Problem 1: (Convexity and Euler’s Equation)

Let $\mathcal{V}$ be a linear vector space and $\mathcal{D}$ a subset of $\mathcal{V}$. A real-valued function $f$ defined on $\mathcal{D}$ is said to be [strictly] convex on $\mathcal{D}$ if

$$f(y + v) - f(y) \geq \delta f(y; v)$$

for all $y, y + v \in \mathcal{D}$,

[with equality if and only if $v = 0$]. Where $\delta f(y; v)$ is the first Gateau variation of $f$ at $y$ on the direction $v$.

a) Prove the following: If $f$ is [strictly] convex on $\mathcal{D}$ then each $x^* \in \mathcal{D}$ for which $\delta f(x^*; y) = 0$ for all $x^* + y \in \mathcal{D}$ minimizes $f$ on $\mathcal{D}$ [uniquely].

Let $f = f(x, y, z)$ be a real value function on $[a, b] \times \mathbb{R}^2$. Assume that $f$ and the partial derivatives $f_y$ and $f_z$ are defined and continuous on $S$. For all $y \in C^1[a, b]$ we define the integral function

$$F(y) = \int_a^b f(x, y(x), y'(x)) \, dx := \int_a^b f[y(x)] \, dx,$$

where $f[y(x)] = f(x, y(x), y'(x))$.

b) Prove that the first Gateau variation of $F$ is given by

$$\delta F(y; v) = \int_a^b \left( f_y[y(x)] v(x) + f_z[y(x)] v'(x) \right) \, dx.$$

c) Let $D$ be a domain in $\mathbb{R}^2$. For two arbitrary real numbers $\alpha$ and $\beta$ define

$$D^{\alpha, \beta}[a, b] = \{ y \in C^1[a, b] : y(a) = \alpha, y(b) = \beta, \text{ and } (y(x), y'(x)) \in D \forall x \in [a, b] \}.$$

Prove that if $f(x, y, z)$ is convex on $[a, b] \times D$ then

1. $F(y)$ defined above is convex on $D$ and
2. each $y \in D$ for which

$$\frac{d}{dx} f_z[y(x)] = f_y[y(x)]$$

on $(a, b)$ satisfies $\delta F(y, v) = 0$ for all $y + v \in D$.

Conclude that such a $y \in D$ that satisfies [*] minimizes $F$ on $D$. That is, extremal solutions are minimizers.

Problem 2: (du Bois-Reymond’s Lemma)

The proof of Euler’s equation uses du Bois-Reymond’s Lemma:
If $h \in C[a, b]$ and $\int_a^b h(x)v'(x) \, dx = 0$

for all $v \in D_0 = \{ v \in C^1[a, b] : v(a) = v(b) = 0 \}$
then \( h = \text{constant on } [a, b] \). Using this lemma prove the more general results.

a) If \( g, h \in C[a, b] \) and \( \int_a^b [g(x)v(x) + h(x)v'(x)] \, dx = 0 \)
for all \( v \in D_0 = \{ v \in C^1[a, b] : v(a) = v(b) = 0 \} \)
then \( h \in C^1[a, b] \) and \( h' = g \).

b) If \( h \in C[a, b] \) and for some \( m = 1, 2, \ldots \) we have \( \int_a^b h(x)v^{(m)}(x) \, dx = 0 \)
for all \( v \in D_0^{(m)} = \{ v \in C^m[a, b] : v^{(k)}(a) = v^{(k)}(b) = 0, k = 0, 1, 2, \ldots, m - 1 \} \)
then on \([a, b]\), \( h \) is a polynomial of degree \( \leq m - 1 \).

**Problem 3:**

Suppose you have inherited a large sum \( S \) and plan to spend it so as to maximize your discounted cumulative utility for the next \( T \) units of time. Let \( u(t) \) be the amount that you expend on period \( t \) and let \( \sqrt{u(t)} \) the the instantaneous utility rate that you receive at time \( t \). Let \( \beta \) be the discount factor that you use to discount future utility, i.e., the discounted value of expending \( u \) at time \( t \) is equal to \( \exp(-\beta t) \sqrt{u} \). Let \( \alpha \) be the risk-free interest rate available on the market, i.e., one dollar today is equivalent to \( \exp(\alpha t) \) dollars \( t \) units of time in the future.

a) Formulate the control problem that maximizes the discounted cumulative utility given all necessary constraints.

b) Find the optimal expenditure rate \( \{u(t)\} \) for all \( t \in [0, T] \).

**Problem 4:** (Production-Inventory Problem)

Consider a make-to-stock manufacturing facility producing a single type of product. Initial inventory at time \( t = 0 \) is \( I_0 \). Demand rate for the next selling season \([0, T]\) is know and equal to \( \lambda(t) \) \( t \in [0, T] \). We denote by \( \mu(t) \) the production rate and by \( I(t) \) the inventory position. Suppose that due to poor inventory management there is a fixed proportion \( \alpha \) of inventory that is lost per unit time. Thus, at time \( t \) the inventory \( I(t) \) increases at a rate \( \mu(t) \) and decreases at a rate \( \lambda(t) + \alpha I(t) \). Suppose the company has set target values for the inventory and production rate during \([0, T]\). Let \( \bar{I} \) and \( \bar{P} \) be these target values, respectively. Deviation from these values are costly, and the company uses the following cost function \( C(I, P) \) to evaluate a production-inventory strategy \( (P, I) \):

\[
C(I, P) = \int_0^T \left[ \beta^2 (\bar{I} - I(t))^2 + (\bar{P} - P(t))^2 \right] \, dt.
\]

The objective of the company is to find and optimal production-inventory strategy that minimizes the cost function subject to the additional condition that \( I(T) = \bar{I} \).

a) Rewrite the cost function \( C(I, P) \) as a function of the inventory position and its first derivative only.

b) Find the optimal production-inventory strategy.