Electronic Companion—“Dynamic Pricing for Nonperishable Products with Demand Learning” by Victor F. Araman and René Caldentey,
Operations Research, doi 10.1287/opre.1090.0725.
APPENDIX A: Main Proofs

A1. Proof of Proposition 1

- Existence and Uniqueness: The existence and uniqueness of a solution $W(n; \theta)$ follows by noticing that recursion (10) is equivalent to $F(W(n; \theta)) = W(n - 1; \theta)$, where the function

$$F(z) \triangleq z + \Phi \left( \frac{r}{\theta} z \right)$$

is continuous, strictly increasing and ranges from $[\Phi(0), \infty)$. This follows from the fact that $\Phi(z)$ is continuous and strictly increasing and non-negative in the domain $[\Phi(0), \infty)$. Since $\Phi(0) \leq 0 \leq R$, it follows that $W(n; \theta)$ is uniquely determined through the recursion

$$W(0; \theta) = R, \quad W(n; \theta) = F^{-1}(W(n - 1; \theta)), \quad n = 1, 2, \ldots.$$  

- Monotonicity on $\theta$: To prove the monotonicity of $W(n; \theta)$ on $\theta$ we use induction over $n$. First note that $W(1; \theta)$ solves

$$rW(1; \theta) = \theta \Psi(R - W(1; \theta)).$$

Since the function $h_1(z, \theta) = \theta \Psi(R - z)$ is increasing in $\theta$ (because $\Psi$ is nonnegative) and decreasing in $z$ it follows that $W(1; \theta)$ increases with $\theta$. Let us assume that $W(n - 1; \theta)$ is increasing in $\theta$ for some $n$. Now, $W(n; \theta)$ solves

$$rW(1; \theta) = \theta \Psi(R - W(1; \theta)).$$

Again, the function $h_n(z, \theta) = \theta \Psi(R - z)$ is increasing in $\theta$ and decreasing in $z$. We conclude that $W(n; \theta)$ is also increasing with $\theta$.

- Monotonicity and Concavity/Convexity on $n$: We now prove the monotonicity and concavity of $W(n; \theta)$ for the case $\theta \geq 1$. The proof in the case $\theta \leq 1$ uses the same line of arguments and it is left to the reader.

Suppose that $\theta \geq 1$. We proof the monotonicity of $W(n; \theta)$ in $n$ by induction.

I) First, for $n = 1$ we have that $W(n - 1; \theta) = W(0; \theta) = R$. Suppose, by contradiction, that $W(1; \theta) < W(0; \theta)$. Under this hypothesis, condition (10) implies $\Phi \left( \frac{rW(1; \theta)}{\theta} \right) > 0$. In addition, by construction $\Phi(z) > 0$ implies $z > c^*$ and so $rW(1; \theta) > \theta c^*$. But this last inequality implies that $W(1; \theta) > \theta R \geq R = W(0; \theta)$, since $c^* = rR$ and $\theta \geq 1$. Therefore, we conclude that $W(1; \theta) \geq W(0; \theta) = R$.

II) Suppose that $W(k; \theta) \geq W(k - 1; \theta)$ for all $k = 1, \ldots, n - 1$, some $n \geq 1$.

III) Let us prove that $W(n; \theta) \geq W(n - 1; \theta)$. Again, by contradiction, let us suppose that $W(n; \theta) < W(n - 1; \theta)$. Condition (10) implies $\Phi \left( \frac{rW(n; \theta)}{\theta} \right) > 0$ and so we must have $\theta R < W(n; \theta) < W(n - 1; \theta)$. In addition, by condition (10) we also have that

$$W(n - 1) = W(n - 2) - \Phi \left( \frac{rW(n - 1; \theta)}{\theta} \right).$$
Since \( \Phi(z) \) is monotonically increasing and \( W(n - 1; \theta) > \theta R \) we conclude
\[
W(n - 1; \theta) < W(n - 2; \theta) - \Phi(r R) = W(n - 2; \theta),
\]
which contradicts the induction step (II). We conclude that \( W(n; \theta) \geq W(n - 1; \theta) \).

To prove the concavity of \( W(n; \theta) \) simply note that condition (10) implies
\[
W(n; \theta) - W(n - 1; \theta) = -\Phi\left(\frac{r W(n; \theta)}{\theta}\right).
\]
Since both \( \Phi(z) \) and \( W(n; \theta) \) are monotonically increasing in their corresponding arguments, we conclude that the right hand side above is monotonically decreasing in \( n \) and so \( W(n; \theta) \) is concave.

- **Limiting Behavior:** Finally, to prove the asymptotic behavior of \( W(n; \theta) \), we first note that \( W(n; \theta) \) is bounded. In fact, for the case \( \theta \leq 1 \) the boundedness follows since \( W(n) \) is decreasing and nonnegative and so \( W(n; \theta) \in [0, W(0; \theta)] \). On the other hand, for the case \( \theta \geq 1 \), \( W(n; \theta) \) is increasing in \( n \) and so by condition (10) and the monotonicity of \( \Phi(z) \) it follows that \( r W(n; \theta)/\theta \leq c^* \), or equivalently, \( W(n; \theta) \leq \theta R \). Given that \( W(n; \theta) \) is bounded and monotonic (either increasing if \( \theta \geq 1 \) or decreasing if \( \theta \leq 1 \)), we have that \( \lim_{n \to \infty} W(n; \theta) \) exists. If we denote by \( W(\infty; \theta) \) this limit, then letting \( n \to \infty \) in condition (10) and using the continuity of \( \Phi(z) \), we conclude that \( \Phi(\frac{r W(\infty; \theta)}{\theta}) = 0 \) or \( W(\infty; \theta) = \theta c^*/r = \theta R. \)

**A2. Proof of Proposition 2**

Combining equations (10) and (11), it follows that
\[
s^*(n; \theta) = \theta \zeta \circ \Phi\left(\frac{r W(n; \theta)}{\theta}\right),
\]
where \( \zeta \circ \Phi \) is the composition of \( \Phi \) and \( \zeta \).

Our assumption that \( \lambda^2 p'(\lambda) \) is decreasing in \( \lambda \) implies that the function \( \lambda^2 p'(\lambda) \) is also decreasing in \( \lambda \). Because \( \lambda \in [0, \Lambda] \), we denote by \( \bar{z} \triangleq \Lambda^2 p'(\Lambda) \) its minimum value. The following lemma will be useful.

**Lemma 1** *The function \( \zeta \circ \Phi \) satisfies*

\[
\zeta \circ \Phi(z) = \begin{cases} 
\lambda \text{ solution to } \lambda^2 p'(\lambda) = -z & \text{if } 0 \leq z \leq -\bar{z} \\
\Lambda & \text{otherwise.}
\end{cases}
\]

It follows from Lemma 1 that if \( r W(n; \theta)/\theta \geq -\bar{z} \) then \( \zeta \circ \Phi(r W(n; \theta)/\theta) = \Lambda \) in which case \( s^*(n \theta) \) is trivially (locally) increasing in \( \theta \). Let us then assume then that \( r W(n; \theta)/\theta < -\bar{z} \). According to Lemma 1, the optimal demand intensity \( \lambda^*(n; \theta) \) satisfies
\[
(\lambda^*(n; \theta))^2 p'(\lambda^*(n; \theta)) = -\frac{r W(n; \theta)}{\theta},
\]
which implies
\[
s^*(n; \theta) = -\frac{r W(n; \theta)}{\lambda^*(n; \theta) p'(\lambda^*(n; \theta))}.
\]
To complete the proof note that (i) $W(n; \theta)$ increases with $\theta$ (by Proposition 1), $\lambda p'(\lambda)$ decreases with $\lambda$ (by assumption), and (iii) $\lambda^*(n; \theta)$ decreases with $\theta$ *(by Corollary 1). □

A3. Proof of Proposition 3

We recall that $D_t = N_0 - N_t$ is the cumulative demand up to time $t$, which has a Poisson distribution with mean $\theta I_\lambda(t)$. Recall that $I_\lambda(t) = \int_0^t \lambda_s \, ds$. The function $\lambda_t = \lambda(p_t)$ is the unscaled demand intensity at time $t$ given the pricing policy $p_t$ selected by the seller. Using the Poisson distribution of cumulative demand in $[0, t]$ and Bayes’ rule we get that

$$q_t = P_q(\theta = \theta_L | \mathcal{F}_t) = \frac{q \cdot (\theta_L \lambda(t))^{D_t} \exp(-\theta_L I_\lambda(t))/D_t!}{q \cdot (\theta_L \lambda(t))^{D_t} \exp(-\theta_L I_\lambda(t))/D_t! + (1 - q) \cdot (\theta_H \lambda(t))^{D_t} \exp(-\theta_H I_\lambda(t))/D_t!} = \frac{q}{q + (1 - q)(\theta_H/\theta_L)^{D_t} \exp-(\theta_H - \theta_L) I_\lambda(t)}.$$  

(a1)

The second equality follows from the Markov property of the demand process. We can now obtain the dynamics of the seller’s belief process ($q_t : t \geq 0$). For that, we write $q_t = f(Y_t)$, where $Y_t = \ln(\theta_H/\theta_L) D_t - (\theta_H - \theta_L) I_\lambda(t)$ is an $\mathcal{F}_t$-semimartingale and $f$ is a twice differentiable and bounded function given by $f(n) \triangleq \frac{q_0}{q_0 + (1 - q_0) \exp(n)}$. From Itô’s lemma (e.g., Ethier and Kurtz (1986)) and the fact that $Y_t$ is a finite variation process (which follows from the fact that $D(t)$ is a pure-jump process and $I_\lambda(t)$ is non-decreasing), we get

$$dq_t = f'(Y_{t-}) \, dY_t + f(Y_t) - f(Y_{t-}) - f'(Y_{t-}) \, \Delta Y_t.$$  

Taking advantage of the pure-jump nature of $D_t$ and the continuity of $I_\lambda(t)$, we have $dD_t = \Delta D_t$, $dY_t = \Delta Y_t - (\theta_H - \theta_L) \, dI_\lambda(t)$, and $f(Y_t) - f(Y_{t-}) = [f(Y_{t-} + \ln(\theta_L/\theta_H)) - f(Y_{t-})] \, dD_t$, so that

$$dq_t = -f'(Y_{t-})(\theta_H - \theta_L) \, dI_\lambda(t) + [f(Y_{t-} + \ln(\theta_L/\theta_H)) - f(Y_{t-})] \, dD_t$$

$$= (\theta_H - \theta_L) \frac{q(1 - q) \exp(Y_{t-})}{(q + (1 - q) \exp(Y_{t-}))^2} \, dI_\lambda(t)$$

$$+ \left[ \frac{q}{q + (1 - q) \exp(Y_{t-})} - \frac{q}{q + (1 - q) \exp(Y_{t-})} \right] \, dD_t$$

$$= -\eta(q_{t-}) \left[ dD_t - (\theta_L q_{t-} + \theta_H (1 - q_{t-})) \, dI_\lambda(t) \right], \quad \text{where} \quad \eta(q_t) \triangleq \frac{q(1 - q)(\theta_H - \theta_L)}{\theta_L q_t + \theta_H (1 - q_t)}. \quad \Box$$

(a2)

A4. Proof of Proposition 5

The monotonicity and boundedness of $V(n, q)$ are proven in the proof of proposition 4 in Appendix B. To prove the convexity of $V(n, q)$ with respect to $q$, we define

$$J_\lambda(n, \theta) \triangleq \int_0^\tau \exp(-rt) \theta c(\lambda_t) \, dt + \exp(-r\tau) R, \quad \tau = \inf\{t \geq 0 : N_t = 0\},$$

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for an arbitrary policy $\lambda \in \mathcal{A}$. We consider a pair of beliefs $q_1, q_2 \in [0, 1]$ and set $q = \alpha q_1 + (1 - \alpha) q_2$ for some $\alpha \in [0, 1]$. Then, convexity follows from

$$V(n, q) = \sup_{\lambda \in \mathcal{A}} \left\{ q \mathbb{E}_{\theta_\lambda} [J_\lambda(n, \theta)] + (1 - q) \mathbb{E}_{\theta_H} [J_\lambda(n, \theta)] \right\}$$

$$= \sup_{\lambda \in \mathcal{A}} \left\{ (\alpha q_1 + (1 - \alpha) q_2) \mathbb{E}_{\theta_\lambda} [J_\lambda(n, \theta)] + (1 - \alpha q_1 - (1 - \alpha) q_2) \mathbb{E}_{\theta_H} [J_\lambda(n, \theta)] \right\}$$

$$\leq \alpha \sup_{\lambda \in \mathcal{A}} \left\{ \mathbb{E}_{\theta_\lambda} [J_\lambda(n, \theta)] \right\} + (1 - \alpha) \sup_{\lambda \in \mathcal{A}} \left\{ \mathbb{E}_{\theta_H}[J_\lambda(n, \theta)] \right\}$$

$$= \alpha V(n, q_1) + (1 - \alpha) V(n, q_2).$$

Finally, to prove the uniform convergence of $V(n, q)$, let $\tau_n$ be the time it takes to sell $n$ units under an optimal pricing policy. Similarly, let $\tau_n(\lambda)$ be the time to deplete $n$ units while keeping the demand rate constant at $\lambda$. Observe that

$$R \bar{\theta}(q) = \max_{0 \leq \lambda \leq \Lambda} \mathbb{E}_q \left[ \int_0^\infty \exp(-rt)\bar{\theta}(q_t)c(\lambda_t)dt \right].$$

To see this note that the Bounded Convergence Theorem allows an interchange of the expected value and the integral. It is then clear that the LHS is an upper bound of the RHS and is achieved for $\lambda_t \equiv \lambda^*$. Hence,

$$R \bar{\theta}(q) \leq \max_{0 \leq \lambda \leq \Lambda} \mathbb{E}_q \int_0^{\tau_n} \exp(-rt)\bar{\theta}(q_t)c(\lambda_t)dt + \max_{0 \leq \lambda \leq \Lambda} \mathbb{E}_q \int_{\tau_n}^\infty \exp(-rt)\bar{\theta}(q_t)c(\lambda_t)dt.$$
\( V \) on \([0,1]\) as \( n \to \infty \). This is in agreement with the limiting differential equation obtained from relation (17) by letting \( n \) goes to infinity

\[
V(q) = V(q - \eta(q)) + \eta(q) V(q) - \Phi\left(\frac{r V(q)}{\theta(q)}\right)
\]

with \( V(0) = R\theta_H \). (a3)

The linear function \( R\theta(q) \) is indeed the unique solution of this ODE. □

A5. Proof of Proposition 6

Suppose that \( \lambda^*_c(n,q) \) is locally decreasing in \( q \) then it follows trivially that \( s^*(n,q) \) is also locally decreasing in \( q \). So, let us assume that \( \lambda^*_c(n,q) \) is locally increasing in \( q \). According to equation (18), the selling rate \( s^*(n,q) \) satisfies

\[
s^*(n,q) = \bar{\theta}(q) \zeta \circ \Phi\left(\frac{r V(n,q)}{\theta(q)}\right).
\]

From here, we can use exactly the same steps as in the proof of Proposition 2 replacing \( W(n;\theta) \) by \( V(n,q) \). □

A6. Proof of Proposition 8

We start by studying the difference \( W_c(n,\theta) = W(n,\theta) - R\theta \). We observe based on the recursion (10) and a first order Taylor expansion that

\[
W_c(n-1;\theta) = W_c(n;\theta) + \Phi\left(\frac{r W_c(n;\theta)}{\theta} + c^*\right)
\]

\[
= W_c(n;\theta)(1 + \frac{c}{\theta} \Phi'(c^*) + o(1)).
\]

It is then easily seen that

\[
\frac{W(n;\theta_L) - R\theta_L}{R\theta_H - W(n;\theta_H)} \sim \alpha \cdot c^{-n},
\]

as \( n \to \infty \); where \( \alpha > 0, c > 1 \) and \( f(x) \sim g(x) \) as \( x \to \infty \) means that \( f(x)/g(x) \to 1 \) as \( x \to \infty \).

Now notice that

\[
\tilde{V}(n,q) - R\bar{\theta}(q) = q W_c(n;\theta_L) - (1-q) W_c(n;\theta_H)
\]

\[
\sim (q c^{-n} - (1-q)) W_c(n;\theta_H)
\]

which is negative for large \( n \). Finally, considering the linear approximation it is easy to see that \( \tilde{V}(n,q) > \tilde{V}(n-1,q) \) if and only if \( \Phi\left(\frac{\tilde{V}(n,q)}{\theta(q)}\right) < 0 \) or equivalently \( \tilde{V}(n,q) - R\bar{\theta}(q) < 0 \) which completes the proof. □

A7. Proof of Proposition 9

- We start with (ii). The convexity proof follows exactly the same steps as in the case of \( V(n,\cdot) \) in Proposition 5. Similarly, the monotonicity in \( q \) follows from the same arguments used in the lemma B2 in Appendix B restricted to \( (0,q_n^*) \).

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For part (i) Observe that the first equation in (23) with the border condition \( U(n, q_n^*) = R \) defines an ODE where a classical Lipschitz argument proves existence and uniqueness of a continuously differentiable solution on \([0, q_n^*]\). Clearly, \( U(n, q_n^*) = R \) on \([q_n^*, 1]\). \( U(n, \cdot) \) is continuous on \( q_n^* \) and so it remains to study the continuity of \( q_n^* \). For that let \( \epsilon > 0 \), define \( q_n(\epsilon) = q_n^* + \epsilon \) and \( q_n'(\epsilon) = q_n^* - \epsilon \). Note that \( U(n, q_n(\epsilon)) > R \) while \( U(n, q_n(\epsilon)) = R \). By taking the difference between the equations in the previous system (23) at the point \( q_n^* \), and letting \( \epsilon \) goes to zero, we obtain by continuity of the functions \( U(n, \cdot) \) and \( U(n-1, \cdot) \) that

\[
0 \leq \eta(q_n^*)U(n, q_n^{*-}).
\]

The function \( U_q \) being non-positive, we conclude that \( U_q(n, q_n^{*-}) = U_q(n, q_n^{*+}) = 0 \), and \( U(n, \cdot) \) is continuously differentiable on \([0, 1]\). Putting \( U(n, q) = R \) we get that \( q_n^* \) is the unique solution of \( R + \Phi\left(\frac{rR}{\theta(q)}\right) = U(n-1, q - \eta(q)) \).

– For part (iii) Fix an initial belief \( q \). With the option of stopping available, a retailer with \( n + 1 \) units can follow the same policy than with \( n \) units and so \( U(n + 1, q) \geq U(n, q) \). As we saw before, this is not necessarily true if the option of stopping is not available. Finally, the bounds are straightforward.

– The proof of (iv) is essentially the same as in Proposition 6.

– To conclude, the prove of part (v) follows from the inequality \( V(n, q) \leq U(n, q) \), the identities

\[
\lambda^*_U(n, q) = \zeta \circ \Phi\left(\frac{rV(n, q)}{\theta(q)}\right) \quad \text{and} \quad \lambda^*_U(n, q) = \zeta \circ \Phi\left(\frac{rU(n, q)}{\theta(q)}\right),
\]

and the fact that \( \zeta \circ \Phi \) is an increasing function. \( \square \)

A8. Proof of Proposition 10

The monotonicity of \( q_n^* \) is a direct consequence of the monotonicity of \( U(n, q) \) in \( n \). To prove that the limiting value \( q_\infty^* \) is less than 1 we note that \( R + \Phi\left(\frac{rR}{\theta_\infty^*}\right) = U_\infty(q_\infty^* - \eta(q_\infty^*)) \) and \( R + \Phi\left(\frac{rR}{\theta(1)}\right) > R = U_\infty(1) = U_\infty(1 - \eta(1)) \). Hence, we must have \( q_\infty^* < 1 \).

The prove that \( q := \frac{\theta_H - 1}{\theta_H - \theta_L} \) is a lower bound for \( q_n^* \) we note that \( R + \Phi\left(\frac{rR}{\theta(q)}\right) < R \) for all \( q < q \).

Since by definition \( q_n^* \) satisfies \( R + \Phi\left(\frac{rR}{\theta(q_n^*)}\right) = U_\infty(q_n^* - \eta(q_n^*)) \geq R \), it follows that \( q_n^* \geq q \).

To derive the upper bound, let us first define the linear function \( U(n, q) := qR + (1 - q)W(n, \theta_H) \).

From the convexity of \( U(n, q) \) as a function of \( q \) and the fact that \( U(n, 0) = W(n, \theta_H) \) and \( U(n, 1) = R \) if follows that

\[
U(n, q) \leq U(n, q) \quad \text{for all } q \in [0, 1].
\]

Since \( R + \Phi\left(\frac{rR}{\theta(q)}\right) \) is an increasing function of \( q \), it follows that \( q_n^* \) solution of \( R + \Phi\left(\frac{rR}{\theta(q)}\right) = U(n - 1, q - \eta(q)) \) must be bounded above by \( \bar{q}_n \) solution of \( R + \Phi\left(\frac{rR}{\theta(q)}\right) = U(n - 1, q - \eta(q)) \). \( \square \)

A9. Proof of Proposition 11

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The bounds on $U(n, q)$ follows from its convexity and the definition of $\bar{q}_n$. The uniform convergence is due to Dini’s Theorem. (Dini’s Theorem states that if a monotone sequence of continuous real-valued functions converge pointwise on a compact set to a continuous function, then the convergence is uniform, see Cheney (2001).) The bounds on $U_\infty(q)$ are again due to the convexity of $U_\infty$ preserved by the uniform convergence. □
APPENDIX B: Proof of Proposition 4

In this appendix we investigate the existence and uniqueness of a solution to the following systems of ODEs.

\[
V(0, q) = R, \quad V(n, q) + \Phi \left( \frac{r V(n, q)}{\theta(q)} \right) - \eta(q) V_q(n, q) = V(n-1, q - \eta(q)),
\]

with border conditions \( V(n, 0) = W(n; \theta_H) \) and \( V(n, 1) = W(n; \theta_L). \) We approach this task recursively. That is, we will assume that we have a solution to \( V(n-1, q) \) with the desired properties and use it to compute \( V(n, q). \)

In what follows, we will drop the dependence on \( n \) and use the notation \( F(q) = V(n-1, q - \eta(q)), \) \( G(q) = V(n, q), \) \( G_0 = W(n, \theta_H) \) and \( G_1 = W(n, \theta_H). \) Also, and due to some mathematical technicalities, we will solve the following weaker version of the problem.

**Problem-L:** Consider two continuously differentiable functions \( F(q) \) and \( \Phi(q). \) \( F(q) \) is decreasing and \( \Phi(q) \) is increasing in \( q \in (0, 1]. \) We are interested to find a continuously differentiable function \( G(q) \) in \( (0, 1] \) that solves the ODE

\[
G'(q) = \frac{G(q) - F(q) + \Phi \left( \frac{r G(q)}{\theta(q)} \right)}{\eta(q)}, \quad q \in (0, 1) \tag{b1}
\]

with boundary condition

\[
\lim_{q \to 1} G(q) = G_1, \quad \text{where } G_1 \text{ solves } G_1 - F(1) + \Phi \left( \frac{r G_1}{\theta(1)} \right) = 0. \tag{b2}
\]

We note that we have replaced the original border conditions at \( q = 0 \) and \( q = 1 \) by (b2) which is only a limiting condition at \( q = 1. \) Fortunately, we will show that any solution to this weaker Problem-L satisfies the original border conditions at both \( q = 0 \) and \( q = 1. \)

For completeness, we also define \( G_0 \) to be the unique root of

\[
G_0 - F(0) + \Phi \left( \frac{r G_0}{\theta(0)} \right) = 0.
\]

The monotonicity of the function \( h(x) = x + \Phi \left( \frac{r x}{\theta} \right) \) guarantees that both \( G_0 \) and \( G_1 \) are uniquely defined.

To avoid confusion we will use the following terminology. We will say that \( G(q) \) is a solution to the ODE if it solves (b1) in \((0, 1). \) We say that \( G(q) \) is a solution to Problem-L if is a solution to the ODE that satisfies the boundary condition (b2).

In what follows, we will prove that there exists a unique solution to Problem-L. This solution will also satisfy the border condition at \( q = 0. \) Also, because we are solving the system of ODE recursively, the solution \( G(q) \) becomes the function \( F(q) \) in the next iteration. Hence, we also need to show that \( G(q) \) is continuously differentiable and decreasing in \((0, 1]. \) The following three sections address these issues of the existence, uniqueness and differentiability and monotonicity of a solution to Problem-L, respectively.
B1 Existence

Before discussing the existence of a solution to problem-L, let us first prove three lemmas.

Lemma B1 Let $G(q)$ be a solution to the ODE. Let $\bar{q} \in (0, 1)$ and $\bar{G} = G(\bar{q})$.

i) If $\bar{G} < G_1$ then $\lim_{q \uparrow 1} G(q) < G_1$.

ii) If $\bar{G} > G_0$ then $\lim_{q \downarrow 0} G(q) > G_0$.

Proof: We prove only part (ii). The proof of (ii) uses the same arguments. Suppose $\bar{G} \leq G_1$ then by the definition of $G_1$ we get

$$G'(q) = \frac{\bar{G} - F'(\bar{q}) + \Phi(\frac{r G_1}{\bar{G}(q)})}{\eta(q)} < \frac{G_1 - F'(\bar{q}) + \Phi(\frac{r G_1}{\bar{G}(1)})}{\eta(q)}$$

The last inequality follows from the monotonicity of $F$ and $\Phi$ and the fact that $\bar{\theta}(\bar{q}) \geq \bar{\theta}(1)$. Then, $G(q)$ is decreasing at $q = \bar{q}$ and so by its continuity we conclude that $\lim_{q \uparrow 1} G(q) < G_1$. □

Lemma B2 Let $G(q)$ be a solution to the ODE. If there is $q_0 \in (0, 1)$ such that $G'(q_0) \geq 0$ then $G'(q) \geq 0$ for all $q \geq q_0$.

Proof: The result follows from noticing that the function

$$h(q, x) := x - F'(q) + \Phi(\frac{r x}{\eta(q)})$$

is increasing in both $x$ and $q$. That is, $h_q(q, x) \geq 0$ and $h_x(q, x) \geq 0$ for all $(q, x)$, where $h_q$ and $h_x$ are the partial derivatives of $h(q, x)$ with respect to the first and second argument respectively. Hence,

$$G'(q) = \frac{1}{\eta(q)} h(q, G(q)) = \frac{1}{\eta(q)} \left( \eta(q_0) G'(q_0) + \int_{q_0}^{q} [h_q(s, G(s)) + h_x(s, G(s)) G'(s)] \, ds \right) \geq 0,$$

where the inequality follows from the assumption $G'(q_0) \geq 0$. □

Lemma B3 Let $G(q)$ be a bounded solution to the ODE. If $\lim_{q \downarrow 0} |G(q)| < \infty$ then $\lim_{q \downarrow 0} G(q) = G_0$. Similarly, if $\lim_{q \uparrow 1} |G(q)| < \infty$ then $\lim_{q \uparrow 1} G(q) = G_1$.

Proof: We prove only the limit at $q = 0$. The argument in limit at $q = 1$ is similar and it is left to the reader. Because $G(q)$ solves the ODE it follows that

$$\eta(q) G'(q) = G(q) - F'(q) + \Phi(\frac{r G(q)}{\theta(q)})$$

for all $q \in (0, 1)$. (b3)
Suppose that \( \lim_{q \downarrow 0} G(q) = \hat{G} \) for some real \( \hat{G} \). We will show, by contradiction, that \( \hat{G} = G_0 \). Let us assume that \( \hat{G} \neq G_0 \). Because of the continuity and boundedness of \( G(q), F(q) \) and \( \Phi(q) \), it follows from condition (b3) that there is constant \( K \neq 0 \) such that
\[
\lim_{q \downarrow 0} \eta(q) G'(q) = \hat{G} - F(0) + \Phi \left( \frac{r \hat{G}}{\theta(0)} \right) = K.
\]
The fact that \( K \neq 0 \) follows from the definition (and uniqueness) of \( G_0 \) and the assumption \( \hat{G} \neq G_0 \).

Suppose \( K > 0 \) (the case \( K < 0 \) uses similar arguments). Because \( \eta(q) \sim q \) around \( q = 0 \) and \( K \neq 0 \), the limit above implies that \( G'(q) \sim q^{-1} \) or equivalently \( G(q) \sim \ln(q) \) which violates the assumption \( \lim_{q \downarrow 0} |G(q)| < \infty \). We conclude that \( \hat{G} = G_0 \). □

We can move to the proof of existence of a solution to Problem-L. For this, we define three families of solutions to the ODE in (b1).

\[
\begin{align*}
\mathcal{G} &:= \left\{ G(q) \text{ solution to the ODE in (b1) such that } \lim_{q \uparrow 1} G(q) > G_1 \right\}, \\
\mathcal{G} &:= \left\{ G(q) \text{ solution to the ODE in (b1) such that } \lim_{q \uparrow 1} G(q) = G_1 \right\}, \\
\mathcal{G}_{\leq} &:= \left\{ G(q) \text{ solution to the ODE in (b1) such that } \lim_{q \uparrow 1} G(q) < G_1 \right\}.
\end{align*}
\]

Proving existence of a solution requires showing \( \mathcal{G} \neq \emptyset \). Suppose, by contradiction that \( \mathcal{G} = \emptyset \).

We now define the following auxiliary functions
\[
\tilde{G}(q) = \inf_{G \in \mathcal{G}} \left\{ G(q) \right\} \quad \text{and} \quad \underline{G}(q) = \sup_{G \in \mathcal{G}_{\leq}} \left\{ G(q) \right\}, \quad \text{for all } q \in (0, 1),
\]
these are the lower and upper envelopes of the set \( \mathcal{G} \) and \( \mathcal{G}_{\leq} \), respectively. Note that Lemma B1 guarantees that both \( \tilde{G} \) and \( \underline{G} \) are nonempty and so the infimum and supremum are well defined.

**Proposition B1** Suppose \( \mathcal{G} = \emptyset \), then for any \( q \in (0, 1) \)
\[
G_1 \leq \underline{G}(q) = \tilde{G}(q) \leq G_0.
\]

Define, \( \bar{G}(q) = \underline{G}(q) \), then \( \bar{G}(q) \) is a solution to the ODE and satisfies
\[
\lim_{q \downarrow 0} \bar{G}(q) = G_0 \quad \text{and} \quad \lim_{q \uparrow 1} \bar{G}(q) = G_1.
\]

**Proof:** The lower bound on \( \underline{G}(q) \) and upper bound on \( \tilde{G}(q) \) follow from Lemma B1. The equality \( \underline{G}(q) = \tilde{G}(q) \) follows from the assumption \( \mathcal{G} = \emptyset \). To prove that \( \bar{G}(q) \) satisfies the ODE, let \( q_0 \in (0, 1) \) and \( \hat{G}(q) \) be the solution of the ODE passing through \( (q_0, \hat{G}(q_0)) \). We will show that \( \bar{G}(q) = \hat{G}(q) \) for all \( q \in (0, 1) \) and so \( \bar{G}(q) \) satisfies the ODE.

Because we are assuming that \( \mathcal{G} = \emptyset \), we must have \( \hat{G} \in \tilde{G} \) or \( \hat{G} \in \underline{G} \). We consider only the case \( \hat{G} \in \tilde{G} \), the proof in the other case follows the same steps. Suppose \( \hat{G}(q) \neq \bar{G}(q) \) then (by the
fact that $\tilde{G}(q)$ is the lower envelope of the set $\tilde{G}$ there exists $\tilde{G}(q) \in \tilde{G}$ such that $\tilde{G}(q) < \tilde{G}(q)$ for all $q \in (0, 1)$. But at $q_0$ the following holds $\tilde{G}(q_0) \leq \tilde{G}(q_0) < \tilde{G}(q_0) = \tilde{G}(q_0)$. This contradiction implies that $\tilde{G}(q) = \tilde{G}(q)$ as required.

Next, we need to show that $\lim_{q \downarrow 0} \tilde{G}(q) = G_0$ and $\lim_{q \uparrow 1} \tilde{G}(q) = G_1$. We start showing that these limits exists. For the right limit at $q = 1$, note that $\tilde{G}(q)$ is continuous in $(0, 1)$; this follows from the fact that $\tilde{G}(q)$ satisfies the ODE. Now if there is $q_0 \in (0, 1)$ such that $\tilde{G}'(q_0) \geq 0$ then by Lemma B2 the function $\tilde{G}(q)$ is increasing in $[q_0, 1)$. Furthermore, by the first part of this proposition, $\tilde{G}(q)$ is also bounded. Hence, the limit (as $q \downarrow 1$) of an increasing and bounded function always exists. On the other hand, if for all $q \in (0, 1)$ $\tilde{G}'(q) < 0$ then $\tilde{G}(q)$ is decreasing and bounded in $(0, 1)$ and, therefore, it must have a limit as $q \downarrow 1$.

For the left limit at $q = 0$, we use a similar argument. Suppose there exists a $q_0 \in (0, 1)$ such that $\tilde{G}'(q_0) < 0$. Then, by Lemma B2 and the fact that $\tilde{G}(q)$ satisfies the ODE, we have that $\tilde{G}'(q) < 0$ for all $q \in (0, q_0]$. Hence by the boundedness of $\tilde{G}(q)$ we conclude that $\lim_{q \downarrow 0} \tilde{G}(q)$ exists. On the other hand, if for all $q \in (0, 1)$ $\tilde{G}'(q) \geq 0$ then this monotone condition and the boundedness of $\tilde{G}(q)$ imply again that $\lim_{q \downarrow 0} \tilde{G}(q)$ exists.

Finally, the desired limits $\lim_{q \downarrow 0} \tilde{G}(q) = G_0$ and $\lim_{q \uparrow 1} \tilde{G}(q) = G_1$ follow from (i) the boundedness of $\tilde{G}(q)$, (ii) the fact that $\tilde{G}$ solves the ODE, and (iii) Lemma B3. □

The proposition shows that $\tilde{G}(q)$ is a solution to Problem-L and we must have $G \neq \emptyset$.

**B2 Uniqueness**

From the previous section, we already know that there exists a solution $\tilde{G}(q)$ to Problem-L that it is bounded and decreasing in $(0, 1)$ and satisfies $\lim_{q \downarrow 0} \tilde{G}(q) = G_0$ and $\lim_{q \uparrow 1} \tilde{G}(q) = G_1$. We can then extend the domain of $\tilde{G}(q)$ to $[0, 1]$ by defining $\tilde{G}(0) = G_0$ and $\tilde{G}(1) = G_1$.

In order to prove the uniqueness of this function $\tilde{G}(q)$, we need the following result.

**Lemma B4** Let $\tilde{G}(q)$ be a solution to Problem-L. Then, $\lim_{q \uparrow 1} \tilde{G}'(q)$ exists and

$$
\tilde{G}'(1) = \frac{F'(1) + \left(\frac{r}{\tilde{\theta}(1)} \tilde{G}(1)\right) \Phi'\left(\frac{r \tilde{G}(1)}{\tilde{\theta}(1)}\right)}{1 + \left(\frac{r}{\tilde{\theta}(1)}\right) \Phi'\left(\frac{r \tilde{G}(1)}{\tilde{\theta}(1)}\right) - \eta'(1)}.
$$

**(b4)**

**Proof:** Let us suppose, by contradiction, that $\lim_{q \uparrow 1} \tilde{G}'(q)$ does not exist. Since $\tilde{G}(q)$ is decreasing, this is equivalent to assume that $\lim_{q \uparrow 1} \tilde{G}'(q) = -\infty$.

Let us define the auxiliary function $h(q) := \eta(q) \tilde{G}'(q)$ and note that $h(1) = 0$ and

$$
h'(q) = \tilde{G}'(q) - F'(q) + \Phi'\left(\frac{r \tilde{G}(q)}{\tilde{\theta}(q)}\right) \left(\frac{r \tilde{G}'(q) \tilde{\theta}(q) - r \tilde{G}(q) \tilde{\theta}'(q)}{\tilde{\theta}^2(q)}\right).
$$

By assumption, $F'(q)$ is bounded and $\Phi'(x)$ is nonnegative. Furthermore, by proposition B1 $\tilde{G}(q)$ is also bounded. Hence, the assumption $\lim_{q \uparrow 1} \tilde{G}'(q) = -\infty$ implies that there exists $q_0 \in (0, 1)$ such that $h'(q) < 0$ for all $q \geq q_0$. 43
Take $\epsilon > 0$ such that $q_0 \leq 1 - \epsilon$. Then, from a first order Taylor expansion we get

$$h(q_0) = h(1 - \epsilon) - \int_{q_0}^{1-\epsilon} h'(q) \, dq.$$ 

Since this is true for any $\epsilon > 0$ and $h(1) = 0$ and $h'(q) < 0$, it follows that $h(q_0) > 0$. We conclude that

$$\tilde{G}'(q_0) = \frac{h(q_0)}{q_0} > 0.$$

This is not possible because $\tilde{G}(q)$ is decreasing. We conclude that the assumption $\lim_{q \to 1} \tilde{G}'(q) = -\infty$ cannot hold. That is, $\tilde{G}(q)$ admits a left derivative at $q = 1$. We can use L'Hôpital's rule to compute $\tilde{G}'(1)$.

$$\tilde{G}'(1) = \lim_{q \to 1^+} \frac{h(q)}{q(1-q)} - \lim_{q \to 1^+} \frac{h'(q)}{\eta'(q)} = \frac{1}{\eta'(1)} \left[ \tilde{G}'(1) - F'(1) + \Phi' \left( \frac{r \tilde{G}(1)}{\theta(1)} \right) \left( \frac{r \tilde{G}'(1) \theta(1) - r \tilde{G}(1) \theta'(1)}{\theta^2(1)} \right) \right].$$

Solving for $G'(1)$ we get condition (b4). \(\square\)

The lemma asserts that any solution $\tilde{G}(q)$ to Problem-L must have bounded derivative in $(0, 1]$ where $\tilde{G}'(1)$ is understood to be the left derivative at $q = 1$.

Now, let us suppose that we have two bounded solutions $\tilde{G}(q)$ and $\tilde{g}(q)$ to Problem-L. Without lost of generality let us suppose that $\tilde{g}(q) \leq \tilde{G}(q)$ in $(0, 1)$. Otherwise, if $\tilde{g}(q_0) = \tilde{G}(q_0)$ for some $q_0 \in (0, 1)$ then they must agree in the entire $(0, 1)$ as they solve the same ODE in (b1).

Since both $\tilde{G}(p)$ and $\tilde{g}(p)$ satisfy the ODE it follows that for every $q$

$$-[\tilde{G}(q) - \tilde{g}(q)] = \int_q^1 \left[ \tilde{G}'(x) - \tilde{g}'(x) \right] \, dx = \int_q^1 \frac{1}{\eta(x)} \left[ \tilde{G}(x) - \tilde{g}(x) + \Phi \left( \frac{r \tilde{G}(x)}{\theta(x)} \right) - \Phi \left( \frac{r \tilde{g}(x)}{\theta(x)} \right) \right] \, dx. \tag{b5}$$

The monotonicity of $\Phi(x)$ and the boundedness of $\tilde{G}(q)$ and $\tilde{g}(q)$ imply that there exists a bounded and nonnegative function $\xi(q)$ such that

$$\Phi \left( \frac{r \tilde{G}(q)}{\theta(q)} \right) - \Phi \left( \frac{r \tilde{g}(q)}{\theta(q)} \right) = \xi(q) (\tilde{G}(q) - \tilde{g}(q)).$$

Then,

$$-[\tilde{G}(q) - \tilde{g}(q)] = \int_q^1 (\tilde{G}(x) - \tilde{g}(x)) \left[ \frac{1 + \xi(x)}{\eta(x)} \right] \, dx. \tag{b6}$$

By assumption, the left-hand side is nonpositive and the right-hand side is nonnegative. Hence, they must be equal to zero. We conclude then that $\tilde{G}(q) = \tilde{g}(q)$ which shows uniqueness. \(\square\)

**B3 Differentiability and Monotonicity**

In order to solve recursively the control problem for $V(n, q)$ using Problem-L, we need to show that any solution $\tilde{G}(q)$ to this problem is continuously differentiable and decreasing in $(0, 1]$. 

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The fact that $\tilde{G}(q)$ is continuously differentiable in $(0, 1]$ follows from proposition B1 and lemma B4. The monotonicity of $\tilde{G}(q)$ follows directly from proposition B1 and lemma B2. In fact, from proposition B1 we know that any solution $\tilde{G}(q)$ to Problem-L is a bounded solution to the ODE and satisfies $G_1 \leq G(q) \leq G_0$ and $\lim_{q \downarrow 0} G(q) = G_0$ and $\lim_{q \uparrow 1} G(q) = G_1$. By lemma B2 if $\tilde{G}(q)$ is non-decreasing at any $q_0 \in (0, 1)$ then it is non-decreasing at any $q \geq q_0$. Combining these properties of $\tilde{G}(q)$ it follows that it must be decreasing in $(0, 1]$. 
APPENDIX C: Supplements

C1. Three Examples of Demand Functions

– **Exponential Demand Model:** Consider the case in which the demand intensity is well approximated by the following exponential demand model

\[ \lambda(p) = \Lambda \exp(-\alpha p), \quad p \geq 0. \]

For this demand model, the corresponding inverse demand function and corresponding revenue rates are given by

\[ p(\lambda) \triangleq \frac{1}{\alpha} \ln \left( \frac{\Lambda}{\lambda} \right) \quad \text{and} \quad c(\lambda) \triangleq \lambda p(\lambda) = \frac{\lambda}{\alpha} \ln \left( \frac{\Lambda}{\lambda} \right), \quad \lambda \in [0, \Lambda] \]

respectively. The demand rate that maximizes \( c(\lambda) \) is \( \lambda^* = \Lambda \exp(-1) \) and \( c^* = \lambda^*/\alpha \). In addition, the Fenchel-Legendre transform \( \Psi \) of \( c(\lambda) \) satisfies

\[ \Psi(z) \triangleq \max_{0 \leq \lambda \leq \Lambda} \{ \lambda z + c(z) \} = \begin{cases} \frac{\Lambda}{\alpha} \exp(\alpha z - 1) & \text{if } z \leq \frac{1}{\alpha} \\ \Lambda z & \text{if } z \geq \frac{1}{\alpha}. \end{cases} \]

The corresponding maximizer is

\[ \zeta(z) \triangleq \arg \max_{0 \leq \lambda \leq \Lambda} \{ \lambda z + c(z) \} = \begin{cases} \Lambda \exp(\alpha z - 1) & \text{if } z \leq \frac{1}{\alpha} \\ \Lambda & \text{if } z \geq \frac{1}{\alpha}. \end{cases} \]

The function \( \Psi(z) \) is continuously differentiable, increasing and convex. The associated inverse function satisfies

\[ \Phi(z) \triangleq \begin{cases} \frac{1}{\alpha} \left[ 1 + \ln \left( \frac{\alpha z}{\Lambda} \right) \right] & \text{if } 0 < z \leq \frac{\Lambda}{\alpha} \\ \frac{z}{\Lambda} & \text{if } z \geq \frac{\Lambda}{\alpha}. \end{cases} \]

Similarly, this function is continuously differentiable, increasing and concave. Note also that \( \Phi(c^*) = 0 \).

– **Linear Demand Model:** Consider the case in which the demand intensity is given by the following linear demand model

\[ \lambda(p) = \Lambda - \alpha p, \quad 0 \leq p \leq \frac{\Lambda}{\alpha}, \]

For this demand model, the corresponding inverse demand function and corresponding revenue rates are given by

\[ p(\lambda) \triangleq \frac{\Lambda - \lambda}{\alpha} \quad \text{and} \quad c(\lambda) \triangleq \lambda p(\lambda) = \frac{\lambda(\Lambda - \lambda)}{\alpha}, \quad \lambda \in [0, \Lambda] \]

respectively. The demand rate that maximizes \( c(\lambda) \) is \( \lambda^* \triangleq \frac{\Lambda}{2} \) and \( c^* \triangleq (\lambda^*)^2/\alpha \). In addition, the Fenchel-Legendre transform \( \Psi \) of \( c(\lambda) \) satisfies

\[ \Psi(z) \triangleq \max_{0 \leq \lambda \leq \Lambda} \{ \lambda z + c(z) \} = \begin{cases} 0 & \text{if } z \leq -\frac{\Lambda}{\alpha} \\ \frac{(\Lambda + \alpha z)^2}{4\alpha} & \text{if } -\frac{\Lambda}{\alpha} \leq z \leq \frac{\Lambda}{\alpha} \\ \Lambda z & \text{if } z \geq \frac{\Lambda}{\alpha}. \end{cases} \]
The corresponding maximizer is
\[
\zeta(z) \triangleq \arg \max_{0 \leq \lambda \leq \Lambda} \{ \lambda z + c(z) \} = \begin{cases} 
0 & \text{if } z \leq -\frac{\Lambda}{\alpha} \\
\frac{\alpha z + \Lambda}{\alpha} & \text{if } -\frac{\Lambda}{\alpha} < z \leq \frac{\Lambda}{\alpha} \\
\Lambda & \text{if } z \geq \frac{\Lambda}{\alpha}.
\end{cases}
\]

The function $\Psi(z)$ is continuously differentiable, nondecreasing and convex. In the domain $z \geq -\frac{\Lambda}{\alpha}$, $\Psi(z)$ admits an inverse function
\[
\Phi(z) \triangleq \begin{cases} 
\frac{\sqrt{4\alpha z - \Lambda}}{\alpha} & \text{if } 0 \leq z \leq \frac{\Lambda^2}{\alpha} \\
\frac{\Lambda}{z} & \text{if } z \geq \frac{\Lambda^2}{\alpha}.
\end{cases}
\]

Similarly, this function is continuously differentiable, increasing and concave. Note also that $\Phi(c^*) = 0$.

**Quadratic Demand Model**: Consider the case in which the demand intensity is given by the following quadratic demand model
\[
\lambda(p) = \sqrt{\Lambda^2 - \alpha p}, \quad 0 \leq p \leq \frac{\Lambda^2}{\alpha}.
\]

For this demand model, the corresponding inverse demand function and corresponding revenue rates are given by
\[
p(\lambda) \triangleq \frac{\Lambda^2 - \lambda^2}{\alpha} \quad \text{and} \quad c(\lambda) \triangleq \lambda p(\lambda) = \frac{\lambda(\Lambda^2 - \lambda^2)}{\alpha}, \quad \lambda \in [0, \Lambda]
\]
respectively. The demand rate that maximizes $c(\lambda)$ is $\lambda^* \triangleq \frac{\Lambda}{\sqrt{3}}$ and $c^* \triangleq 2(\lambda^*)^3/\alpha$. In addition, the Fenchel-Legendre transform $\Psi$ of $c(\lambda)$ satisfies
\[
\Psi(z) \triangleq \max_{0 \leq \lambda \leq \Lambda} \{ \lambda z + c(z) \} = \begin{cases} 
0 & \text{if } z \leq -\frac{\Lambda^2}{\alpha} \\
\frac{2(\Lambda^2 + \alpha z)\frac{3}{\sqrt{3}}}{3 \sqrt{3} \alpha} & \text{if } -\frac{\Lambda^2}{\alpha} \leq z \leq 2 \frac{\Lambda^2}{\alpha} \\
\Lambda z & \text{if } z \geq 2 \frac{\Lambda^2}{\alpha}.
\end{cases}
\]

The corresponding maximizer is
\[
\zeta(z) \triangleq \arg \max_{0 \leq \lambda \leq \Lambda} \{ \lambda z + c(z) \} = \begin{cases} 
0 & \text{if } z \leq -\frac{\Lambda^2}{\alpha} \\
\frac{\sqrt{\Lambda^2 + \alpha z}}{\alpha} & \text{if } -\frac{\Lambda^2}{\alpha} \leq z \leq 2 \frac{\Lambda^2}{\alpha} \\
\Lambda & \text{if } z \geq 2 \frac{\Lambda^2}{\alpha}.
\end{cases}
\]

The function $\Psi(z)$ is continuously differentiable, nondecreasing and convex. In the domain $z \geq -\frac{\Lambda^2}{\alpha}$, $\Psi(z)$ admits an inverse function
\[
\Phi(z) \triangleq \begin{cases} 
\frac{3(\alpha z/2)^3 - \Lambda^2}{\alpha} & \text{if } 0 \leq z \leq 2 \frac{\Lambda^2}{\alpha} \\
\frac{\Lambda^3}{3 \sqrt{3} \alpha} & \text{if } z \geq 2 \frac{\Lambda^2}{\alpha}.
\end{cases}
\]

Similarly, this function is continuously differentiable, increasing and concave. Note also that $\Phi(c^*) = 0$. 

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C2. Derivation of the HJB optimality condition

We consider the stochastic control problem

\[
V(N_0, q) = \sup_{\lambda \in \mathcal{A}} \mathbb{E}_q \left[ \int_0^\tau \exp(-rt) \theta c(\lambda_t) dt + \exp(-r\tau) R \right]
\]

subject to \( N_t = N_0 - \int_0^t dD(I_\lambda(s)), \)

\[
dq_t = \eta(q_{t-}) \left[ dD_t - (\theta_L q_{t-} + \theta_H (1 - q_{t-})) \lambda_t dt \right], \quad q_0 = q,
\]

\[
\tau = \inf \{ t \geq 0 : N_t = 0 \}.
\]

The dynamic programming equation for this infinite horizon is

\[
r V(n, q) = \max_{\theta \in \mathcal{A}} \left[ -\mathcal{G}^\lambda V(n, q) + \bar{\theta}(q)c(\lambda) \right],
\]

where \( \mathcal{G}^\lambda \) is the infinitesimal generator of \((N_t, q_t)\) given the control \( \lambda \), which following the notations of Fleming and Soner (1993) is defined by

\[
\mathcal{G}^\lambda V(n, q) \triangleq \lim_{h \to 0} h^{-1}\left[ \mathbb{E}_q V(n(h), q(h)) - V(n, q) \right].
\]

To compute this last term we apply Itô’s lemma to the function \( V \), while noticing that both processes \( N_t \) and \( q_t \) have finite variation. Using the fact that \( N_t \) is a pure-jump process and so \( dN_t = \Delta N_t \), we obtain

\[
dV(N_t, q_t) = V_q(N_{t-}, q_{t-})dq_t + V(N_t, q_t) - V(N_{t-}, q_{t-}) - V_q(N_{t-}, q_{t-})\Delta q_t,
\]

where the notation \( V_q \) stands for \( \frac{\partial V}{\partial q} \). From the dynamics of \( q_t \) in (14), it follows that \( N_t = N_0 - D_t \) and \( q_t \) have common jumps of size -1 and \(-\eta(q_t)\), respectively. From this observation, it follows that

\[
V(N_t, q_t) - V(N_{t-}, q_{t-}) = -\left[ V(N_{t-} - 1, q_{t-} - \eta(q_{t-})) - V(N_{t-}, q_{t-}) \right] dN_t
\]

and so for a fixed control \( \lambda \)

\[
dV(N_t, q_t) = q_t(1 - q_t)(\theta_H - \theta_L) V_q(N_t, q_t) \lambda dt - \left[ V(N_{t-} - 1, q_{t-} - \eta(q_{t-})) - V(N_{t-}, q_{t-}) \right] dN_t.
\]

We define \( \kappa(q) \triangleq q(1 - q)(\theta_H - \theta_L) \) and write

\[
h^{-1} \mathbb{E}_q[V(n(h), q(h)) - V(n, q)]
\]

\[
= h^{-1} \mathbb{E}_q \left[ \int_0^h \kappa(q_t)V_q(N_t, q_t) \lambda dt - \int_0^h \left[ V(N_{t-} - 1, q_{t-} - \eta(q_{t-})) - V(N_{t-}, q_{t-}) \right] dN_t \right]
\]

\[
= h^{-1} \mathbb{E}_q \left[ \int_0^h \kappa(q_t)V_q(N_t, q_t) \lambda dt + \int_0^h \bar{\theta}(q_t) \left[ V(N_{t-} - 1, q_{t-} - \eta(q_t)) - V(N, q_t) \right] \lambda dt \right],
\]

where the second equality follows from the fact that for a given control \( \lambda \) the process \( N_t + \bar{\theta}(q_t)\lambda t - N_0 \) is an \( \mathcal{F}_t \)-martingale. Finally, letting \( h \downarrow 0 \) we conclude that

\[
\mathcal{G}^\lambda V(n, q) = -\lambda \left( \kappa(q) V_q(n, q) + \bar{\theta}(q)[V(n - 1, q - \eta(q)) - V(n, q)] \right).
\]
APPENDIX D: HJB Numerical Solution

In this appendix we describe the algorithm that we use in our numerical computations. The method used to solve the HJB optimality conditions is essentially the same whether the optimal stopping option is available or not. The algorithm is based on a finite-difference scheme in which the belief space \( \{ q \in [0, 1] \} \) is partitioned using a mesh \( \mathcal{M}(\Delta q) := \{ q_0, \ldots, q_M \} \) such that \( q_0 = 0 \), \( q_M = 1 \) and \( q_j - q_{j-1} = \Delta q \) for all \( j = 1, \ldots, M \). In our computations, the size of the mesh was chosen equal to \( \Delta q = 10^{-3} \).

For simplicity, in what follows we will use the index \( i \) to refer to the belief \( q_i \) and the index \( n \) to refer to the level of inventory. In addition, we will use the same notation \( U \) and \( V \) for the (numerically computed) value function for the case with and without the stopping option, respectively. For example \( V(n, i) \) is the (numerically computed) value function when the inventory is \( n \), the belief is \( q_i \) and the option to stop is not available.

D1. Algorithm Without the Optimal Stopping Option

Suppose the system is in state \( (n, i) \) and we choose a demand intensity equal to \( \lambda \). Given our discrete mesh \( \mathcal{M} \), we will keep this value of \( \lambda \) until (i) a sale occurs and the state jumps to \( (n - 1, i - \eta(i)) \) or (ii) the belief process moves up to \( q_{i+1} \) and the state becomes \( (n, i + 1) \).

Let us denote by \( \Delta t_i^\lambda \) the length of the time interval during which the control \( \lambda \) is kept constant if there is no sale. We need to write this time as a function of the initial belief \( q_i \) and the control \( \lambda \) in order to ensure local consistency between this discrete approximation and the actual continuous time evolution of \( q_t \). In fact, suppose that \( q_t = q_i \) and the control is fixed at \( \lambda \) then by definition \( \Delta t_i^\lambda \) should satisfy

\[
q_{t + \Delta t_i^\lambda} = q_{i+1} \quad \text{conditional on the fact that there is no sale in the time interval } [t, t + \Delta t_i^\lambda).
\]

From equation (14) it follows that in the absence of sales and for a fixed \( \lambda \), \( q_t \) evolves according to the deterministic ODE

\[
dq_t = \lambda q_t (1 - q_t) (\theta_H - \theta_L) \, dt.
\]

After integration we can show that

\[
\Delta t_i^\lambda = \frac{1}{\lambda (\theta_H - \theta_L)} \ln \left( \frac{q_{i+1} (1 - q_i)}{q_i (1 - q_{i+1})} \right).
\]

Let us denote by \( \tau_i^\lambda \) the random time at which a sale occurs if we kept the demand intensity \( \lambda \) fixed when the initial belief is \( q_i \). Then, the discrete version of the HJB optimality condition for \( V(n, q) \) in equation (16) takes the form

\[
V(n, i) = \max_{0 \leq \lambda \leq \Lambda} \mathbb{E}_q \left[ \mathbf{1}_{\{ \tau_i^\lambda \leq \Delta t_i^\lambda \}} e^{-r \tau_i^\lambda} \left( p(\lambda) + V(n - 1, i - \eta(i)) + \mathbf{1}_{\{ \tau_i^\lambda > \Delta t_i^\lambda \}} e^{-r \Delta t_i^\lambda} V(n, i + 1) \right) \right],
\]

\[\text{(d1)}\]

with border conditions \( V(0, i) = R \) and \( V(n, M) = W(n, \theta_L) \), for all \( i = 1, \ldots, m \) and all \( n \).

It follows from the Markovian dynamics of \( q_t \) that

\[
E_{q_t} \left[ \mathbb{I}_{\{\tau^+_i > \Delta t^i_\lambda\}} \right] = P(\tau^+_i > \Delta t^i_\lambda) = \exp \left( -\lambda \int_0^{\Delta t^i_\lambda} \theta(q_t) \, dt \right),
\]

where \( q_t \) satisfies

\[
dq_t = q_t \left( 1 - q_t \right) (\theta_H - \theta_L) \, dt, \quad q_0 = q_i, \quad \text{for all } t \in [0, \Delta t^i_\lambda).
\]

The following result follows after integration and we omit the details.

**Lemma B5** For all \( i = 0, \ldots, M - 1 \) and \( \lambda \in [0, \Lambda] \),

\[
E_{q_t} \left[ \mathbb{I}_{\{\tau^+_i \leq \Delta t^i_\lambda\}} e^{-r \tau^+_i} \right] = \int_0^{\Delta t^i_\lambda} e^{-r t} dF^\lambda_i(t) = e^{-r \Delta t^i_\lambda} F^\lambda_i(\Delta t^i_\lambda) + \int_0^{\Delta t^i_\lambda} F^\lambda_i(t) \, dt,
\]

where \( F^\lambda_i(t) \) is the cumulative distribution function of \( \tau^+_i \). For all \( t \in [0, \Delta t^i_\lambda) \) we have that \( \bar{\theta}(i + 1) \leq \bar{\theta}(q_t) \leq \bar{\theta}_i \) and it follows that \( 1 - \exp(-\lambda \bar{\theta}_{i+1} t) \leq F^\lambda_i(t) \leq 1 - \exp(-\lambda \bar{\theta}_i t) \). As result, we get the inequalities

\[
E_{q_t} \left[ \mathbb{I}_{\{\tau^+_i \leq \Delta t^i_\lambda\}} e^{-r \tau^+_i} \right] \leq e^{-r \Delta t^i_\lambda} F^\lambda_i(\Delta t^i_\lambda) + \frac{\lambda \bar{\theta}_i}{r + \lambda \bar{\theta}_i} - \left( 1 - \frac{r}{r + \lambda \bar{\theta}_i} \right) e^{-\lambda \bar{\theta}_i \Delta t^i_\lambda} e^{-r \Delta t^i_\lambda},
\]

\[
E_{q_t} \left[ \mathbb{I}_{\{\tau^+_i \leq \Delta t^i_\lambda\}} e^{-r \tau^+_i} \right] \geq e^{-r \Delta t^i_\lambda} F^\lambda_i(\Delta t^i_\lambda) + \frac{\lambda \bar{\theta}_{i+1}}{r + \lambda \bar{\theta}_{i+1}} - \left( 1 - \frac{r}{r + \lambda \bar{\theta}_{i+1}} \right) e^{-\lambda \bar{\theta}_{i+1} \Delta t^i_\lambda} e^{-r \Delta t^i_\lambda}.
\]

Note that we can write \( F^\lambda_i(\Delta t^i_\lambda) = 1 - E_{q_0} \left[ \mathbb{I}_{\{\tau^+_i > \Delta t^i_\lambda\}} \right] \) and use Lemma B5 to get its value.

If the mesh size \( \Delta q \) is small then the upper and lower bound above are closed to each other. Hence, we can get an asymptotically optimal approximation of \( E_{q_t} \left[ \mathbb{I}_{\{\tau^+_i \leq \Delta t^i_\lambda\}} e^{-r \tau^+_i} \right] \) using one of the two bounds. In our computations, we use

\[
E_{q_t} \left[ \mathbb{I}_{\{\tau^+_i \leq \Delta t^i_\lambda\}} e^{-r \tau^+_i} \right] \approx \frac{\lambda \bar{\theta}_i}{r + \lambda \bar{\theta}_i} - \frac{q_i}{q_{i+1}} \frac{\theta_H - \theta_L}{\theta_H - \theta_L} \left( 1 - \frac{r}{r + \lambda \bar{\theta}_i} \right) e^{-r \Delta t^i_\lambda}.
\]

After plugging the expressions in equations (d2) and (d3) back into the DP (d1) we get the recursion that we use to solve numerically the value function \( V(n, i) \). A few technical remarks about this DP recursion are now in order.

- First, note that if a sale occur within the time interval \( \Delta t^i_\lambda \) then the belief process will jump backward from \( q_i \) to \( q_i - \eta(q_i) \). Because of the discreteness of our mesh this value \( q_i - \eta(q_i) \) might not be a member of \( \mathcal{M} \). In this case, we replace \( q_i - \eta(q_i) \) by the closest value \( q_j \in \mathcal{M} \). Because \( V(n, q) \) is continuous and uniformly bounded on \( q \), this approximation is asymptotically exact as \( \Delta q \downarrow 0 \).
Second, the value of $\Delta t^\lambda_i$ above is equal to infinity for $i = 0$ and $i = M - 1$. This is a consequence of the fact that the HJB has two singularities at $q = 0$ and $q = 1$. Because the DP recursion works backward on $q$, we are only concerned with the singularity at $q = 1$ that defines one of the border conditions. To bypass this technical problem, we use the (left) continuity of $V(n, q)$ at $q = 1$ and the border condition $V(n, 1) = W(n, \theta_L)$ to define a new border condition at $q_{M-1} = 1 - \Delta q$ such that

$$V(n, M - 1) = W(n, \theta_L) - \Delta q \frac{dV(n, q)}{dq} \bigg|_{q=1}.$$ 

By Lemma B4 in Appendix B, we can compute recursively the derivative of $V(n, q)$ as follows.

$$\frac{dV(n, q)}{dq} \bigg|_{q=1} := V_q(n, 1) = \frac{V_q(n - 1, 1) - r (\theta_H - \theta_L) W(n, \theta_L) \phi' \left( \frac{r W(n, \theta_L)}{\theta_L} \right)}{1 + r \phi' \left( \frac{r W(n, \theta_L)}{\theta_L} \right)}$$

and $V_q(0, 1) = 0$.

In terms of computational complexity, we solve the optimization in (d1) using a line search.

**D2. Algorithm With the Optimal Stopping Option**

In this case, the only change that we need to introduce in the previous algorithm is to change the recursion in equation (d1) to

$$U(n, i) = \max \left\{ R, \max_{0 \leq \lambda \leq \Lambda} \mathbb{E}_{q_i} \left[ \mathbb{1}_{\{\tau^\lambda_i \leq \Delta t^\lambda_i\}} e^{-r \tau^\lambda_i} (p(\lambda) + U(n - 1, i - \eta(i))) + \mathbb{1}_{\{\tau^\lambda_i > \Delta t^\lambda_i\}} e^{-r \Delta t^\lambda_i} U(n, i + 1) \right] \right\},$$

with border conditions $U(0, i) = R$ and $U(n, M) = R$, for all $i = 1, \ldots, m$ and all $n$. Once $U(n, i)$ has been computed for all $i$, we can determine the threshold $q^*_n$ to be equal to $i^* \Delta q$, where $i^* = \min\{i : U(n, i) = R\}$. 

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