Online Auction and List Price Revenue Management

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We analyze a revenue management problem in which a seller facing a Poisson arrival stream of consumers operates an online multiunit auction. Consumers can get the product from an alternative list price channel. In the first variant, the list price is an external channel run by another firm. In the second one, the seller manages both the auction and the list price channels.

Each consumer, trying to maximize his own surplus, must decide either to buy at the posted price and get the item at no risk, or to join the auction and wait until its end, when the winners are revealed and the auction price is disclosed.

Our approach consists of two parts. First, we study structural properties of the problem, and show that the equilibrium strategy for both versions of this game is of the threshold type, meaning that a consumer will join the auction only if his arrival time is above a function of his own valuation. This consumer’s strategy can be computed using an iterative algorithm in a function space, provably convergent under some conditions. Unfortunately, this procedure is computationally intensive.

Second, and to overcome this limitation, we formulate an asymptotic version of the problem, in which the demand rate and the initial number of units grow proportionally large. We obtain a simple closed-form expression for the equilibrium strategy in this regime, which is then used as an approximate solution to the original problem. Numerical computations show that this heuristic is very accurate. The asymptotic solution culminates in simple and precise recipes of how bidders should behave, as well as how the seller should structure the auction, and price the product in the dual-channel case.

Key words: revenue management; online auction; dual channel; strategic behavior; asymptotic analysis

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1. Introduction

In the last few years, Revenue Management (RM) has widened its focus from capacity control and dynamic pricing to alternative selling mechanisms proposed by electronic commerce, such as group purchasing, online negotiations, and auctions. (See Talluri and van Ryzin 2004 for a reference on RM methods and applications, or the survey by Bitran and Caldentey 2003 for an overview of dynamic pricing models.) Although list pricing is probably still the most familiar and used pricing mechanism, online auctions are certainly an increasing phenomenon.

Nowadays, a huge variety of products is sold simultaneously through online posted price and auction channels, allowing consumers to compare prices and bid states easily across different channels in real time. This boost in market information and the corresponding reduction in search costs have a significant impact on consumers’ purchasing behavior and should be considered by a seller when designing online sales mechanisms.

In this paper, we address the problem of an online seller who is endowed with a fixed initial inventory and faces a stochastic arriving stream of strategic consumers. To capture different e-business environments, we consider two alternative formulations. First, we consider the case in which the seller controls an online auction exclusively and competes with a third party that manages a list price channel. We refer to this case as the single-auction channel model. In the second formulation, the dual-channel model, the seller is a monopolist and controls both the auction and list price channels.

For these two scenarios, we are interested in answering the following questions: How should strategic consumers behave, that is, which channel should each consumer choose? What should the bidding strategy be of those consumers that enter the auction? Given consumers’ behavior, how should the seller manage an online auction to maximize revenue? What should the length of the auction and the reservation price (i.e., minimum acceptable bid) be? How should the seller manage parallel online auction and list price channels to maximize revenue? In particular, the dual-channel case motivates an important managerial question: How should the seller design both channels to segment the population of consumers to extract as much revenue as possible from each...
segment? A seller does not want to offer a business model that cannibalizes itself, that is, if she offers a high posted price, she narrows the list price channel, and middle-to-high valuation consumers are tempted to join the auction, which could eventually close at a low price. On the other hand, if she posts a low list price, she widens the list price channel, pooling together low- and high-valuation consumers. In both cases, the seller runs the risk of decreasing revenues. Our purpose is to shed some light on the trade-offs that are inherent to this business environment.

More formally, we analyze a single-period model in which a seller operates a multiunit, uniform price, online auction, offering multiple units of a homogeneous good. The seller announces the inventory put online auction, offering multiple units of a homogeneous good. The seller announces the inventory put up for sale $Q_0$, the auction duration $T$, and the auction reservation price $v^R$. Consumers with single-unit demand arrive according to a Poisson process. They have a private value for the product, independently drawn from a continuous distribution. They must decide whether to bid and wait for the auction outcome at time $T$, or buy at the posted price $\hat{P}$, and get the unit instantaneously. As mentioned above, we consider two variants of this problem:

- In the single-auction channel case, the fixed-price channel is external and run by another firm, which we assume has unlimited inventory. Hence, if a bidder is not among the winners of the auction, he can always buy the item at the posted price at a later time $T$, although his utility is discounted.
- In the dual-auction and list price channel case, the seller is a monopolist who manages both the auction and the list price channels. In this case, an auction takes place at time $T$ only if there are units left unsold by then. Hence, supply is limited, and bidders that lose in the auction have no alternative market in which to buy the product.

The consumers’ strategy consists of two decisions: (i) whether or not to join the auction, and (ii) in the case of joining the auction, what bidding strategy to use. The supply size under both scenarios produces different bidding behaviors. Regarding the first decision, we prove that a symmetric equilibrium strategy exists in both variations of the problem, and it is characterized by a threshold function in the space (valuation, time): For a consumer arriving at time $t$ with valuation $v$, there is a threshold $H(v)$ such that if $H(v) \leq t$, then he will participate in the auction. Otherwise, he will buy at the posted price $\hat{P}$. This participation strategy can be computed using an iterative algorithm (in an appropriate function space) provably convergent under some special conditions. Unfortunately, this procedure is computationally intensive and does not lead to simple managerial insights. To overcome this limitation, we formulate an asymptotic version of the problem, in which the demand rate and the initial number of units grow proportionally large. We get a simple closed-form expression for the equilibrium strategy in this limiting regime, which is then used as an approximated solution for the original problem. Numerical computations show that this heuristic is very accurate.

Finally, we analyze the seller’s optimization problem in both the single- and dual-channel settings, and plug the consumer’s asymptotic participation strategy into them to compute the optimal values of the parameters $Q_0$, $T$, $v^R$ (and $\hat{P}$ in the dual channel case), that the seller must announce to maximize revenues. We can then assert that the asymptotic solution culminates in precise and simple guidelines for how bidders should behave and how the seller should design the auction and list price channels.

The main insights that we obtained are the following. For the single-auction channel case, we find that the optimal number of units to offer is a nonmonotonic function of the external fixed price, $\hat{P}$, and that it is bounded above by 80% of the average demand. In addition, the optimal duration of the auction is an increasing function of $\hat{P}$. In the dual auction and list price channel case, we find that if the seller’s initial inventory endowment is small or if her discount factor is large, then she does not have enough economic incentives to run a terminal auction; a single fixed-price channel is the optimal selling mechanism. On the other hand, if the endowment is large, or the seller discount factor is small, or buyers are impatient, then running both channels in parallel is optimal. In any of these cases, we show that a dual-channel operation can have a significant impact on revenues compared to a single fixed-price channel. The magnitude of the increase in revenues can be as large as 33% for the case of uniformly distributed valuations.

1.1. Literature Review

Auctions have been extensively studied in the economic literature (e.g., see the survey by Klemperer 1999 or the recent book by Krishna 2002). Price discrimination has been argued as one of the main reasons for using them (see Bulow and Roberts 1989). Maskin and Riley (1989) proved the optimality of the uniform price mechanism for the single-period multiunit auction.

Few papers have put auctions in an operational perspective. Specifically, regarding its connection with RM, Pinker et al. (2000) study how to run a sequence of standard multiunit auctions, using bidding information to learn about the consumer’s valuation distribution. Vulcano et al. (2002) characterize an optimal dynamic auction for a firm selling a fixed capacity over a finite horizon.
The firm’s choice between auctions and posted prices for the single-channel case has also been addressed (e.g., see Vany 1987, Wang 1993, Harstad 1990).

The problem of jointly managing auction and list price channels has not received much attention in the literature. New features like the buy now prices have been addressed by Budish and Takeyama (2001), although their model is limited to two bidders and two valuation types. Within the business-to-consumer (B2C) framework, the empirical study of Vakrat and Seidmann (1999) compares prices paid through online auctions and catalogs for the same product. They observe that auctions result in average prices 25% below the catalog ones. They build a simple model of single-unit auctions with a deterministic number of bidders, but ignoring consumer choice behavior. In the infinite-horizon model of van Ryzin and Vulcano (2004, §3.3), the seller operates auctions and posted prices simultaneously, and replenishes her stock in every period. However, the streams of consumers for both channels are independent, and the seller decides how many units to allocate to each of the channels separately.

Our research is mainly motivated by the work of Etzion et al. (2006). They analyze simultaneous online auctions and list price channels in a multiperiod B2C framework, where a seller with infinite supply maximizes her average expected revenue. Consumers arrive according to a Poisson process, and decide which channel to join. They found two optimal auction design strategies: short single-unit auctions and long multiunit auctions.

Our work differs from theirs in the way we model the supply side, because in our dual channel case scarcity plays a critical role, as is usually the case in RM: Given the risk that potentially no item could remain available for the auction by time $T$ (which occurs when all the inventory is depleted through the list price channel), what should the consumer’s participation strategy be? In the case of going for the auction, scarcity induces the standard dominant “bid your own value” strategy for multiunit uniform price auctions (see Krishna 2002, §2.2 for a comprehensive study of bidding behavior). The situation is different in the single-auction channel case, where the infinite supply (as is the case in Etzion et al. 2006) induces no bidder to bid higher than the posted price: In case he loses, he always has the chance to pay that price at time $T$. Now, given both settings, how should the seller structure the business, accounting for consumers’ strategic behavior?

Etzion et al. (2006) work with additive consumer utility functions, and characterize the consumer equilibrium bidding strategy with a single value $\hat{v}$, such that all consumers with valuation below the posted price, and those arriving later than the threshold $\hat{v}$ with valuation above the posted price, will join the auction. All other consumers will go to the online catalog. Our equilibrium participation strategy turns out to be more complex because it is based on a multiplicative utility function, and as we said above, is defined by a continuous threshold function in the space (valuation, time). Furthermore, when computing the participation strategy in their paper, Etzion et al. assume that the total number of competing consumers is deterministic. We instead embed the random nature of the arrival process in the computation of the consumer’s participation strategy.$^1$

A distinguishing characteristic of our research is the asymptotic analysis of the game. We show that the complex threshold that describes the strategic behavior of the consumers can be easily computed in the limiting regime where the consumers’ arrival rate and the number of units offered grow proportionally large, without missing the predictive power of the model.

Overall, we believe that the two models share some features, but contrast in these important dimensions, which are worth exploring.

Finally, the problem of analyzing the equilibrium of a system where consumers arrive during a time window has been addressed by few papers, but they are oriented to the characterization of the arrival pattern (e.g., Glazer and Hassin 1983 or Lariviere and Van Mieghem 2004). In our setting, the arrival process is exogenous, and we concentrate on characterizing the Nash equilibrium (in pure strategies) of the participation behavior of the consumers.

The remainder of this paper is organized as follows. We introduce the model for both variants of the problem in §2. In §§3 and 4, we study the consumers’ problem of selecting an optimal participation and bidding strategy for both the single-auction and dual-channel models, respectively. We prove the existence of a symmetric equilibrium within a large class of participation strategies and use asymptotic analysis to characterize this equilibrium. The analysis in §§3 and 4 assumes that the initial inventory $Q_0$, the auction bidding period $T$, the reservation price $v^R$, and the posted price $\hat{P}$ are fixed. We turn to the seller’s revenue maximization problem of optimally choosing $Q_0$, $v^R$, $T$, and $\hat{P}$ in §5. Finally, §6 summarizes our concluding remarks.

The paper includes two online appendices (provided in the e-companion).$^2$

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$^1$ In the final version of their paper, however, which was later than our first draft, they included a section in the appendix where they discussed the stochastic number of bidders assumption.

$^2$ An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.
the proofs. Appendix B provides a detailed mathematical description of the dual-channel model.

2. Model Description

We study the problem faced by a firm (seller) endowed with an initial inventory $\bar{Q}$ of a homogeneous product. We take an RM point of view and assume that the seller cannot replenish her inventory throughout the selling season. We do, however, allow the seller to ration by choosing the quantity $Q_0 \leq \bar{Q}$ to put up for sale. The remaining quantity $\bar{Q} - Q_0$ is discarded at no extra cost or salvage value.

We discuss two variants of this problem. In the first variant, the seller manages a single-auction channel, and there is an external market with infinite supply where the same product is available at a fixed price $\bar{P}$. In the second one, the seller is a monopolist managing both the auction and the list price channels simultaneously. During the selling process, she allocates units through the list price channel. Therefore, for the seller at time $t = 0$ and for any arriving consumer, the effective number of units to be auctioned at time $T$ is described by a random variable $Q_T$, with support $\{0, \ldots, Q_0\}$.

We also assume that the seller announces on her website the time left for the auction completion, the reservation price $\bar{v}$, and visit the seller’s website following a Poisson process with constant intensity $\lambda$. On the demand side, consumers have single-unit requests, and visit the seller’s website following a Poisson process with constant intensity $\lambda$. They are characterized by two quantities: (i) their arrival time, (ii) their private valuation, their knowledge of the arrival time, and their private valuation for the product. For notational convenience, we denote the private valuation of a consumer arriving at time $t$ by $v_t$. We assume that this notation is well defined because, with probability one, the Poisson process has at most one arrival at any given time. We also assume that the cumulative probability distribution $F$ of the random variable $v_t$ is a time-homogeneous and differentiable function with support $\gamma' \triangleq [0, \bar{v}]$ that admits a density function $f(v)$. Both $\lambda$ and $F$ are common knowledge. Without loss of generality, we assume from now on that $\bar{v} = 1$, that is, we scale all prices in this economy by $\bar{v}$.

When visiting the website, consumers must choose either to bid or to buy the product at a posted price $\bar{P}$ to maximize their own surplus. We assume that they are sensitive to delay, and denote by $u(t, \tau, v - p)$ the quasilinear discounted utility function of a consumer arriving at time $t$ with valuation $v$ who eventually gets a unit of product at time $\tau$ at a price $p$ (paid at the moment of getting it). If the consumer never gets the object, we use the convention $\tau = \infty$. In particular, we consider an exponentially discounted utility function of the form:

$$u(t, \tau, v - p) = (v - p) \exp(-w(\tau - t)), \quad (1)$$

where $w$ is a fixed constant shared by all consumers that captures the consumers’ disutility for waiting. As a side remark, we note that our main theoretical results are not especially tight to the functional form of the utility in (1) as long as it remains increasing in $v - p$ and decreasing in delay $\tau - t$.

We assume that a consumer arriving at $t \in [0, T]$ chooses at this time whether to enter the auction or to buy at the fixed price. He bases this decision on his private valuation, his knowledge of the arrival rate $\lambda$ and the distribution of valuations $F$, the initial inventory $Q_0$, the auction remaining time $T - t$, the reservation price $\bar{v}$, and the posted price $\bar{P}$. This participation strategy can be characterized by a threshold function $H(\cdot)$ such that a consumer with valuation $v_t$ enters the auction only if his arrival time $t$ exceeds $H(v_t)$. Pictorially, $H(v_t)$ divides consumers’ type space (valuation, arrival time) into two regions—as shown in Figure 1—one corresponding to the posted-price buyers and the other to the auction bidders.

The seller’s problem is to design a single-auction channel in the first model (by setting a value for $Q_0$, $\bar{P}$, and $\bar{v}$), and a dual auction and list price channel in the second model (by setting also a value $\bar{P}$) to maximize her expected revenue, which is also exponentially discounted over time.

2.1. Discussion of the Model

On a theoretical level, our model is in many ways a variation of the classical, single-period, private-value
auction price for this realization of demand. There are 4 winning bids, and 5 losing bids. The vertical dashed line represents the distribution assumptions that bidders are symmetric or that the limitations of the auction literature, such as the coexistence of the alternative list price channel.

On a practical level, our model shares some of the limitations of the auction literature, such as the assumptions that bidders are symmetric or that the distribution $F$ and the arrival rate $\lambda$ are common knowledge (see Pinker et al. 2003 for further discussion). On the other hand, our model does capture important features of the real game. For instance, our model is able to explain the surge in bids close to the end of an auction with a sharp deadline, a phenomenon commonly observed in practice (e.g., Roth and Ockenfels 2002). Our result in Proposition 3 supports this fact (see also Figure 1), showing that the proportion of consumers that go to the auction is increasing in the elapsed time $t$.

The fact that when consumers arrive at the website they are only informed about the initial number of units $Q_0$ is reasonable in the single-auction channel, but it is certainly a limitation of the dual-channel model. This contrasts with the business practice of posting buy now prices while running an auction (e.g., eBay.com): Consumers are informed about the number of remaining units $Q_t$ when visiting the website at time $t$. This assumption is made for mathematical tractability, because including this factor in the consumer choice behavior would add an extra dimension to the current space (valuation, time) for the participation decision.\(^6\) Nevertheless, we think that the kind of arguments and technical tools used in this paper can be applied to this extension as well. In particular, we show that in an asymptotic sense (that we make precise in §§3 and 4), our proposed solution under partial information is also an equilibrium for the game in which the seller provides full information about the inventory level and bidding process over time. Hence, although restrictive, our model can be viewed as an asymptotically exact first-step approximation of the more complex game with full information.

Because there is no information update on the remaining number of units, consumers have no incentive to wait for a time window between arrival at the website and the final participation decision; otherwise, they would just be incurring a discount for waiting.

We model the auction as a multiunit uniform-price sealed-bid auction. In standard single-period auction theory, this auction format turns out to be strategically equivalent to the English auction\(^7\) (e.g., see Krishna 2002). Actually, the English auction is by far the most common online auction format (see Pinker et al. 2003). Some empirical research supports the theoretical equivalence between sealed-bid uniform-price and open English auctions in the online practice (e.g., Lucking-Reiley 1999).

3. Single-Auction Channel
In this section, we study consumers’ optimal purchasing strategy (i.e., channel selection and bidding) for the single-auction model. We assume that the external list price $\hat{P}$ has been posted and the seller (i.e., the auctioneer in this case) has already announced the parameters of the auction $(Q_0, T, v^\delta)$. The auctioneer’s problem of optimally designing the auction is postponed to §5.

Let us start discussing the optimal bidding strategy. In our setting, where the sealed $(Q_0 + 1)$-price auction\(^8\) operates in parallel to an infinite supply fixed-price market, it is a weakly dominant strategy for bidders with valuations below the posted price to bid their true values, and it is dominant for bidders above the posted price to bid the posted price. In other words, if a consumer with valuation $v$ decides to enter the auction, then he will bid $b(v)$, where

$$b(v) = \begin{cases} v & \text{if } v < \hat{P} \\ \hat{P} & \text{if } v \geq \hat{P} \end{cases} = \min\{v, \hat{P}\}. \quad (2)$$

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\(^6\) Gupta and Gallien (2007) provide an online auction model where real-time information of $Q_t$ is revealed, but they need to restrict the analysis to single-unit auctions.

\(^7\) In a multiunit English auction, bids are open for all to see. As in the multiunit uniform-price sealed-bid auction, if $Q$ units are auctioned, the $Q$ highest bidders win, and all the winners pay the highest losing bid.

\(^8\) In this single-auction channel, $Q_t = Q_0$ because there are no items sold during the time window $[0, T]$. 

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*Figure 1 An Example of a Participation Strategy $H(v)$ for the Case $T = 1$ and $v^\delta = 0$*

<table>
<thead>
<tr>
<th>Arrival Time ($t$)</th>
<th>Valuation ($v$)</th>
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<td>0.9</td>
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Note. The dots represent a sample path of the arrival process. In this case, $Q_0 = 6$. Two consumers buy at the posted price; and nine submit bids. There are 4 winning bids, and 5 losing bids. The vertical dashed line represents the auction price for this realization of demand.
The optimality of \( b(v) = v \) for a low-valuation bidder with \( v < \hat{P} \) follows from noticing that for this consumer the auction is the only profitable channel from which he can get the object. In addition, it is well known that for the \((Q_T+1)\)-price auction mechanism, bidding the true valuation is a dominant strategy, i.e., the strategy \( b(v) = v \) maximizes the low-valuation bidder’s expected utility independently of other bidders’ strategies. For high-valuation consumers with \( v \geq \hat{P} \), the posted-price channel has unlimited supply, and bidders with \( v \geq \hat{P} \) know that they can always get a unit at this fixed price \( \hat{P} \). Therefore, these high-valuation consumers will never bid above the posted price, that is, \( b(v) \leq \hat{P} \). In addition, a bidding strategy with \( b(v) < \hat{P} \) is also suboptimal because under the \((Q_T+1)\)-price auction mechanism this strategy reduces the bidder probability of winning the auction (with respect to the strategy \( b(v) = \hat{P} \)) and at the same time does not affect the auction price in the case the bidder actually wins the auction. Thus, the high-valuation bidder is better off choosing \( b(v) = \hat{P} \).

Interestingly, if we compare the bidding strategy \( b(v) = \min\{v, \hat{P}\} \) of this model and the traditional bidding strategy \( b(v) = v \) of the standard \((Q_T+1)\)-price auction, we conclude that, from the auctioneer’s point of view, the presence of an uncapacitated fixed-price channel is equivalent to collapsing the range of valuations \([\hat{P}, \bar{v}]\) into a single value \( \hat{P} \). For further details about the optimality of \( b(v) = \min\{v, \hat{P}\} \), we refer the reader to Etzion et al. (2006, Lemma 1).

Equation (2) characterizes the dominant bidding strategy for all bidders. The probability distribution of the number of bidders and their valuations is the only piece of information that we need to fully characterize the output of the auction. We address this issue in the remainder of this section. Because \( \rho^R \) is kept constant for the rest of this section, we set it equal to zero \( (\rho^R = 0) \), and rescale the bidder’s valuations (and the corresponding probability distribution) accordingly. That is,

\[
\begin{align*}
\tilde{v} &\leftarrow v - \rho^R, \\
\tilde{\hat{P}} &\leftarrow \frac{\hat{P} - \rho^R}{1 - \rho^R}, \\
F(\tilde{v}) &\leftarrow \frac{F(v) - F(v + \rho^R)}{1 - F(v^R)}, \\
\lambda &\leftarrow \lambda(1 - F(v^R)).
\end{align*}
\]

This transformation achieves two objectives. First, only those consumers with valuation greater than or equal to the auction reservation price are considered, the others are discarded. This is true without loss of generality because discarded consumers have no real impact on the auction output. The second objective is that under this scaling, the range of valuations of the (nondiscarded) consumers remains \([0, 1]\). The corresponding scaling of the posted price \( \hat{P} \), distribution \( F \), and arrival rate \( \lambda \) follows from these two conditions.

### 3.1. Participation Strategy and Auction Price

#### Probability Distribution

Because information about the number of units \( Q_0 \) to auction, as well as the probability distribution of consumer valuations \( F \), are common knowledge, we can characterize the decision of a consumer arriving at time \( t \), with private valuation \( v \), by a threshold function \( H(v) \) such that the consumer will place a bid if and only if \( t \geq H(v) \). The fact that we can represent the participation strategy for all \( v \)-consumers (i.e., those consumers with valuation \( v \)) by a single threshold \( H(v) \) is a consequence of the monotonicity of the utility function in the waiting time. In other words, if it is optimal for a \( v \)-consumer arriving at time \( t \) to wait \((T - t)\) time units for the auction, then it is also optimal for any other \( v \)-consumer arriving after \( t \).

Two assumptions are used in this representation of the participation strategy. First, this characterization is based on the notion of a symmetric equilibrium in which all consumers use the same threshold function \( H(v) \). In addition, we assume that a consumer arriving at time \( t \) is incapable of observing the number of bids already in the system. That is, we assume that the only information that a consumer uses to decide whether or not to enter the auction—besides \( \lambda \), \( T \), \( Q_0 \), and \( F \)—are his arrival time and private valuation.

We will denote by \( \mathcal{H} \) the set of participation thresholds from which consumers choose their strategies. To keep our formulation reasonably simple, avoiding measure theory technicalities, we restrict consumers to the use of “well-behaved” participation strategies. Specifically, we assume that \( \mathcal{H} \subseteq \mathcal{D} \), the set of piecewise continuous functions with right and left limits. Although a restriction, we believe this set \( \mathcal{D} \) is large enough to include most strategies that are reasonable from a practical standpoint.\(^9\) In Proposition 1, we will show that the set \( \mathcal{D} \) is larger than necessary in the sense that in equilibrium any symmetric participation strategy \( H \in \mathcal{D} \) is actually continuous. Note that by our definition, the elements of \( \mathcal{H} \) are functions taking values in \([0, T]\). Furthermore, for any \( H \in \mathcal{H} \) and any valuation \( v \in [0, \hat{P}] \), we must have \( H(v) = 0 \) and \( b(\bar{v}) = \bar{v} \). This reflects the fact that any consumer with valuation in this range cannot afford to buy the product in the external market; the auction is his only

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\(^9\) For instance, it includes the set of càdlàg functions; those that are right continuous and have left limit. The problem of characterizing other types of equilibria that are not in \( \mathcal{D} \) is certainly interesting from a theoretical perspective, but goes beyond the scope of this paper.
potentially profitable channel, no matter his arrival time. In summary, we define the set of potential bidding strategies as the set of functions

$$\mathcal{H} = \{ H \in \mathcal{D}, \ H : [0, 1] \rightarrow [0, T],$$

such that $$H(v) = 0$$ for all $$v \in [0, \hat{P}]\}.$$ We still need to characterize what the equilibrium $$H(v) \in \mathcal{H}$$ looks like for $$v \in (\hat{P}, 1]$$. Consumers in this range will participate in the auction (and bid $$b(v) = \hat{P}$$) only if the expected auction price is small enough to compensate for the disutility associated with waiting for the auction closing time $$T$$.

For any $$H \in \mathcal{H}$$, let us define a useful random variable.

- $$P_H(v)$$: the auction price given that (i) there is a $$v$$-consumer that has joined the auction, and (ii) all other consumers use the participation strategy $$H \in \mathcal{H}$$. To compute the probability distribution of this price we need to estimate the number of bidders and their corresponding valuations. For this, consider $$H \in \mathcal{H}$$ and let us define for all $$v \in \mathcal{V}$$

$$\Lambda_H(v) \triangleq \lambda \int_v^1 (T - H(x)) \, dF(x) \triangleq \lambda T \eta_H(v),$$

where $$\eta_H(v) \triangleq \int_v^1 \left(1 - \frac{H(x)}{T}\right) \, dF(x).$$ (4)

By the definition of the bidding function $$H(v)$$, the function $$\Lambda_H(v)$$ represents the average number of bidders with valuation in $$[v, 1]$$. Similarly, $$\eta_H(v)$$ represents the average fraction of arrivals with valuation in this range who choose the auction channel. We also note that the restriction $$\mathcal{H} \subset \mathcal{D}$$ ensures that $$\Lambda_H(v)$$ and $$\eta_H(v)$$ are well-defined functions continuous in $$v$$.

Let us denote by $$B(\Lambda_H(v))$$ the random number of bidders with valuation greater than or equal to $$v$$. Because consumers arrive according to a Poisson process with rate $$\lambda$$, $$B(\Lambda_H(v))$$ has a Poisson distribution with mean $$\Lambda_H(v)$$. We can now compute the probability distribution of $$P_H(v)$$ given a symmetric participation strategy $$H \in \mathcal{H}$$. Under a $$(Q_0 + 1)$$-price auction, $$P_H(v) < x$$ if and only if (i) $$x > \hat{P}$$, or (ii) the number of bidders with valuation greater than or equal to $$x$$ is less than or equal to the number of objects in the auction minus $$\mathbb{1}(x \leq v)$$. That is,

$$\mathbb{P}(P_H(v) < x) = \begin{cases} 1 & \text{if } x > \hat{P} \\ \mathbb{P}(B(\Lambda_H(x)) + \mathbb{1}(x \leq v) \leq Q_0) & \text{if } x \leq \hat{P} \\ \sum_{k=0}^{Q_0-\mathbb{1}(x \leq v)} \left(\Lambda_H(x)\right)^k \exp\left(-\Lambda_H(x)\right) \frac{1}{k!} & \text{if } x \leq \hat{P}, \end{cases}$$ (5)

where $$\mathbb{1}(E)$$ is the indicator function of event $$E$$. This also follows from the continuity of $$\Lambda_H(x)$$ that $$\mathbb{P}(P_H(v) < x)$$ is continuous in $$x \in [0, 1] - [\hat{P}, \hat{v} \wedge \hat{P}]$$.

### 3.2. Characterization of a Symmetric Participation Equilibrium $$H(v)$$

To characterize a symmetric participation equilibrium (SPE) $$H \in \mathcal{H}$$, we use the following two-step approach. First, we look at a consumer’s best-response participation strategy assuming that other consumers use a fixed strategy $$H \in \mathcal{H}$$. We denote by $$\mathcal{R}(H) \in \mathcal{H}$$ this best-response participation strategy and refer to $$\mathcal{R}$$ as the best-response mapping on $$\mathcal{H}$$. Second, we impose the equilibrium condition $$\mathcal{R}(H') = H'$$. Before moving into this analysis, we recall that the optimal strategy for consumers with valuation $$v \leq \hat{P}$$ (independent of $$H$$) is to enter the auction, and so we must have $$\mathcal{R}(H)(v) = 0$$ for all $$v \in [0, \hat{P}]$$.

Suppose a consumer—referred to as consumer $$\tau$$—arrives at time $$\tau$$ with private valuation $$v_\tau > \hat{P}$$, and suppose that every other consumer is using the participation strategy $$H$$. If consumer $$\tau$$ decides not to bid and buy a unit through the external fixed-price channel, then his expected utility would be $$u(\tau, T, v_\tau - \hat{P})$$. On the other hand, if he decides to bid, then his profit would be $$u(\tau, T, v_\tau - \hat{P})$$ if he does not get the object (because he can always buy the product in the external market at time $$T$$), and $$u(\tau, T, v_\tau - P_H(v_\tau))$$ if he indeed does get the object. Thus, a rational consumer $$\tau$$ enters the auction only if

$$u(\tau, T, v_\tau - \hat{P})(1 - \mathbb{P}(P_H(v_\tau) < \hat{P})) + \mathbb{E}[u(\tau, T, v_\tau - P_H(v_\tau)) | P_H(v_\tau) < \hat{P}] \mathbb{P}(P_H(v_\tau) < \hat{P}) \geq u(\tau, T, v_\tau - \hat{P}),$$ (6)

where $$\mathbb{P}(P_H(v_\tau) < \hat{P})$$ is the probability that bidder $$\tau$$ gets one of the auctioned objects at a price strictly less than the posted price. We still need to explicitly characterize this participation constraint in terms of the function $$H(\cdot)$$, but at this stage note that $$H(\cdot)$$ is embedded in Condition (6) through the random variable $$P_H(v_\tau)$$ (see Equation (5)).

Without loss of generality, we assume that $$\hat{P} > 0$$ (otherwise, the auction is meaningless because consumers will go directly to the external fixed price channel and get a unit at no risk). This is also consistent with the scaling in (3) where we set $$v^R = 0$$, because all the bidders with valuations smaller than $$v^R$$ will have zero utility, and will quit without purchasing in any of the channels.

We compute the best-response strategy $$\mathcal{R}(H)$$ for consumer $$\tau$$ by looking at the threshold function that is consistent with (6). First, note that in our setting, where consumers have the exponentially discounted utility function defined in Equation (1), Condition (6) is equivalent to

$$\frac{\hat{P} - \mathbb{E}[P_H(v_\tau) | P_H(v_\tau) < \hat{P}]}{v_\tau - \hat{P}} \mathbb{P}(P_H(v_\tau) < \hat{P}) \geq \exp(\omega(T - \tau)) - 1.$$ (7)
From this condition, we conclude that, in equilibrium \( E[P_H(v_\tau) | P_H(v_\tau) < \hat{P}] < \hat{P} \). That is, no consumer would have an incentive to bid if the expected auction price in case of winning is at least what he would have paid in the external market, at no risk. The following proposition characterizes \( R(H) \).

**Proposition 1.** For the exponential utility function (1), Condition (7) is equivalent to
\[
\tau \geq T - \frac{1}{w} \ln \left( 1 + \int_0^{\hat{P}} \frac{\mathbb{P}(P_H(v_\tau) < x)}{v_\tau - \hat{P}} \, dx \right).
\]
(8)

Thus, a consumer arriving at time \( \tau \) with valuation \( v_\tau \), enters the auction if and only if \( \tau \geq R(H)(v_\tau) \), where
\[
R(H)(v_\tau) \triangleq \begin{cases} 
0 & \text{if } v_\tau \in [0, \hat{P}] \\
T - \frac{1}{w} \ln \left( 1 + \int_0^{\hat{P}} \frac{\mathbb{P}(P_H(v_\tau) < x)}{v_\tau - \hat{P}} \, dx \right) & \text{if } v_\tau \in (\hat{P}, 1].
\end{cases}
\]
(9)

This best-response mapping \( R(H)(v_\tau) \) is continuous in \( v_\tau \).

Because the best-response strategy \( R(H)(v) \) is continuous in \( [0, 1] \), it follows that \( R \) effectively maps \( \mathcal{H} \) into \( \mathcal{H} \). Furthermore, because an SPE is characterized by the fixed-point condition \( R(H) = H \), we conclude that a symmetric participation equilibrium of this game \( H^* \in \mathcal{H} \) is in fact continuous. Our next result extends this conclusion and shows that the best-response strategies are \( K \)-Lipschitz continuous functions in \( [0, 1] \), for an appropriate constant \( K > 0 \). This additional property of the bidding strategies becomes relevant in our proof of existence of an equilibrium. However, before we formally address this issue we need the following lemma.

**Lemma 1.** For all \( H \in \mathcal{H} \) there is a valuation \( v_H > \hat{P} \) such that \( R(H)(v) = 0 \) for all \( v \in [0, v_H] \). The infimum of \( v_H \) over \( H \) satisfies
\[
\hat{v} = \inf_{H \in \mathcal{H}} \{ v_H \} \\
\geq \min \left\{ 1; \hat{P} + \exp(-wT) \left( 1 + \int_0^{\hat{P}} \mathbb{P}(B(\lambda T[1 - F(x)]) \leq Q_0 - 1) \, dx \right) \right\}
\]

In particular, this means that in equilibrium, all consumers with valuations slightly above the posted price (i.e., with valuation \( v \in [\hat{P}, v_H] \)) will join the auction regardless of their arrival time. Also, note that for some instances of the problem (e.g., when \( wT \) is small) it could be that \( \hat{v} = 1 \). In these cases, we must have \( R(H)(v) = 0 \) for all \( v \in (\hat{P}, 1] \) and so \( H(v) = 0 \) is the unique SPE.

**Proposition 2.** For the exponential utility function (1) and for all \( H \in \mathcal{H} \), there is a positive constant \( K \) (independent of \( H \)) such that the best-response strategy \( R(H)(v) \) is a \( K \)-Lipschitz continuous function that satisfies \( R(H)(v) = 0 \) for all \( v \in [0, \hat{v}] \). In addition, if \( \hat{v} = 1 \), then it is optimal for every consumer, independent of his arrival time and private valuation, to enter the auction. That is, if \( \hat{v} = 1 \), then \( H(v) = 0 \) for all \( v \in [0, 1] \) is the unique (symmetric) participation strategy equilibrium.

Proposition 2 characterizes an SPE for those special cases in which every consumer enters the auction. For the general case, finding a symmetric equilibrium \( H(v) \) or even proving its existence is not an easy task. A standard way to approach the existence problem is to prove that the set of bidding strategies \( \mathcal{H} \) has the fixed-point property (see Cheney 2001, §7.1, for details.)\(^{11}\) and that the best-response mapping \( R \) is continuous in \( \mathcal{H} \). We will take this approach here, although we first need to slightly modify our set of strategies \( \mathcal{H} \). From our previous discussion and the result in Proposition 2, we can restrict the search of a symmetric equilibrium to those strategies \( H \) that are \( K \)-Lipschitz continuous and satisfies \( H(v) = 0 \) in \( [0, \hat{v}] \). For this reason, we redefine \( \mathcal{H} \) to be this set:
\[
\mathcal{H} \triangleq \{ H: [0, 1] \rightarrow [0, T] \text{ s.t. } H \text{ is } K \text{-Lipschitz continuous and } H(v) = 0 \text{ in } v \in [0, \hat{v}] \}.
\]

Note that by Proposition 2, \( R \) is a well-defined mapping from \( \mathcal{H} \) to \( \mathcal{H} \).

**Theorem 1.** The set of strategies \( \mathcal{H} \) equipped with the uniform norm \( \|X\| = \sup_{v \in [0, 1]} |X(v)| \) in \( [0, 1] \) exhibits the fixed-point property. In addition, for all \( H, \widetilde{H} \in \mathcal{H} \), the mapping \( R \) satisfies:
\[
\|R(H) - R(\widetilde{H})\| \leq \int_0^{\hat{P}} \frac{\mathbb{P}(B(Q_0 - 1) = Q_0 - 1)}{w} \left( \int_0^{\hat{P}} \frac{1 - F(x)}{\hat{v} - \hat{P}} \, dx \right) \|H - \widetilde{H}\|.
\]

Therefore, \( R \) is a continuous mapping and there always exists an SPE. In addition, if
\[
\int_0^{\hat{P}} \frac{\mathbb{P}(B(Q_0 - 1) = Q_0 - 1)}{w} \left( \int_0^{\hat{P}} \frac{1 - F(x)}{\hat{v} - \hat{P}} \, dx \right) < 1,
\]
then \( R \) is a contraction. In this case, the fixed point \( \mathcal{R}(H^*) = H^* \) is guaranteed to be unique in \( \mathcal{H} \) and can be

\(^{11}\) A set \( \mathcal{H} \) has the fixed-point property if every continuous mapping \( R: \mathcal{H} \rightarrow \mathcal{H} \) has a fixed point.
found through the iteration $H^{n+1} = \mathcal{R}(H^n)$ starting at an arbitrary $H^1 \in \mathcal{R}$.

We conclude our characterization of an SPE with the following proposition.

**Proposition 3.** For the exponential utility function (1), an SPE $H^*(v)$ is an increasing and concave function of $v \in [v_{\min}, 1]$.

The monotonicity of $H^*(v)$ implies that in equilibrium consumers with large private valuation are less likely to enter the auction. High-value consumers lose more by waiting and thus are less likely to choose the auction; this leads to adverse selection and a poorer distribution of values among auction bidders.\(^{12}\)

Another interesting property of $H^*(v)$ has to do with the resulting bidding pattern it induces. Let us define $\lambda_H(t)$ to be the equilibrium bidding rate at time $t$. Because of the concavity of $H^*(v)$, it follows that $\lambda_H(t)$ is increasing and convex in the range $t \in [0, H^*(1)]$ and remains constant afterwards at its maximum value $\lambda$. This bid-speeding-up feature (sometimes referred to as late bidding or sniping) has been empirically observed in online auctions with a rigid deadline, like the ones conducted through eBay (see, e.g., Roth and Ockenfels 2002, Shmueli et al. 2004).\(^{13}\)

In this respect, our model provides a simple description of this phenomenon based on consumers’ sensitivity to delay.

Note that extending our analysis to include randomized strategies could extend the scope of our results. However, in our setting there is no need for this additional degree of complexity.

**Proposition 4.** Without loss of generality, we can restrict our equilibrium analysis to pure participation strategies.

**Theorem 2.** Suppose the participation strategy $H(v)$ is fixed. Then, in the limit as $n \to \infty$, the auction price $P_H^n(v)$ converges weakly\(^{15}\) to the constant $P_H^\infty(v) = \min\{\bar{P}, \eta_H^{-1}(\rho)\}$, where $\eta_H^{-1}(\rho) = \min\{v \in [0, 1] : \eta_H(v) \leq \rho\}$.

\(^{12}\) We thank one of the referees for pointing out this observation.

\(^{13}\) In a different auction-ending rule, bids are accepted after the original deadline, as long as there is some recent bidding activity, say within the last 10 minutes. This is the type of closing rule followed at Amazon.

3.3. Asymptotic Analysis

In this section, we characterize the outcome of the auction using asymptotic analysis. In particular, we consider the limiting case in which both the number of units $Q_0$ and the average arrival rate $\lambda$ grow proportionally large. In this regime, we show that characterizing the consumers’ strategy (i.e., the threshold function $H(v)$) is equivalent to solving a deterministic problem, which we can do efficiently (Theorem 3).

Consider a sequence of instances of the problem indexed by $n$ and let $Q_0^n$ and $\lambda^n$ be the corresponding number of units to auction and demand rate for instance $n$, respectively. All other parameters are kept independent of $n$. The asymptotic regime that we consider is defined by

$$\lim_{n \to \infty} \frac{Q_0^n}{n} = Q_0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\lambda^n}{n} = \lambda.$$

For each instance $n$ of the problem, we let $\rho^n \triangleq Q_0^n/(\lambda^n T)$. Then, $\lim_{n \to \infty} \rho^n = \rho$, for $\rho \triangleq Q_0/(\lambda T)$. For future reference, we refer to $\rho$ as the excess-supply ratio, which can be viewed as a proxy for the average number of units available per arriving consumer. In this respect, the case $\rho < 1$ reflects the most interesting situation in which, on average, there are fewer units than consumers.

We denote by $P_H^n(v)$ the auction price for instance $n$ given that a $v$-consumer enters the auction. The distribution of this price is given by Equation (5) replacing $Q_0$ by $Q_0^n$ and $\lambda$ by $\lambda^n$. We are now ready to characterize the asymptotic regime.

**Theorem 2.** Suppose the participation strategy $H(v)$ is fixed. Then, in the limit as $n \to \infty$, the auction price $P_H^n(v)$ converges weakly\(^{15}\) to the constant $P_H^\infty(v) = \min\{\bar{P}, \eta_H^{-1}(\rho)\}$, where $\eta_H^{-1}(\rho) = \min\{v \in [0, 1] : \eta_H(v) \leq \rho\}$.

\(^{15}\) In practice, we have been able to find an SPE for all instances that we have tested using the following small step-size version of the iteration in Theorem 1 (see Bertsekas and Tsitsiklis 1996, Chapter 4):

$$H^{n+1} = a H^n + (1-a) \mathcal{R}(H^n),$$

where $a \in [0, 1)$ is empirically selected.

\(^{15}\) A sequence of distribution functions is said to converge weakly to a limit $F$ (written $F_n \Rightarrow F$) if $F_n(y) \to F(y)$ for all $y$ that are continuity points of $F$. A sequence of random variables $X_n$ is said to converge weakly or converge in distribution to a limit $X_\infty$ (written $X_n \Rightarrow X_\infty$) if their distribution functions $F_n(x) \triangleq P(X_n \leq x)$ converge weakly (see Durrett 1996, §2.2).
It follows from Theorem 2 that the probability distribution of the limiting auction price is
\[ P(P_H^n(v) < x) = \begin{cases} 1 & \text{if } x > \hat{P}, \text{ or } x \leq \hat{P} \text{ and } \rho > \eta_H(x) \\ 0 & \text{if } \rho < \eta_H(x) \text{ and } x \leq \hat{P}. \end{cases} \]

Suppose \( x \leq \hat{P} \) and \( \rho < \eta_H(x) \), that is, there are on average more bidders with valuation at least \( x \) than units to auction, \( Q_0 < \Lambda_H(x) \). In this situation, there is a scarcity of units for bidders with valuation greater than or equal to \( x \). Note that the monotonicity of \( \eta_H(x) \) implies \( x = \min\{\hat{P}, \eta_H(\rho)\} \), and it follows from Theorem 2 that the auction price will be higher than \( x \). On the other hand, when there are more units to auction than bidders with valuation of at least \( x \), that is, \( \rho > \eta_H(x) \), the final auction price will be lower than \( x \) with certainty.

Note that the limiting auction price \( P_H^n(v) \) does not depend on \( v \) because as the number of units and consumers goes to infinity, the auction price is unaffected by the decision of one particular bidder. In other words, we can think of this asymptotic regime as one in which we have a continuum of marginal consumers, each one having no impact on the overall outcome of the auction. In this respect, the asymptotic regime under consideration is of the fluid type. For notational convenience, let us define \( P_H \equiv P_H^n(v) \).

Suppose consumer \( \tau \) with valuation \( v_\tau \) arrives at time \( \tau \) and suppose that every other consumer is using the participation strategy \( H \).

- If \( v_\tau \in [0, \hat{P}] \), the consumer enters the auction independently of the auction price because the posted price exceeds his valuation. Thus, we have that \( \mathcal{R}(H)(v_\tau) = 0 \) for all \( v_\tau \in [0, \hat{P}] \).
- If \( v_\tau \in (\hat{P}, 1] \), then it is optimal for consumer \( \tau \) to participate in the auction only if
  \[ (v_\tau - P_H)(\exp(-w(T - \tau)) \geq v_\tau - \hat{P}, \]
  or equivalently \( \tau \geq h(v_\tau) \) (11)
for the auxiliary threshold function \( h(v_\tau) = T - 1/w \cdot \ln((v_\tau - P_H)/(v_\tau - \hat{P})) \). Because the logarithm in \( h(v_\tau) \) goes to \( \infty \) as \( v_\tau \downarrow \hat{P} \), consumers with valuation greater, but close to \( \hat{P} \), will enter the auction. In fact, let us define \( v_H \) to be the root of the equation
\[ T - \frac{1}{w} \ln\left(\frac{v - P_H}{v - \hat{P}}\right) = 0 \]
which is equivalent to
\[ v_H \triangleq \min\left\{ \hat{P} \exp(wT) - P_H, \frac{1}{\exp(wT) - 1} \right\}. \] (12)

Then, by Condition (11) consumer \( \tau \) with valuation \( v_\tau \in (\hat{P}, v_H] \) will always enter the auction independently of his arrival time \( \tau \). On the other hand, consumer \( \tau \) with valuation \( v_\tau \) greater than \( v_H \) will enter the auction only if his arrival time is greater than \( \mathcal{R}(H)(v_\tau) \), where \( \mathcal{R}(H)(v_\tau) = T - 1/w \cdot \ln((v_\tau - P_H)/(v_\tau - \hat{P})) \), for \( v_\tau \in [v_H, 1] \). Clearly, if \( v_H = 1 \), then every consumer will enter the auction independently of his arrival time, in which case \( H^*(v) = 0 \) for all \( v \in [0, 1] \).

In summary, given \( H \in \mathcal{H} \) and the associated auction price \( P_H \), the best-response participation strategy \( \mathcal{R}(H) \) satisfies
\[ \mathcal{R}(H)(v) = \begin{cases} 0 & \text{if } v \in [0, v_H] \\ T - \frac{1}{w} \ln\left(\frac{v - P_H}{v - \hat{P}}\right) & \text{if } v \in [v_H, 1]. \end{cases} \] (13)

We note that \( \mathcal{R}(H)(v) \) is increasing and concave in \( v \), for all \( v \geq v_H \).

To determine the equilibrium value of the auction price \( P_H \) and the corresponding participation strategy \( H^*(v) \), we have to impose the equilibrium condition \( \mathcal{R}(H^*) = H^* \). In this asymptotic regime, we can solve this fixed-point condition efficiently using Equation (13) and Theorem 2.

**Theorem 3.** In the asymptotic regime under consideration, the auction price \( P_H \) is the unique solution in \([0, \hat{P}]\) to the equation
\[ F(v_H) - F(P_H) + \frac{1}{w} \int_{v_H}^{1} \ln\left(\frac{v - P_H}{v - \hat{P}}\right) dF(v) = \min\{\rho, \eta_H(0)\}, \] (14)
where \( v_H = \min\{\hat{P} \exp(wT) - P_H, (\exp(wT) - 1), 1\} \) and the SPE strategy \( H^*(v) \) is given by
\[ H^*(v) = \begin{cases} 0 & \text{if } v \in [0, v_H] \\ T - \frac{1}{w} \ln\left(\frac{v - P_H}{v - \hat{P}}\right) & \text{if } v \in [v_H, 1]. \end{cases} \]

The next result characterizes two extreme outputs of the game and follows directly from Theorem 3.

**Corollary 1.** In the asymptotic regime, it follows that
(a) the auction price equals the reservation price, that is, \( P_H = 0 \), if
\[ F(v_H) + \frac{1}{w} \int_{v_H}^{1} \ln\left(\frac{v}{v - \hat{P}}\right) dF(v) \leq \rho, \]
where \( v_H = \min\{\hat{P} \exp(wT) - P_H, (\exp(wT) - 1), 1\} \) and
\[ F^{-1}(1 - \rho) \leq 1 - \exp(wT)(1 - \hat{P}), \]
in which case \( P_H = F^{-1}(1 - \rho) \).

(b) all arriving consumers enter the auction, that is, \( H^*(v) = 0 \) for all \( v \in [0, 1] \), if
Using a slight abuse of notation, let $H^*(Q_v, \rho)$ be the optimal participation strategy if the auctioneer has $Q_v$ units to auction and the excess supply ratio ($Q_v/\lambda T$) is equal to $\rho$. In Figure 2, we compare the optimal asymptotic participation strategy $H^*(\infty, 0.6)$ (computed using Theorem 3) to four optimal bidding strategies: $H^*(1, 0.6)$, $H^*(5, 0.6)$, $H^*(10, 0.6)$, and $H^*(20, 0.6)$ (computed numerically using the iteration in Theorem 1). From the graph on the top of Figure 2, we can see that the asymptotic approximation mimics quite closely consumers’ participation strategy even for small values of $Q_v$. As a matter of fact, for values of $Q_v$ greater than 10 units, the bidding strategies $H^*(Q_v, 0.6)$ and $H^*(\infty, 0.6)$ are almost indistinguishable.

The table on the bottom of Figure 2 compares the expected price of the auction $\bar{P}_H$ with the approximated value $P_H^{\text{Approx}}$ obtained using the asymptotic participation strategy. In other words, $P_H^{\text{Approx}}$ is the expected auction price if every consumer uses the participation strategy $H^*(\infty, \rho)$. As we can see, for a single-unit auction the error on the estimate is about 7%. However, for moderate multiunit auctions (with five or more items) the quality of the approximation improves considerably quickly. With 20 items the approximation is almost exact. As we expect, both the auction price $\bar{P}_H$ and the approximated auction price $P_H^{\text{Approx}}$ converge to the asymptotic price $P_H^\infty$ as $Q_v$ goes to infinity. We note that the quality of results reported in this example were systematically replicated in all instances of the problem that we considered.

We conclude this section by applying our results to a particular instance of the problem with uniform valuations, a distribution widely considered in the auction literature. In §5, we will use this example to study the seller’s optimization problem.

**Example (Uniform Distribution Case).** Suppose the valuations are uniformly distributed in $[0, 1]$, that is, $F(v) = v$. In this case, we can apply Theorem 3 to get the following cases:

- If $\hat{P} = \hat{P}_1(\rho, \omega T)$ where $\hat{P}_1(\rho, \omega T)$ solves
  \[
  \frac{1}{\omega T} \left[ (1 - \hat{P}_1) \ln(1 - \hat{P}_1) + \hat{P}_1 \ln \frac{\hat{P}_1}{\exp(\omega T) - 1} \right] = \min \{1, \rho\},
  \]
  then $P_{HI}^* = 0$ and $H^*(v) = [T - (1/\omega) \ln(v/(v - \hat{P}_1))]^+$. If $\hat{P}_1(\rho, \omega T) < \hat{P} < \bar{P}_1(\rho, \omega T) \triangleq 1 - \exp(-\omega T) \cdot (1 - (1 - \rho)^+)$, then the auction price $P_{HI}^* \in (0, \hat{P})$ solves $\eta_{HI}^*(P_{HI}^*) = \min \{1, \rho\}$, which in this case is the same as
  \[
  \frac{1}{\omega T} \left[ (1 - P_{HI}^*) \ln(1 - P_{HI}^*) - (1 - \hat{P}) \ln(1 - \hat{P}) \right] - (\hat{P} - P_{HI}^*) \ln \frac{\hat{P}_1 - P_{HI}^*}{\exp(\omega T) - 1} = \min \{1, \rho\}.
  \]
- If $\hat{P} \geq \bar{P}_2(\rho, \omega T)$, then $H^* = 0$ and $P_{HI}^* = (1 - \rho)^+$. Note that in this case $\eta_{HI}^*(P_{HI}^*) = \min \{1, \rho\}$.

In other words, $\hat{P}_1(\rho, \omega T)$ is a lower bound on the posted price over which there will be enough people in the auction such that the resulting auction price will be positive. $\bar{P}_2(\rho, \omega T)$ is the minimum posted price, such that all consumers will participate in the auction. Figure 3 plots the auction price as a function of the posted price. As we can see, for low values of the posted price $\hat{P}$, the auction price coincides with the reservation price at zero. The intuition in this case is that a low posted price will induce a large fraction of high-value consumers to purchase at the posted price—resulting in fewer bidders left on the auction; bidders that, moreover, have a low valuation. As we can see from Figure 3, $\hat{P}(\rho, \omega T)$ increases, with $\rho$ reflecting the fact that the higher the number of units in the auction, the lower the value of the bid of the last winning bidder. On the other extreme, $\bar{P}_2(\rho, \omega T)$ decreases with $\rho$; that is, the higher the $\rho$, the lower the posted price needed to induce all the players to participate in the auction. In summary, the auction price $P_{HI}^*$ is a nondecreasing function of $\hat{P}$ given by

\[
P_{HI}^* = \Phi(\hat{P}, \rho, \omega T)
\]

where

\[
\Phi(\hat{P}, \rho, \omega T) \triangleq \begin{cases} 
0 & \text{if } \hat{P} \in [0, \hat{P}_1(\rho, \omega T)] \\
\eta_{HI}^*(P_{HI}^*) = \min \{1, \rho\} & \text{if } \hat{P} \geq \bar{P}_1(\rho, \omega T).
\end{cases}
\]
The corresponding number of units sold in the auction $Q_{H^*}$ satisfies

$$Q_{H^*} = \lambda T \mathbb{E}[(\hat{P}, \rho, wT)],$$

where $\mathbb{E}[(\hat{P}, \rho, wT)] \triangleq \begin{cases} 
\eta_{H^*}(0) & \text{if } \hat{P} \in [0, \hat{P}_1(\rho, wT)] \\
\min[1, \rho] & \text{if } \hat{P} \in [\hat{P}_1(\rho, wT), 1],
\end{cases}$

and

$$\eta_{H^*}(0) = \begin{cases} 
\frac{1}{wT} \left[ \hat{P} \ln \left( \frac{(1 - \hat{P})(\exp(wT) - 1)}{\hat{P}} \right) - \ln(1 - \hat{P}) \right] & \text{if } \hat{P} \exp(wT) < \exp(wT) - 1 \\
1 & \text{if } \hat{P} \exp(wT) \geq \exp(wT) - 1.
\end{cases}$$

4. Dual Channel with Static List Price

In this section, we study the consumers’ optimal buying strategy in the face of a monopolistic seller who has $Q_0$ units to sell, using two parallel channels: a list price channel in which she sets a constant list price $\hat{P}$ for the entire selling horizon, and the auction that will take place at time $T$ with the remaining $Q_T$ units. Most of the analysis of this dual channel mimics the steps we used in the previous section. For this reason, we summarize only the main results here, skipping many of the mathematical details. A complete analysis of this dual channel model can be found in online Appendix B.

One of the important differences of this model with respect to the one discussed in the previous section is that the fixed-price channel now has a limited supply of $Q_0$ units. Therefore, bidders that lose in the auction have no alternative market in which to buy the product. Hence, to decide whether to enter the auction or buy at the fixed price, an arriving consumer must estimate the joint probability distribution of the auction price ($P_{H^*}$) and the number of units that will be left unsold for the auction ($Q_T$). In general, this could be a rather complicated (path-dependent) task because the state of the system changes continuously. However, under our assumption that the seller does not reveal any information about the inventory position or bidding process over time (see §2 for more details), the actual computation of $P_{H^*}$ and $Q_T$ simplifies considerably. This assumption is certainly restrictive, and getting rid of it would be an important extension to our analysis (again, see Gupta and Gallien 2007 for some preliminary steps in this direction). We will argue at the end of this section, however, that in an asymptotic sense our equilibrium under partial information is also an equilibrium for the general case with full information. This is a positive conclusion that validates in part our (simplifying) informational assumptions.

Another important difference of this model is on the bidding side. In this new setting the optimal strategy for bidders is $b(v) = \tilde{V}$ as opposed to the strategy $b(v) = \min[\hat{P}, v]$ of the previous section. To see this, note that in this case a high-valuation consumer (i.e., with $v > \hat{P}$) who enters the auction and loses will not get the object at all, because all the units are cleared at $T$ among buyers with valuations above $v^R$. Hence, once this high-valuation consumer decides to enter the auction, the subgame he plays is exactly equivalent to a $(Q_T + 1)$-price auction for which we know the optimal strategy is to bid the true valuation, $b(v) = v$.\(^{16}\)

As in the previous section, we analyze the consumers’ participation problem under the scaling in (3), which we revert to in §5 when we study the seller’s optimization problem. For simplicity, we will keep the same notation that we use in the single-channel case, with the understanding that the value of some quantities (such as $\tilde{V}$, $v_{H^*}$, or $P_{H^*}$) will be slightly different.

Suppose $\hat{P}$ has been selected. As in the previous section, we restrict consumers’ participation strategies to the set

$$\mathcal{X} = \{ H \in \mathcal{D} : [0, 1] \rightarrow [0, T] \text{ such that } H(v) = 0, \text{ for all } v \in [0, \hat{P}] \}.$$\(^{16}\)

Note that in this case $Q_T$ is not a fixed quantity, but a random variable that depends on the initial number of units $Q_0$ and the number of consumers that select the fixed-price channel during $[0, T)$. The strategy is dominant even for this case, where the number of units to auction $Q_0$ is uncertain. The argument is similar to the one used for the standard uniform price auction.
and we characterize the decision of a consumer by means of a threshold function \( H \in \mathcal{H} \), such that a consumer arriving at time \( t \) with valuation \( v \), will join the auction only if \( H(v) \leq t \). The following is our main result from the point of view of the consumers’ game.

**Theorem 4.** For the dual-channel model with exponential utility function (1), a \( K \)-Lipschitz continuous symmetric equilibrium always exists in \( \mathcal{H} \). In addition, for any symmetric equilibrium \( H^* \) there is a valuation \( v_{H^*} > \tilde{P} \) such that \( H^*(v) = 0 \) for all \( v \leq v_{H^*} \).

The existence of \( v_{H^*} \) implies that there is a range of valuations above the list price \( \tilde{P} \), such that consumers with valuations in this range join the auction regardless of their arrival time.

Theorem 4 is important from a theoretical standpoint, but it does not lead to a simple way of characterizing a symmetric participation equilibrium \( H^* \). As in §3.3, we can tackle the problem of determining \( H^* \) using asymptotic analysis when both the initial number of units \( Q_0^e \) and the arrival rate \( \lambda^e \) grow proportionally large. Specifically, we consider a sequence of instances of the dual-channel problem indexed by \( n \) and let \( Q_0^n \) and \( \lambda^n \) be the corresponding initial number of units and demand rate for instance \( n \). The asymptotic regime that we consider is again given by Equation (10).

For a given participation strategy \( H \in \mathcal{H} \), we define the auxiliary function

\[
\eta_{H^-}(x) \equiv \int_x^1 \frac{H(v)}{T} dF(v) = \bar{F}(x) - \eta_H(x),
\]

where \( \bar{F}(x) \) stands for the tail distribution of the valuations. We also define \( P_H^e(v) \) to be the auction price for instance \( n \), given that a bidder with valuation \( v \) enters the auction, and \( Q_T^e \) to be the final random number of units to auction. The following result characterizes the asymptotic regime.

**Theorem 5.** Suppose the participation strategy \( H(v) \) and the static price \( \tilde{P} \) are given. Then, in the limit as \( n \to \infty \):

(i) The rescaled number of units \( Q_T^n/n \) to sell through the auction converges weakly to a constant \( Q_T \equiv (Q_0 - \lambda T \eta_{H^-}(0))^+ \).

(ii) If a final auction takes place (i.e., \( Q_T > 0 \)), its price \( P_H^e(v) \) converges weakly to a constant \( P^*_H \equiv \min(\rho - \eta_H(0), 1) \), where \( \rho = Q_0/(\lambda T) \).

We can use this result to characterize a symmetric participation equilibrium, \( H^* \), in this asymptotic regime. Of course, we already know that \( H^*(v) = 0 \) in \( v \in [0, \tilde{P}] \), so we only need to find the behavior of \( H^*(v) \) in \( v \in [\tilde{P}, 1] \). We distinguish two cases.

**Case 1:** Limited-Supply. Suppose that the initial supply of units is limited in the sense that \( \rho \leq 1 - F(\tilde{P}) \). In this situation, consumers with valuation greater than \( \tilde{P} \) have no incentive to enter the auction because the auction price is guaranteed to be greater than or equal to \( \tilde{P} \). Therefore, the resulting participation strategy is \( H^*(v) = T \) for all \( v \geq \tilde{P} \), and the auction never takes place because all the units will be bought at the posted price (i.e., \( Q_T = 0 \)).

We note that \( H^*(v) = T \) is the only SPE in this case. In fact, let us define \( \tau^* = Tp(1 - F(\tilde{P}))^{-1} \). Then, any \( H \) of the form \( H(v) = (\tau^* + h(v))T \) for an arbitrary nonnegative and bounded function \( h(v) \leq T - \tau^* \), is an SPE. In fact, for such an \( H \), the initial \( Q_0^e \) units will be depleted by time \( \tau^* \) (i.e., \( Q_{\tau^*} = 0 \), because \( \tau^* \lambda(1 - F(\tilde{P})) = Q_0^e \)). Therefore, any consumer arriving after \( \tau^* \) will never get a unit, and so he becomes indifferent between the two channels.

**Case 2:** Abundant-Supply. Suppose that initial supply is abundant in the sense that \( \rho > 1 - F(\tilde{P}) \). In this case, \( Q_T > 0 \), and some consumers with valuation smaller than \( \tilde{P} \) get units through the auction. From part (ii) in Theorem 5, it is not hard to see that in this abundant-supply case, the auction price satisfies \( P_H^e = F^{-1}(1 - \rho) \). Therefore, a consumer arriving at time \( T \) with valuation \( v \geq \tilde{P} \) enters the auction only if \( v - \tilde{P} \leq \exp(-w(T - \tau^*))/\eta_H(0) - P_H^e \). We conclude that in this abundant case the unique SPE \( H^*(v) \) is given by

\[
H^*(v) = \begin{cases} 
0 & \text{if } v \in [0, v_{H^*}] \\
T - \frac{1}{w} \ln\left(\frac{v - P_H^e}{v - \tilde{P}}\right) & \text{if } v \in [v_{H^*}, 1], \\
\text{where } v_{H^*} = \min\left\{ \tilde{P}, \exp( -w(T - \tau^*))/\eta_H(0) - P_H^e \right\} 
\end{cases}
\]

Figure 4 compares the optimal asymptotic strategy \( H^*(\infty, \rho) \) to four optimal participation strategies \( H^*(Q_0, \rho) \) (\( Q_0 = 1, 5, 10, 20 \)) for the case of \( \rho = 0.5 \). As in the single-auction channel case, the asymptotic approximation is very accurate even for small values of \( Q_0 \), and almost identical to the optimal strategy for values of \( Q_0 \) greater than 10 units. The table on the bottom compares the expected value of the auction price \( P_H^e \) and the approximation \( P_H^\text{approx} \) obtained using the asymptotic participation strategy \( H^*(\infty, 0.5) \).
5. Seller’s Optimization Problem

In the previous two sections we have studied the optimal buying strategy from the consumers side, assuming that the parameters of the auction and list price channels are fixed. In this section, we turn to the seller’s problem and derive some managerial guidelines to support an optimal design of the auction and the list price channels. For simplicity, and given the accuracy of the asymptotic approximation, we will work under this limiting regime. Furthermore, we will assume that the original distribution of valuations is a standard uniform, that is, \( F \triangleq \text{Unif}[0, 1] \). We use this uniform distribution to make the exposition cleaner, but we note that the general case uses exactly the same line of arguments.

As with most asymptotic analysis, the results in this section have to be interpreted with caution. This is because our fluid-type scaling in (10) has the property of “washing away” the stochasticity of the problem, capturing only first-order effects. Nevertheless, our numerical experiments (together with the weak convergence results) suggest that this deterministic approximation is indeed robust even for small inventory levels.

The first step in formulating the seller’s problem is to revert to the scaling in (3) and write the optimization in terms of the original parameters. The unscaled arrival rate, excess-supply ratio, and posted price are

\[
\lambda \rightarrow \lambda (1 - v^R), \quad \rho \rightarrow \frac{Q_0}{\lambda T (1 - v^R)} = \frac{\rho}{1 - v^R},
\]

In the asymptotic regime under consideration, however, it turns out that there is no loss of generality in taking \( v^R = 0 \). This follows from: (i) the fact that in this asymptotic limit the final auction price \( (P_H, R) \) is a deterministic function of \( (P_{\text{tr}}, Q_0) \); and (ii) the seller’s ability to ration the number of units to auction. Therefore, for those instances in which \( 0 < v^R = P_{\text{tr}} \), there is a revenue-equivalent solution with \( v^R = 0 \) and the same auction price \( P_{\text{tr}} \), but with (possibly) fewer units being auctioned. \(^{18}\)

We will take advantage of this property to simplify the analysis (and notation) of the seller’s optimization problem.

5.1. Single-Auction Channel: Auctioneer’s Optimization Problem

In the single-auction channel, the auctioneer has control over the duration of the auction \( T \) and the number of units to allocate in the auction \( Q_0 \). We assume that the auctioneer is endowed with \( \bar{Q} \) units and

\(^{17}\) These are the only instances in which the choice of the reservation price affects the seller’s revenue.

\(^{18}\) From a practical standpoint, we can implement the asymptotic solution by setting the reservation price \( v^R \) equal to the resulting auction price minus a safety factor \( SF > 0 \). That is, \( v^R = P_{\text{tr}} - SF \).
uses a discount factor $\alpha$ to penalize future payoffs. Hence, the auctioneer is interested in solving (possibly numerically) the following problem:

$$V_A(\bar{Q}) = \max_{T, \rho} \left\{ e^{-\alpha T} P_{1T}(\rho, T, \bar{P}) Q_{1T}(\rho, T, \bar{P}) \right\}$$

subject to $0 \leq \rho \leq \frac{\bar{Q}}{\lambda T}$, (18)

where $P_{1T}(\rho, T, \bar{P})$ and $Q_{1T}(\rho, T, \bar{P})$ are the auction price and number of units sold in the auction as a function of $\rho = Q_0/(\lambda T)$, $T$, and the fixed price $\bar{P}$. These values are computed in Equations (15) and (16), respectively.

As we can see, the optimal choice of $T$ trades off the time value of money (captured by the discount factor $\exp(-\alpha T)$) with the corresponding volume of demand and auction payoff. On the other hand, the optimal choice of $\rho$ (or equivalently $Q_0$) balances the auction price and demand. This rationing decision is due to the fact that (under a multiunit uniform price auction) expected revenues are not guaranteed to be increasing in the number of units. We will solve the optimization problem in (18) in two steps. First, we discuss this rationing decision for a given $T$ and then determine the optimal value of $T$.

In Proposition 5 below, we show that the normalized revenue $P_{1T}(\rho, T, \bar{P}) \times Q_{1T}(\rho, T, \bar{P})$ is unimodal in $\rho$ for every $T$ and $\bar{P}$. This allows us to drop the supply constraint $\rho \leq \bar{Q}/(\lambda T)$ with the understanding that if the unconstrained solution $\rho^*$ exceeds this upper bound, then we will have to truncate the solution to $\rho^* = \bar{Q}/(\lambda T)$.

In the asymptotic regime under consideration, condition (15) implies that the auctioneer must restrict the choice of $\rho$ so that $\bar{P} \geq \bar{P}(\rho, wT)$. Hence, we can simplify the search of an optimal solution imposing the condition $\eta_{1T}(P_{1T}) = \min[1, \rho]$. In addition, condition (16) implies that $Q_{1T} = \lambda T \min[1, \rho]$. Finally, we note that if $\rho \geq 1$, then $P_{1T} = b^1$ and so $P_{1T}$ is unimodal in $\rho$. This implies that at optimality, $Q_{1T} = \lambda T \rho$.

The following proposition characterizes an optimal solution $(\rho^*, P_{1T})$ for a fixed $T$. Before we state the result, we define $\bar{P}(\bar{P})$ as the unique root in $[\bar{P}_0, \bar{P}]$ of the equation

$$\left(\frac{(e^{wT} - 1)(1 - \bar{P})^{1-2\beta}}{\bar{P} - \bar{P}}\right)^{1-2\beta} \left(\frac{(e^{wT} - 1)(1 - \bar{P})^{1-\beta}}{\bar{P} - \bar{P}}\right)^{1-\beta} = \left(\frac{(e^{wT} - 1)(1 - \bar{P})^{1-2\beta}}{\bar{P} - \bar{P}}\right)^{1-\beta}$$

where $\bar{P}_0 = (\bar{P} + 1 - \exp(wT))^+$.

**Proposition 5.** The revenue $\lambda T \rho^* P_{1T}(\rho^*, T, \bar{P})$ is unimodal in $\rho$ for every $T$ and $\rho$. In addition, the optimal (unconstrained) excess-supply ratio $\rho^*$ and corresponding auction price $P_{1T}$ satisfy:

Case 1. If $\bar{P} < 1 - e^{-wT}/2$, then

$$\rho^* = \frac{1}{2wT}\ln\left(\frac{1 - \bar{P}(\rho^*)}{1 - \bar{P}}\right) + \bar{P}\ln\left(\frac{e^{wT} - 1(1 - \bar{P})}{\bar{P} - \bar{P}(\rho^*)}\right)$$

and $P_{1T} = \bar{P}(\rho^*)$.

Case 2. If $\bar{P} \geq 1 - e^{-wT}/2$, then $\rho^* = P_{1T} = 1/2$.

Note that in Case 2 the price $\bar{P}$ is sufficiently large that no arriving consumer will buy in this channel. Hence, in equilibrium, the resulting game looks exactly like a standard multiunit uniform-price auction. The corresponding $\rho^*$ and $P_{1T}$ are the optimal design parameters for this auction in our large-capacity and sales volume setting.19

The following corollary is immediate.

**Corollary 2.** It follows from Proposition 5 that in the limit as consumers become increasingly time sensitive or the auction time grows large, $\lim_{wT \to \infty} \rho^* = \lim_{wT \to \infty} P_{1T} = \bar{P}/2$. On the other hand, in the limit as consumers become increasingly patient or the auction time is short, then $\lim_{wT \to 0} \rho^* = 1 - \bar{P}$ and $\lim_{wT \to 0} P_{1T} = \bar{P}$.

Figure 5 shows the behavior of $\rho^*$, $P_{1T}$, and the auctioneer’s revenue as a function of $\bar{P}$. As expected, both the auction price and revenues are nondecreasing in $\bar{P}$. On the other hand, the optimal $\rho$ (or equivalently, the optimal number of units $Q_0$ to offer) is nonmonotonic in $\bar{P}$. Although we cannot get a closed-form solution for the maximum value of $\rho^*$ as a function of $\bar{P}$ and $wT$, we can numerically show that $\rho^*$ is always bounded above by 0.8. Hence, independent of $\bar{P}$, $w$, and $T$, the optimal number of units to auction should never exceed 80% of the average demand $\lambda T$. We also note that, consistent with Case 2 in the proposition, when the fixed price is sufficiently large (i.e., $\bar{P} \geq 1 - \exp(-wT)/2$) the fixed-price channel has no effect on the choice of $\rho^*$ and the auction output.

We conclude this subsection by discussing the optimal choice of $T$. Figure 6 depicts the auctioneer’s optimal revenue as a function of $T$ for three values of $\bar{P}$. Naturally, revenues increase with $\bar{P}$. Less obvious is the fact that the optimal $T^+$ is bounded above by $\alpha^{-1}$ (in Figure 6, this value is represented by the dashed line). To see this, note that (i) the optimal normalized revenue $\rho^* P_{1T}(\rho^*, T, \bar{P})$ is nonincreasing in $T$ for every $\bar{P}$, and (ii) the function $\lambda T \exp(-\alpha T)\rho^* P_{1T}(\rho^*, T, \bar{P})$ is maximized at $T = \alpha^{-1}$. Hence, the auctioneer’s revenue function $\lambda T \exp(-\alpha T)\rho^* P_{1T}(\rho^*, T, \bar{P})$ attains its maximum in $[0, \alpha^{-1}]$. Furthermore, Figure 6 also reveals that $T^+$ is increasing in $\bar{P}$. In other words, the higher the fixed price, the longer the duration of the auction.

19This outcome is also equivalent to the optimal design parameters of a list price channel (see Talluri and van Ryzin 2004, §6.2.6.2).
From Case 2 in Proposition 5, it follows that the upper bound $T^* = \alpha^{-1}$ is attained if $\hat{P} \geq 1 - \exp(-w/\alpha)/2$.

5.2. Dual Channel: Seller’s Optimization Problem

In the dual-channel case, the seller must solve (possibly numerically) a more complex problem than the one in §5.1, as she also controls the list price channel $\hat{P}$. To write down the seller’s optimization problem, we introduce one more piece of notation. We denote by $\lambda_{F}(t, \rho, T, \hat{P})$ the demand intensity for the fixed-price channel at time $t$ given $(\rho, T, \hat{P})$. Combining the results in §4 and the uniform distribution assumption, it is not hard to show that (see also the online Appendix B)

$$
\lambda_{F}(t, \rho, T, \hat{P}) = \begin{cases}
\lambda(1 - \hat{P})1(t(1 - \hat{P}) \leq \rho T) & \text{if } \rho \leq 1 - \hat{P} \\
\lambda \left[1 - \frac{\hat{P}\exp(w(T-t)) - (1 - \rho)^+}{\exp(w(T-t)) - 1}\right]^{+} & \text{if } \rho \geq 1 - \hat{P}.
\end{cases}
$$

The seller’s optimization problem is given by

$$
V_{D}(\tilde{Q}) = \max_{T, \rho, \hat{P}} \left\{ \int_{0}^{T} e^{-\alpha t} \lambda_{F}(t, \rho, T, \hat{P}) \hat{P} \, dt + e^{-\alpha T} P_{H}(\rho, T, \hat{P}) Q_{H}(\rho, T, \hat{P}) \right\}
$$

subject to $0 \leq \rho \leq \frac{\tilde{Q}}{\lambda T}$. (19)

Unfortunately, the optimization problem above does not admit a simple analytical solution and we must rely on numerical computations to derive an optimal solution $(\rho^*, \hat{P}^*, T^*)$. Nevertheless, we have been able to identify some useful properties of an optimal solution that we summarize in the following proposition.

Proposition 6. (a) Suppose that $T$ is fixed and define $\delta = \alpha T \exp(-\alpha T)/(1 - \exp(-\alpha T))$. Then, the optimal solution $(\rho^*, \hat{P}^*)$ satisfies $1 - \hat{P}^* \leq \rho^* \leq (1 - \tilde{P}^*) \exp(wT)$. Furthermore,

Case 1. If $\tilde{Q}/(\lambda T) \leq (1 - \delta)/(2 - \delta)$, then

$$
V_{D} = (\lambda T \exp(-\alpha T)/\delta)\rho^* \hat{P}^*
$$

where $\rho^* = 1 - \hat{P}^* = \tilde{Q}/(\lambda T)$. 
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that the seller chooses (suboptimally) to operate only as consumers become increasingly patient. Intuitively, the first asymptotic result implies that maximum possible revenue

bound

The optimal time $T^*$ is bounded above by the unique nonnegative root of

$$
\frac{\bar{Q}}{\lambda T} = \frac{1 - \exp(-\alpha T) - \alpha T \exp(-\alpha T)}{2 - \exp(-\alpha T) - \alpha T \exp(-\alpha T)}.
$$

The optimality condition $\rho^* = 1 - \bar{\rho}^*$ in Case 1 implies that the auction price equals the fixed price, and so all units are sold through the fixed-price channel. Note that $\delta$ is a decreasing function of $\alpha T$, and so the condition $\bar{Q}/(\lambda T) \leq (1 - \delta)/(2 - \delta)$ tends to be satisfied for small values of $\bar{Q}$ and large values of $\alpha T$. If $\bar{Q}$ is small, then the seller can target this limited inventory to high-valuation buyers, and there is no need for an auction. On the other hand, if $\alpha T$ is large, then any revenue collected in the auction will be penalized by a small discount factor $\exp(-\alpha T)$, and so the seller has no economic incentive to run an auction.

In Case 2, when $\bar{Q}/(\lambda T) \geq (1 - \delta)/(2 - \delta)$, it is in the seller’s best interest to operate both channels. In this case, we cannot get a closed-form solution, but we derive upper and lower bounds for the seller’s revenue. The lower bound $V_1$ is derived, assuming that the seller chooses (suboptimally) to operate only the fixed-price channel (i.e., $\rho_1 = 1 - \bar{\rho}_1$). The upper bound $V_2$ is obtained, assuming that all buyers with valuation greater than the posted price will buy in this channel, independently of the expected auction price.

Interestingly, these bounds are asymptotically optimal. Intuitively, the first asymptotic result implies that as consumers become increasingly patient ($w \downarrow 0$), the seller is unable to segment the population using the two channels. Consumers are willing to wait for the auction if the auction price is below the fixed price. Hence, in the limit $w \downarrow 0$, the auction price must equal the fixed price, and all units are sold through the fixed-price channel. This is the worst possible scenario for the seller because the revenue reaches the lower bound $V_1$. From a practical standpoint, we can interpret this result as suggesting that the seller should choose a fixed price and initial inventory so that most of this inventory is sold through the fixed-price channel.

On the other hand, if consumers become increasingly impatient ($w \rightarrow \infty$), then the seller can perfectly segment these consumers among those that can afford to pay the fixed price and those that cannot. This gives the seller the ability to separate consumers exclusively based on their valuation and to optimally design the fixed price and the auction to achieve the maximum possible revenue $V_2$. The auction in this case is an effective selling mechanism that complements the fixed-price operation well.

The previous discussion raises an important question regarding the value of a dual-channel operation versus a traditional fixed-price channel. To address this question, we will compare the expected revenue collected using both channels, $V_D$, and the one collected using the fixed-price channel only. Note that if the seller uses only the fixed-price channel, she will choose a fixed price equal to $\max[1/2; 1 - \bar{Q}/(\lambda T)]$ and collect a revenue equal to $V_1$ in Proposition 6. Figure 7 plots the ratio $V_D/V_1$ as a function of $\bar{Q}/(\lambda T)$ (left panel) and $w$ (right panel) for three different values of $\alpha$. Consistent with the results in Proposition 6, when $\bar{Q}/(\lambda T)$ or $w$ are small or $\alpha$ is large, the ratio $V_D/V_1$ is close to one. Hence, in these cases there is little advantage in running an auction. On the other hand, for small values of $\alpha$ or large values of $\bar{Q}/(\lambda T)$ or $w$, the seller can significantly increase her discounted revenue by operating both channels (close to 22% in the examples in Figure 7). We can get an upper bound on this relative revenue increase by replacing $V_D$ by the upper bound $V_2$. After some straightforward manipulations, combining Case 2 in Proposition 6 and the fact that $\delta \leq 1$, we get that

$$
1 \leq \frac{V_D}{V_1} \leq \frac{V_2}{V_1} \leq \frac{4}{4 - \delta} \leq \frac{4}{3},
$$

that is, the seller can increase her revenues by as much as 33% by adding a terminal auction to a fixed-price operation. This upper bound is reached in the limit as $w \rightarrow \infty$ and $\bar{Q} \rightarrow \lambda T$.

6. Concluding Remarks

In this paper, we have proposed a model to analyze the problem faced by a seller when designing a single-channel online auction, or when managing a dual online auction and list price channel. The key to building this model is to understand consumers’ strategic behavior, provided they choose to either join the auction or buy the product at the posted price.
The private information of the consumers has two dimensions: the arrival time and the private value for one of the units being offered. Using a time-sensitive utility function, we showed that their participation equilibrium strategy is of the threshold type, that is, a consumer will join the auction if and only if his arrival time is higher than a function of his own valuation. Of course, for consumers with values below the list price, the optimal strategy is always to participate in the auction. For consumers with higher values, the threshold is nondecreasing in their own valuation. Interestingly, we found that there is always a range of values greater than the fixed price for which it is also optimal to enter the auction, regardless of the arrival time. We also found that, in equilibrium, the auction bidding rate is an increasing function of time (sniping); a phenomenon that is frequently observed in practice. In this respect, our model provides a simple explanation for this late bidding behavior based on consumers’ sensitivity to delay.

At a theoretical level, we proved that a symmetric participation equilibrium always exists. We also proposed a contraction algorithm in a function space to find this threshold and proved its convergence under some conditions. When these conditions are not satisfied, we managed to find the fixed point of the algorithm by slightly perturbing the values of each iteration.

In general, the exact algorithm is computationally intensive and does not lead to clean managerial insights. To overcome these limitations, we proposed an asymptotic analysis for both settings. In this asymptotic regime the initial number of units and the demand rate grow proportionally large. In the limit, we showed that there is weak convergence to a unique equilibrium that we were able to characterize in closed form. Using numerical experiments, we also showed that this solution is indeed a good approximation even for moderate values of the inventory and arrival rate. Because of the simplicity and accuracy of the asymptotic analysis, we proposed to solve the seller’s optimization problem using this limiting regime. In the case where the seller controls only the auction, we found that the optimal number of units to offer is a nonmonotonic function of the fixed price $\hat{P}$. Although we could not characterize this optimal supply in closed form, we derived a simple analytical expression for it and showed that it is always bounded above by 80% of the average demand. In terms of the optimal duration of the auction, we found that this optimal time is an increasing function of the fixed price, and it is bounded above by $\alpha^{-1}$ (the inverse of the seller discount factor). This upper bound is attained if $\hat{P} \geq 1 - \exp(-w/\alpha)/2$.

In the case in which the seller is a monopolist controlling both channels, we found that the auction is only useful if the excess-supply ratio $\hat{Q}/(\lambda T)$ exceeds the threshold $(1 - \delta)/(2 - \delta)$ (see Proposition 6). Hence, if the initial endowment $\hat{Q}$ is small or the discount factor $\alpha$ is large, then the seller does not have enough economic incentives to run a terminal auction; a single fixed-price channel is the optimal selling mechanism. On the other hand, if the excess-supply ratio is sufficiently large, then running both channels in parallel is optimal. In the latter case, we showed that a dual-channel operation can have a significant impact on revenues compared to a single fixed-price channel. The magnitude of the increase in revenues can be as large as 33% (for the case of uniformly distributed valuations), an upper bound that is reached in the limit as consumers’ sensitivity to delay $w$ grows large.

We believe that the techniques used to derive the equilibrium participation strategy here, in particular the asymptotic analysis, can also be used when extending our model to incorporate the number of remaining units as part of the information structure of the bidders. That would mean adding one coordinate to the threshold, leading to a three-dimensional surface, but we have not explored this direction in detail.
Other possible extensions are related to the seller’s optimization problem. In our formulation, we have included the capital cost. However, for example, in the dual-channel case, one can easily add some holding cost for keeping the units until the end of the time horizon, such that the seller has an incentive to give more units through the list price channel. One could also add a penalty cost for keeping units at the end of the horizon, or a shortage cost for not being able to serve a consumer if the units are depleted before the time scheduled for the auction.

7. Electronic Companion
An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

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References
Online Appendices

Appendix A. Proofs
The following lemma will be used a few times in this appendix.

**Lemma A1.** Let $B(x)$ be a Poisson random variable with mean $x > 0$. Then, for a nonnegative integer $n$

$$
\frac{d}{dx} \mathbb{P}(B(x) \leq n) = -\mathbb{P}(B(x) = n).
$$

**Proof.**

$$
\frac{d}{dx} \mathbb{P}(B(x) \leq n) = \sum_{k=0}^{n} \frac{d}{dx} \left( \frac{x^k \exp(-x)}{k!} \right) = \sum_{k=1}^{n} \frac{k x^{k-1} \exp(-x)}{k!} - \sum_{k=0}^{n} \frac{x^k \exp(-x)}{k!} = \frac{n}{x} \mathbb{P}(B(x) = n).
$$

**Proof of Proposition 1.** Using the fact that $P_H(v_r)$ is a random variable with positive mass at $v_R > \hat{P}$ and the properties of conditional expectation, we have that

$$
\mathbb{E}[P_H(v_r) | P_H(v_r) < \hat{P}] = \mathbb{E}[P^R | P_H(v_r) < \hat{P}] + \mathbb{E}[P_H(v_r) | v_R < P_H(v_r) < \hat{P}] \mathbb{P}(P_H(v_r) > v_R | P_H(v_r) < \hat{P})
$$

$$
= \left( \int_{0}^{\hat{P}} \mathbb{P}(P_H(v_r) \geq x | 0 < P_H(v_r) < \hat{P}) dx \right) \mathbb{P}(P_H(v_r) > v_R | P_H(v_r) < \hat{P})
$$

$$
= \int_{0}^{\hat{P}} \frac{\mathbb{P}(x \leq P_H(v_r) < \hat{P})}{\mathbb{P}(P_H(v_r) < \hat{P})} dx. \tag{EC.A1}
$$

Combining this expression and (7), after some algebra we get the participation constraint:

$$
\tau \geq T - \frac{1}{w} \ln \left( \hat{P} \mathbb{P}(P_H(v_r) < \hat{P}) - \int_{0}^{\hat{P}} \mathbb{P}(x \leq P_H(v_r) < \hat{P}) dx \right) \frac{1}{v_r - \hat{P}} + 1
$$

$$
= T - \frac{1}{w} \ln \left( 1 + \int_{0}^{\hat{P}} \frac{\mathbb{P}(P_H(v_r) < x)}{v_r - \hat{P}} dx \right) \triangleq h_H(v_r),
$$

which proves (8). From this last inequality, it follows that the right-hand side $h_H(v_r)$ is the natural candidate for best-response strategy for buyer $\tau$ with valuation $v_r > \hat{P}$. However, for an arbitrary $H \in \mathcal{H}$, $h_H(v_r)$ is not guaranteed to be nonnegative, and so $h_H$ is not necessarily a well-defined participation strategy in $\mathcal{H}$. Fortunately, note that we can correct this problem by simply setting the best-response strategy $\hat{H}(H)(v_r) = (h_H(v_r))^+$, which is equivalent to (9) and consistent with (7).
To prove the continuity of $\mathcal{R}(H)(v)$, first note that $\mathcal{R}(H)(v)$ is trivially continuous in $v \in [0, \hat{P}]$ as well as continuous in $v \in (\hat{P}, 1]$.\(^{EC\text{I}}\) In addition, as $v \downarrow \hat{P}$ we have that $\mathcal{R}(H)(v) \to 0 = \mathcal{R}(H)(\hat{P})$. This observation follows by noting that

$$\lim_{v_i \downarrow \hat{P}} \int_{0}^{\hat{P}} \frac{P(H(v_i) < x)}{v_i - \hat{P}} \, dx = +\infty,$$

because the numerator of the integrand is strictly positive and bounded away from zero. Therefore, for any $H \in \mathcal{H}$, the best-response strategy $\mathcal{R}(H)(v)$ is in fact continuous in $[0, 1]$. \(\square\)

**Proof of Lemma 1.** First of all, note that $\mathcal{R}(H)(v) > 0$ if

$$1 + \int_{0}^{\hat{P}} \frac{P(H(v) < x)}{v - \hat{P}} \, dx < \exp(wT).$$

Because the left-hand side goes to infinity as $v \downarrow \hat{P}$, we can unambiguously define for every $H \in \mathcal{H}$

$$v_H \triangleq 1 \wedge \arg \min_{v \geq \hat{P}} \left\{ 1 + \int_{0}^{\hat{P}} \frac{P(H(v) < x)}{v - \hat{P}} \, dx \leq \exp(wT) \right\}. \quad (EC.A2)$$

Note that by construction, for any $H \in \mathcal{H}$ we must have $\mathcal{R}(H)(v) = 0$ for all $v \in [0, v_H]$. To obtain a lower bound on $v_H$ independent of $H$, we can solve

$$\bar{v} \triangleq \inf_{H \in \mathcal{H}} \{v_H\}. \quad (EC.A3)$$

Unfortunately, this is not a straightforward optimization problem for which we can compute the optimal solution $\bar{v}$. However, we can obtain a lower bound on $\bar{v}$ by means of the following inequality.

For all $H \in \mathcal{H}$,

$$1 + \int_{0}^{\hat{P}} \frac{P(H(v) < x)}{v - \hat{P}} \, dx \geq 1 + \int_{0}^{\hat{P}} \frac{P(\hat{P}(v) < x)}{v - \hat{P}} \, dx,$$

where $P_0(v)$ stands for the auction price when $H = 0$. Because for $x < \hat{P}$, $P(\hat{P}(v) < x) = P(B(\Lambda_0(x)) \leq Q_0 - 1) = P(B(\Lambda T[1 - F(x)]) \leq Q_0 - 1)$, we get a lower bound on $\bar{v}$ solving for $\bar{v}$ in

$$1 + \int_{0}^{\hat{P}} \frac{P(B(\Lambda T[1 - F(x)]) \leq Q_0 - 1)}{v - \hat{P}} \, dx = \exp(wT),$$

or equivalently,

$$\bar{v} = \hat{P} + \exp(-wT) \left( 1 + \int_{0}^{\hat{P}} \frac{P(B(\Lambda T[1 - F(x)]) \leq Q_0 - 1)}{v - \hat{P}} \, dx \right). \quad \square$$

**Proof of Proposition 2.** Note that we only need to prove the K-Lipschitz property of $\mathcal{R}(H)(v)$; the rest of the proposition follows directly from the definition of $\bar{v}$.

To prove that $\mathcal{R}(H)(v)$ is K-Lipschitz continuous, we will make use of the following lemma (not hard to prove).

**Lemma A2.** For arbitrary reals $a$, $b$, and $c$: $|a - b|^+ - |a - c|^+ \leq |b - c|$. For arbitrary nonnegative reals $x$, $y \geq 1$: $|\ln(x) - \ln(y)| \leq |x - y|$.

Recall from the definition of $v_H$ that $\mathcal{R}(H)(v) = 0$ for all $v \in [0, v_H]$. Therefore, we can concentrate on proving the K-Lipschitz property on the interval $[v_H, 1]$. In this range, condition (9) implies that

$$\mathcal{R}(H)(v) = \left[ T - \frac{1}{w} \ln(Z_H(v)) \right]^+$$

where $Z_H(v) \triangleq 1 + \int_{0}^{\hat{P}} \frac{P(H(v) < x)}{v - \hat{P}} \, dx$.\(^{EC\text{I}}\) This follows from the continuity of $\Lambda_H(x)$, which implies the continuity of $P(P_H(v) < x)$ for $x \in [0, \hat{P})$ (see Equation (5)).
Therefore, based on the previous lemma, we have that for arbitrary $v_1, v_2 \in [v_H, 1]$

$$|\mathcal{R}(H)(v_1) - \mathcal{R}(H)(v_2)| = \left| \left[ T - \frac{1}{w} \ln(Z_H(v_1)) \right]^+ - \left[ T - \frac{1}{w} \ln(Z_H(v_2)) \right]^+ \right|$$

$$\leq \frac{1}{w} |\ln(Z_H(v_1)) - \ln(Z_H(v_2))| \leq \frac{1}{w} |Z_H(v_1) - Z_H(v_2)|.$$  

From the definition of $Z_H(v)$ and the fact that $\int_0^\hat{P} \mathbb{P}(P_H(v) < x) \, dx$ is independent of $v$ for $v \geq v_H > \hat{P}$ (because bidders with valuation in this range bid $b(v) = \hat{P}$ independent of $v$), we have that

$$|Z_H(v_1) - Z_H(v_2)| = \left| \int_{v_1}^{v_2} \frac{d}{dv} Z_H(v) \, dv \right|$$

$$\leq \left| \int_{v_1}^{v_2} \left( - \int_0^{\hat{P}} \frac{d}{dv} \mathbb{P}(P_H(v) < x) \, dx \right) \, dv \right| \leq \left| \int_{v_1}^{v_2} \left[ \frac{\hat{P}}{(\hat{P} - P)^2} \right] \, dv \right| \quad \text{(using } \hat{\delta} \leq v \leq 1)$$

$$= \left[ \frac{\hat{P}}{(\hat{P} - P)^2} \right] |v_1 - v_2|.$$  

We conclude that for arbitrary $v_1, v_2 \in [v_H, 1]$

$$|\mathcal{R}(H)(v_1) - \mathcal{R}(H)(v_2)| \leq \frac{1}{w} \left[ \frac{\hat{P}}{(\hat{P} - P)^2} \right] |v_1 - v_2| \triangleq K |v_1 - v_2|.$$  

The constant $K$ is guaranteed to be finite because $\hat{\delta} \geq \bar{v} > \hat{P}$. \hfill \Box

**Proof of Theorem 1.** To prove that $\mathcal{R}$ has the fixed-point property, we apply the Schauder-Tychonoff Fixed-Point Theorem (see Cheney 2001, Chapter 7 for details). For this, we need to show that $\mathcal{H}$ is a compact convex set. Convexity is immediate from the definition of $\mathcal{H}$. To check compactness, we apply the Arzelà-Ascoli Theorem II (Cheney 2001, Chapter 7), that is, we need to show that $\mathcal{H}$ is closed, bounded, and equicontinuous. Take a sequence $\{H^n\}_{n \geq 1}$ of strategies in $\mathcal{H}$ that converges pointwise to $H$. For all $v \in [0, \bar{v}]$, $H^n(v) = 0$, and so $H(v) = 0$ as well. In addition, to verify the $K$-Lipschitz property of $H$, note that from Proposition 2, for $n \geq 1$ and $v_1, v_2 \in [0, 1]$,

$$|H^n(v_1) - H^n(v_2)| \leq K |v_1 - v_2|.$$  

By the continuity of the absolute value and the pointwise convergence of $H^n$ to $H$, we conclude

$$|H(v_1) - H(v_2)| \leq K |v_1 - v_2|,$$

which proves the closedness of $\mathcal{H}$. The boundedness of $\mathcal{H}$ follows from the fact $H(v) \in [0, T]$ for all $H \in \mathcal{H}$. Equicontinuity, on the other hand, follows directly from the fact that the elements of $\mathcal{H}$ are $K$-Lipschitz continuous. In fact, to prove equicontinuity we need to show that for $\epsilon > 0$ there is $\delta > 0$ such that

For all $H \in \mathcal{H}$ and $v_1, v_2 \in \mathcal{V}$ such that $|v_1 - v_2| < \delta$, then $|H(v_1) - H(v_2)| < \epsilon$.  

For this, take $\delta = \epsilon/K$ and use the $K$-Lipschitz continuity of $\mathcal{H}$ as follows:

For all $H \in \mathcal{H}$ and $v_1, v_2 \in \mathcal{V}$ such that $|v_1 - v_2| < \delta$, $|H(v_1) - H(v_2)| \leq K |v_1 - v_2| < K\delta = \epsilon$.

This proves that $\mathcal{H}$ has the fixed-point property.

We now prove that the best-response $\mathcal{R}$ mapping is continuous. Note that from the definitions of the mapping $\mathcal{R}$ and $\hat{\delta}$, we need only to prove the result for the restriction of $\mathcal{R}$ to the interval $(\hat{\delta}, 1]$. For the proof, we will require the following lemma.

**Lemma A3.** Let $a$ and $b$ be two nonnegative reals and $N \geq 1$ and integer. Then, there is $0 \leq \beta(N) \leq 1$ such that

$$|\mathbb{P}(B(b) \leq N) - \mathbb{P}(B(a) \leq N)| \leq \beta(N)|b - a|.$$
Proof. Because \( P(B(a) \leq N) \) is a continuous and differentiable function of \( a \), we have that

\[
P(B(b) \leq N) = P(B(a) \leq N) + \int_a^b \frac{d}{dx} P(B(x) \leq N) \, dx,\]

which is straightforward to prove (using Lemma A1) that it is equivalent to

\[
P(B(b) \leq N) = P(B(a) \leq N) - \int_a^b P(B(x) = N) \, dx.
\]

Therefore,

\[
|P(B(b) \leq N) - P(B(a) \leq N)| = \int_{a}^{x} P(B(x) = N) \, dx \\
\leq |b - a| P(B(N) = N),
\]

where the inequality follows from the fact that \( P(B(x) = N) \) is maximized at \( x = N \). Therefore, by setting

\[
\beta(N) \triangleq P(B(N) = N)
\]

the proof of the lemma is completed. \( \square \)

Based on Lemma A2, we have that for any \( \nu \in (\bar{\nu}, 1] \),

\[
|\mathcal{R}(H)_\nu - \mathcal{R}(\bar{H})_\nu| = \left| \left[ T - \frac{1}{\nu} \ln \left( 1 + \int_0^\beta \frac{\mu(H(u) < x)}{\nu - \bar{P}} \, dx \right) \right]^+ - \left[ T - \frac{1}{\nu} \ln \left( 1 + \int_0^\beta \frac{\mu(H(u) < x)}{\nu - \bar{P}} \, dx \right) \right]^+ \right|
\]

\[
\leq \frac{1}{\nu} \left( \int_0^\beta \frac{|\mu(H(u) < x) - \mu(\bar{H}(u) < x)|}{\nu - \bar{P}} \, dx \right).
\]

Now, by Lemma A3,

\[
|\mu(H(u) < x) - \mu(\bar{H}(u) < x)| = |\mu(B(\Lambda(u)) \leq Q_0 - 1) - \mu(B(\bar{\Lambda}(u)) \leq Q_0 - 1)|
\]

\[
\leq \beta(Q_0 - 1)|\Lambda(u) - \bar{\Lambda}(u)| = \beta(Q_0 - 1) |\lambda \int_x^1 (\bar{H}(y) - H(y)) \, dF(y)|
\]

\[
\leq \lambda \beta(Q_0 - 1)(1 - F(x)) \| H - \bar{H} \|
\]

Finally, from this inequality we get that for all \( \nu \in (\bar{\nu}, 1] \)

\[
|\mathcal{R}(H)_\nu - \mathcal{R}(\bar{H})_\nu| \leq \frac{\lambda \beta(Q_0 - 1)}{w} \left( \int_0^\beta (1 - F(x)) v - \bar{P} \, dx \right) \| H - \bar{H} \|
\]

\[
\leq \frac{\lambda \beta(Q_0 - 1)}{w} \left( \int_0^\beta (1 - F(x)) \frac{v}{\bar{P} - \bar{P}} \, dx \right) \| H - \bar{H} \|
\]

where the second inequality follows from the fact that \( v \in (\bar{\nu}, 1] \). From this result, we conclude that \( \mathcal{R} \) is continuous, which together with the fixed-point property of the set \( \mathcal{R} \) guarantees the existence of an SPE. \( \square \)

Proof of Proposition 3. Given the equilibrium \( H^* \), we define \( Z_{H^*}(v) \) by

\[
Z_{H^*}(v) \triangleq 1 + \int_0^\beta \frac{\mu(H^*(v) < x)}{v - \bar{P}} \, dx, \quad v \in [v_H, 1].
\]

Note that in the range \([0, v_H]\) the function \( H^* \) is constant at zero.

Based on the definition of the best-response mapping \( \mathcal{R} \), we can write the fixed-point condition \( \mathcal{R}(H^*) = H^* \) satisfied by \( H^*(v) \) as

\[
H^*(v) = T - \frac{1}{w} \ln(Z_{H^*}(v)), \quad \text{for all } v \in [v_H, 1].
\]
Taking the derivative with respect to \( v \) and using the fact that

\[
\chi \triangleq \int_{0}^{\hat{p}} \frac{d}{dx} P(P_{\hat{p}}^{\mu}(v) < x) \, dx
\]

is independent of \( v \) for \( v \geq v_{H} \), we get that

\[
\frac{d}{dv} H^{*}(v) = -\frac{1}{w} \frac{1}{Z_{H}^{*}(v)} \frac{d}{dv} Z_{H}^{*}(v) = \frac{\chi}{w} \left( \frac{1}{(v - \hat{p})(v - \hat{p} + \chi)} \right) > 0,
\]

and we conclude that \( H^{*}(v) \) is increasing in \( v \) for all \( v \geq v_{H} \). Finally, taking a second derivative we get

\[
\frac{d^{2}}{dv^{2}} H^{*}(v) = -\frac{\chi}{w} \left( \frac{2(v - \hat{p}) + \chi}{(v - \hat{p})^{2}(v - \hat{p} + \chi)^{2}} \right) < 0,
\]

and we conclude that \( H^{*}(v) \) is concave in the range \( v \in [v_{H}, 1] \). \( \square \)

**Proof of Theorem 2.** We need the following preliminary result.

**Lemma A4.** Let \( B_{i}(\mu_{n}) \) be a sequence of i.i.d Poisson random variables with mean \( \mu_{n} \), and let \( y_{n} \) be an increasing sequence of nonnegative integers. For each \( n \), define \( S_{y_{n}} \triangleq \sum_{i=1}^{y_{n}} B_{i}(\mu_{n}) \). Suppose that \( \lim_{n \to \infty} y_{n} = \infty \) and \( \lim_{n \to \infty} \mu_{n} = \mu \). Then, the moment-generating function of the r.v. \( Y_{n} \triangleq S_{y_{n}} / y_{n} \) converges to a constant \( \exp(\theta \mu) \), and hence \( Y_{n} \) converges weakly to the constant \( \mu \).

**Proof of Lemma A4.** First, note that \( S_{y_{n}} \sim \text{Poisson}(y_{n} \mu_{n}) \). Using the moment-generating function for the Poisson,

\[
\mathbb{E}\exp\left( \frac{\theta}{y_{n}} S_{y_{n}} \right) = \mathbb{E}\exp\left( \frac{\theta}{y_{n}} y_{n} \mu_{n}(e^{\theta} - 1) \right).
\]

\[
= \exp\left( y_{n} \mu_{n} \left( \frac{\theta}{y_{n}} + o(1/y_{n}) \right) \right) = \exp(\mu_{n} \theta) + o(1).
\]

Hence,

\[
\lim_{n \to \infty} \mathbb{E}\exp\left( \frac{\theta}{y_{n}} S_{y_{n}} / y_{n} \right) \to \exp(\theta \mu),
\]

the moment-generating function of the constant \( \mu \). This guarantees convergence in distribution. \( \text{EC2} \) \( \square \)

We now prove the theorem. From the definitions of \( \Lambda_{H}^{n}(x) \) and \( \eta_{H}(x) \), and the relationship between \( \lambda^{n} \) and \( Q_{0}^{n} \), we have that

\[
\Lambda_{H}^{n}(x) = \frac{\eta_{H}(x)}{\rho^{n}} Q_{0}^{n}.
\]

Now, let \( \{B_{i}(\eta_{H}(x)(\rho^{n})^{-1}); i = 1, \ldots, Q_{0}^{n}\} \) be a sequence of i.i.d Poisson r.v. with mean \( \eta_{H}(x)(\rho^{n})^{-1} \). It follows that \( B(\Lambda_{H}^{n}(x)) \) has the same distribution as the sum of the \( B_{i}(\eta_{H}(x)(\rho^{n})^{-1}) \) from \( i = 1 \) to \( Q_{0}^{n} \). Therefore, for a given \( n \),

\[
\mathbb{P}(P_{\hat{p}}^{\mu}(v) < x) = 1, \quad \text{if } x > \hat{p} \quad \text{or} \quad \mathbb{P}(P_{\hat{p}}^{\mu}(v) < x) = \mathbb{P}(B(\Lambda_{H}^{n}(x)) \leq Q_{0}^{n} - 1(x \leq v)) \]

\[
= \mathbb{P}\left( \sum_{i=1}^{Q_{0}^{n}} B_{i}(\eta_{H}(x)(\rho^{n})^{-1}) \leq Q_{0}^{n} - 1(x \leq v) \right) = \mathbb{P}\left( \frac{\sum_{i=1}^{Q_{0}^{n}} B_{i}(\eta_{H}(x)(\rho^{n})^{-1})}{Q_{0}^{n}} \leq 1 - \frac{1}{Q_{0}^{n}}(x \leq v) \right), \quad \text{if } x \leq \hat{p}. \quad \text{(EC.A4)}
\]

\[\text{EC2} \] See, for example, §30 in Billingsley (1995).
Taking the left-hand side of the last inequality inside the parentheses, we define
\[
\mathcal{B}^n(x) \triangleq \frac{\sum_{k=1}^{\infty} B_k(\eta_H(x)(\rho^*)^{-1})}{Q_n^{\rho}}.
\]

From Lemma 4, \( \mathcal{B}^n(x) \) converges in distribution to the constant \( \eta_H(x)\rho^{-1} \). Moreover, it is clear that the right-hand side of the inequality in (EC.A5) converges to one. Therefore, by focusing on the continuity points, the distribution of \( P_H^n(v) \) converges weakly to the distribution:
\[
P(P_H^n(v) < x) = \begin{cases} 
1 & \text{if } x > \hat{P} \\
1 & \text{if } \rho > \eta_H(x) \text{ and } x \leq \hat{P} \\
0 & \text{if } \rho < \eta_H(x) \text{ and } x \leq \hat{P}.
\end{cases} \quad \text{(EC.A6)}
\]

This corresponds to the distribution of the constant \( \min\{\hat{P}, \eta_H^{-1}(\rho)\} \) at its continuity points, and so \( \mathbb{P}(P_H^n(v) < x) \Rightarrow \mathbb{P}(P_H^\infty(v) < x) \). Thus, \( P_H^n(v) \Rightarrow P_H^\infty(v) \), for \( P_H^\infty(v) = \min\{\hat{P}, \eta_H^{-1}(\rho)\} \).

**Proof of Theorem 3.** From Theorem 2, we have that \( P_{H^*} = \min\{v \in [0, 1]: \eta_H(v) \leq \rho\} \). Because the function \( \eta_H(v) \) is monotonically decreasing, we conclude that \( \eta_H(P_{H^*}) = \min\{\rho, \eta_H(0)\} \). This condition, together with Equation (13) and the equilibrium condition \( H^* = \mathcal{R}(H^*) \), implies condition (14). The value of \( v_{H^*} \) and \( H^* \) follow from conditions (12) and (13), respectively. □

**Proof of Proposition 4.** Suppose that there exists a randomized equilibrium that is characterized by \( n \) different participation strategies \( \{H_i(v) \in \mathcal{H}\}_{i=1}^n \) and \( n \) probability mappings \( \{\gamma_k(v) \in [0, 1]\}_{k=1}^n \), such that \( \sum_{k=1}^{\infty} \gamma_k(v) = 1 \) for all \( v \in [0, 1] \). In this equilibrium, a consumer with valuation \( v \in [0, 1] \) arriving at time \( \tau(v) \in [0, T] \) will use the participation strategy \( H_k(v) \) with probability \( \gamma_k(v) \). Note that we only consider randomization in the participation decision (buy now or bid in the auction), because on the bidding side it is a dominant strategy to bid \( b(v) = \min\{v, \hat{P}\} \).

To show that such a randomized equilibrium is not possible, let us take two participation strategies \( H_i(v) \) and \( H_j(v) \) such that there exists a \( \tilde{v} \in [0, 1] \), such that \( H_i(\tilde{v}) < H_j(\tilde{v}) \) and \( \gamma_i(\tilde{v}) > 0 \) and \( \gamma_j(\tilde{v}) > 0 \). Naturally, such a pair \((i, j)\) and valuation \( \tilde{v} \) must exist in order to have a randomized equilibrium. Also, note that \( \tilde{v} > \hat{P} \) because \( H_i(v) = H_j(v) = 0 \) for \( v \leq \hat{P} \).

Consider a consumer with valuation \( \tilde{v} \) arriving at time \( \tau(\tilde{v}) \in (H_i(\tilde{v}), H_j(\tilde{v})) \). Because this consumer assigns positive probabilities to both participation strategies \( H_i(v) \) and \( H_j(v) \), he must be indifferent between the expected payoff generated by these two strategies.

- If he selects strategy \( H_i(v) \), then he will purchase on the fixed-price channel (because \( \tau(\tilde{v}) > H_i(\tilde{v}) \)), obtaining a payoff of \( \tilde{v} - \hat{P} \).
- If he selects strategy \( H_j(v) \), then he will enter the auction and bid \( \hat{P} \). The corresponding expected payoff is
\[
\mathbb{E}[e^{-\omega(T - \tau(\tilde{v}))}(\tilde{v} - \hat{P})],
\]
where \( \hat{P}_A \) is the auction price given that consumer \( \tilde{v} \) enters the auction and all other consumers use the randomized strategy \( \{(H_k(v), \gamma_k(v))\}_{k=1}^n \).

Hence, consumer \( \tilde{v} \) will be willing to randomize between \( H_i(v) \) and \( H_j(v) \) only if
\[
\tilde{v} - \hat{P} = \mathbb{E}[e^{-\omega(T - \tau(\tilde{v}))}(\tilde{v} - \hat{P}_A)]. \quad \text{(EC.A7)}
\]

Note that the left-hand side of this equality is independent of the arrival time \( \tau(\tilde{v}) \in (H_i(\tilde{v}), H_j(\tilde{v})) \). On the other hand, it is not hard to see that \( \hat{P}_A \) is also independent of \( \tau(\tilde{v}) \), and so the right-hand side is a strictly increasing function of \( \tau(\tilde{v}) \in (H_i(\tilde{v}), H_j(\tilde{v})) \). We conclude that the randomization condition (EC.A7) cannot be sustained for all \( \tau(\tilde{v}) \in (H_i(\tilde{v}), H_j(\tilde{v})) \) unless \( H_i(\tilde{v}) = H_j(\tilde{v}) \). This argument shows that there cannot be a randomized equilibrium. (We note that the result in this Proposition 4 and its proof here are also valid for the dual-channel case.) □

**Proof of Proposition 5.** The proof uses extensively the notation and results of the uniform distribution example in §3.3 in the main paper. We suggest that the reader review this example before...
going over the following proof. Also, because $T$ is fixed in this proof, we will drop the dependence of $Q_{Ht'}(\rho, T, \hat{P})$ and $P_{Ht'}(\rho, T, \hat{P})$ on $T$.

First of all, we note that given the optimality condition $\rho < 1$, it follows that $Q_{Ht'}(\rho, \hat{P}) = \rho$ for every $T$ and $\hat{P}$. Hence, the seller’s optimization problem reduces to maximizing the function

$$\max_{0 < x < Q/A} \{\lambda T \exp(-\alpha T)xP_{Ht'}(\rho, \hat{P})\},$$

where $P_{Ht'}(\rho, \hat{P})$ is the resulting auction price given $\rho$ and $\hat{P}$. To simplify the exposition, we will use Figure EC.A1 as a guide to support the argument of the proof.

The figure depicts the value of the auction price $P_{Ht'}$ as a function of $\rho$ and $\hat{P}$. The threshold functions $\rho_1$ and $\rho_2$ are derived from the definition of $\hat{P}_1(\rho, wT)$ and $\hat{P}_2(\rho, wT)$, respectively. Specifically, we have that

$$\rho_1(\hat{P}) = \frac{-1}{wT} \left[ (1 -(\hat{P} \wedge P_1)) \ln(1 -(\hat{P} \wedge P_1)) + (\hat{P} \wedge P_1) \ln \left( \frac{(\hat{P} \wedge P_1)}{\exp(wT)} - 1 \right) \right], \quad \text{and} \quad \rho_2(\hat{P}) = (1 - \hat{P}) \exp(wT),$$

where $P_1 = 1 - \exp(-wT)$ and $x \wedge y$ stands for $\min(x, y)$. Figure EC.A1 distinguishes three regions in the $(\rho, \hat{P})$ space, which are defined as follows:

Region I $\triangleq \{(\rho, \hat{P}): \rho \leq \min(\rho_1(\hat{P}), \rho_2(\hat{P}))\}$, \quad Region II $\triangleq \{(\rho, \hat{P}): \rho_2(\hat{P}) \leq \rho \leq \rho_1(\hat{P})\}$, \quad and \quad Region III $\triangleq \{(\rho, \hat{P}): \rho_1(\hat{P}) \leq \rho\}$.

In Region III, the auction price is zero, $P_{Ht'} = 0$, and so it never optimal for the seller to choose $\rho \geq \rho_1(\hat{P})$. Hence, the optimal $\rho^*$ as a function of $\hat{P}$ must lie in Region I or in Region II. In order to characterize this optimal $\rho^*$, we will use the following intermediate result.

**Lemma A5.** For any $\hat{P} \in [0, 1]$, the function $\rho P_{Ht'}(\rho, \hat{P})$ is unimodal in $\rho \in [0, \rho_1(\hat{P})]$, and so it admits a unique local maximum.

**Proof of the Lemma.** We divide the proof into two cases.

**Case 1.** $\hat{P} \leq 1 - \exp(-wT)$. In this case, we will prove that the function $\rho P_{Ht'}(\rho, \hat{P})$ is in fact concave, from which the unimodality follows directly. The value of $P_{Ht'}(\rho, \hat{P})$ is the maximal root of the equation

$$\frac{1}{wT} \left[ (1 - P_{Ht'}) \ln(1 - P_{Ht'}) + (1 - \hat{P}) \ln(1 - \hat{P}) - (\hat{P} - P_{Ht'}) \ln \left( \frac{\hat{P} - P_{Ht'}}{\exp(wT)} - 1 \right) \right] = \rho. \tag{EC.A8}$$

From this condition we get $P_{Ht'}$ as a function of $\rho$. It is not hard to show that the left-hand side is a decreasing function of $P_{Ht'}$ in the range $P_{Ht'} \in [\bar{P}_0, \hat{P}]$, where

$$\bar{P}_0 \triangleq \frac{(\hat{P} + 1 - \exp(wT))^+}{2 - \exp(wT)}.$$

Hence, for any $\rho \in [0, \rho_1(\hat{P})]$, the solution $P_{Ht'}(\rho, \hat{P})$ is the unique root of (EC.A8) in $[\bar{P}_0, \hat{P}]$. Now, taking derivative of (EC.A8) with respect to $\rho$, we get that

$$\frac{dP_{Ht'}}{d\rho} = \frac{wT}{\ln(\hat{P} - P_{Ht'}) - \ln((\exp(wT) - 1)(1 - P_{Ht'}))}.$$
which is negative in $P_{H^*} \in [\tilde{P}_1, \tilde{P}]$. Differentiating one more time, we get

$$\frac{d^2 P_{H^*}}{d\rho^2} = \frac{wT(1 - \tilde{P})}{(\tilde{P} - P_{H^*})(1 - P_{H^*})} \frac{dP_{H^*}}{d\rho},$$

which is also negative in $[\tilde{P}_1, \tilde{P}]$.

The concavity of $\rho P_{H^*}(\rho, \tilde{P})$ then follows from the fact that

$$\frac{d^2 \rho P_{H^*}(\rho, \tilde{P})}{d\rho^2} = 2 \frac{dP_{H^*}}{d\rho} + \rho \frac{d^2 P_{H^*}}{d\rho^2},$$

which is negative for any $\rho \in [0, \rho_2(\tilde{P})]$.

**Case 2.** $\tilde{P} > 1 - \exp(-wT)$. In this case, the function $\rho P_{H^*}(\rho, \tilde{P})$ has two pieces, depending on whether $\rho \in [0, \rho_2(\tilde{P})]$ or $\rho \in [\rho_2(\tilde{P}), 1]$.

The arguments used in Case 1 extend in this case to the range $\rho \in [0, \rho_2(\tilde{P})]$, and so $\rho P_{H^*}(\rho, \tilde{P})$ is concave in this range. On the other hand, for $\rho \in [\rho_2(\tilde{P}), 1]$ the revenue function is $\rho P_{H^*}(\rho, \tilde{P}) = \rho(1 - \rho)$, which is trivially concave. Hence, in order to show unimodality it suffices to show that the two pieces match smoothly at $\rho = \rho_2(\tilde{P})$. This is equivalent to showing

$$\lim_{\rho \to \rho_2(\tilde{P})} P_{H^*} = 1 - \rho_2(\tilde{P}) \quad \text{and} \quad \lim_{\rho \to \rho_2(\tilde{P})} \frac{dP_{H^*}}{d\rho} = -1.$$

It is a matter of simple calculations to verify these two conditions, and we leave completing this final step to the reader. □

Based on this lemma, the rest of the proof of the proposition follows directly. In fact, because of the unimodality (as a function of $\rho$) of the revenue $\rho P_{H^*}(\rho, \tilde{P})$ for every $\tilde{P}$ there is a unique solution $\rho^*$.

For $\tilde{P} \geq 1 - \exp(-wT)/2$, it is not hard to show that the unconstrained solution is $\rho^* = 1/2$. This follows from the unimodality of the revenue function and the fact the revenue function is increasing at $\rho = \rho_2(\tilde{P})$ if $\tilde{P} \geq 1 - \exp(-wT)/2$.

On the other hand, for $\tilde{P} \leq 1 - \exp(-wT)/2$ the revenue function is decreasing at $\rho = \rho_2(\tilde{P})$, and so the optimal solution lies in Region I. In this region, the first-order condition is

$$\frac{d\rho P_{H^*}(\rho, \tilde{P})}{d\rho} = 0 \implies P_{H^*} + \rho \ln(\tilde{P} - P_{H^*}) - \ln((\exp(wT) - 1)(1 - P_{H^*})) = 0.$$

We use condition (EC.A8) to replace $\rho$ in terms of $P_{H^*}$. Rearranging, we get that the solution $P_{H^*}$ satisfies

$$\frac{(e^{wT} - 1)(1 - P_{H^*})}{\tilde{P} - P_{H^*}}^{1 - 2\rho_{H^*}} = \left[\frac{(e^{wT} - 1)(1 - \tilde{P})}{\tilde{P} - P_{H^*}}\right]^{1 - \rho_{H^*}},$$

which completes the proof. □

**Proof of Proposition 6.** From the asymptotic analysis in §4, it follows that at optimality $\rho \leq 1$ (or, equivalently, $Q_0 \leq \lambda T$). We will assume that this optimality condition is satisfied in the remainder of this proof.

Let us first show that at optimality $1 - \tilde{P}^* \leq \rho^* \leq (1 - \tilde{P}^*) \exp(wT)$.

Suppose that $\rho \leq 1 - \tilde{P}$. This corresponds to the case of limited supply, and in equilibrium all units are sold through the fixed-price channel, that is, $Q_{H^*}(\rho, T, \tilde{P}) = 0$. The seller’s revenue is given by

$$\lambda(1 - \tilde{P})\tilde{P}\left(\frac{1 - \exp(-\alpha\rho T/(1 - \tilde{P}))}{\alpha}\right).$$

It is not hard to see that in the region $\{(\rho, \tilde{P}): \rho \leq 1 - \tilde{P} \text{ and } \rho \leq \bar{Q}/\lambda T\}$, this revenue function is maximized at $\rho_1 = \min\{1/2; \bar{Q}/\lambda T\}$, $\tilde{P}_1 = 1 - \rho_1$, and it is equal to

$$V_1 = \lambda\rho_1(1 - \rho_1)\left(\frac{1 - \exp(-\alpha T)}{\alpha}\right).$$

This proves that $V_1$ is a lower bound on the optimal seller’s revenue and that at optimality $1 - \tilde{P}^* \leq \rho^*$. 
Suppose now that $\rho \geq (1 - \hat{P}) \exp(wT)$. In this case, all units are sold in the auction in equilibrium. The seller’s revenue is given by

$$\lambda T p(1 - \rho) \exp(-\alpha T).$$

In the region $\{(\rho, \hat{P}): \rho \geq (1 - \hat{P}) \exp(wT) \text{ and } \rho \leq \bar{Q}/\lambda T, \}$, this revenue function is maximized at $\rho = \rho_1$, and any $\hat{P} \geq 1 - \exp(-wT)\rho_2$ and equals

$$V_1 = \lambda \rho_1 (1 - \rho_1) T \exp(-\alpha T).$$

Because $(1 - \exp(-\alpha T))/\alpha \geq T \exp(-\alpha T)$ for all $T \geq 0$ and $\alpha \geq 0$, we conclude that $V_1 \geq V_T$. Therefore, it is never optimal to choose a solution $(\rho^*, \hat{P}^*)$ in the region $\rho \geq (1 - \hat{P}) \exp(wT)$. We conclude that at optimality $1 - \hat{P}^* \leq \rho^* \leq (1 - \hat{P}^*) \exp(wT)$ and that the seller’s optimal revenue $V_1$ is bounded below by $V_1$.

Let us now derive the upper bound for $V_D$. For any $\rho$ and $\hat{P}$, the seller’s revenue can be bounded above by assuming that every buyer with valuation greater than $\hat{P}$ buys the product in the fixed-price channel, and the remaining $\lambda T [(\rho - (1 - \hat{P})]$ units are sold in the auction at a price $P_H = 1 - \rho$. Note that the optimality condition $1 - \hat{P}^* \leq \rho^*$ guarantees the nonnegativity of the number of units sold in the auction. The corresponding upper bound on the seller’s revenue is given by

$$\bar{V}(\hat{P}, \rho) = \lambda \hat{P} (1 - \hat{P}) \left(1 - \exp(-\alpha T)\right) + \lambda (1 - \rho)(\hat{P} - (1 - \rho)) T \exp(-\alpha T).$$

In order to maximize $\bar{V}(\hat{P}, \rho)$, let us first fix $\rho$ and optimize over $\hat{P}$ under the constraint $\hat{P} \geq 1 - \rho$. It follows that the optimal $\hat{P}(\rho)$, as a function of $\rho$, satisfies

$$\hat{P}(\rho) = \begin{cases} 1 - \rho & \text{if } \rho \leq \frac{1 - \delta}{2 - \delta} \\ \frac{1}{2} (1 + (1 - \rho)\delta) & \text{if } \rho \geq \frac{1 - \delta}{2 - \delta}. \end{cases}$$

Recall that $\delta = \alpha T \exp(-\alpha T)/(1 - \exp(-\alpha T))$. From this solution, $\hat{P}(\rho)$, we recover the two cases in the proposition.

Case 1. Suppose $\bar{Q}/\lambda T \leq (1 - \delta)/(2 - \delta)$; then any feasible $\rho$ also satisfies $\rho \leq (1 - \delta)/(2 - \delta)$ and the upper bound $\bar{V}$ equals

$$\bar{V}(\hat{P}(\rho), \rho) = \lambda \rho (1 - \rho) \left(1 - \exp(-\alpha T)\right).$$

Because $(1 - \delta)/(2 - \delta) \leq 1/2$, it follows that $\bar{V}$ is maximized at $\rho = \bar{Q}/\lambda T$ and $\bar{V} = V_1$ because $\rho_1 = \bar{Q}/\lambda T$ in this case. Because the upper and lower bounds are equal, we conclude that $V_D = V_1$.

Case 2. Suppose $\bar{Q}/\lambda T \geq (1 - \delta)/(2 - \delta)$ and the remaining $\lambda T [\rho - (1 - \hat{P})]$ units are sold in the auction at a price $P_H = 1 - \rho$. On the other hand, as $w \to \infty$, all buyers with valuation greater than $\hat{P}$ will buy the product on the fixed-price channel, and so $V_D = V_2$. Finally, let us derive the upper bound on $T^*$. Suppose the seller chooses a time $T$ so that the condition

$$\bar{Q}/\lambda T \leq \frac{1 - \delta}{2 - \delta}$$

is satisfied. Then, because the left-hand side decreases with $T$ and the right-hand side increases with $T$, the inequality will also hold for any $T' \geq T$. 


According to Case 1 in Proposition 6, this inequality implies that the seller’s revenue is given by

\[ V_D = \frac{\lambda T \exp(-a T) Q}{\delta} \left( 1 - \frac{\bar{Q}}{\lambda T} \right) = \frac{\bar{Q}(1 - \exp(-a T))(\lambda T - \bar{Q})}{\alpha \lambda T^2}. \]

Taking a first derivative with respect to \( T \), and after some manipulations, we get

\[ \frac{d}{dT} V_D = \frac{\bar{Q}}{\alpha T^2} \left[ \frac{\bar{Q}}{\lambda T} (2 \exp(-2a T) - a T \exp(-a T)) - (1 - \exp(-a T) - a T \exp(-a T)) \right]. \]

It is not hard to show that under the assumption \( \bar{Q}/\lambda T \leq (1 - \delta)/(2 - \delta) \), the derivative above is nonpositive. Hence, the seller will never choose \( T \) so that \( \bar{Q}/\lambda T \leq (1 - \delta)/(2 - \delta) \) is satisfied. This observation, together with the discussion in the first paragraph of this proof, implies that an upper bound for \( T \) is given by the unique nonnegative solution of

\[ \frac{\bar{Q}}{\lambda T} = \frac{1 - \delta}{2 - \delta} \quad \text{or, equivalently,} \quad \frac{\bar{Q}}{\lambda T} = \frac{1 - \exp(-a T) - a T \exp(-a T)}{2 - 2 \exp(-a T) - a T \exp(-a T)}. \]

The uniqueness follows because of the (opposed) monotonicity in \( T \) of the left- and right-hand sides of the equality. \( \square \)

**Appendix B. Dual Channel with Static List Price**

This appendix is an extended version of §4 in the main paper. It contains a detailed mathematical description of the dual-channel model, including some numerical examples that complement the discussion in the main paper.

Recall that in this dual channel, a monopolistic seller has \( Q_0 \) units to sell through two different channels: a list price channel, in which she sets a constant list price \( \hat{P} \) that will be kept during the whole horizon of length \( T \); and the auction that will take place at the end, with the remaining \( Q_T \) units.

We also recall that (because of the finite number of units in the market) the optimal bidding strategy for bidders is \( b(v) = v \) (see §4 in the main paper for more details).

**EC.B1. Characterization of a Symmetric Participation Equilibrium \( H(v) \)**

Suppose \( \hat{P} \) has been selected. We restrict buyers’ participation strategies to the set

\[ \mathcal{H} = \{ H \in \mathcal{D} : [0, 1] \to [0, T] \text{ such that } H(v) = 0, \text{ for all } v \in [0, \hat{P}] \} , \]

and we characterize the decision of a buyer by means of a threshold function \( H \in \mathcal{H} \), such that a buyer arriving at time \( t \) with valuation \( v \), will join the auction only if \( H(v) \leq t \). Because we have assumed that arriving buyers do not get any information regarding number of units sold or outstanding bids, the participation strategy \( H(v) \) does not depend explicitly on these quantities. That is, at any time \( t \in [0, T] \), buyer \( t \) knows only the initial quantity \( Q_0 \).

We analyze the buyer’s problem at his arriving time in order to compute \( \mathcal{B}(H) \), the best-response participation strategy given that all other buyers use the strategy \( H \). Define the random variable \( P_H(v) \) as follows:

\[ P_H(v) \] is the (random) auction price given that (i) there is a \( v \)-buyer that has joined the auction, (ii) all other buyers use the participation strategy \( H \), and (iii) the (random) number \( Q_T \) of items left for the auction is equal to the difference between the initial value \( Q_0 \) and the number of items purchased through the fixed-price channel during \([0, T]\). We also use the convention \( P_H(v) = 1 \) if \( Q_T = 0 \).

As we will see shortly, an SPE in this dual-channel case depends on the probability distribution of \( P_H(v) \) and the value of \( P(Q > 0 | H) \), the probability that at time \( \tau \) there are still unsold units given that buyers use the participation strategy \( H \). In order to compute these quantities, we follow an approach similar to the one developed for the single auction channel: We fix \( H \), and in addition we define

\[ \Lambda_{H^+}(\tau) \defeq \lambda T \eta_{H^+}(\tau), \quad \text{for } \eta_{H^+}(\tau) \defeq \int_0^\tau \min\left\{ \frac{\tau}{T}, \frac{H(v)}{T} \right\} dF(v), \quad \text{and} \]

\[ \Lambda_{H^-}(x) \defeq \lambda T \eta_{H^-}(x), \quad \text{for } \eta_{H^-}(x) \defeq \int_x^1 \frac{H(v)}{T} dF(v) = \bar{F}(x) - \eta_{H^+}(x), \]
we have that
\[ \mathbb{P}(Q_\tau > 0 \mid H) = \mathbb{P}(B(\Lambda_{H+}(\tau)) \leq Q_0 - 1) = \sum_{k=0}^{Q_0-1} \frac{(\Lambda_{H+}(\tau))^k \exp(-\Lambda_{H+}(\tau))}{k!}. \]

On the other hand, \( \Lambda_{H-}(x) \) represents the average number of fixed-price buyers (i.e., those below the threshold \( H \)) with valuation greater than or equal to \( x \), and \( \eta_{H-}(x) \) is the fraction of arrivals with valuation greater than or equal to \( x \) who go for the fixed-price channel.

The random variable \( B(\Lambda_{H-}(x)) \) represents the number of fixed-price buyers with valuation greater than or equal to \( x \). Again, given that customers arrive according to a Poisson process with rate \( \lambda \), we have that \( B(\Lambda_{H-}(x)) \) has a Poisson distribution with mean \( \Lambda_{H-}(x) \). One important member of this family of random variables is \( B(\Lambda_{H-}(0)) \), which represents the total number of buyers that have selected the fixed-price channel. Therefore, we can define the number of units left for the auction as \( Q_T = (Q_0 - B(\Lambda_{H-}(0)))^+ \). From this observation and condition (5), we obtain

\[
\mathbb{P}(P_H(v) < x) = \sum_{k=1}^{Q_0} \mathbb{P}(B(\Lambda_H(x)) + 1(x \leq v) \leq k) \mathbb{P}(Q_T = k)
= \sum_{k=1}^{Q_0} \sum_{n=0}^{Q_0-k} \frac{(\Lambda_H(x))^n \exp(-\Lambda_H(x)) (\Lambda_{H-}(0))^{Q_0-k} \exp(-\Lambda_{H-}(0))}{n!} (Q_0 - k)!
= \left[ \sum_{k=0}^{Q_0-1} \frac{(\Lambda_H(x) + \Lambda_{H-}(0))^k}{k!} + \mathbb{I}(x > v) \frac{(\Lambda_H(x) + \Lambda_{H-}(0))^{Q_0} - (\Lambda_{H-}(0))^{Q_0}}{Q_0!} \right],
\]

\[
\cdot \exp(-\Lambda_H(x) + \Lambda_{H-}(0)).
\]

We note that for \( x \leq v \), the distribution of \( P_H(v) \) reduces to

\[
\mathbb{P}(P_H(v) < x) = \sum_{k=0}^{Q_0-1} \frac{(\Lambda_H(x) + \Lambda_{H-}(0))^k}{k!} \exp(-\Lambda_H(x) + \Lambda_{H-}(0))
= \mathbb{P}(B(\Lambda_H(x) + \Lambda_{H-}(0)) \leq Q_0 - 1). \quad (EC.B2)
\]

To get some intuition about this condition (EC.B2), note that \( \Lambda_H(x) + \Lambda_{H-}(0) \) represents the average number of buyers that either enter the auction bidding more than \( x \) (first summand) or buy the object directly from the fixed-price channel (second summand).

We are now ready to characterize the best-response mapping \( \mathcal{R} \) in this dual-channel case. Consider buyer \( \tau \) arriving at time \( \tau \) with valuation \( v_\tau \). If \( v_\tau \leq \tilde{P} \), then the auction is his only profitable channel, and so he enters the auction independently of \( \tau \). On the other hand, if \( v_\tau > \tilde{P} \), then both channels are potentially profitable. If he decides to buy a unit through the fixed-price channel, his expected utility is zero if \( Q_\zeta = 0 \) (that is, there are no units left) or equals \( u(\tau, \tau, v_\tau - \tilde{P}) \) if \( Q_\zeta > 0 \). Thus, the expected utility if he selects the fixed-price channel is given by \( (v_\tau - \tilde{P}) \mathbb{P}(Q_\zeta > 0 \mid H) \). On the other hand, if buyer \( \tau \) decides to bid and gets one object, then his utility is \( u(\tau, T, v_\tau - P_H(v_\tau)) \), and zero otherwise.\(^{EC3}\) Therefore, buyer \( \tau \) enters the auction if his expected utility from bidding exceeds his expected utility from the fixed-price channel. From the exponentially discounted utility function (1) that we consider, this participation condition is equivalent to

\[
\exp(-w(T - \tau))(v_\tau - E[P_H(v_\tau) \mid P_H(v_\tau) < v_\tau]) \Pr(P_H(v_\tau) < v_\tau) \geq (v_\tau - \tilde{P}) \mathbb{P}(Q_\zeta > 0 \mid H),
\]

which we can rewrite for the case \( v_\tau > \tilde{P} \) in the more convenient form (see Equation (EC.A1)):

\[
\frac{1}{v_\tau - \tilde{P}} \int_0^{v_\tau} \mathbb{P}(P_H(v_\tau) < x) \, dx \geq \exp(w(T - \tau)) \mathbb{P}(Q_\zeta > 0 \mid H). \quad (EC.B3)
\]

\(^{EC3}\) Note that in this dual-channel setting, the seller clears \( \min\{Q_\zeta, B(AT)\} \) units through both channels by time \( T \).
Condition (EC. B1) implies that for every $H \in \mathcal{H}$ the function $\mathbb{P}(Q_\tau > 0 \mid H)$ is continuous and non-increasing in $\tau \in [0, T]$. Therefore, the function $\mathcal{F}(H)(\tau) \equiv \exp(w(T-\tau))\mathbb{P}(Q_\tau > 0 \mid H)$ is monotonically decreasing in $\tau$ and admits a continuous decreasing inverse function $\mathcal{F}(H)^{-1}$ in the domain $[\mathbb{P}(Q_T > 0 \mid H), \exp(wT)]$. We find it convenient to (continuously) extend this domain of $\mathcal{F}(H)^{-1}$ to the entire $\mathbb{R}^+$ as follows:

$$\mathcal{F}(H)^{-1}(x) = T, \quad x \in [0, \mathbb{P}(Q_T > 0 \mid H)] \quad \text{and} \quad \mathcal{F}(H)^{-1}(x) = 0, \quad x \geq \exp(wT).$$

Although a closed-form expression for $\mathcal{F}(H)^{-1}$ is not available, its existence is all that we need to establish the following result.

**Proposition B1.** In the dual-channel case, for any strategy $H \in \mathcal{H}$, the corresponding best-response participation strategy $\mathcal{R}(H) \in \mathcal{H}$ is continuous and satisfies

$$\mathcal{R}(H)(v) = \begin{cases} 0 & \text{if } v \in [0, \hat{P}] \\ \mathcal{F}(H)^{-1}\left(\int_0^{v} \mathbb{P}(P_H(v_t) < x) \, dx \right) & \text{if } v \in (\hat{P}, 1]. \end{cases}$$

The proof of the proposition is omitted because it follows directly from the participation condition (EC. B3) and the extended definition of $\mathcal{F}(H)^{-1}$ above. Only the continuity of $\mathcal{R}(H)(v)$ at $v = \hat{P}$, as it is required by the condition $\mathcal{R}(H) \in \mathcal{H}$, deserves some attention. For this, note that for all $H \in \mathcal{H}$ we have that

$$\lim_{v \to \hat{P}} \int_0^{v} \mathbb{P}(P_H(v_t) < x) \, dx \, \frac{1}{v - \hat{P}} \to +\infty.$$ 

Continuity at $\hat{P}$ now follows from the fact that $\mathcal{F}(H)^{-1}(x) = 0$ for all $x \geq \exp(wT)$. Using a similar argument, we also note that for every $H \in \mathcal{H}$ there is a $v_H > \hat{P}$ such that $\mathcal{R}(H)(v) = 0$ for all $v \in [\hat{P}, v_H]$. As in Equation (EC. A3), we define

$$\bar{v} \triangleq \inf_{H \in \mathcal{H}} \{v_H\}.$$ 

We can get a lower bound on $\bar{v}$ from the fact that

$$\int_0^{v} \mathbb{P}(P_H(v_t) < x) \, dx \, \frac{1}{v - \hat{P}} \geq \int_0^{v} \mathbb{P}(B(\lambda T(1 - F(\hat{P}))) \leq Q_0 - 1) \, dx \, \frac{v \mathbb{P}(B(\lambda T(1 - F(\hat{P}))) \leq Q_0 - 1)}{v - \hat{P}}.$$ 

The lower bound $\bar{v}$ is obtained by solving

$$\frac{v \mathbb{P}(B(\lambda T(1 - F(\hat{P}))) \leq Q_0 - 1)}{v - \hat{P}} = \exp(wT),$$

that is,

$$\bar{v} = \frac{\exp(wT)\hat{P}}{\exp(wT) - \mathbb{P}(B(\lambda T(1 - F(\hat{P}))) \leq Q_0 - 1)}.$$

As in the single-channel case, the existence of $\bar{v} > \hat{P}$ guarantees that the best-response strategy $\mathcal{R}(H)$ is $K$-Lipschitz continuous for an appropriate constant $K$. Therefore, we can redefine the space of strategies to be

$$\mathcal{H} \triangleq \{H: [0, 1] \to [0, T] \text{ s.t. } H \text{ is } K\text{-Lipschitz continuous and } H(v) = 0 \text{ in } v \in [0, \bar{v}]\}.$$ 

The following result formalizes this claim and proves the existence of a symmetric participation equilibrium (SPE) for this dual-channel case.

**Theorem B1.** For the exponential utility function (1) and for all $H \in \mathcal{H}$, there is a positive constant $K$ (independent of $H$) such that the best-response strategy $\mathcal{R}(H)(v)$ is a $K$-Lipschitz continuous function that satisfies $\mathcal{R}(H)(v) = 0$ for all $v \in [0, \bar{v}]$. In addition, the best-response mapping $\mathcal{R}$ is continuous in $\mathcal{H}$ equipped with the uniform norm, and so a symmetric equilibrium always exists in $\mathcal{H}$.

The proof of this theorem can be found at the end of this appendix.

Again, as in the end of §3.2, we point out the existence of $v_H$ and its implication: There is always a range of buyer valuations above the list price $\hat{P}$, such that those buyers will join the auction regardless of their arrival time.
EC.B2. Asymptotic Analysis

In this section, we analyze the limiting regime for the dual-channel setting when both the initial number of units $Q_0$ and the arrival rate $\lambda$ grow proportionally large (see Equation (10) in the main paper). The auction price—for instance, $n$ given that a bidder with valuation $v$ enters the auction—is $P_n^H(v)$, and the final random number of units to auction is $Q_n^\tau$. The next theorem characterizes the asymptotic regime:

**Theorem B2.** Suppose that the participation strategy $H(v)$ and the static price $\hat{P}$ are given. Then, in the limit as $n \to \infty$:

(i) The rescaled number of units $Q_n^\tau/n$ to sell through the auction converges weakly to a constant $Q_T \triangleq (Q_0 - \lambda T \eta_{H*.}(0))^\ast$.

(ii) If a final auction takes place (i.e., $Q_T > 0$), its price $P_n^H(v)$ converges weakly to a constant $P_H^\infty \triangleq \min\{v \in [0, 1]; \eta_H(v) \leq \rho - \eta_{H*.}(0)\}$, where $\rho = Q_0/\lambda T$.

The proof of the theorem can be found at the end of this appendix.

Following the argument in §3.3, in order to determine the value of the auction price $P_H$ and the corresponding participation strategy $H(v)$, we have to impose the equilibrium condition $\hat{\beta}(H^*).$ Because $\hat{\beta}(H)(v) = 0$ for all $v < \hat{P}$, we must have $H^*(v) = 0$ in $v \in [0, \hat{P})$. In other words, buyers with valuation smaller than the posted price $\hat{P}$ have no other choice but entering the auction. To describe the behavior of $H^*(v)$ in $v \in [\hat{P}, 1]$, we need to distinguish two cases:

**Case 1.** Suppose that the initial supply of units is limited in the sense that $\rho \leq 1 - F(\hat{P})$. In this situation, buyers with valuation greater than $\hat{P}$ have no incentive to enter the auction because the auction price is guaranteed to be greater than or equal to $\hat{P}$. Therefore, the resulting participation strategy is $H^*(v) = T$ for all $v \geq \hat{P}$, and the auction never takes place because all the units will be bought at the posted price (i.e., $Q_T = 0$).

We note that $H^*(v) = T \mathbb{1}(v \geq \hat{P})$ is not the only SPE in this case. In fact, let us define $\tau^* \triangleq T \hat{P}(1 - F(\hat{P}))^{-1}$. Then, any $H$ of the form $H(v) = (\tau^* + h(v)) \mathbb{1}(v \geq \hat{P})$ for an arbitrary nonnegative and bounded function $h(v) \leq T - \tau^*$ is an SPE. In fact, for such an $H$ the initial $Q_0$ units will be depleted by time $\tau^*$ (i.e., $Q_{\tau^*} = 0$, because $\tau^* \lambda (1 - F(\hat{P})) = Q_0$). Therefore, any buyer arriving after $\tau^*$ will never get a unit, and so he becomes indifferent between the two channels.

**Case 2.** Suppose that initial supply is abundant in the sense that $\rho > 1 - F(\hat{P})$. In this case, $Q_T > 0$, and some buyers with valuation smaller than $\hat{P}$ get units through the auction. It is not hard to see that in this case the auction price is given by $P_{H^*} = F^{-1}(1 - \rho)$. Therefore, buyer $\tau$ arriving at time $\tau$ with valuation $v_\tau \geq \hat{P}$ enters the auction only if $v_\tau - \hat{P} \leq \exp(-w(T - \tau))(v_\tau - P_{H^*})$. We conclude that in this abundant case the unique SPE $H^*(v)$ is given by:

$$H^*(v) = \begin{cases} 0 & \text{if } v \in [0, v_{H^*}] \\ T - \frac{1}{\lambda} \ln \left( \frac{v - P_{H^*}}{v - \hat{P}} \right) & \text{if } v \in [v_{H^*}, 1] \end{cases}$$

where $v_{H^*} = \min \left\{ \hat{P} \frac{\exp(wT) - P_{H^*}}{\exp(wT) - 1}, 1 \right\}$.

Figure EC.B1 compares the optimal asymptotic strategy $H^*(\infty, \rho)$ to four optimal participation strategies $H^*(Q_0, \rho)$ ($Q_0 = 1, 5, 10, 20$) for the case of $\rho = 0.5$. As in the single-channel case, the asymptotic strategy is almost identical to the optimal strategy for values of $Q_0$ greater than 10 units. The table on the right compares the expected value of the auction price $P_{H^*}$ and the approximation $P_{H^*}^{\text{approx}}$ obtained using the asymptotic participation strategy $H^*(\infty, 0.5)$. Similarly to the single auction channel, the asymptotic approximation is very accurate even for small values of $Q_0$.

We conclude this appendix by specializing the asymptotic results to the case of uniformly distributed valuations.

**Examples**

**Uniform Distribution Case**

Suppose buyers’ valuations are uniformly distributed in $[0, 1]$. Under this assumption, we will characterize the auction price $P_{H^*}$ as well as the number of units sold in the auction $Q_{H^*}$, the cumulative
number of units sold in the fixed-price channel during \([0, T]\) \(Q_{F*}\), and the corresponding rate \(\lambda_{F*}(t)\) at which these units are sold. In the asymptotic regime under consideration we have that

\[
Q_{F*} = \int_0^T \lambda_{F*}(t) \, dt.
\]

Depending on the values of \(\rho\) and \(\hat{\rho}\), we distinguish two cases.

**Limited-Supply Case.** \(\rho \leq 1 - \hat{\rho}\)

In this case, all units are depleted through the fixed-price channel, that is, \(Q_{F*} = Q_0\) and \(Q_{H*} = 0\). In addition, the fixed-price channel demand rate satisfies

\[
\lambda_{F*}(t) = \lambda(1 - \hat{\rho}) \mathbb{1}(t \leq \tau^*), \quad t \in [0, T],
\]

where \(\tau^* = Q_0 / (\lambda(1 - \hat{\rho})) \leq T\) is the time at which all \(Q_0\) units are sold.

**Abundant-Supply Case.** \(\rho > 1 - \hat{\rho}\)

In this case, there is a positive number of units that are sold through the auction at a price \(P_{H*} = (1 - \rho)^+\). The number of units auctioned and sold through the fixed-price channel are

\[
Q_{H*} = \lambda T \eta_{H*}(P_{H*}) \quad \text{and} \quad Q_{F*} = \lambda T (1 - \eta_{H*}(P_{H*})),
\]

respectively, where

\[
\eta_{H*}(P_{H*}) = \int_{P_{H*}}^1 \left(1 - \frac{1 - \frac{1}{wT} \ln \left(\frac{v - (1 - \rho)^+}{v - \hat{\rho}}\right)^+}{\exp(wT - t) - 1}\right) \, dv.
\]

In this case, the fixed-price channel demand rate satisfies

\[
\lambda_{F*}(t) = \lambda H^{\tau^*}(t)
\]

\[
= \lambda \left[1 - \frac{\hat{\rho} \exp(w(T - t)) - (1 - \rho)^+}{\exp(w(T - t)) - 1}\right]^+.
\]

**Proofs**

Proof of Theorem B1. We first prove \(K\)-Lipschitz continuity of \(\mathcal{B}(H)(v)\) in \((\bar{v}, 1]\). Observe that due to the shape of the function \(\mathcal{F}_H^{-1}(x)\) (flat at \(T\) in the range \([0, \mathbb{P}(Q_T > 0 | H)]\), and flat at zero
for $x \geq \exp(wT)$, it is enough to prove this property in the range $(\mathbb{P}(Q_T > 0 \mid H), \exp(wT))$, where $\mathcal{R}(H)(v)$ is differentiable. Let us define

$$Z_H(v) \triangleq \frac{1}{v - \bar{P}} \int_0^v \mathbb{P}(P_H(v) < x) \, dx.$$ 

Now, for any pair $v_1, v_2 \in (\bar{v}, 1)$ we have that

$$|\mathcal{R}(H)(v_1) - \mathcal{R}(H)(v_2)| = |\mathcal{T}_H^{-1}(Z_H(v_1)) - \mathcal{T}_H^{-1}(Z_H(v_2))|$$

$$= \left| \int_{Z_H(v_1)}^{Z_H(v_2)} \frac{d}{dx} \mathcal{T}_H^{-1}(x) \, dx \right| = \left| \int_{Z_H(v_1)}^{Z_H(v_2)} \left( \frac{d}{d\tau} \mathcal{T}_H(\tau) \right)^{-1} \right|_{\tau = Z_H(v)} \, d\tau.$$

Note that the differentiability of $\mathcal{T}_H^{-1}(x)$ follows from the fact that the function $\mathcal{T}_H(\tau)$ is differentiable. In fact, from Lemma A1 we have that

$$\frac{d}{d\tau} \mathcal{T}_H(\tau) = -\exp(w(T - \tau)) \left[ w\mathbb{P}(Q_T > 0 \mid H) + \mathbb{P}(B(\Lambda_H(\tau)) = Q_0 - 1) \frac{d}{d\tau} \Lambda_H(\tau) \right],$$

where

$$\frac{d}{d\tau} \Lambda_H(\tau) = \lambda \frac{d}{d\tau} \int_0^1 \min(\tau, H(v)) \, dF(v) = \lambda \int_0^1 I(\tau \leq H(v)) \, dF(v).$$

Using the fact that $\mathbb{P}(Q_T > 0 \mid H) \geq \mathbb{P}(Q_T > 0 \mid H) \geq \mathbb{P}(B(\lambda T) \leq Q_0 - 1)$, that $0 \leq (d/d\tau) \Lambda_H(\tau) \leq \lambda$, and that $1 \leq \exp(w(T - \tau)) \leq \exp(wT)$, we get that

$$w\mathbb{P}(B(\lambda T) \leq Q_0 - 1) \leq \left| \frac{d}{d\tau} \mathcal{T}_H(\tau) \right| \leq \exp(wT)[w + \lambda],$$

and so

$$|\mathcal{R}(H)(v_1) - \mathcal{R}(H)(v_2)| = |\mathcal{T}_H^{-1}(Z_H(v_1)) - \mathcal{T}_H^{-1}(Z_H(v_2))|$$

$$\leq \left| w\mathbb{P}(B(\lambda T) \leq Q_0 - 1) \right| |Z_H(v_1) - Z_H(v_2)|. \quad (EC.B5)$$

K-Lipschitz continuity follows now, combining this inequality and the following:

$$|Z_H(v_1) - Z_H(v_2)| = \left| \int_{v_1}^{v_2} \frac{d}{dv} Z_H(v) \, dv \right| = \left| \int_{v_1}^{v_2} \left[ \frac{\mathbb{P}(P_H(v) < x)}{v - \bar{P}} - \int_0^v \frac{\mathbb{P}(P_H(v) < x)}{(v - \bar{P})^2} \, dx \right] \, dv \right|$$

$$\leq \left| \int_{v_1}^{v_2} \left[ \frac{1}{\bar{v} - \bar{P}} + \frac{1}{(\bar{v} - \bar{P})^2} \right] \, dv \right| = \left| \frac{1 + \bar{v} - \bar{P}}{(\bar{v} - \bar{P})^2} \right| |v_1 - v_2|.$$ 

The constant $K$ equals $(w\mathbb{P}(B(\lambda T) \leq Q_0 - 1))^{-1}((1 + \bar{v} - \bar{P})/(\bar{v} - \bar{P})^2)$, and it is well defined because $\bar{v} \geq \bar{P}$.

To prove the continuity in $\mathcal{H}$ of the mapping $\mathcal{R}$, we first note that the mapping $\mathcal{T}(H)$ is continuous in $\mathcal{H}$. In fact,

$$|\mathcal{T}(H)(\tau) - \mathcal{T}(\bar{H})(\tau)| = \exp(w(T - \tau))|\mathbb{P}(Q_T > 0 \mid H) - \mathbb{P}(Q_T > 0 \mid \bar{H})|$$

$$= \exp(w(T - \tau))|\mathbb{P}(B(\Lambda_H(\tau)) \leq Q_0 - 1) - \mathbb{P}(B(\Lambda_{\bar{H}}(\tau)) \leq Q_0 - 1)|$$

$$\leq \exp(w(T - \tau))|\mathbb{P}(B(Q_0 - 1) = Q_0 - 1)| \Lambda_H(\tau) - \Lambda_{\bar{H}}(\tau)|$$

$$\leq \lambda \exp(w(T - \tau))|\mathbb{P}(B(Q_0 - 1) = Q_0 - 1)| \|H - \bar{H}\| \triangleq K_{\lambda} \|H - \bar{H}\|,$$

where the first inequality follows from Lemma A3, and the second one follows from the definition of $\Lambda_H(\tau)$ and the property $|\min(\tau, a) - \min(\tau, b)| \leq |a - b|$. The continuity of the mapping $\mathcal{R}(H) = \mathcal{T}(H)^{-1}$ follows now from

$$|\mathcal{R}(H)(v) - \mathcal{R}(\bar{H})(v)| = |\mathcal{T}(H)^{-1}(Z_H(v)) - \mathcal{T}(\bar{H})^{-1}(Z_{\bar{H}}(v))|$$

$$\leq |\mathcal{T}(H)^{-1}(Z_H(v)) - \mathcal{T}(\bar{H})^{-1}(Z_H(v))| + |\mathcal{T}(\bar{H})^{-1}(Z_H(v)) - \mathcal{T}(\bar{H})^{-1}(Z_{\bar{H}}(v))|. \quad (EC.B6)$$
Regarding the first term in (EC.B6), from condition (EC.B5), we have that
\[ |\mathcal{F}(H)^{-1}(Z_H(v)) - \mathcal{F}(H)^{-1}(Z_H(v))| \leq (w\mathbb{P}(B(\lambda T) \leq Q_0 - 1))^{-1}|Z_H(v) - Z_H(v)|. \]
As in the proof of Theorem 1, we can prove that
\[ |Z_H(v) - Z_H(v)| \leq K_Z ||H - \tilde{H}|| \]
for an appropriate constant $K_Z$.

Now we focus on the second term in (EC.B6). Without loss of generality, suppose $\mathcal{F}(H)^{-1}(Z_H(v)) \leq \mathcal{F}(H)^{-1}(Z_H(v))$. Using the continuity of $\mathcal{F}$ in $\mathcal{R}$ that we just proved, it follows that
\[ \mathcal{F}(H)(\mathcal{F}(H)^{-1}(Z_H(v))) \leq Z_H(v) + K_{\gamma} ||H - \tilde{H}||. \]
Applying $\mathcal{F}(H)^{-1}$ in both sides, and given that $\mathcal{F}(H)^{-1}(v)$ is nonincreasing in $v$, we have that
\[ \mathcal{F}(H)^{-1}(Z_H(v)) = \mathcal{F}(H)^{-1}(\mathcal{F}(H)(\mathcal{F}(H)^{-1}(Z_H(v)))) \geq \mathcal{F}(H)^{-1}(Z_H(v) + K_{\gamma} ||H - \tilde{H}||). \]
From the assumption $\mathcal{F}(H)^{-1}(Z_H(v)) \leq \mathcal{F}(H)^{-1}(Z_H(v))$ we get
\[ |\mathcal{F}(H)^{-1}(Z_H(v)) - \mathcal{F}(H)^{-1}(Z_H(v))| \leq |\mathcal{F}(H)^{-1}(Z_H(v) + K_{\gamma} ||H - \tilde{H}||) - \mathcal{F}(H)^{-1}(Z_H(v))| \]
\[ \leq (w\mathbb{P}(B(\lambda T) \leq Q_0 - 1))^{-1}K_{\gamma} ||H - \tilde{H}||, \]
where the second inequality follows from condition (EC.B5) above.

Therefore, using the bounds for the two absolute terms in (EC.B6), we conclude that
\[ |\mathcal{R}(H)(v) - \mathcal{R}(H)(v)| \leq (w\mathbb{P}(B(\lambda T) \leq Q_0 - 1))^{-1}(K_Z + K_{\gamma} ||H - \tilde{H}||), \]
which proves the continuity of $\mathcal{R}$ in $\mathcal{R}$.

Finally, as in Theorem 1, the existence of a symmetric equilibrium follows again from the Schauder-Tychonoff Fixed-Point Theorem. \(\square\)

**Proof of Theorem B2.** Recall that $B(\Lambda_{H^-}(x))$ is a Poisson r.v. with mean $\Lambda_{H^-}(x)$. To prove (i), we start by rewriting $\Lambda_{H^-}(x)$ as
\[ \Lambda_{H^-}(x) = \frac{\eta_{H^-}(x)}{\rho^n}Q^n_0. \]
Let $\{B_i(\eta_{H^-}(x)(\rho^n)^{-1}) : i = 1, \ldots, Q^n_0\}$ be a sequence of i.i.d Poisson random variables with mean $\eta_{H^-}(x)(\rho^n)^{-1}$. The random variable $B(\Lambda_{H^-}(x))$ has the same distribution as the sum of the $B_i(\eta_{H^-}(x)(\rho^n)^{-1})$, $1 \leq i \leq Q^n_0$. For a fixed $0 \leq \alpha \leq 1$,
\[ \mathbb{P}(Q^n_T \geq \alpha Q^n_0) = \mathbb{P}(B(\Lambda_{H^-}(0)) \leq Q^n_0(1 - \alpha)) \]
\[ = \mathbb{P}\left( \sum_{i=1}^{Q^n_0} B_i(\eta_{H^-}(0)(\rho^n)^{-1}) \leq Q^n_0(1 - \alpha) \right) \]
\[ = \mathbb{P}\left( \frac{\sum_{i=1}^{Q^n_0} B_i(\eta_{H^-}(0)(\rho^n)^{-1})}{Q^n_0} \leq 1 - \alpha \right), \]
where the first equality follows from the fact that all the units put into the auction are the remaining ones from the list price channel. Let
\[ \mathcal{B}^n(0) \triangleq \sum_{i=1}^{Q^n_0} B_i(\eta_{H^-}(0)(\rho^n)^{-1}) \frac{Q^n_0}{Q^n_0}. \]
From Lemma A4, $\mathcal{B}^n(0)$ converges in distribution to the constant $\eta_{H^-}(0)\rho^{-1}$. Given that for $n$ sufficiently large $Q^n_0 = nQ_0 + o(n)$, then by focusing on the continuity points, the tail distribution of $Q^n_T/n$ converges weakly to the tail distribution:
\[ \bar{F}_{Q_T}(\alpha Q_0) = \begin{cases} 1 & \text{if } \eta_{H^-}(0) < \rho(1 - \alpha) \\ 0 & \text{if } \eta_{H^-}(0) > \rho(1 - \alpha). \end{cases} \]
In other words, the first case corresponds to \( \alpha Q_0 < Q_0 - \lambda T \eta_{H-}(0) \), that is, there are more units available in the auction channel than the requested \( \alpha Q_0 \); the second case is the opposite. This is the distribution of the constant \((Q_0 - \lambda T \eta_{H-}(0))^+\) at its continuity points, and so \( Q^n_T/n \Rightarrow Q_T \).

For part (ii), we have:

\[
\mathbb{P}(P_H^n(v) < x) = \mathbb{P}(B(\Lambda_H^n(x)) \leq Q^n_T - 1(x \leq v)) \\
= \mathbb{P}\left(\sum_{i=1}^{Q^n_T} B_i(\eta_H(x)(\rho^n)^{-1}) \leq Q^n_T - 1(x \leq v)\right) \\
= \mathbb{P}\left(\frac{\sum_{i=1}^{Q^n_T} B_i(\eta_H(x)(\rho^n)^{-1})}{Q^n_0} \leq \frac{Q^n_T - 1(x \leq v)}{Q^n_0}\right). \\
\tag{EC.B7}
\]

A similar argument to the one above shows that as \( n \to \infty \),

\[
\mathcal{B}^n(x) \triangleq \frac{\sum_{i=1}^{Q^n_T} B_i(\eta_H(x)(\rho^n)^{-1})}{Q^n_0} \Rightarrow \eta_H(x)^{-1}.
\]

Regarding the right-hand side in (EC.B7), from part (i) if a final auction occurs, for \( n \) large enough, we have \( Q^n_T \approx Q^n_0 - \lambda^n T \eta_{H-}(0) \). Then, as \( n \to \infty \),

\[
\frac{Q^n_T - 1(x \leq v)}{Q^n_0} \to 1 - \frac{\eta_{H-}(0)}{\rho}.
\]

By focusing on the continuity points, the distribution of \( P_H^n(v) \) converges weakly to the distribution:

\[
\mathbb{P}(P_H^n(v) < x) = \begin{cases} 
1 & \text{if } \eta_H(x) < \rho - \eta_{H-}(0) \\
0 & \text{if } \eta_H(x) > \rho - \eta_{H-}(0).
\end{cases}
\]

This corresponds to the distribution of the constant \( P_H^\infty = \min\{v \in [0,1]: \eta_H(v) \leq \rho - \eta_{H-}(0)\} \) at its continuity points, and so \( P_H^n(v) \Rightarrow P_H^\infty \). □

**References**

*See reference list in the main paper.*