting numbers are assumed to be produced by *any set of* accounting methods as long as the clean surplus relation holds.

The ARR over a one-period time segment that starts at the end of period $t-1$ (beginning of period $t$) and ends at the end of period $t$, $a_{t-1,t}$, is defined as the value of $a_{t-1,t}$ that solves

$$A_{t-1} = \frac{C_t + A_t}{1 + a_{t-1,t}}. \tag{1}$$

Solving equation (1) for $a_{t-1,t}$ gives the standard definition of the ARR as the ratio of the income in a period to the book value at the beginning of the period:

$$a_{t-1,t} = \frac{C_t + A_{t-1} - A_t}{A_{t-1}} = \frac{I_t}{A_{t-1}}. \tag{2}$$

- Property 1: Constant ARR

If the ARR is constant over T periods and $a_{0,1} = a_{1,2} = ... = a_{T-1,T}$, the ARR is equal to the value of $a_{0,T}$ that solves

$$A_0 = \frac{C_1}{(1 + a_{0,T})^1} + \frac{C_2}{(1 + a_{0,T})^2} + ... + \frac{C_T}{(1 + a_{0,T})^T} + \frac{A_T}{(1 + a_{0,T})^T}. \tag{3}$$

Proof: The proof is given for T = 3. The generalization follows immediately. Define the constant one-period ARR as $k$. Then, from equation (1),

$$A_2 = \frac{C_3 + A_3}{1 + k}.$$

Continuing recursively,

$$A_1 = \frac{C_2 + A_2}{1 + k} = \frac{C_2 + \frac{C_3 + A_3}{1+k}}{1 + k} = \frac{C_2}{1 + k} + \frac{C_3 + A_3}{(1 + k)^2},$$

and

$$A_0 = \frac{C_1 + A_1}{1+k} = \frac{C_1 + \frac{C_2}{1+k} + \frac{C_3+A_3}{(1+k)^2}}{1+k},$$

or

$$A_0 = \frac{C_1}{1+k} + \frac{C_2}{(1+k)^2} + \frac{C_3+A_3}{(1+k)^3}.$$

Therefore, if the one-period ARRs are a constant, $k$, $k = a_{0,T}$.

This proof is similar to the example in Vatter (1965) but his example does not explicitly use the definition of the ARR in equation (1) so the logic is not apparent. Peasnell's (1982) proof of Property 1 is inductive, not deductive. The one-period ARRs need not be constant to show that the net dividends and terminal book value always discount back to $A_0$. Whether or not the ARRs are equal, the interim accounting book values over the T periods can be defined as the present discounted value of future dividends plus the discounted terminal book value. It is assumed throughout that equation (3) has a unique solution, but the question of uniqueness is addressed later.

Property 1 implies that the only constant ARR consistent with a given set of values for net dividends, $C_t$, and opening and closing books values, $A_o$ and $A_T$, is the ARR defined in equation (3).

The definition of the ARR as a multi-period rate of return in Equation (3) was first explicitly discussed in Brief, Merino and Weiss (1980). However, the idea can be traced to Kay (1976) who, in effect, defined the constant ARR equal to the IRR. In Kay and Mayer (1986) and Edwards, Kay and Mayer (1987), the ARR was given a separate identity. Brief and Lawson (1992) and Brief (1996) also distinguished between the IRR and the ARR defined in equation (3). The constant ARR and the book values and income numbers associated with this constant have a number of analytical properties which are discussed below.

- Property 2: The ARR as a Weighted Average

The ARR, $a_{0,T}$, can be expressed as a weighted average of one-period ARRs:

$$a_{0,T} = \frac{\sum_{t=1}^{T} \frac{a_{t-1,t}A_{t-1}}{(1+a_{0,T})^t}}{\sum_{t=1}^{T} \frac{A_{t-1}}{(1+a_{0,T})^t}},$$

or, equivalently,

$$a_{0,T} = \sum_{t=1}^{T} w_t a_{t-1,t}, \tag{4}$$

where

$$w_t = \frac{\frac{A_{t-1}}{(1+a_{0,T})^t}}{\sum_{t=1}^{T} \frac{A_{t-1}}{(1+a_{0,T})^t}}.$$

Proof: A simple proof is suggested by Franks and Hodges (1984). Assuming the clean surplus relation holds, substitute the accounting equivalent of net dividends, $C_t = I_t - (A_t - A_{t-1}) = (1 + a_{t-1,t})A_{t-1} - A_t$ into equation (3) and simplify in several steps.

First,

$$A_0 = \frac{(1+a_{0,1})A_0 - A_1}{(1+a_{0,T})^1} + \frac{(1+a_{1,2})A_1 - A_2}{(1+a_{0,T})^2} + ... + \frac{(1+a_{T-1,T})A_{T-1} - A_T}{(1+a_{0,T})^T} + \frac{A_T}{(1+a_{0,T})^T}$$

or

$$0 = -A_0 + \frac{A_0 + a_{0,1}A_0 - A_1}{(1+a_{0,T})^1} + \frac{A_1 + a_{1,2}A_1 - A_2}{(1+a_{0,T})^2} + ... + \frac{A_{T-1} + a_{T-1,T}A_{T-1}}{(1+a_{0,T})^T}.$$

Simplifying and collecting terms,

$$0 = \sum_{t=1}^{T} \frac{A_{t-1}}{(1+a_{0,T})^t} + \sum_{t=1}^{T} \frac{a_{t-1,t}A_{t-1}}{(1+a_{0,T})^t} - \sum_{t=1}^{T} \frac{A_{t-1}}{(1+a_{0,T})^{t-1}}.$$

Multiplying the numerator and denominator of the third term on the right hand side of the equation by $(1 + a_{0,T})$,

$$0 = \sum_{t=1}^{T} \frac{A_{t-1}}{(1+a_{0,T})^t} + \sum_{t=1}^{T} \frac{a_{t-1,t}A_{t-1}}{(1+a_{0,T})^t} - (1+a_{0,T}) \sum_{t=1}^{T} \frac{A_{t-1}}{(1+a_{0,T})^t}$$

which simplifies to

$$0 = \sum_{t=1}^{T} \frac{a_{t-1,t}A_{t-1}}{(1+a_{0,T})^t} - a_{0,T} \sum_{t=1}^{T} \frac{A_{t-1}}{(1+a_{0,T})^t}.$$

Solving for $a_{0,T}$ gives Property 2.

Since equation (4) defines $a_{0,T}$ as a weighted average ARR, the maximum and minimum values of the one-period ARR set the bounds on $a_{0,T}$ (assuming all $w_t > 0$). Property 2 also can be used to show that if the one-period ARRs are constant, $k$, the constant must equal $a_{0,T}$. Substitute $k$ for $a_{t-1,t}$ in equation (4). Since $\sum_{t=1}^{T} w_t = 1$, $a_{0,T} = k$.

- Property 3: Relationship Between $a_{0,T}$ and $\bar{a}$

a. Assume the growth of book value is constant in each one-period time segment and let the constant growth rate equal $g_{0,T}^{A}$. By definition, $g_{0,T}^{A} = (\frac{A_T}{A_0})^{\frac{1}{T}} - 1$. If $g_{0,T}^{A} = a_{0,T}$, then a simple average of the one-period segment returns, $\bar{a}$, equals $a_{0,T}$ :

$$\bar{a} = a_{0,T}.$$

Proof: If $g_{0,T}^{A} = a_{0,T}$, $w_t = \frac{A_0}{\sum_{t=1}^{T} A_0} = \frac{1}{T}$. Property 3 immediately follows from the definition of $a_{0,T}$ in equation (4). Further insight into the conditions under which Property 3 holds is given in Property 11c(1) and 11c(2).

b. If growth, $g_{0,T}^{A}$, of book value is constant in each one-period time segment and if $g_{0,T}^{A} > a_{0,T}$ (or $g_{0,T}^{A} < a_{0,T}$), then

$$a_{0,T} \neq \bar{a}.$$

Proof: When $g_{0,T}^{A} > a_{0,T}$ ($g_{0,T}^{A} < a_{0,T}$), the weights, $w_t$ are increasing (decreasing) over time and the later (earlier) one-period ARRs are given greater weight.

## Smoothed Time Series Derived from the Constant ARR

- Property 4: Using the ARR to Smooth a Time Series

Any time series of accounting income numbers, $I_t$, and book values, $A_t$, can be restated, without loss of generality, into an imputed series, $I_t^*$ and $A_t^*$. The process of restating the accounting numbers can be viewed as the smoothing, standardizing and/or normalizing of an accounting time series.

The derivation of the imputed series depends only on the information in equation (1). First, compute $a_{0,T}$. Then, compute $I_1^* = a_{0,T}A_0$ and then $A_1^* = A_0 + I_1^* - C_1$, $I_2^* = a_{0,T}^{*}A_1^*$, etc. These imputed accounting numbers have properties that are similar to Hotelling (1925) and Hicksian (1939) concepts of economic depreciation, income and wealth. The smoothing procsss also can be viewed as a variant of the “effective interest method” which has a long history.[1] The critical question is what purpose would be served by this smoothing process? Would this smoothing or standardizating procedure produce accounting numbers that are more “useful” than the original time series? The question of purpose is paramount.

The argument here is that this accounting model has desirable analytical properties that provide a framework for analysis. First, the smoothed series mimics economic concepts. Therefore, the comparison of this model with Hicksian or present value models makes many relationships transparent. Second, under certain conditions, the smoothed time series also mimics steady state conditions. Third, the smoothed time series has other attractive properties. Fourth, while some important analytical relationships that have been derived in accounting are based on some of these properties, this connection has generally not been recognized.

- Property 5: The ARR of the Smoothed Time Series

The one-period ARRs of the imputed series, $a^*_{t-1,t}$, are constant and

$$a^*_{t-1,t} = a_{0,T}.$$

Proof: Since $I^*_t = a_{0,T}A^*_t$ and $a^*_{t-1,t} = I^*_t/A^*_t$, it immediately follows that $a^*_{t-1,t} = a_{0,T}$.

It follows from equation (1) with a constant IRR that

$$A_0 = \frac{I^*_1}{a_{0,T}}$$

which means that the book value at the beginning of any period can be written as the present discounted value of a perpetual annuity of $I^*_1$ :

$$\sum_{t=1}^{\infty} \frac{I^*_1}{(1 + a_{0,T})^t}.$$

- Property 6: Smoothed Book Values, $A^*_t$

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[1] Properties of the effective interest method have recently been discussed by Penman (1998) in connection with accounting for financial assets.

The book values[2] imputed in any period j by the process described above are equal to the present discounted value of the remaining stream of benefits if the discount rate is $a_{0,T}$:

$$A_j^* = \sum_{t=j+1}^{T} C_t/(1+a_{0,T})^{t-j} + A_T/(1+a_{0,T})^{T-j}. \tag{6}$$

Proof: Equation (6) follows immediately from the proof of Property 1. Another proof is more direct. Since, by definition, $A_1^* = A_0 + I_1^* - C_1 = (1+a_{0,T})A_0 - C_1$,

$$\begin{aligned} A_1^* &= (1+a_{0,T})(\sum_{t=1}^{T} C_t/(1+a_{0,T})^t + A_T/(1+a_{0,T})^T) - C_1 \\ &= \sum_{t=2}^{T} C_t/(1+a_{0,T})^{t-1} + A_T/(1+a_{0,T})^{T-1}). \end{aligned}$$

Similarly, $A_2^* = A_1^* + I_2^* - C_2 = (1+a_{0,T})A_1^* - C_2$. Therefore,

$$\begin{aligned} A_2^* &= (1+a_{0,T})(\sum_{t=2}^{T} C_t/(1+a)^{t-1} + A_T/(1+a)^{T-1}) - C_2 \\ &= \sum_{t=3}^{T} C_t/(1+a_{0,T})^{t-2} + A_T/(1+a_{0,T})^{T-2}). \end{aligned}$$

The results generalize to equation (6). Since, under steady state conditions, the ARR is constant, the book values under steady state conditions are the smoothed book values defined in equation (6).

- Property 7: Depreciation (Appreciation) of $A_t^*$

For a going concern, the concept of depreciation (appreciation) relates to the decrease (increase) in the firm's book. (We assume that terminal book value, $A_T$, is positive.) The accounting equivalent of Hotelling or Hicksian depreciation (appreciation) for the going concern, i.e., , is

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[2] Note the the initial book value, $A_0$, and the terminal book value, $A_T$, are the book values for *any* accounting method. The set of imputed (smoothed) book values, $A_t^*$, $t = 1, 2, ..., T-1$, are substituted for the infinitely many possible sets of interim unsmoothed book values.

$$D_t^* = A_{t-1}^* - A_t^* = C_t - a_{0,T} A_{t-1.}^* \tag{7}$$

Proof: Since $(1 + a_{0,T})A_{t-1} = A_t + C_t$, $D_t = A_t^* - A_t^* = C_t - a_{0,T}A_{t-1.}^*$

Depreciation (appreciation) in smoothed book value from the end of period $t-1$ to $t$ equals net dividends in period $t$ minus imputed income. It immediately follows that

$$\sum_{t=1}^{T} D_t^* = A_0 - A_T = \sum_{t=1}^{T} C_t - a_{0,T} \sum_{t=1}^{T} A_{t-1.}^* \tag{8}$$

Instead of making the assumption that the firm is a going concern, notional liquidation of the firm at the end of period T can be assumed. Equivalently, the accounting unit could be a project with a life of T periods and the project is written off at the end of period T. In both of these cases, $A_T = 0$. If depreciation (appreciation) is expressed as a fraction, $p_t^{A^*}$, of opening book value, $A_0$, where $p_t^{A^*} = (A_{t-1}^* - A_t^*)/A_0$,

$$\sum_{t=1}^{T} D_t^* = \sum_{t=1}^{T} p_t^{A^*} A_0 = A_0. \tag{9}$$

It immediately follows that $\sum_{t=1}^{T} p_t^{A^*} = 1$.

- Property 8: Total Income of Smoothed Time Series, $I_t^*$

Total income for the smoothed time series is

$$\sum_{t=1}^{T} I_t^* = \sum_{t=1}^{T} a_{0,T} A_{t-1}^* = \sum_{t=1}^{T} C_t + A_T - A_0. \tag{10}$$

Proof: Equation (10) follows directly from equation (8).

Even if the unsmoothed series is not based on the clean surplus relation, the smoothed series will have the "additivity property" of clean surplus. Clean surplus means that total income over $T$ periods must be fully allocated. The smoothing process produces clean surplus financial statements because the process of imputing income and book values is equivalent to the allocation of total income, $\sum_{t=1}^{T} C_t + A_T - A_0$, over the $t = 1, 2, ...T$ periods so that the accounting rate of return is constant (Brief and Owen, 1968). This is the same process associated with almost all allocation methods in accounting.

Another property of a clean surplus double-entry system is (again assuming that equation (3) has a unique solution) that it can produce only one constant ARR that is consistent with the net dividend sequence and opening and closing book values. If the accounting rate of return is constant over a multi-period time horizon but does not equal $a_{0,T}$, the clean surplus relation has been violated. In that case, the total income that is reported over the time horizon will not be the same as the total income defined in equation (10).

Accounting-based valuation models require the clean surplus property to hold. (See Property 19.) That means that the only constant ARR that can be used in the valuation process is the constant ARR defined in equation (3). However, this requirement has not been discussed in the valuation literature. For example, Copeland, Koller and Murrin (1995) assume that the ARR is constant, but they do not comment on how this constant is determined. Damodaran (1996) states that to compute the constant ARR, financial analysts calculate the arithmetic mean of one-period ARRs. However, Property 3 shows that the equality between the arithmetic mean and the constant ARR defined in equation (3) is a special case and does not, in general, hold.

- Property 9: $I_t^*$ is positive if $A_T^*$ is positive.

Proof: As long as the $A_t^*$ are positive (we continue to assume that $a_{0,T}$ is positive and unique), losses cannot occur.

Losses often occur in one-period time segments and create a variety of problems in fiancial analysis, e.g., the calculation of growth rates, negative price-earnings ratios, etc. One approach for dealing with one-period losses would be to compute, in the period of the loss, a time series of $A_t^*$ and $I_t^*$ over, say, the last five periods, ending in the current period. This smoothed time series and related growth rates would be much more "representative" of the firm's financial performance than the original time series and growth rates derived from it. As an example, the first two columns in Table I give a time series of unsmoothed book values and income numbers over 5 periods where a loss is reported in period 5. The smoothed series also is provided.

To illustrate the difference in growth rates calculated from the smoothed and unsmoothed time series, consider a method described in Damodaran (1995) to deal with computing the growth rate of income of an unsmoothed time series when there are losses. The first step is to regress the time series, $I_t$, on time over, say, the last five periods; and then divide the regression coefficient ("average" change per period) by the average value of the time series, $\bar{I}$, to get the growth rate. Using this method for the example in Table I, the growth rate of income from periods 1 to 5 is -2/18 or -.11. This compares to the geometric mean growth rate of the smoothed income series .082. The transformation of the conventional time series into a smoothed time series is one way to deal with the general issue of losses.

- Property 10: Finite Time Horizon Equivalent of Steady State

a. If the growth of the smoothed book values, $A_t^*$, in each one-period time segment is constant and is equal to $g_{0,T}^A$, net dividends, $C_t$, will grow at the same rate over the finite time horizon except when all $C_t = 0$.

Proof: Since $I_t^* = a_{0,T} A_{t-1}^* = C_t + g_{0,T}^A A_{t-1}^*$, $C_t = (a_{0,T} - g_{0,T}^A) A_{t-1}^*$.

Constant growth of smoothed book value implies that net dividends and income grow at the same rate unless all $C_t = 0$. This is an attractive property of the smoothed time series of income and book value in that the one-period growth rates of the two series are the same and are equal to the growth rate of net dividends. Therefore, constant growth, $g_{0,T}^A$, of the smoothed book values over the T periods results in the finite time period equivalent of the steady state.

In general, the constant growth of dividends over the period, (0,T) does not imply constant growth of smoothed book values. But there are two exceptions.

b(1). Over an infinite time horizon, constant growth of dividends imply that the smoothed book values will grow at the same rate.

Proof: Let $g_{0,\infty}^C$ be the constant growth of dividends in each period. If $a_{0,\infty} - g_{0,\infty}^C > 0$, $A_0 = \frac{C_1}{a_{0,\infty} - g_{0,\infty}^C}$, $A_1^* = \frac{C_1(1+g_{0,\infty}^C)}{a_{0,\infty} - g_{0,\infty}^C}$, etc. Therefore, smoothed book values grow at the same rate as dividends.

b(2). Over a finite time horizon, $(0, T)$, the smoothed book values will grow at the constant rate, $g_{0,T}^C$, in each period if $g_{0,T}^C = g_{0,T}^A$ (where $g_{0,T}^A$ geometric mean growth rate of book value).

Proof: Assuming that $g_{0,T}^C = g_{0,T}^A$,

$$A_0 = C \frac{1 - (\frac{1+g_{0,T}^C}{1+a_{0,T}})^T}{a_{0,T} - g_{0,T}^A} + A_0 (\frac{1 + g_{0,T}^C}{1 + a_{0,T}})^T$$

which implies that $A_0 = \frac{C_1}{a_{0,T} - g_{0,T}^A}$, $A_2 = \frac{C_1(1+g_{0,T}^C)}{a_{0,T} - g_{0,T}^A}$, etc. Therefore, since, by assumption, $g_{0,t}^C = g_{0,T}^A$, constant growth of dividends implies that $A_t = A_{t-1}(1 + g_{0,T}^A)$.

- Property 11: Constant Growth: Some Special Cases

a. When the $A_t^*$ grow at the constant rate, $g_{0,T}^A$, and when $g_{0,T}^A = a_{0,T}$, all $C_t = 0$.

Proof: By definition, $I_t^* = a_{0,T}A_{t-1}^*$, and given the assumption that $g_{0,T}^A = a_{0,T}$, $I_t^* = g_{0,T}^A A_{t-1}^*$. But $I_t^*$ also equals $C_t + g_{0,T}^A A_{t-1}^*$ which means that all $C_t = 0$.

b. When all $C_t = 0$, the smoothed book values, $A_t^*$ grow at the constant rate, $g_{0,T}^A$. Under these conditions, $g_{0,T}^A = a_{0,T}$.

Proof: When $C_t = 0$, the smoothed book values defined in equation (6) become $A_j^* = A_T/(1+a_{0,T})^{T-j}$. Therefore, $A_{j+1}^*/A_j^* = 1+a_{0,T}$, so the smoothed book values grow at the constant rate $a_{0,T} = g_{0,T}^A$.

c. When $A_t$ (unsmoothed book values) grows at the constant rate, $g_{0,T}^A$, and when $g_{0,T}^A = a_{0,T}$, $\sum_{t=1}^T C_t/(1 + a_{0,T})^t = 0$ (from equation (3)). There are two possibilities: (1) all $C_t = 0$; and (2) some of the terms in $\sum_{t=1}^T C_t/(1+a_{0,T})^t$ offset to make the present value zero.

c(1). If all $C_t = 0$, the smoothed series is identical to the unsmoothed series and $I_t = I_t^*$ and $A_t = A_t^*$;

Proof of c(1): First, if all $C_t = 0$, $A_t^*$ grows at the rate $a_{0,T}$ (from Property 11b) which means that $A_t^*$ and $A_t$ grow at the same rate. Therefore, the smoothed and unsmoothed series of book values must be equal and the ARR is constant.

c(2). If some of the terms in $\sum_{t=1}^T C_t/(1 + a_{0,T})^t$ offset to make the present value zero, then $\bar{a} = a_{0,T} = g_{0,T}^A$.

Proof of c(2): If the terms in $\sum_{t=1}^T C_t/(1 + a_{0,T})^t$ offset to make the present value zero, the one-period accounting rate of return, $a_{t-1,t}$, is $C_t/A_0(1+a_{0,T})^{t-1} + a_{0,T}$. (Since $a_{0,T} = g_{0,T}^A$.). Furthermore, since $(1 + a_{0,T})\sum_{t=1}^T C_t/(1 + a_{0,T})^t = \sum_{t=1}^T C_t/(1 + a_{0,T})^{t-1} = 0$, $\bar{a} = a_{0,T}$

d. If all $C_t = 0$, the growth of the unsmoothed book values, $g_{0,T}^A$, is not, in general, constant. But the geometric mean of the one-period growth rates equal $g_{0,T}^A$.

Proof: The growth rate, $g_{0,T}^A$, equals $(A_T/A_0)^{\frac{1}{T}} - 1$ and, by definition, is the geometric mean of the one-period growth rates when all $C_t = 0$.

Properties 11a, 11b, 11c and 11d have not been made explicit in the literature. The distinction between the properties of the smoothed and unsmoothed series becomes important whenever the constant growth of book value and/or constant ARR is assumed.

- Property 12: Average Life of a Project

Let $p_t$ be any depreciation method e.g., straight-line, sum-of-years digits, etc., where $\sum_{t=1}^{T} p_t = 1$. Define $M = \sum_{t=1}^{T} p_t(t)$ as the average lifetime (measured in time periods) of an investment project (de Wolff, 1966). For example, sum-of-years digits depreciation gives $M = \frac{T+2}{3}$ or, for $T = 10$, $M = 4$.

For a depreciation scheme that mimics Hotelling depreciation,

$$M^* = \sum_{t=1}^{T} p_t^{A^*}(t) = \frac{\sum_{t=1}^{T} A^*_{t-1}}{A_0} = \frac{\sum_{t=1}^{T} C_t - A_0}{a_{0,T} A_0} \tag{11}$$

Proof: Since $\sum_{t=1}^{T} A^*_{t-1} = A_0[(\sum_{t=1}^{T} p_t^{A^*}(t) + \sum_{t=2}^{T} p_t^{A^*}(t) + \sum_{t=3}^{T} p_t^{A^*}(t) + ... + p_T^{A^*}(T)]$, it immediately follows that $\sum_{t=1}^{T} p_t^{A^*}(t) = \frac{\sum_{t=1}^{T} A^*_{t-1}}{A_0}$. Then, from equation (8), $\frac{\sum_{t=1}^{T} A^*_{t-1}}{A_0} = \frac{\sum_{t=1}^{T} C_t - A_0}{a_{0,T} A_0}$.[3] $M$ also can be interpreted as a project's total income, standardized in terms of smoothed income in period 1.

- Property 13: Duration

$$D = \frac{\sum_{t=1}^{T} A^*_{t-1}/(1 + a_{0,T})^{t-1}}{A_0} = \frac{\sum_{t=1}^{T} tC_t(1 + a_{0,T})^{-t} + tA_T(1 + a_{0,T})^{-t}}{\sum_{t=1}^{T} C_t(1 + a_{0,T})^{-t} + A_T(1 + a_{0,T})^{-t}} \tag{12}$$

where $D$ is the accounting equivalent of Duration, a statistic that measures the time weighted average life of a stream of cash flows.

Proof: Each term in the sum, $\sum_{t=1}^{T} A^*_{t-1}/(1 + a_{0,T})^{t-1}$, can be written as

$$\begin{aligned}
A_0/(1 + a_{0,T})^0 &= C_1/(1 + a_{0,T}) + C_2/(1 + a_{0,T})^2 + C_3/(1 + a_{0,T})^3 ... + A_T/(1 + a_{0,T})^T \\
A_1/(1 + a_{0,T})^1 &= C_2/(1 + a_{0,T})^2 + C_3/(1 + a_{0,T})^3 ... + A_T/(1 + a_{0,T})^T \\
A_2/(1 + a_{0,T})^2 &= C_3/(1 + a_{0,T})^3 + ... + A_T/(1 + a_{0,T})^T \\
&\quad ...
\end{aligned}$$

Adding up gives $\sum_{t=1}^{T} tC_t(1 + a_{0,T})^{-t} + tA_T(1 + a_{0,T})^{-t}$.

By multiplying the numerator and denominator of the first term on the right hand side of equation (12) by $a_{0,T}$, Duration also can be interepreted as the

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[3] Brief and Anton (1987) derived $M$ for economic depreciation in connection with a concept which Ijiri (1967) called the PRD (Periodic Reinvestment based on Depreciation). To model economic depreciation, replace the $A^*_t$ with $MV_t$ (economic value) and $a_{0,T}$ with $r_{0,T}$ (economic rate of return). See Property 15.

present discounted value of a project's income stream, standardized in terms of the smoothed income in period 1. It follows that

$$Da_{0,T}A_0 = \sum_{t=1}^{T} a_{0,T}A^*_{t-1}/(1+a_{0,T})^{t-1}$$

so that if smoothed income is constant and $a_{0,T}A^*_{t-1} = I^*_1$, $D = \frac{1}{a_{0,T}}$. In contrast, if net dividends are constant, $D = \frac{1+a_{0,T}}{a_{0,T}}$.

In related work, Brief (1984) showed that

$$D = \frac{1+a_{0,T}}{a_{0,T}}[1 - \sum_{t=1}^{T} p_t^{A^*}(1+a_{0,T})^{-t}]. \tag{13}$$

- Property 14: Uniqueness of $a_{0,T}$

One issue that often is raised in connection with polynomial expressions like equation (3) is whether there will be multiple positive roots. If there are multiple roots, the meaning of the ARR is unclear. Many basic finance texts contains some discussion of the subject. Mao (1969) and Merrett and Sykes (1973) are particularly useful.

The main concern about uniqueness arises when an investment project has a series of both negative and positive cash flows. Negative flows in the later stages of a project are a necessary but not sufficient condition for multiple roots. The possibility of multiple roots in equation (3) led Edwards, Kay and Mayer (1987, p. 38) to focus on the one-period ARR in equation (1) instead of the multi-period ARR in equation (3). They reasoned that the problem of non uniqueness is minimized in a one-period setting because opening and closing book values, after accounting for net dividends, are almost always positive.[4]

However, in the context of accounting for a firm (in contrast to for an investment project), equation (3) will not, in general, have multiple positive roots. The mathematics on which this statement is based is given Soper (1959) who, like most writers on the subject, was concerned about the possibility that an investment project would have multiple positive roots.

First, Soper points out that a necessary but not sufficient condition that there *can be* one positive root in an expression in the form of equation (3) is that sign

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[4] Another reason given for focusing on a single period by Edwards, Kay and Mayer is because the cost of capital required for comparison with the ARR can be obtained more directly.

of the last term is positive. This means that in the situation we are considering, the sum, $C_T + A_T$, must be positive. This will almost always be the case. In the firm situation, negative net dividends get added back to book value; therefore, terminal book value will almost always be larger than these negative cash flows. Second, Soper shows that if an expression like equation (3) *can* have one positive root and

$$A_0 > \sum_{t=1}^{j} \frac{C_t}{(1 + a_{0,T})^t}, \; j = 1, 2, ..., T - 1, \tag{14}$$

then the one positive root is the only positive root.

Equation (14) can be rewritten as

$$A_t^* > 0, \; t = 1, 2, ..., T - 1 \tag{15}$$

which means that as long as the firm's smoothed book values are positive and the sum, $C_T + A_T$, is positive, equation (3) will have one positive root. If $A_0$ and $A_T$ are positive, negative book values will arise only if a firm sustains large write-offs after raising large amounts of capital which is not likely. It is difficult, therefore, to conceive of too many situations where the smoothed book values will be negative and, therefore, the uniqueness of $a_{0,T}$ will generally be assured.

### Economic Models as Special Cases

Two economic models can be viewed as special cases of Equation (3): (1) internal rate of return (IRR) model: and (2) market value (MV) model.

- Property 15: IRR and MV Models

Define $HV_0$ and $HV_T$ as opening and closing Hotelling valuations of the firm and $r_{0,T}$ as the IRR. Hotelling valuations are obtained by discounting future net dividends at the IRR. The IRR model is a special case of equation (3) when $A_0$ is replaced by $HV_0$, $A_T$ by $HV_T$ and $a_{0,T}$ by $r_{0,T}$:

$$HV_0 = \sum_{t=1}^{T} \frac{C_t}{(1 + r_{0,T})^t} + \frac{HV_T}{(1 + r_{0,T})^T}. \tag{16}$$

Along the same lines, the market value (MV) model can be obtained by replacing $A_0$, $A_T$ and $a_{0,T}$ with $MV_0$, $MV_T$ and $k_{0,T}$, the cost of capital.[5]

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[5] Issues related to more precise definitions of "economic value" are not addressed in this paper. For an extensive discussion of the subject, see Edwards, Kay and Mayer (1987).

All of the properties of the ARR previously discussed are applicable to these special cases with one exception. Since $r_{0,T}$ and $k_{0,T}$ are constant, the interim values, $HV_t$ and $MV_t$ are, by definition, smoothed values and are analogous to $A_t^*$.

**Relationship Between IRR and ARR**

- Property 16: Relationship Between IRR and the Sequence of $a_{t-1,t}$

$$r_{0,T} = \frac{\sum_{t=1}^{T} \frac{a_{t-1,t}A_{t-1}}{(1+r_{0,T})^t}}{\sum_{t=1}^{T} \frac{A_{t-1}}{(1+r_{0,T})^t}} + \frac{E_T^{HV} - E_0^{HV}}{\sum_{t=1}^{T} \frac{A_{t-1}}{(1+r_{0,T})^t}}, \tag{17}$$

where $E_T^{HV} = (\frac{HV_T}{1+r_{0,T}})^T - (\frac{A_T}{1+r_{0,T}})^T$ is the present value of the terminal valuation error and $E_0^{HV} = HV_0 - A_0$ is the opening valuation error.

Proof: In equation (16) which defines the IRR, add $-A_0 + A_0$ to the term on the left hand side of the equation, $HV_0$, (so that the term becomes $E_0^{HV} + A_0$) and $A_T - A_T$ to the numerator of the last term on the right hand side of equation (16), $HV_T$ (so that the last term becomes $E_T^{HV} + A_T/(1 + r_{0,T})^T$). The proof of Property 17 follows along the same lines as the proof of Property 2.

Kay (1976) initially assumed that $A_0 = HV_0$ and $A_T = HV_T$ so, in effect, equations (3) and (16) were identical and, therefore, the ARR and IRR were equal. The question of errors in opening and closing valuations was raised in a note by Wright (1978) and a reply by Kay (1978). However, Kay did not actually give the constant ARR a separate identity, i.e., a constant ARR, apart from the IRR, was not defined. Subsequent work by Peasnell (1982) produced equation (17) which was incorporated into later work by Kay and Edwards (1986) and Edwards, Kay and Mayer (1987).[6]

Equation (17) makes it clear that the weighted average single period ARR equals the IRR when $A_0 = HV_0$ and $A_T = HV_T$ and $E_0^{HV} = E_T^{HV} = 0$. This special case will hold over the lifetime of the firm, from its inception to its liquidation. The ARR also is equal to the IRR when $E_0^{HV} = E_T^{HV}$. Under these conditions, equations (3) and (16) are identical so that $a_{0,T} = r_{0,T}$ and, therefore, the weighted average ARR equals $r_{0,T}$.

When $E_T^{MV} \neq E_0^{MV}$, the link between the ARR and IRR is broken and the first term on the right hand side of equation (17), the weighted average, has

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[6] The constant ARR was given a separate identity (apart from the IRR) in Kay and Edwards (1986) and Edwards, Kay and Mayer (1987)

no obvious interpretation. The weighted average and the error term could have infinitely many values, depending on the accounting methods employed. However, since $a_{0,T}$ is unambiguously defined in equation (3), whether or not the error terms offset, equation (17) can, without loss of generality, be restated by replacing the weighted average ARR with $a_{0,T}$.

- Property 17: Relationship Between IRR and Constant ARR

$$r_{0,T} = a_{0,T} + \frac{E_T^{MV} - E_0^{MV}}{\sum_{t=1}^{T} \frac{A_{t-1}^*}{(1+r_{0,T})^t}}. \tag{18}$$

Proof: Equation (18) is a special case of equation (17) for a constant ARR.

Equation (18) has interesting implications for government contracts. If a government contract requires a payment that provides for a constant ARR on the book value of a project, the ARR equals $a_{0,T}$, regardless of the accounting methods employed. Then, if the error terms in equation (18) are zero, $r_{0,T} = a_{0,T}$ and, whatever accounting methods are used to derive the project's book value, the depreciation method will be Hotelling depreciation.[7] The same argument would apply (assuming zero errror) in a situation where the government permitted a firm to earn a fair (constant) rate of return. The fair rate of return would equal the IRR, regardless of the accounting methods adopted.[8] But accounting methods do matter in the sense that they would influence the amount and timing of the project's cash flows.

- Property 18: The IRR as an Average ARR[9]

$$r_{0,T} = \frac{1}{T}\sum_{t=1}^{T} a_{t-1,t} = \bar{a} \tag{19}$$

Three conditions are required for equation (19) to hold: (1) book values grow at a constant rate in each period; (2) the growth rate is equal to the economic

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[7] This is the case considered by Edwards, Kay and Mayer (1987). But they don't seem to recognize that if the ARR is constant and the error in equation (18) is zero, the depreciation method used is, by definition, Hotelling depreciation.

[8] However, if the firm were a continuing enterprise, then the error terms would need to be assessed before the relationship between the ARR and IRR could be determined.

[9] Property 18 is Peasnell's (1982) Corollary 6. The book values referred to below are book values for *any* clean surplus method since the accounting model was not restricted.

rate of return ($g^A_{0,T} = r_{0,T}$); and (3) $E^{HV}_T - E^{HV}_0 = 0$. Under these conditions, $w_t = 1/T$ so from Property 3 and equation (17), $r_{0,T} = \bar{a}$.

The implications of Property 18 become clearer when the economic conditions implied by the assumptions are spelled out in detail. First, if $E^{HV}_T - E^{HV}_0 = 0$, then $a_{0,T} = r_{0,T}$. (from equation (18)). Second, if $g^A_{0,T} = r_{0,T} = a_{0,T}$, then from Property 11c, $\sum_{t=1}^{T} C_t/(1 + a_{0,T})^t = 0$. There are two possibilities. First, if all $C_t = 0$ and the smoothed book values grow at the constant rate, $g^A_{0,T}$, the single-period ARRs, $a_{t-1,t}$, are constant and equal to $a_{0,T}$ (from Property 11c(1) so the average of the constant ARRs, $\bar{a}$, obviously must equal $a_{0,T} = r_{0,T.}$ Second, the other possibility is that the terms in the sum, $\sum_{t=1}^{T} C_t/(1 + a_{0,T})^t$, offset so that the sum is zero in which case $\bar{a} = a_{0,T} = r_{0,T}$ (from Property 11c(2)).

## Residual Income Valuation (RIV) Model

- Property 19: Residual Income Valuation Model

$$MV_0 = A_0 + \sum_{t=1}^{T} \frac{(a_{t-1,t} - k_{0,T})A_{t-1}}{(1 + k_{0,T})^t} + E^{MV}_T \tag{20}$$

or, with a constant ARR,

$$MV_0 = A_0 + (a_{0,T} - k_{0,T}) \sum_{t=1}^{T} \frac{A^*_{t-1}}{(1 + k_{0,T})^t} + E^{MV}_T \tag{21}$$

where $E^{MV}_T = \frac{MV_T - A_T}{(1+k_{0,T})^T}$ is the present value of the valuation error at the end of period T.[10]

Proof: The RIV model is, in effect, an indirect comparison of the accounting model in equation (3) with the MV model. In equations (17) and (18), replace $A_0$ by $MV_0$, $A_T$ by $MV_T$ and $r_{0,T}$ by $k_{0,T}$ and solve for $MV_0$.

## Direct Comparison of Accounting Model and Economic Models

- Property 20: Comparison of Accounting and IRR Models.

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[10] $E^{MV}_T$ is usually referred to as the "terminal value," not the valuation error. In equation (20), $E^{MV}_T$ is equal to $\sum_{t=T+1}^{\infty} \frac{(a_{t-1,t} - k_{0,T})A^*_{t-1}}{(1+k_{0,T})^t}$. In equation (21), $E^{MV}_T$ is equal $(a_{0,T} - k_{0,T}) \sum_{t=T+1}^{\infty} \frac{A^*_{t-1}}{(1+k_{0,T})^t}$.

The direct comparison of equations (3) and (16) gives

$$\frac{HV_0}{A_0} = \frac{\sum_{t=1}^{T} \frac{C_t}{(1+r_{0,T})^t}}{\sum_{t=1}^{T} \frac{C_t}{(1+a_{0,T})^t}} \times \frac{1-(\frac{1+g^{A}_{0,T}}{1+a_{0,T}})^T}{1-(\frac{1+g^{HV}_{0,T}}{1+r_{0,T}})^T}, \tag{22}$$

where, as before, the growth rates are defined by identity relationships, $g^{A}_{0,T} = (A_T/A_0)^{\frac{1}{T}} - 1$ and $g^{HV}_{0,T} = (HV_T/HV_0)^{\frac{1}{T}} - 1$.

Proof: The proof is obvious.

Equation (22) can be redefined in terms of MV model with appropriate substitutions.

Previous work by Kay (1976), Peasnell (1992), Brief and Lawson (1992) and others compared the accounting and IRR (or MV) models using equations (17), (18), (20) or (21) or some equivalent form of these expressions. Here, the accounting and economic models are compared directly by dividing equation (16) by equation (3).

When all $C_t = 0$, equation (22) is not defined. But from Property 11b (when all $C_t = 0$), $a_{0,T} = g^{A}_{0,T}$ and $r_{0,T} = g^{HV}_{0,T}$. When, in addition, $r_{0,T} = g^{A}_{0,T}$, then $a_{0,T} = r_{0,T}$. This is one of the classic results in the literature. Solomon (1966, 1970) and others have considered this situation in the steady state where the ARR was constant and $g^{A}_{0,T} = g^{HV}_{0,T}$, obtaining the same results. Whittington (1979), commenting on Solomon result's (which Solomon called "remarkable"), said that this was the "only important generalization" derived in some of the early research on the ARR. The result is transparent, given equations (3) and (16) with all $C_t = 0$. It holds over a finite time horizon and in the steady state where the constant ARR is defined by equation (3).

Equation (22), which has not previously appeared, has a number of interesting special cases.

- Property 21: Special Case of Equation (22) for $T = 1$.

When the time horizon is one period,

$$\frac{HV_0}{A_0} = \frac{a_{0,1} - g^{A}_{0,1}}{r_{0,1} - g^{HV}_{0,1}}. \tag{23}$$

Proof: The proof is obvious.

Equation (23) was derived by Franks and Hodges (1984). The derivation from equation (22) is more direct.

- Property 22: Special Case of Equation (22) for Constant Growth of Dividends

An important special case of equation (22) is obtained by assuming that net dividends grow at a constant rate, $g_{0,T}^{C}$, in each period:

$$\frac{HV_0}{A_0} = \frac{a_{0,T} - g_{0,T}^{C}}{r_{0,T} - g_{0,T}^{C}} \times \frac{1 - (\frac{1+g_{0,T}^{C}}{1+r_{0,T}})^{T}}{1 - (\frac{1+g_{0,T}^{C}}{1+a_{0,T}})^{T}} \times \frac{1 - (\frac{1+g_{0,T}^{A}}{1+a_{0,T}})^{T}}{1 - (\frac{1+g_{0,T}^{HV}}{1+r_{0,T}})^{T}}. \tag{24}$$

Proof: This proof follows directly after summing the geometric dividend series.

Equation (24) has not previously appeared. The core term of equation (24) is $\frac{a_{0,T}-g_{0,T}^{C}}{r_{0,T}-g_{0,T}^{C}}$ which, as will be shown, has previously appeared in connection with certain steady state results. The important point to note is that the growth rate in this core term is the growth rate of net dividends (which may or may not equal $g_{0,T}^{A}$ or $g_{0,T}^{HV}$ (or both).

- Property 23: Special Case of Equation (22) for $g_{0,T}^{C} = g_{0,T}^{HV} = g_{0,T}^{A}$

$$\frac{HV_0}{A_0} = \frac{a_{0,T} - g_{0,T}^{C}}{r_{0,T} - g_{0,T}^{C}}. \tag{25}$$

Proof: The proof is obvious.

In the general case over a finite time horizon, the three growth rates in equation (24) are not equal since constant growth of net dividends does not imply constant growth of either the smoothed economic values or book values. However, in this special case, which applies to a finite time horizon, the three growth rates are assumed to be equal. Equation (25) also can be derived from equation (24) if it is assumed that $T \to \infty$ providing that $a_{0,T} > g_{0,T}^{A}$ and $r_{0,T} > g_{0,T}^{HV}$.

Other work in this area assumes steady-state economic conditions, so dividends are assumed to grow forever at a constant rate and the ARR is assumed to be a constant. Under these conditions, the Hotelling values and the smoothed book values also will grow at the same rate (Property 10). However, the unsmoothed book values need not grow at this rate, even under steady state conditions.

Equation (25) is well-known, particularly in the valuation literature which is based on the MV model where $HV_0$ is replaced $MV_0$ and $r_{0,T}$ is replaced by $k_{0,T}$. However, while the form of equation (25) that appears elsewhere is the same, our version is the finite time equivalent of the steady state. This makes equation (25) more general since its use is not restricted to situations where convergence in the "long-run" is assumed.

Equation (25) has appeared in many different forms. For example, the basic formula for estimating a firm's continuing value that appeared in Copeland, Koller and Murrin (1996) is

$$Continuing\ Value = \frac{NOPLAT(1 - g/r)}{WACC - g}.$$

Converting to our notation, $Continuing\ Value = MV_0$, $NOPLAT = a_{0,T}A_0$, $g = g_t^C$, and $r = k_{0,T}$ which means that this formula is identical to equation (23) redefined to compare MV with the accounting model. One problem in the literature is that there are many different ways to express equation (25) and the basic assumptions underlying this relationship, particularly the assumption of a constant ARR and steady-state conditions, are not always made explicit.

- Property 24: Special Case of Equation (22) for MV Model when $g_{0,T}^A = g_{0,T}^C$

$$\frac{MV_0}{A_0} = \frac{a_{0,T} - g_{0,T}^C}{k_{0,T} - g_{0,T}^C} \times \frac{1 - (\frac{1+g_{0,T}^A}{1+k_{0,T}})^T}{1 - (\frac{1+g_{0,T}^{MV}}{1+k_{0,T}})^T}. \tag{26}$$

Proof: The proof is obvious.

Equation (26) can be rewritten as the RIV model with a constant ARR and constant growth of book value:

$$PV_0 = A_0 + \frac{a_{0,T} - k_{0,T}^C}{k_{0,T} - g_{0,T}^C} A_0(1 - (\frac{1 + g_{0,T}^A}{1 + k_{0,T}})^T) + \frac{PV_0(1 + g_{0,T}^{PV})^T}{(1 + k_{0,T})^T} - \frac{A_0(1 + g_{0,T}^A)^T}{(1 + k_{0,T})^T}. \tag{27}$$

Others like Brief and Lawson (1992) derived equations (26) and (27) along lines that were similar to the derivation of equation (17).

The accounting model in equation (3) provides a unifying framework because the two economic models, the MV model and the HV model, have the same form as the accounting model. An example illustrating the application of equations (26) and (27) can be found in a case study in a textbook by Revsine, Collins and Johnson (1999, p. 262). The case problem requires the computation of $MV_0$ for Allied Signal. The facts of the case are that over the relevant 10-year horizon, the ARR is assumed to be constant and $a_{0,10} = .32$. Growth of book value and dividends also are assumed constant and $g^A_{0,10} = g^C_{0,10} = .24$ (deduced from the assumption of constant payout of .25). $k_{0,10}$ is assumed to be .14. The terminal valuation error in equation (20) is to be ignored so

$$MV_0 = A_0 + (a_{0,T} - k_{0,T}) \sum_{t=1}^{T} \frac{A^*_{t-1}}{(1 + k_{0,T})^t}. \tag{28}$$

The equivalent form of equation (28) with constant growth of the smoothed book values $(g^A_{0,T} = g^C_{0,t})$ is

$$MV_0 = A_0 + A_0 \frac{a_{0,T} - k^C_{0,T}}{k_{0,T} - g^C_{0,T}}_0 \, (1 - (\frac{1 + g^A_{0,T}}{1 + k_{0,T}})^T) \tag{29}$$

Substituting the the numbers for the variables,

$$MV_0 = 10.54 + 10.54(\frac{.32 - .14}{.14 - .24})(1 - (\frac{1.24}{1.14})^{10}) = 35.55.$$

However, ignoring the terminal value is not the complete story. In principle, the terminal value can be ignored when it is assumed to be equal to zero and that will be the case when $MV_0(1 + g^{MV}_{0,T})^T = A_0(1 + g^A_{0,T})^T$. Solving for $g^{MV}_{0,T}$ in the equation, $35.55(1 + g^{MV}_{0,10})^{10} = 10.54(1.24)^{10}$, $g^{mV}_{0,10} = .09805$. So the calculation of $MV_0$ from equation (29) implicitly assumes a value of $g^{MV}_{0,10}$. That means that the same answer for $MV_0$ can be obtained using equation (26) by explicitly recognizing that assuming that the terminal value is zero is equivalent to assuming that $g^{MV}_{0,10} = .09805$. From equation (26),

$$MV_0 = 10.54(\frac{.32 - .24}{.14 - .24}) \times \frac{1 - (\frac{1.24}{1.14})^{10}}{1 - (\frac{1.09805}{1.14})^{10}} = 35.55.$$

The fact that the growth rate of book value is assumed to be .24 and the growth rate of MV is assumed to be nearly .10 might be of interest to the financial analyst. Beyond this, the suggestion by Collins, Revsine and Johnson (p. 262) that the error in valuation, $E^{MV}_T$, over time horizons of 10-15 years has a trivial effect on the estimate of $MV_0$ is questionable. This can be shown by substituting explicit growth rates of $MV_0$ into equation (26). For example, for growth rates of .08, .09, .11, .12, and .13, the present values are: 26.61, 30.76, 47.49, 68.53 and 131.80. $MV_0$ is very sensitive to small differences in the estimate of the growth rate of market value. The virtue of equation (26) is that it does not allow the user to the terminal value is zero.

- Property 25: Special Case of Equation (22). Book Value Grows at a Constant Rate and Ratio of Book Value to Hotelling Value is Constant

$$r_{0,T} = a_{0,T} + \frac{HV_0 - A_0}{A_0}(g^A_{0,T} - r_{0,T}). \tag{27}$$

Proof: If the growth rate of book value in each period is constant and if the ratio of book value to Hotelling value also is constant, the growth rate of the book values must equal the growth of Hotelling values. Further, from Property 10, net dividends grow at the same rate as Hotelling values. Therefore, the book values must be the smoothed book values since the book values have the same growth rate as dividends. Under the conditions implied by in this situation, i.e., a constant ARR, $a_{0,T}$, and $g^C_{0,T} = g^{HV}_{0,T} = g^A_{0,T}$), equation (25) is a special case of equation (22). To derive equation (27) from equation (25) subtract 1 from both sides of the equation and solve for $r_{0,T}$.

Property 25 is Peasnell's (1982) Corollary 9. His main concern was in deriving the error term directly from equation (17). He did not address the question of whether the assumptions implied that the ARR was constant.

- Property 26: Estimate of ARR in Steady-State: Fisher and McGowan (1983)

In their influential paper, Fisher and McGowan (1983) used simulation techniques to show that the ARR is a biased estimate of the IRR. The simulation assumed that the firm was a collection of identical projects which had a 6-year life and after-tax IRR of .15.[11]

Assume that the firm is a collection of identical projects, each with a life of T periods. Assume further that the firm invests in one project per period. The steady state with zero growth will be reached at the end of T periods.

An approximate comparison of the ARR and IRR can be obtained from:

$$a_{T,\infty} \approx (r_{T,\infty} - g^C_{T,\infty})\frac{\sum_{t=1}^{T} HV_{t-1}}{\sum_{t=1}^{T} A_{t-1}} + g^C_{T,\infty}. \tag{28}$$

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[11] The project's cash flows were -100, 23.3 + 2.4, 35, 37.1, 28.7, 15.4 and 6.5 or a net total of 48.4. There was an error in the original article. Cash flows in year 1 should have included the tax refund of 2.4 due to the operating loss. Therefore, 2.4 is added to the first period's cash flows. This error was confirmed in private correspondence by Franklin Fisher and he indicated that the error had been corrected in a reprint of the article in Fisher (1991). The after-tax IRR was .15.

Proof: This approximate result is based on equation (25), where book values and Hotelling values are steady state values.[12]

In the Fisher and McGowan example, $\sum_{t=1}^{T} HV_{t-1}$ can be computed directly from equation (11), modified to reflect the Hotelling model:

$$\sum_{t=1}^{T} HV_{t-1} = \frac{\sum_{t=1}^{T} C_t - HV_0}{r_{T,\infty}} = \frac{48.4}{.15} = 322.67.$$

The value of $\sum_{t=1}^{T} A_{t-1}$, based on sum-of-years digits depreciation, is the steady state book value associated with a constant ARR, $a_{T,\infty}$.

$$\sum_{t=1}^{T} A_{t-1} = \sum_{t=1}^{T} tp_t A_0 = \sum_{t=1}^{6} t\left(\frac{6-t+1}{\sum_{t=1}^{6} t}\right)100 = 266.67.$$

Therefore,

$$\frac{\sum_{t=1}^{T} HV_{t-1}}{\sum_{t=1}^{T} A_{t-1}} = \frac{322.67}{266.67} = 1.21.$$

The accounting rate of return can be estimated by from equation (26) with $r_{T,\infty} = .15$, $\frac{\sum_{t=1}^{T} HV_{t-1}}{\sum_{t=1}^{T} A_{t-1}} = 1.21$ for the various values of $g_{T,\infty}^{C}$ assumed by Fisher and McGowan.

Table II compares the estimated ARR with the ARR obtained in Fisher and McGowan's simulation. This computation assumes zero growth whereas the simulation assumes various levels of growth. This introduces an error in the ratio but this error will not be material unless there is a large difference between economic depreciation and the accountant's depreciation method. For the Fisher and McGowan data, it is clear that almost no error is introduced by assuming zero growth in deriving the ratio of the firm's steady-state economic value to the firm's steady-state book value.

## Conclusion

While financial analysts and accounting theorists often assume that the ARR is constant to facilitate analysis, the implications of this assumption have not been fully explored. In this paper, which is closely related to prior research that

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[12] The steady state book values and Hotelling values assume zero growth and therefore a bias is introduced. Nevertheless, the approximation is very accurate.

focused on basic analytical relationships in accounting that can be derived when the clean surplus relation holds, 26 properties related to a constant ARR are enumerated and the usefulness of these properties is illustrated. Several points are worth noting.

First, the analytical relationships derived in this paper hold for a finite time horizon and have more generality than many of the results previously derived which assume either an infinite time horizon and/or steady-state conditions.

Second, the concept of a smoothed or standardized accounting time series plays an important role in explicating the properties of a constant ARR. This concept should have a more prominent place in accounting.

Third, the idea that the economic model is a special case of the accounting model is a break with tradition. Rather than fit the accounting model into economics, we do the reverse. The accounting model pinpoints the critical role of opening and closing book values in the determination of profitability (Cf. Edwards, Kay and Mayer, 1987)).

Fourth, a natural extension of this work is the comparison of two accounting policies, which would also depend only on opening and closing book values. An analyst might want to compare alternative accounting policies like, for example, purchase accounting vs. pooling of interests. Or an analyst might want to compare the performance of two firms. In both cases, equation (3) provides a framework for analysis and the process of smoothing or standardizing an accounting time series would facilitate these comparisons by putting the financial statements related to the comparison of two policies or entities on the same footing.

Fifth, structuring the accounting model in terms of a present value model makes many analytical relationships transparent. This transparency provides insights into the accounting process that cannot be as readily obtained in more traditional analysis which embeds the accounting model into the economic model by replacing net dividends in the economic model with its accounting equivalent.

Sixth, the twenty-six properties of a constant ARR provide a framework for analyzing financial statements over a series of time periods. While the current approach for multi-period analysis is to look at the time series of accounting variables associated with one-period financial statements, the focus here is on standardizing or smoothing the time series. The smoothed series has attractive properties.

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## **Table I**

Loss Example

| Period | $A_t$ | $I_t$ | $C_t$ | $A_t^*$ | $I_t^*$ |
|---|---|---|---|---|---|
| 0 | 100 | - | - | 100 | - |
| 1 | 102 | 10 | 8 | 107.36 | 15.36 |
| 2 | 114 | 20 | 8 | 115.85 | 16.49 |
| 3 | 136 | 30 | 8 | 125.65 | 17.80 |
| 4 | 168 | 40 | 8 | 136.95 | 19.30 |
| 5 | 150 | -10 | 8 | 149.88 | 21.03 |

Table II
Comparison of ARR Based on Simulation
(Fisher and McGowan, 1983) vs Estimated ARR

| Growth, $g_t^C$ | $a_{T,\infty}$(simulation) | $a_{T,\infty}$(estimate) | Error |
|---|---|---|---|
| 0 | .181 | .182 | +.001 |
| .05 | .170 | .171 | +.001 |
| .10 | .159 | .161 | +.002 |
| .15 | .15 | .15 | 0 |
| .20 | .141 | .140 | -.001 |
| .25 | .133 | .129 | -.004 |
| .30 | .126 | .119 | -.007 |