Insider Trading with Stochastic Valuation†

René Caldentey
Stern School of Business, New York University,
44 West Fourth Street, Suite 8-77, New York, NY 10012.

Ennio Stacchetti
Department of Economics, New York University,
19 West Fourth Street, 6th floor, New York, NY 10012.

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Abstract

This paper studies a model of strategic trading with asymmetric information of an asset whose value follows a Brownian motion. An insider continuously observes a signal that tracks the evolution of the asset fundamental value. At a random time a public announcement reveals the current value of the asset to all the traders. The equilibrium has two regimes separated by an endogenously determined time T. In $[0, T)$, the insider gradually transfers her information to the market and the market's uncertainty about the value of the asset decreases monotonically. By time T all her information is transferred to the market and the price agrees with the market value of the asset. In the interval $[T, \infty)$, the insider trades large volumes and reveals her information immediately, so market prices track the market value perfectly. Despite this market efficiency, the insider is able to collect strictly positive rents after T.

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1 Introduction

This paper studies a model of strategic trading with asymmetric information of an asset whose value follows a Brownian motion. An insider receives a flow of (noisy) signals that tracks the evolution of the asset value. Other traders receive no signals and can only observe the total volume of trade. There is uncertainty about the value of the asset before the insider gets the first signal, hence the first signal generates a lumpy informational asymmetry between the insider and the rest of the market participants. The signals the insider receives later are equally informative, but they contribute only marginally to the informational asymmetry. The information advantage continues until an unpredictable time when a public announcement reveals the current value of the asset to all the traders.

Kyle (1985) introduced a dynamic model of insider trading where an insider receives only one signal and the fundamental asset value does not change over time. Through trade, the insider progressively releases her private information to the market as she exploits her informational advantage. The market is also populated by many liquidity traders that are uninformed and trade randomly. At time 0, the insider observes the value of an asset. The same information is publicly released later, at time 1, to all market participants. In each trading period in the time interval $[0, 1]$, traders submit order quantities to a risk-neutral market maker who sets prices competitively and trades in his own account to clear the market. The market maker cannot observe individual trades, but can observe the total volume of trade in each trading period. The market maker also knows (in equilibrium) the strategy of the informed trader, and sets prices efficiently conditional on past and present volumes of trade.

Kyle constructs a linear equilibrium where in each period the price adjustment is proportional to the volume of trade, and the volume the insider trades is proportional to the gap between the asset value and the current market price. The market maker's estimate of the asset value, reflected in the current market price, improves over time. As the public announcement date approaches, this estimate converges to the value of the asset and the insider trades frantically in her desire to exploit any price differential.

Our model differs from Kyle's model in three important ways. First, the fundamental value of the asset follows a Brownian motion and therefore changes continuously over time. Second, in addition to the initial observation, the insider continuously receives a signal of the current fundamental value of the asset. Third, the public announcement date is unpredictable: it has an exponential distribution.

The first difference by itself is irrelevant. In Kyle's model it makes no difference whether at time 0 the insider observes the true value of the asset or just an unbiased signal. Moreover, the model where the insider observes the true value and the value of the asset follows a Brownian motion is formally equivalent to a model where the initial observation is an unbiased signal of the final value of the asset. But this feature of our model becomes important when it is combined with the second feature. Finally, the third feature removes the pressure in Kyle's model behind the trade frenzy that occurs as the announcement date approaches. In our model, where the announcement date is not deterministic, the insider has no urgency to exhaust all arbitrage opportunities, and release all her private information in the process, by a particular deadline. Thus, while it is evident that in Kyle's model the price will become efficient (in the sense that it incorporates all the available information) as time reaches the announcement date, it is unclear whether in our model the insider will ever fully reveal her private information.

Our model is not the first to introduce a public announcement with random time. Back and Baruch (2004) compare the models of Kyle (1985) and Glosten and Milgrom (1985). To facilitate the comparison, they adopt a Glosten and Milgrom model with a single long-lived insider (who times her transactions strategically) and a Kyle model with a random terminal time and a risky asset that takes only the values 0 or 1.

Our model includes various special cases. The value of the asset remains constant over time if the variance of its Brownian motion is reduced to 0. Since in our model the insider observes the initial value without noise, the signals that track the value of the asset over time becomes superfluous. This version of our model is similar to Kyle's model, where the insider is endowed only with an initial piece of private information, but with a random end time. Alternatively, we can specialize our model to give the insider no initial informational advantage. This is accomplished by informing *all* traders of the initial value of the asset. In this version of the model, the insider's informational advantage arises exclusively from her ability to observe the evolution of the asset value. This is an important model in its own right. An interesting question in this model is how the insider 'manages' the information asymmetry. For example, the insider could let the information asymmetry (the variance of the uninformed traders' estimate of the current value) grow to reach asymptotically a certain limit or without bound. The larger is the information asymmetry, the more likely it is that the market price will diverge substantially from the actual value of the asset, and therefore, the larger are the profitable arbitrage opportunities. Thus, in this model as well it is not evident how much of the insider's information is incorporated in the market price and how quickly this happens. We study this special case in the process of constructing an equilibrium for our general model. It turns out that in equilibrium the insider fully reveals her information as soon as she receives it. Hence, the market price equals the asset value at all times. Yet, the insider makes strictly positive profits. In independent work, Chau and Vayanos (2006) reach the same conclusion (for this case without initial informational asymmetry) in a slightly different model. They assume that the insider receives a flow of information, the asset pays a dividend, and there is no public announcement. In addition, they assume that the market maker continuously observes a noisy signal of the value of the asset. In the absence of this noisy signal, their model would be formally equivalent to ours. Chau and Vayanos (2006) limit attention to the steady state of their model and do not study how the equilibrium approaches the steady state. One implication of our results is that in the absence of an initial information asymmetry, the steady state is reached 'immediately' (as the period length goes to 0), so although Chau and Vayanos (2006) assume that trading has been taking place indefinitely, this is not needed.

We pause now to discuss related results in some of the seminal papers in the literature. In their celebrated paper, Grossman and Stiglitz (1980) study a trade model with asymmetric information, where consumers can acquire costly signals before they trade. They demonstrate that a rational expectations equilibrium does not exist if the cost of information is relatively low and there are no other sources of uncertainty besides the value of the risky asset (they also consider a model with supply uncertainty that does have an equilibrium). In a rational expectations equilibrium, the price is a sufficient statistic for the information of all the informed traders. Therefore, the informed traders enjoy no informational advantage and do not get compensated for the costly signals they acquire. Thus, in equilibrium, no consumer would incur the cost of acquiring information. But then, unexpectedly acquiring information would be profitable. Grossman and Stiglitz (1980) analyze a static general equilibrium model. Hellwig (1982) introduces a dynamic general equilibrium model with a risky asset whose dividends follow a Brownian motion. In order to achieve Walrasian market clearing while escaping the problematic features identified by Grossman and Stiglitz, he assumes that agents condition their demands on the current price, but that they *ignore* the informational content of that price (using only past prices to make inferences). Hellwig shows that in this model, an equilibrium exists. When the length of the period converges to 0, and therefore the price for the previous period contains almost as much information as the price for the current period, the informed traders' rents remain bounded away from zero. Moreover, as in our special case with no initial informational asymmetry, the price incorporates all the available information with (almost) no delay. Thus, in Hellwig's model, the informed traders get compensated and in equilibrium a fraction of them acquire costly information. With our simple demand protocol, with agents placing orders before learning the price, there is no need to resort to Hellwig's device of having consumers respond less than rationally to the current price. Like Hellwig, we find that information rents are bounded uniformly away from zero as the period length converges to zero, even though the difference in the information contained in this period's and last period's prices is also converging to zero. However, we do not assume perfect competition (our insider is a monopolist), and our model has a second source of uncertainty, the amount traded by liquidity traders, which is not present in Hellwig's model. So, while in Hellwig's model the total volume of trade perfectly reveals the information of the informed traders, in our model the liquidity traders' orders provide camouflage for the insider to conceal her trades. But in equilibrium, she does not.

The equilibrium of our general model has a striking feature. There is a time T, endogenously determined in equilibrium, by which the insider reveals all her information (if the public announcement has not yet occurred). Thus, even though there is no deterministic deadline, the price converges to the asset value at time T. Moreover, time T divides the equilibrium into two phases. As long as the public announcement does not occur, in the interval $[0, T)$ the insider gradually transfers her information to the market and the market's uncertainty about the value of the asset decreases to 0 monotonically. In the interval $[T, \infty)$, the insider trades large volumes and reveals her information immediately, so market prices track the asset value perfectly. Nevertheless, as we explained above, after T the insider collects strictly positive rents. In $[0, T)$ the insider is indifferent about her order quantities, though she trades according to a deterministic function of the current price and value of the asset. Therefore, she is indifferent about purchasing an additional share of the asset now or in the future, even though she discounts future payoffs. This is so because the market compensates her more generously in the future for any price differential. In $[T, \infty)$, her compensation, as a function of the price differential, is constant over time, and thus she is eager to cash in her rents as soon as arbitrage opportunities materialize.

We conclude the Introduction by discussing a small subset of the vast literature on insider trading.[1] Two of the most influential papers in the area of strategic trading with asymmetric information are Kyle (1985) and Glosten and Milgrom (1985). These classic papers formalize Bagehot (1971) intuitive story that the market provides a mechanism to compensate informed traders for their superior information, while liquidity traders are willing to make (small) losses for the benefit of carrying out their transactions immediately. Glosten and Milgrom study a market where multiple insiders and noisy traders place orders sequentially (one at a time) to a risk-neutral and competitive specialist, who sets bid and ask prices. If the proportion of insiders is high and/or the quality of their private information is too good then the resulting bid-ask spread is too wide and the market shuts down. However, when there are few insiders with limited private information, the market does operate. Moreover, the bid-ask spread converges to zero as time goes by. Three notable extensions of the Glosten and Milgrom model are Easley and O'Hara (1987) that study the impact of block trading on the bid-ask spread, Glosten (1989) that considers a monopolist specialist that maximizes expected profits, and Dasgupta and Prat (2005) that analyze a model where some insiders receive superior signals and informed traders care about their reputations. In this last paper, in equilibrium, there is herd behavior and prices do not converge to the asset value.

More closely related to our work is the literature that builds upon Kyle (1985). In a continuous-time setting, Back (1992) considers a general distribution for the insider's private signal (Kyle assumes a normal distribution) and prove the existence and *uniqueness* of an equilibrium pricing rule. Holden and Subrahmanyam (1992) and Foster and Viswanathan (1996) consider a market with multiple competing insiders. Holden and Subrahmanyam assume a symmetric model in which the insiders are endowed with the same private information and show that insiders' competition creates a strong-form efficient market almost immediately. Foster and Viswanathan allow for heterogenous information among the insiders and show that this asymmetry reduces the degree of competition among them, and hence, the efficiency of market prices. In a one-period model with heterogeneous insiders, Spiegel and Subrahmanyam (1992) replace Kyle's uninformed liquidity traders (and their exogenous price-inelastic noisy trades) with strategic utility-maximizing agents trading for hedging purposes. They demonstrate that the welfare of uninformed traders *decreases* with the number of insiders. In a multi-period setting, Mendelson and Tunca (2004) propose an alternative endogenous liquidity trading model allowing for various type of market information; some available exclusively to the insider (tractable information) and some unavailable to all market participants (intractable information) that gets partially released over time. In contrast to Kyle's model, Mendelson

[1] For a comprehensive review of this literature, and its connection to the broader market microstructure theory, we refer the reader to O'Hara (1997), Brunnermeier (2001), Biais et al. (2005), Amihud et al. (2006) and references therein.

and Tunca assume that the insider's private information acquisition is costly. The volume of uninformed trades decreases with market uncertainty, forcing the insider to reduce her own volume of trade. As a result, less information is acquired by the insider and information is spread out into the market more slowly.

The rest of the paper is organized as follows. Section 2 introduces the model in full generality and deals with its discrete-time version. We construct the unique linear Markovian equilibrium for this model and derive its asymptotic properties. Here we also study the special case where the fundamental value of the asset is constant over time. In Section 3 we study the limit of the discrete-time equilibrium as the length of a period goes to zero, including the special case when the value of the asset does not change. This exercise suggests an equilibrium for the continuous-time model that we pursue in Section 4. The equilibrium is composed of two distinct phases that we show paste smoothly. In Section 5 we discuss features and extensions of the continuous-time equilibrium, and argue that the continuous-time solution of Section 4 is indeed a good approximation for the equilibrium of the discrete-time model.

2 Discrete Time Model

We introduce first a continuous-time model, where the fundamental value of the asset and the liquidity trader's (target) holding of the asset are described by continuous time stochastic processes. In the discrete time model that we study in this Section, trading orders are restricted to take place only at discrete times; the time between two trading dates is a period. The continuous time model we study in Section 4 removes this institutional constraint. We then construct a linear Markovian equilibrium for the discrete time model.

The market participants are the insider, the market maker and a (large) number of liquidity traders. The insider (and only her) continuously receives private information about the fundamental value of the asset. Every period n, the insider and the liquidity traders place buy/sell orders for a quantity of the asset. An order is a binding contract to buy/sell a quantity of the asset (the 'size of the order') at a price determined by the market maker. At the end of the period, after observing the total volume of trade, the market maker sets the price p_n and trades the necessary quantity to close all orders. This trading process continues until an unpredictable random time τ when the fundamental value of the asset becomes public knowledge. At this time, the market price immediately matches the fundamental value and the insider loses her informational advantage.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with two independent standard Brownian motions B_t^v and B_t^y, where $t \in [0, \infty)$ denotes (calendar) time. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the usual filtration generated by (B^v, B^y). The value of the fundamental at time t is $\bar{V}_t$, which we assume evolves over time as an arithmetic Brownian motion

$$\mathrm{d}\bar{V}_t = \bar{\sigma}_v \, \mathrm{d}B_t^v,$$

for some constant $\bar{\sigma}_v \geq 0$. The initial value $\bar{V}_0$ is drawn from a normal distribution with mean $\bar{v}_0$ and variance $\bar{\Sigma}_0$. The insider alone observes the (stochastic) evolution of $\bar{V}_t$ during $t \in [0, \tau)$.

The market maker and the rest of the market participants only know the distribution of $\bar{V}_0$. The random time τ when the value of the fundamental becomes public knowledge is exponentially distributed with mean $1/\theta$, and is independent of (B^v, B^y).

In the discrete time model the market maker opens the floor for trading only at discrete times $\{t_n\}_{n\geq 0}$. We assume that these trading dates are evenly spaced over time (*e.g.*, once a day) so that $t_n = n\,\Delta$ for some positive constant Δ. The interval of time $[t_n, t_{n+1})$ is called period n. For $t > 0$, let $\lfloor t \rfloor$ denote the largest integer n such that $n\Delta \leq t$. The period when the fundamental value becomes public knowledge is $\nu = \lfloor \tau \rfloor$, and we assume that the announcement always occur at the end of the period. The discrete random variable ν has a geometric distribution with probability of failure $q = e^{-\theta\Delta}$.

During the trading period $[0, \tau)$, the insider and the liquidity traders simultaneously place their orders at the begining of every period. Liquidity trades are not strategic agents and they are motivated to trade for idiosyncratic reasons. They trade so as to match a moving target for their net holding of the asset. Their holding target Y_t at time t follows an arithmetic Brownian motion

$$\mathrm{d}Y_t = \sigma_y \,\mathrm{d}B_t^y$$

for some constant $\sigma_y > 0$. The insider and the market maker know $\bar{\sigma}_v$ and σ_y.

At trading time t_n, the liquidity traders place orders for a total of $y_n = Y_{t_n} - Y_{t_{n-1}}$. While the insider starts trading at time 0, the moment she starts observing her private information, the liquidity traders have been trading prior to this time and at time 0, before they place their orders, they already hold $Y_{-\Delta}$ shares of the asset. Without loss of generality, hereafter we assume that $Y_{-\Delta} = 0$. Given that $\{Y_t\}$ follows a Brownian motion, $\{y_n\}$ is a sequence of i.i.d. normal random variables with mean 0 and variance $\Sigma_y = \sigma_y^2\Delta$. Let x_n denote the order placed by the insider at trading time t_n, and let X_t be her net holding at time t (including the order she placed for the current period). That is, $X_t = 0$ for $t < 0$ and

$$X_t = \sum_{n=0}^{\lfloor t \rfloor} x_n \quad \text{for } t \geq 0.$$

Similarly, let $z_n = x_n + y_n$ denote the total volume of trade at trading time t_n, and let $Z_t = 0$ for $t < 0$ and

$$Z_t = \sum_{n=1}^{\lfloor t \rfloor} z_n \quad \text{for } t \geq 0.$$

Note that at each trading time t_n, $Z_{t_n} = X_{t_n} + Y_{t_n}$ is the total holding of the asset (including the current orders) by the insider and liquidity traders.

At the begining of each period n before the fundamental value becomes public knowledge, the market maker commits to a pricing rule (that is legally binding). The rule specifies the price p_n for the current period's transactions as a function of the total volume of trade z_n. The insider and the liquidity traders place their orders after the rule is announced. All orders are executed at the end of the period. To understand the filtration we define below, note that while the market maker commits to a rule before knowing the current period's volume of trade, the actual price is determined after learning the volume of trade. Let the price process $\{P_t\}$ be defined as follows: $P_t = p_{\lfloor t \rfloor}$ for $t \in [0, (\nu+1)\Delta)$, and $P_t = \bar{V}_{\lfloor t \rfloor \Delta}$ for $t \in [(\nu+1)\Delta, \infty)$.

The market maker observes the public history of prices and (total) volumes of trade. His information is represented by the filtration $\mathbb{F}^M = \{\mathcal{F}_t^M\}_{t\geq 0}$, where $\mathcal{F}_t^M = \sigma(P_s : 0 \leq s < t) \vee \sigma(Z_s : 0 \leq s \leq t)$ is the sigma algebra generated by the history of prices and holdings up to time t. Since information is only revealed at trading times t_n, in period n, the market maker knows the history $h_n^M = (z_0, p_0, \ldots, z_{n-1}, p_{n-1}, z_n)$.[2] Each period, the market maker learns the volume of trade before he sets the market price. The insider's information includes the public history of prices and trades, and the private history of orders she has placed and fundamental values she has observed. Her information is represented by the filtration $\mathbb{F}^I = \{\mathcal{F}_t^I\}_{t\geq 0}$, where $\mathcal{F}_t^I = \sigma((P_s, X_s, Z_s) : 0 \leq s < t) \vee \sigma(\bar{V}_s : 0 \leq s \leq t)$. That is, at trading time t_n, she knows the history $h_n^I = (\bar{V}_0, x_0, z_0, p_0, \bar{V}_1, \ldots, x_{n-1}, z_{n-1}, p_{n-1}, \bar{V}_n)$. The insider places her order at the begining of the period, after observing the current value of the fundamental.

The insider and the market maker are risk neutral and discount future payoffs by the discount factor $\delta > 0$. Given a trajectory $\{X_t\}$ for the insider's holding and $\{P_t\}$ for market prices, the insider's payoff is

$$\Pi(P, X) = \sum_{n=0}^{\nu} [e^{-\nu\delta\Delta}\bar{V}_{t_{\nu+1}} - e^{-n\delta\Delta}p_n]x_n.$$

With uncertainty, the risk-neutral insider maximizes the expected value of $\Pi(P, X)$. Let V_n denote the insider's expected discounted value of the fundamental value at time τ given that the fundamental value has not been publicly revealed yet and her information at time t_n. That is

$$V_n = \mathbb{E}[e^{-\delta(t_\nu - t_n)}\bar{V}_{t_{\nu+1}} \mid \nu \geq n,\, \bar{V}_{t_n}] = \mathbb{E}[e^{-(\nu-n)\delta\Delta} \mid \nu \geq n]\bar{V}_{t_n} = \left[\frac{1-q}{1-\rho}\right]\bar{V}_{t_n},$$

where $\rho = qe^{-\delta\Delta} = e^{-(\theta+\delta)\Delta}$. V_n represents the current *intrinsic value* of the asset. Let $\sigma_v = \bar{\sigma}_v(1-q)/(1-\rho)$. Then

$$V_{n+1} = V_n + W_n,$$

where $\{W_n\}$ is a sequence of i.i.d. normal random variables with mean 0 and variance $\Sigma_v = \sigma_v^2\Delta$.

Definition 1 *A strategy for the market maker is an $\mathcal{F}_t^M$-adapted process $\{P_t\}_{0\leq t\leq\tau}$, and a strategy for the insider is an $\mathcal{F}_t^I$-adapted process $\{X_t\}_{0\leq t\leq\tau}$. The profile (P, X) is an equilibrium if (i) for any $n \geq 0$*

$$P_{t_n} = \mathbb{E}[e^{-\delta(t_\nu - t_n)}\bar{V}_{t_{\nu+1}} \mid \nu \geq n,\, X,\, \mathcal{F}_{t_n}^M] = \mathbb{E}[V_n \mid X,\, \mathcal{F}_{t_n}^M],$$

and (ii) given P, $\mathbb{E}[\Pi(P, X)]$ is bounded above and $\{X_t\}$ maximizes $\mathbb{E}[\Pi(P, X)]$.

[2] In the tradition of game theory, we are including the trajectory of prices in the market maker's history, though he is not a strategic player. If he were a strategic player, he could deviate and set prices out of equilibrium. In that case, for a given history of trades, he would make different inferences for different trajectory of prices.

We do not model explicitly competition among market makers, but we implicitly assume that our market maker competes in prices with other market makers. In equilibrium, this competition drives the market maker to set the price equal to the expected value of the asset market value at time $t_{\nu+1}$ given the history of information he has observed so far and the insider's trading strategy. The market maker only uses his history to make inferences about the past choices of the insider and therefore, indirectly, about the distribution of $\bar{V}_t$. The insider chooses her strategy so as to maximize her expected discounted profit, given that she knows how the market maker will choose prices.

In equilibrium, the market maker's expected payoff is 0 and the insider's expected payoff is positive. In expectation, the insider's profits are equal to the liquidity traders' losses (see below the third remark after Lemma 3). In our model the liquidity traders are very primitive and are not sensitive to losses. A more realistic assumption would require that the volume they trade decreases with the losses they make.

The model is not exactly a game and our definition of an equilibrium does not coincide with that of a Nash equilibrium. However, Kyle (1985) suggests that this definition would coincide with that of a Nash equilibrium in a game where two market makers simultaneously bid prices after observing the current volume of trade and the winner gets the right to clear the market at the wining price. To avoid collusion, we can assume that there is a large population of market makers and that each market maker participates in the bidding game only once.

We will restrict attention to Markovian equilibria with a particular state space. At the begining of period n, before the market maker observes the volume of trade, the state is $(n, v_{n-1}, \Sigma_{n-1})$, where v_{n-1} is the market maker's estimate of V_n and Σ_{n-1} is the variance of this estimate. Note that since $V_n = V_{n-1} + W_{n-1}$, and W_{n-1} is an independent random variable with mean 0 and variance Σ_v, v_{n-1} coincides with the market maker's estimate of V_{n-1}, but as an estimate of V_{n-1}, the variance is $\Sigma_{n-1} - \Sigma_v$. Since the market maker's estimate of V_n depends on the strategy X of the insider, the state and corresponding Markovian strategy profile need to be specified simultaneously.

Definition 2 *A strategy profile (P, X) is Markovian if for each n, the insider's order x_n and the market maker's price p_n depend only on the current state $(n, v_{n-1}, \Sigma_{n-1})$ and the signals they receive in period n, V_n for the insider and z_n for the market maker. In this case we write $x_n = X_n(v_{n-1}, \Sigma_{n-1}, V_n)$ and $p_n = P_n(v_{n-1}, \Sigma_{n-1}, z_n)$. The state evolves according to the following transition rule*

$$v_n = \mathbb{E}[V_{n+1}|v_{n-1}, \Sigma_{n-1}, z_n, X] \quad \text{and} \quad \Sigma_n = \mathbb{E}[(V_{n+1} - v_n)^2|v_{n-1}, \Sigma_{n-1}, z_n, X], \text{ where}$$
$$v_{-1} = \left[\frac{1-q}{1-\rho}\right] \bar{v}_0 \quad \text{and} \quad \Sigma_{-1} = \left[\frac{1-q}{1-\rho}\right]^2 \bar{\Sigma}_0$$

If (P, X) is a Markovian strategy profile, let

$$\Pi_n(v_{n-1}, \Sigma_{n-1}, V_n) = \mathbb{E}[\sum_{k=n}^{\nu}(V_n - e^{-(k-n)\Delta} p_k) x_k \mid v_{n-1}, \Sigma_{n-1}, V_n, (P, X)]$$

be the insider's expected payoff for the transactions made from period n until the fundamental value is publicly revealed, discounted to the end of period n, when the current state is $(n, v_{n-1}, \Sigma_{n-1})$ and the insider observes V_n. When (P, X) is a Markovian equilibrium, $p_n = v_n$ for all n.

Below we construct linear Markovian equilibria (P, X) such that

$$P_n(v_{n-1}, \Sigma_{n-1}, z_n) = v_{n-1} + \lambda_n(\Sigma_{n-1})z_n \quad \text{and} \quad X_n(v_{n-1}, \Sigma_{n-1}, V_n) = \beta_n(\Sigma_{n-1})(V_n - v_{n-1}), \tag{1}$$

where $\{\lambda_n\}$ and $\{\beta_n\}$ are sequences of functions $\lambda_n, \beta_n : \mathbb{R}_{++} \to \mathbb{R}_+$. In order to analyze these strategies, we need a couple of preliminary results.

Each period n, the market maker uses the new observation z_n to update his prior distribution on V_n. When the insider chooses her order according to the rule $x_n = \beta_n(\Sigma_{n-1})(V_n - p_{n-1})$, (V_n, z_n) has a multinormal joint distribution. The Projection Theorem (see Lemma 2 below) implies that conditional on z_n, V_n has a normal distribution whose variance is *independent* of z_n. Thus, in equilibrium, the trajectory $\{\Sigma_n\}$ is *deterministic* and independent of the history of trades. Therefore the sequences $\{\lambda_n\}$ and $\{\beta_n\}$ are also deterministic and hereafter we drop the arguments Σ_{n-1} (we also drop this argument in the function Π_n and write $\Pi_n(v_{n-1}, V_n)$). Moreover, since in equilibrium $p_n = v_n$ for all n, hereafter we do not differentiate these two variables.

Assume that the market maker's strategy P satisfies (1) for some sequence $\{\lambda_n\} \subset \mathbb{R}_{++}$. Given P, the insider confronts each period n a non-stationary dynamic programming problem. If $\{\lambda_n\}$ satisfies a certain transversality condition, the sequence $\{\Pi_n\}$ satisfies a Bellman equation.

Let $\mathbb{B}$ be the space of continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbb{B}^\infty$ be the space of sequences $\hat{\Pi} = \{\hat{\Pi}_n\}$ such that $\hat{\Pi}_n \in \mathbb{B}$ for all $n \geq 0$. Recall that $\{(y_n, W_n)\}$ is an independent sequence of i.i.d. normal random variables, with 0 mean and covariance matrix

$$\begin{bmatrix} \Sigma_y & 0 \\ 0 & \Sigma_v \end{bmatrix}$$

For any $\hat{\Pi} \in \mathbb{B}^\infty$, $n \geq 0$ and $(p, V) \in \mathbb{R}^2$, let

$$b_n(\hat{\Pi}_{n+1})(p, V) = \max_x \, (V - p - \lambda_n x)x + \rho\mathbb{E}\big[\hat{\Pi}_{n+1}(p + \lambda_n(x + y_n), V + W_n)\big],$$

and let $B(\hat{\Pi})$ be the sequence of functions $\{B(\hat{\Pi})_n\}$, where $B(\hat{\Pi})_n = b_n(\hat{\Pi}_{n+1})$ for each $n \geq 0$. When $\{\lambda_n\}$ converges to 0 'too fast' (for example, faster than $\{\omega^n\}$ for some $0 \leq \omega < \rho$), $\Pi_n(p, V)$ is unbounded. But if, for example, $\lambda_n \geq \omega^n$ for some $\omega \geq \rho$, each $\Pi_n(p, V)$ is bounded and Π satisfies the Bellman equation $\Pi = B(\Pi)$ (that is, Π is a fixed point of B).

Lemma 1 (Optimal Profits) *Assume that $\{P_n\}$ satisfies (1) for the sequence $\{\lambda_n\} \subset \mathbb{R}_{++}$. Let*

$$S = \sum_{n=1}^{\infty} \frac{\rho^n}{\lambda_n}.$$

If $S = \infty$ then $\Pi_n(p, V) = \infty$ for all $n \geq 0$ and $(p, V) \in \mathbb{R}^2$. If $S < \infty$ and there is $M > 0$ such that $\lambda_n < M$ and $\rho\lambda_n/\lambda_{n+1} \leq 1$ for all $n \geq 0$, then there exist positive sequences $\{\alpha_n\}$ and $\{\gamma_n\}$ such that $\lambda_n\alpha_{n+1} \leq 1/2$ and $\rho\Pi_n(p, V) = \alpha_n(p - V)^2 + \gamma_n$ for all $n \geq 0$ and $(p, V) \in \mathbb{R}^2$, and $\Pi = B(\Pi)$.

If for some $\omega \in (\rho, 1]$, the sequence $\{\lambda_n\}$ satisfies $\lambda_{n+1}/\lambda_n \geq \omega$ for all $n \geq 0$, then $\sum \rho^n/\lambda_n < \infty$ and Π_n is well defined for all $n \geq 0$. However, the condition $\lambda_{n+1}/\lambda_n > \rho$ for all $n \geq 0$ may not be sufficient. For example, if $\lambda_n = \rho^n(1 + [n + 1]^{-1})$ for all $n \geq 0$, then $\sum \rho^n/\lambda_n = \infty$ and $\Pi_n \equiv \infty$ for all $n \geq 0$, even though $\lambda_{n+1}/\lambda_n > \rho$ for all $n \geq 0$.

Lemma 2 (Projection Theorem for Normal Random Variables) *Consider a normally distributed two-dimensional random vector (ξ, η). Then, ξ admits the following factorization*

$$\xi = \mathbb{E}[\xi] + \frac{\mathbb{C}\mathrm{ov}[\xi, \eta]}{\mathbb{V}\mathrm{ar}[\eta]} (\eta - \mathbb{E}[\eta]) + \epsilon,$$

where ϵ is a normally distributed random variable independent of η with mean $\mathbb{E}[\epsilon] = 0$ and variance $\mathbb{V}\mathrm{ar}[\epsilon] = \mathbb{V}\mathrm{ar}[\xi]\,(1-r^2)$, and r is the correlation coefficient between ξ and η. It follows that

$$\begin{aligned} \mathbb{E}[\xi|\eta = z] &= \mathbb{E}[\xi] + \frac{\mathbb{C}\mathrm{ov}[\xi, \eta]}{\mathbb{V}\mathrm{ar}[\eta]} (z - \mathbb{E}[\eta]) \quad \textit{and} \\ \mathbb{V}\mathrm{ar}[\xi|\eta = z] &= \mathbb{V}\mathrm{ar}[\epsilon] = \mathbb{V}\mathrm{ar}[\xi]\,(1 - r^2). \end{aligned}$$

An important conclusion of the Projection Theorem is that $\mathbb{V}\mathrm{ar}[\xi|\eta = z]$ is independent of z. In the context of our linear Markovian equilibrium, this fact implies that the evolution of the variance Σ_n is independent of the volumes of trade and the insider's trading decisions.

Theorem 1 *There exist unique sequences $\{\lambda_n\}$, $\{\beta_n\} \in \mathbb{R}_{++}$ such that the linear strategy profile (P, X) defined by (1) is a Markovian equilibrium. In equilibrium, $\{\Sigma_n\}$ is a deterministic trajectory that is not affected by the (stochastic) choices of the insider and the market maker. Furthermore, there exist sequences $\{\alpha_n\}, \{\gamma_n\} \subset \mathbb{R}_{++}$ such that the insider's expected payoff for (P, X) satisfies*

$$\rho\,\Pi_n(p, \Sigma, V) = \alpha_n \left(V - p\right)^2 + \gamma_n \qquad \textit{for all } n \geq 0. \tag{2}$$

Proof: The proof requires to establish three facts: (i) assuming that X_n satisfies (1) for some β_n, there exists a constant λ_n such that $\mathbb{E}[V_{n+1} \mid v_{n-1}, \Sigma_{n-1}, z_n, X_n] = v_{n-1} + \lambda_n z_n$; (ii) assuming that $\{P_n\}$ satisfies (1) for some sequence $\{\lambda_n\}$, $\{\Pi_n\}$ satisfies (2) for some sequence $\{(\alpha_n, \gamma_n)\}$ and $\{X_n\}$ satisfies (1) for some sequence $\{\beta_n\}$; and (iii) there are unique sequences $\{\lambda_n\}$ and $\{\beta_n\}$ such that the corresponding strategy profile (P, X) defined by (1) is a Markovian equilibrium.

Assume that X_n is given by (1) for some constant β_n, and that $p_{n-1} = v_{n-1}$. Define the random variables $\xi = V_n - p_{n-1}$ and $\eta = \beta_n(V_n - p_{n-1}) + y_n$. Conditional on (v_{n-1}, Σ_{n-1}), the vector (ξ, η) is normally distributed with

$$\begin{aligned}
\mathbb{E}[\xi] &= 0, \quad \mathbb{V}\text{ar}[\xi] = \Sigma_{n-1} \\
\mathbb{E}[\eta] &= 0, \quad \mathbb{V}\text{ar}[\eta] = \beta_n^2\Sigma_{n-1} + \Sigma_y \\
\mathbb{C}\text{ov}(\xi, \eta) &= \mathbb{E}[\xi(\beta_n\xi + y_n)] = \beta_n\Sigma_{n-1}, \quad \text{and} \quad r = \frac{\beta_n\sqrt{\Sigma_{n-1}}}{\sqrt{\beta_n^2\Sigma_{n-1} + \Sigma_y}} .
\end{aligned}$$

By the Projection Theorem,

$$\begin{aligned}
v_n &= \mathbb{E}[V_{n+1} \mid \eta = z_n] = p_{n-1} + \mathbb{E}[(V_n - p_{n-1}) + W_n \mid \eta = z_n] = p_{n-1} + \frac{\beta_n\Sigma_{n-1}}{\beta_n^2\Sigma_{n-1} + \Sigma_y} z_n \\
\mathbb{V}\text{ar}[V_n \mid \eta = z_n] &= \mathbb{V}\text{ar}[\xi \mid \eta = z_n] = \Sigma_{n-1}\left[1 - \frac{\beta_n^2\Sigma_{n-1}}{\beta_n^2\Sigma_{n-1} + \Sigma_y}\right] = \frac{\Sigma_{n-1}\Sigma_y}{\beta_n^2\Sigma_{n-1} + \Sigma_y} .
\end{aligned}$$

Therefore,

$$\Sigma_n = \mathbb{V}\text{ar}[V_{n+1} \mid \eta = z_n] = \mathbb{V}\text{ar}[V_n + W_n \mid \eta = z_n] = \Sigma_v + \frac{\Sigma_{n-1}\Sigma_y}{\beta_n^2\Sigma_{n-1} + \Sigma_y} , \tag{3}$$

and Σ_n is independent of z_n. Since in equilibrium, $p_n = P_n(p_{n-1}, \Sigma_{n-1}, z_n) \equiv v_n$ it follows that $P_n(p_{n-1}, \Sigma_{n-1}, z_n)$ satisfies (1) with

$$\lambda_n = \frac{\beta_n\Sigma_{n-1}}{\beta_n^2\Sigma_{n-1} + \Sigma_y} . \tag{4}$$

Now assume that $\{P_n\}$ satisfies (1) for some sequence $\{\lambda_n\}$ such that $\sum \rho^n/\lambda_n < \infty$ and $\rho\lambda_n/\lambda_{n+1} \leq 1$ for all $n \geq 1$. Then, by Lemma 1, there exist a sequence $\{(\alpha_n, \gamma_n)\}$ such that $\{\Pi_n\}$ satisfies (2). Therefore, in period n, the insider's expected value $\Pi_n(p_{n-1}, \Sigma_{n-1}, V_n)$ is

$$\begin{aligned}
& \max_x \ \mathbb{E}\left[(V_n - P_n(p_{n-1}, \Sigma_{n-1}, x + y_n))x + \rho\Pi_{n+1}(P_n(p_{n-1}, \Sigma_{n-1}, x + y_n), \Sigma_n, V_{n+1}) \mid V_n\right] \\
= & \max_x \ \mathbb{E}\left[(V_n - p_{n-1} - \lambda_n(x + y_n))x + \alpha_{n+1}(V_n + W_n - p_{n-1} - \lambda_n(x + y_n))^2 + \gamma_{n+1}\right] \\
= & \max_x \ \left[(V_n - p_{n-1} - \lambda_n x)x + \alpha_{n+1}(\lambda_n^2 x^2 - 2\lambda_n x(V_n - p_{n-1})) + C\right],
\end{aligned} \tag{5}$$

where $C = \alpha_{n+1}((V_n - p_{n-1})^2 + \Sigma_v + \lambda_n^2\Sigma_y) + \gamma_{n+1}$ is independent of x. This is the Bellman equation for period n; the right-hand side of (5) is precisely $b_n(\Pi_{n+1})(p_{n-1}, V_n)$. By Lemma 1, $\lambda_n\alpha_{n+1} < 1$, so the quadratic objective function is a concave function of x and the optimal solution is obtained from the first-order condition:

$$x^* = \beta_n(V_n - p_{n-1}) \quad \text{where} \quad \beta_n = \frac{1 - 2\lambda_n\alpha_{n+1}}{2\lambda_n(1 - \lambda_n\alpha_{n+1})} . \tag{6}$$

Thus X_n defined by (1) is indeed the insider's best reply function.

Equations (5) and (6) imply that

$$\Pi_n(p_{n-1}, \Sigma_{n-1}, V_n) = \frac{(V_n - p_{n-1})^2}{4\lambda_n(1 - \lambda_n\alpha_{n+1})} + \alpha_{n+1}(\Sigma_v + \lambda_n^2\Sigma_y) + \gamma_{n+1}.$$

That is

$$\frac{\alpha_n}{\rho} = [4\lambda_n(1-\lambda_n\alpha_{n+1})]^{-1} \quad (7)$$

$$\frac{\gamma_n}{\rho} = \gamma_{n+1} + \alpha_{n+1}(\Sigma_v + \lambda_n^2\Sigma_y). \quad (8)$$

Conditions (3), (4) and (6) – (8) define recursively the sequence $\{(\Sigma_n, \lambda_n, \beta_n, \alpha_n, \gamma_n)\}$. As we will see below, given Σ_{-1}, each sequence is uniquely identified by the choice of β_0. However, the sequence becomes infeasible (for example, $\beta_n < 0$ for some n) if β_0 is not chosen properly. There is a unique choice β_0^* that leads to a feasible sequence that also satisfies $\sum \rho^n/\lambda_n < \infty$. By Lemma 1, in this case Π satisfies (2) and therefore the linear Markovian strategy (P, X) corresponding to $\{(\lambda_n, \beta_n)\}$ is an equilibrium. All other choices of β_0 lead to infeasible sequences or to sequences that satisfy $\sum \rho^n/\lambda_n = \infty$, and therefore, by Lemma 1, are not consistent with equilibrium. ▮

Starting from (Σ_{-1}, β_0), we now recursively construct the sequence $\{(\Sigma_n, \beta_{n+1})\}$ and establish the properties invoked at the end of the previous proof. Equations (6) and (4) imply that

$$\alpha_{n+1} = \frac{1-2\lambda_n\beta_n}{2\lambda_n(1-\lambda_n\beta_n)} = \frac{\Sigma_y^2 - \beta_n^4\Sigma_{n-1}^2}{2\beta_n\Sigma_{n-1}\Sigma_y}.$$

Combining this equation with (7) and (4), we obtain

$$\alpha_n = \frac{\rho}{4\lambda_n(1-\lambda_n\alpha_{n+1})} = \frac{\rho(1-\lambda_n\beta_n)}{2\lambda_n} = \frac{\rho\Sigma_y}{2\beta_n\Sigma_{n-1}}. \quad (9)$$

The last two equations (with the time index shifted by 1) imply that

$$\frac{\Sigma_y^2 - \beta_n^4\Sigma_{n-1}^2}{2\beta_n\Sigma_{n-1}\Sigma_y} = \frac{\rho\Sigma_y}{2\beta_{n+1}\Sigma_n} \quad \text{or} \quad \beta_{n+1}\Sigma_n = \rho\beta_n\Sigma_{n-1}\left[\frac{\Sigma_y^2}{\Sigma_y^2 - \beta_n^4\Sigma_{n-1}^2}\right]. \quad (10)$$

Equations (3) and (10) define (Σ_n, β_{n+1}) as a function of (Σ_{n-1}, β_n). The sequence $\{(\lambda_n, \alpha_n, \gamma_n)\}$ can be derived afterwards, using equations (4), (7) and (8), once the whole sequence $\{(\Sigma_n, \beta_{n+1})\}$ has been computed first. To compute the sequence $\{(\Sigma_n, \beta_{n+1})\}$ recursively, it is convenient to introduce the following change of variables

$$A_n = \frac{\Sigma_{n-1}}{\Sigma_v} \quad \text{and} \quad B_n = \frac{\beta_n\Sigma_{n-1}}{\sqrt{\Sigma_y\Sigma_v}}.$$

Then, equations (3) and (10) imply that $(A_{n+1}, B_{n+1}) = F(A_n, B_n)$, where

$$F_A(A_n, B_n) = 1 + \frac{A_n^2}{A_n + B_n^2} \qquad \text{and} \qquad F_B(A_n, B_n) = \rho\left[\frac{A_n^2\, B_n}{A_n^2 - B_n^4}\right].$$

Let

$$G_1(A) = \sqrt{\frac{A}{A-1}}, \quad G_2(A) = \sqrt{A}\,[1-\rho]^{1/4} \quad \text{and} \quad G_3(A) = \sqrt{A}.$$

Since in equilibrium $\beta_n > 0$ for all n, a point (A, B) is *feasible* only if $F_B(A, B) \geq 0$, that is, only if $B \leq G_3(A)$. The function G_1 is defined so that $F_A(A, G_1(A)) = A$. If $B > G_1(A)$,

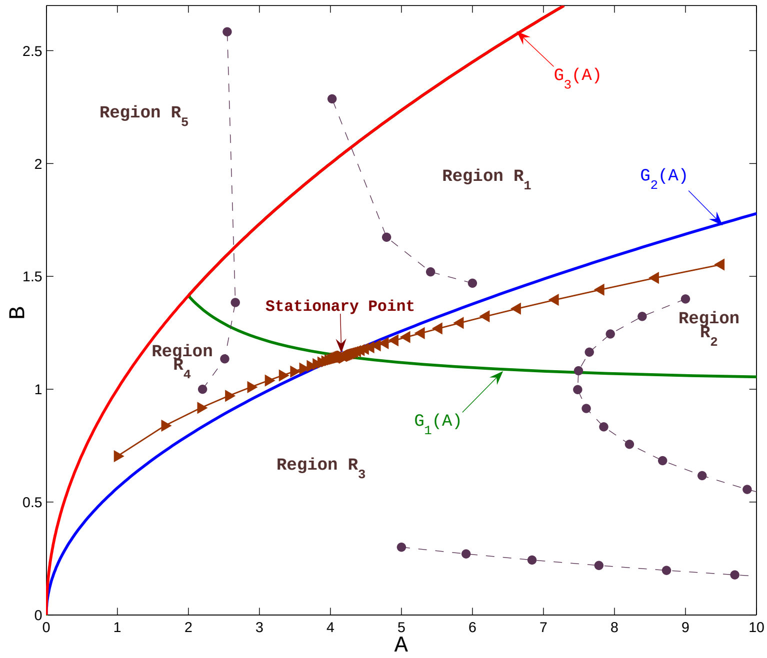

Figure 1: Partition induced by the functions G_1, G_2 and G_3.

then $F_A(A, B) < A$, and if $B < G_1(A)$, then $F_A(A, B) > A$. Similarly, the function G_2 is defined so that $F_B(A, G_2(A)) = B$. If $B > G_2(A)$, then $F_B(A, B) > B$, and if $B < G_1(A)$, then $F_B(A, B) < B$. As Figure 1 above shows, the graphs of these functions partition the (A, B) space into 5 regions. In R_1, $F(A, B)$ is always to the left and higher than (A, B), and any sequence $\{(A_n, B_n)\}$ with initial point (A_0, B_0) in this region eventually crosses the graph of G_3 and becomes infeasible. In R_2, $F(A, B)$ is always to the left and lower than (A, B). In R_3, $F(A, B)$ is always to the right and lower than (A, B). In R_4, $F(A, B)$ is always to the right and higher than (A, B). R_5 is the region of infeasible points. In Figure 1 we have also plotted four sequences, each starting in a different region. A sequence that remains feasible must start in R_2, R_3 or R_4, and any sequence that starts in R_3 always remain feasible. But not all sequences that start in R_2 or R_4 remain feasible. Sequences that start in R_1 always become infeasible.

By definition, the intersection of the graphs of G_1 and G_2 define a stationary point $(\hat{A}, \hat{B})$ such that $(\hat{A}, \hat{B}) = F(\hat{A}, \hat{B})$. This stationary point is

$$\hat{A} = \frac{1 + \sqrt{1-\rho}}{\sqrt{1-\rho}} \qquad \text{and} \qquad \hat{B} = \sqrt{1 + \sqrt{1-\rho}}\,.$$

The corresponding $(\hat{\Sigma}, \hat{\beta}, \hat{\lambda}, \hat{\alpha}, \hat{\gamma})$ associated with $(\hat{A}, \hat{B})$ is

$$\hat{\Sigma} = \hat{A}\Sigma_v, \quad \hat{\beta} = \frac{\hat{B}}{\hat{A}}\sqrt{\frac{\Sigma_y}{\Sigma_v}}, \quad \hat{\lambda} = \frac{1}{\hat{B}}\sqrt{\frac{\Sigma_v}{\Sigma_y}}, \quad \hat{\alpha} = \frac{\rho}{2\hat{B}}\sqrt{\frac{\Sigma_y}{\Sigma_v}} \quad \text{and} \quad \hat{\gamma} = \frac{\rho\hat{\alpha}(\Sigma_v + \hat{\lambda}^2\Sigma_y)}{1-\rho}\,,$$

where we used the definitions of A_n and B_n to compute $\hat{\Sigma}$ and $\hat{\beta}$; (4) and the identities $\hat{A} = F_A(\hat{A}, \hat{B})$ and $\hat{B} = G_1(\hat{A})$ to compute $\hat{\lambda}$; (9) to compute $\hat{\alpha}$; and (8) to compute $\hat{\gamma}$. If $(A_0, B_0) = (\hat{A}, \hat{B})$, then $(A_n, B_n) = (\hat{A}, \hat{B})$ for all $n \geq 1$. Therefore, if $(\Sigma_{-1}, \beta_0) = (\hat{\Sigma}, \hat{\beta})$, then

$(\Sigma_{n-1}, \beta_n, \lambda_n, \alpha_n, \gamma_n) = (\hat{\Sigma}, \hat{\beta}, \hat{\lambda}, \hat{\alpha}, \hat{\gamma})$ for all $n \geq 0$. Thus, if $\Sigma_{-1} = \hat{\Sigma}$, there is a stationary Markovian equilibrium, where

$$P_n(p_{n-1}, \Sigma_{n-1}, z_n) = p_{n-1} + \hat{\lambda} z_n \quad \text{and} \quad X_n(p_{n-1}, \Sigma_{n-1}, V_n) = \hat{\beta}(V_n - p_{n-1}) \quad \text{for all } n \geq 0.$$

In this equilibrium, the variance of the market maker's estimate remains constant: $\Sigma_n = \hat{\Sigma}$ for all $n \geq 0$. Along the stochastic equilibrium path, the fundamental value and price evolve until time τ according with the processes

$$V_{n+1} = V_n + W_n \quad \text{and} \quad p_{n+1} = \left[\frac{\sqrt{1-\rho}}{1+\sqrt{1-\rho}}\right] V_n + \left[\frac{1}{1+\sqrt{1-\rho}}\right] p_n + \left[\frac{\Sigma_v}{\Sigma_y(1+\sqrt{1-\rho})}\right]^{\frac{1}{2}} y_n.$$

By continuity of the vector field F, there exists a curve $\mathcal{C}$, contained in $R_2 \cup R_4$ and passing through $(\hat{A}, \hat{B})$, such that $F(A, B) \in \mathcal{C}$ for all $(A, B) \in \mathcal{C}$. That is, $\mathcal{C}$ is the largest subset of $\mathbb{R}^2$ such that $F(\mathcal{C}) \subset \mathcal{C}$ and $(\hat{A}, \hat{B}) \in \mathcal{C}$. We do not have an analytical representation for $\mathcal{C}$, but we can approximate it numerically. This curve is strictly increasing, and it approaches the origin to the left (but it does not contain it). Therefore, there exists a strictly increasing function $\psi : (0, \infty) \to (0, \infty)$, such that $(A, B) \in \mathcal{C}$ if and only if $B = \psi(A)$. For any initial $A_0 > 0$, let $B_0 = \psi(A_0)$. Then the sequence $\{(A_n, B_n)\}$, where $(A_{n+1}, B_{n+1}) = F(A_n, B_n)$ for each n, is contained in $\mathcal{C}$ (that is, $B_n = \psi(A_n)$ for all $n \geq 0$) and therefore remains feasible forever. Moreover, $(A_n, B_n) \to (\hat{A}, \hat{B})$ as $n \to \infty$. When $A_0 < \hat{A}$ (respectively, $A_0 > \hat{A}$), $B_0 < \hat{B}$ ($B_0 > \hat{B}$) and $\{(A_n, B_n)\}$ is monotonically increasing (decreasing). Since ψ is concave and

$$\lambda_n = \frac{1}{B_n}\sqrt{\frac{\Sigma_v}{\Sigma_y}} \quad \text{and} \quad \beta_n = \frac{B_n}{A_n}\sqrt{\frac{\Sigma_y}{\Sigma_v}},$$

the sequence $\{(\lambda_n, \beta_n)\}$ is also monotone and $(\lambda_n, \beta_n) \to (\hat{\lambda}, \hat{\beta})$. Therefore, $\lambda_n \geq \min\{\lambda_0, \hat{\lambda}\}$ for all $n \geq 1$, and for *any* $\omega \in (\rho, 1)$ there exists $\ell > 0$ so that $\lambda_n \geq \omega^n/\ell$. Hence, by Lemma 1, $\{\Pi_n\}$ satisfies (2) for the sequence $\{(\alpha_n, \gamma_n)\}$ and the linear strategy (P, X) associated with the sequence $\{(\lambda_n, \beta_n)\}$ is an equilibrium. In summary, for any given $\Sigma_{-1} > 0$, if we initialize

$$\beta_0 = \Psi(\Sigma_{-1}) \qquad \text{where} \qquad \Psi(\Sigma_{-1}) = \frac{\sqrt{\Sigma_y \Sigma_v}}{\Sigma_{-1}} \psi\left(\frac{\Sigma_{-1}}{\Sigma_v}\right),$$

we obtain a feasible sequence $\{(\Sigma_{n-1}, \beta_n, \lambda_n, \alpha_n, \gamma_n)\}$, and the corresponding linear strategy (P, X) is a Markovian equilibrium.

For any given $\Sigma_{-1} > 0$, if $\beta_0 > \Psi(\Sigma_{-1})$, the corresponding (A_0, B_0) lies above $\mathcal{C}$ (that is, $B_0 > \psi(A_0)$). In this case, we show below that the sequence $\{(A_n, B_n)\}$ will eventually become infeasible (that is, for some finite n, $(A_n, B_n) \in R_5$). Therefore, such choice of β_0 is not compatible with equilibrium. If $\beta_0 < \Psi(\Sigma_{-1})$ instead, the corresponding (A_0, B_0) lies below $\mathcal{C}$ and the sequence $\{(A_n, B_n)\}$ remains feasible forever. However, in this case we show below that the sequence enters region R_3 and remains there forever afterwards. Lemma 3 then establishes that $\sum \rho^n/\lambda_n = \infty$. Therefore, by Lemma 1, the sequence $\{\lambda_n\}$ is not consistent with equilibrium. Thus, the only feasible choice is $\beta_0 = \Psi(\Sigma_{-1})$, leading to the Markovian equilibrium described above.

Lemma 3 *If* $\beta_0 < \Psi(\Sigma_{-1})$ *then* $\sum \rho^n/\lambda_n = \infty$.

Lemma 3 provides an alternative characterization of the equilibrium curve $\mathcal{C}$. Given Σ_{-1}, every $\beta_0 > 0$ uniquely defines a sequence $\{(\Sigma_{n-1}, \beta_n)\}$ through the recursions (3) and (10). However, some of these sequences are infeasible in the sense that $\beta_n < 0$ for some $n \geq 0$ (that is, $(A_n, B_n) \in R_5$). By our previous discussion regarding Figure 1, an alternative characterization of infeasability is $\Sigma_{n-1}\beta_n < \Sigma_n\beta_{n+1}$ for some $n \geq 0$ (that is, $(A_n, B_n) \in R_1$). Let us define $\mathcal{B}(\Sigma_{-1})$ to be the set of all $\beta_0 > 0$ such that the sequence $\Sigma_{n-1}\beta_n$ is (weakly) decreasing. Then, Lemma 3 and the discussion that follows Figure 1 imply that the equilibrium profile is obtained setting $\beta_0 = \Psi(\Sigma_{-1}) = \sup\{\beta \in \mathcal{B}(\Sigma_{-1})\}$. We will make use of this alternative characterization in section 5 to discuss the connection between the discrete-time and continuous-time equilibria.

Remarks:

- Despite the fact that insider's trades are informative and reduce the market uncertainty, when the initial variance $\Sigma_0 < \hat{\Sigma}$, Σ_n ends up *increasing* with n. In this case, the variance reduction induced by insider trading is insufficient to compensate for the additional uncertainty generated by the evolution of $\{V_n\}$.

- To carry the analysis above we had to assume a state $(n, v_{n-1}, \Sigma_{n-1})$. But, in equilibrium, $\{\Sigma_n\}$ is a monotone sequence and there is a one-to-one relationship between n and Σ_n. Hence, we can reduce the state variables to (v_{n-1}, Σ_{n-1}). Indeed, the equilibrium is stationary. The continuation value for the insider in period n, for example, does not depend on n and could be written as $\Pi(v_{n-1}, \Sigma_{n-1}, V_n)$ instead of $\Pi_n(v_{n-1}, \Sigma_{n-1}, V_n)$. Put a different way, if we consider another problem where the initial variance is Σ_{n-1}, its equilibrium would coincide with the continuation equilibrium from period n onward of the equilibrium where the initial variance is Σ_{-1}. Similarly, we could write $\beta(\Sigma_{n-1})$ instead of β_n (and the same is true for the other sequences that define the equilibrium).

- In our definition of an equilibrium we have ruled out profiles (P, X) for which $\mathbb{E}[\Pi(P, X)] = \infty$. For each $\beta_0 \in (0, \Psi(\Sigma_1))$ there is a profile (P, X) such that $\mathbb{E}[\Pi(P, X)] = \infty$ but that otherwise satisfies all the conditions for an equilibrium (see Lemma 3). For those profiles, one can show that liquidity traders always have bounded (positive or negative) payoffs, independent of whether the insider's payoff is finite or not. Hence, it is the market maker that finances the insider's infinite expected rents. When the market maker sets the price equal to the expected value of the asset (as required in equilibrium), $\mathbb{E}[\Pi(P, X)] < \infty$ implies that he makes 0 profits. But when $\mathbb{E}[\Pi(P, X)] = \infty$, he makes infinite losses. We require that in equilibrium $\mathbb{E}[\Pi(P, X)] < \infty$ because an outcome where the market maker incurs infinite losses would not be sustainable.

- Any sequence $\{(A_n, B_n)\}$ with $(A_0, B_0) \in R_2$ cannot jump to R_4 (and vice-versa). That is, if the sequence abandons the region R_2, it must go to regions R_1 or R_3. As a result, the equilibrium sequence $\{(A_n, B_n)\}$ is monotone. Indeed, let $(A, B) \in R_2$, $(A', B') =$

$F(A,B)$, and $c = \sqrt{1-\rho}$. Then for (A', B') to be in R_4 we must have that $G_2(A') \leq B' \leq G_1(A')$, which implies that $G_2(A') \leq G_1(A')$, or

$$\sqrt{c\left[1 + \frac{A^2}{A+B^2}\right]} = \sqrt{\frac{c(A+A^2+B^2)}{A+B^2}} \leq \sqrt{\frac{1+A^2/(A+B^2)}{A^2/(A+B^2)}} = \sqrt{\frac{A+A^2+B^2}{A^2}}$$

That is, we must have that $c/(A+B^2) \leq 1/A^2$ or $\sqrt{cA}\sqrt{A-1/c} \leq B$. But $(A,B) \in R_2$ implies that $B \leq G_2(A) = \sqrt{cA}$ and $A > \hat{A} = (1+c)/c$. Therefore, $\sqrt{cA}\sqrt{A-1} \leq \sqrt{cA}$, or $A - 1/c \leq 1$, which is a contradiction.

Similarly, a sequence that starts in R_4 cannot jump to R_2. Indeed, let $(A,B) \in R_4$ and $(A', B') = F(A,B)$. For (A', B') to be in R_2 we must have that $G_1(A') \leq B' \leq G_2(A')$, which implies that $G_2(A') \geq G_1(A')$, or $\sqrt{cA}\sqrt{A-1/c} \geq B$. But $(A,B) \in R_4$ implies that $B \geq G_2(A) = \sqrt{cA}$ and $A < \hat{A} = (1+c)/c$. Therefore, $\sqrt{cA}\sqrt{A-1} \geq \sqrt{cA}$, or $A - 1/c \geq 1$, which is a contradiction.

2.1 The Perfect Information Case

A special case of our model is when $\Sigma_v = 0$. In this case the insider knows from the start what the value of the fundamental will be at the time it is revealed. This is the assumption made by Kyle (1985).

Proposition 1 *Suppose $\Sigma_v = 0$ and let Σ_{-1} be given. Then, there exists a linear Markovian equilibrium defined by the sequences*

$$\beta_n = \sqrt{\frac{S\Sigma_y}{\Sigma_{-1}}}\,(1+S)^{\frac{n}{2}} \qquad \text{and} \qquad \lambda_n = \sqrt{\frac{S\Sigma_{-1}}{\Sigma_y}}\,(1+S)^{-\frac{n+2}{2}}, \qquad n \geq 0,$$

where S is the unique root in $(0,1)$ of the equation $(1+S)(1-S)^2 = \rho^2$. The resulting equilibrium satisfies

$$\Sigma_n = \frac{\Sigma_{-1}}{(1+S)^{n+1}}, \quad \alpha_n = \frac{\rho}{2}\sqrt{\frac{\Sigma_y}{S\Sigma_{-1}}}\,(1+S)^{\frac{n}{2}}, \quad \text{and} \quad \gamma_n = \frac{\rho^2}{2}\,\frac{\sqrt{\Sigma_{-1}\,\Sigma_y\,S}}{\sqrt{1+S}-\rho}\,(1+S)^{-\frac{n+2}{2}}.$$

Note that in equilibrium the value of Σ_n converges to 0 as $n \to \infty$. That is, the market is asymptotically efficient as the number of periods goes to infinity. Recall that $\rho = e^{-(\theta+\delta)\Delta}$, and thus ρ is a decreasing function of θ and δ. That is, ρ decreases if the insider becomes more impatient or the public revelation of the fundamental value happens faster. When ρ decreases, S increases, the insider reveals her private information faster, and market efficiency increases.

We will return to this special case in Section 4, where we derive its continuous-time counterpart by letting the period length Δ go to zero.

3 Continuous-Time Trading as a Discrete-Time Limit

In this section, we analyze the discrete-time linear equilibrium in Theorem 1 in the limit as Δ goes to 0. This limit will help us identify heuristically features of the equilibrium which we will use in section 4 to formally derive a continuous-time solution.

First, let us explicitly rewrite the discrete-time model in terms of the calendar time t. Recall that for any time $t \geq 0$ the corresponding trading period is $n = \lfloor t \rfloor$. In the discrete-time model, the insider's trading strategy in period n is $x_n = \beta_n (V_n - p_{n-1})$. To get a continuous-time analogue, we would like to express the insider's strategy in terms of her trading rate per unit time. For this, we define $\beta_t = \beta_n/\Delta$ where $n = \lfloor t \rfloor$. We also define the continuous time extensions $\Sigma_t = \Sigma_n$, $\lambda_t = \lambda_n$, $\alpha_t = \alpha_n$ and $\gamma_t = \gamma_n$ where $n = \lfloor t \rfloor$. Finally, recall that $\Sigma_y = \sigma_y^2 \Delta$, $\Sigma_v = \sigma_v^2 \Delta$ and $\rho = e^{-\mu \Delta}$, where $\mu = \delta + \theta$ and $\sigma_v = \bar{\sigma}_v(1 - e^{-\theta\Delta})/(1 - e^{-\mu\Delta})$. Note that $\sigma_v \to \bar{\sigma}_v \theta/\mu$ as $\Delta \to 0$.

Theorem 1, establishes that there exists a unique equilibrium where the processes λ_t and β_t are deterministic functions of $\Sigma_{t-\Delta}$. This equilibrium is defined by equations (3), (4), (8), (9) and (10) and satisfies

$$\Sigma_t = \sigma_v^2 \Delta + \frac{\sigma_y^2 \, \Sigma_{t-\Delta}}{\beta_t^2 \Sigma_{t-\Delta} \, \Delta + \sigma_y^2}, \quad \lambda_t = \frac{\beta_t \Sigma_{t-\Delta}}{\beta_t^2 \, \Sigma_{t-\Delta} \, \Delta + \sigma_y^2} \quad \text{and} \quad \beta_{t+\Delta} \Sigma_t = \frac{e^{-\mu \Delta} \beta_t \, \sigma_y^4 \, \Sigma_{t-\Delta}}{\sigma_y^4 - \beta_t^4 \Sigma_{t-\Delta}^2 \, \Delta^2}. \quad (11)$$

Furthermore, the insider's expected profit-to-go function at the beginning of period $\lfloor t \rfloor$ satisfies

$$\Pi_{\lfloor t \rfloor}(V, p) = \alpha_t \, (V - p)^2 + \gamma_t, \quad \text{where}$$
$$\alpha_t = \frac{e^{-\mu \Delta} \, \sigma_y^2}{2\beta_t \, \Sigma_{t-\Delta}} \qquad \text{and} \qquad \gamma_t \, e^{\mu \Delta} = \gamma_{t+\Delta} + \alpha_{t+\Delta}(\sigma_v^2 + \lambda_t^2 \, \sigma_y^2) \, \Delta.$$

Figure 2 depicts the values of β_t (left panel) and Σ_t (right panel) for different values of Δ as function of the calendar time t. For any $\Delta' < \Delta$ and for any $t \geq 0$, $\beta_t(\Delta') > \beta_t(\Delta)$ and $\Sigma_t(\Delta') < \Sigma_t(\Delta)$. The intuition for the first inequality is clear. As $\Delta \to 0$, the liquidity traders' volume of trade per period becomes increasingly noisier relative to the insider's volume of trade per period. More precisely, the insider's volume of trade is $\beta_t(V_t - p_t)\Delta$, while the standard deviation of the liquidity traders' volume of trade is $\sigma_y \sqrt{\Delta}$. Since $\sqrt{\Delta}/\Delta \to \infty$ as $\Delta \downarrow 0$, when Δ decreases the insider can afford to trade more without revealing more information. The behavior of β_t (as a function of t) appears to have two distinctive phases as $\Delta \downarrow 0$. First, as $t \downarrow 0$, β_t converges to a fixed value β_0 independent of Δ. Second, as $t \to \infty$, β_t converges to β_∞ which, as a function of Δ, diverges to $+\infty$ when Δ goes to 0. Furthermore, Figure 2 suggests the following stronger result

$$\lim_{\Delta \downarrow 0} \beta_t = \infty, \quad \text{for all } t \geq T,$$

where T is a finite time represented by the vertical dashed line in Figure 2. Recall that in the time interval $[t_n, t_{n+1})$ the insider trades the amount $\beta_{t_n} (V_{t_n} - p_{t_n}) \Delta$. Hence, for $t \geq T$ the insider's trading rate grows arbitrarily large as Δ goes to zero. When β_t is large, the market maker is able to differentiate insider trading from liquidity trading. Hence, when Δ is small and $t \geq T$, the insider is revealing her private information very fast. This effect is captured in the

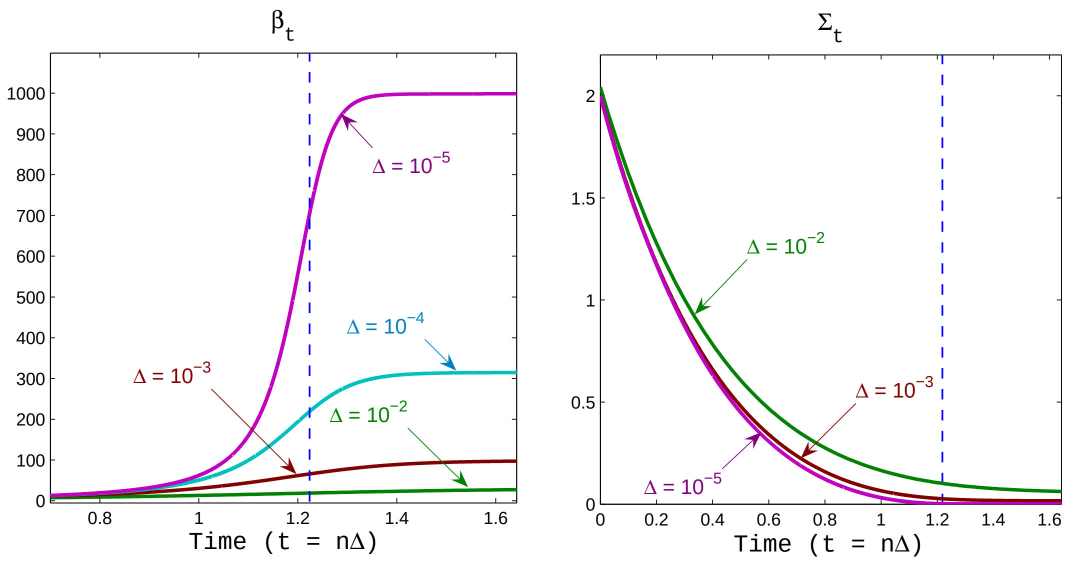

Figure 2: Evolution of β_t (left panel) and Σ_t (right panel) as a function of the calendar time $t = n\,\Delta$ for different values of Δ. (Data: $\sigma_y^2 = 5$, $\sigma_v^2 = 0.5$, $\mu = 1$ and $\Sigma_0 = 2$.)

right panel of Figure 2 that shows Σ_t decreasing monotonically to zero and staying arbitrarily closed to zero for $t \geq T$ as $\Delta \downarrow 0$. In other words, for $t \geq T$, the market is asymptotically efficient as the period length goes to zero.

Recall that for any $\Delta > 0$, the equilibrium $(\Sigma_t, \beta_t, \lambda_t, \alpha_t, \gamma_t)$ converges to a stationary point $(\hat{\Sigma}, \hat{\beta}, \hat{\lambda}, \hat{\alpha}, \hat{\gamma})$ as t goes to infinity. In terms of Δ, this limit is given by

$$\hat{\Sigma} = \left[\frac{1+\sqrt{1-e^{-\mu\,\Delta}}}{\sqrt{1-e^{-\mu\,\Delta}}}\right] \sigma_v^2\,\Delta, \quad \hat{\beta} = \frac{1}{\Delta}\left[\frac{1-e^{-\mu\,\Delta}}{1+\sqrt{1-e^{-\mu\,\Delta}}}\right]^{\frac{1}{2}} \frac{\sigma_y}{\sigma_v}, \quad \hat{\lambda} = \left[\frac{1}{1+\sqrt{1-e^{-\mu\,\Delta}}}\right]^{\frac{1}{2}} \frac{\sigma_v}{\sigma_y},$$

$$\hat{\alpha} = \frac{e^{-\mu\,\Delta}\,\hat{\lambda}}{2}\,\frac{\sigma_y^2}{\sigma_v^2} \qquad \text{and} \qquad \hat{\gamma} = \frac{e^{-\mu\,\Delta}\,\hat{\alpha}}{1-e^{-\mu\,\Delta}}\,(\sigma_v^2 + \hat{\lambda}^2\,\sigma_y^2)\,\Delta.$$

If we let $\Delta \downarrow 0$, the stationary equilibrium converges to

$$\lim_{\Delta\downarrow 0}\,(\hat{\Sigma}, \hat{\beta}, \hat{\lambda}, \hat{\alpha}, \hat{\gamma}) = \left(0, \infty, \frac{\sigma_v}{\sigma_y}, \frac{\sigma_y}{2\,\sigma_v}, \frac{\sigma_y\,\sigma_v}{\mu}\right). \tag{12}$$

The first two limits are consistent with our previous discussion: When $\Delta \downarrow 0$, the market becomes asymptotically efficient ($\hat{\Sigma} = 0$ and the insider trading rate grows arbitrarily large ($\hat{\beta} = \infty$) as $t \to \infty$. The limit above also shows that the insider makes positive profits in the limiting regime since $\hat{\alpha}$ and specially $\hat{\gamma}$ are both positive.

4 Continuous Time Model

In this section, we derive an "equilibrium" for the model in which trading occurs continuously over time. More precisely, the strategy profile we construct is not an equilibrium of a continuous-time game; instead, it is the limit of equilibria for a family of continuous time models where the insider's trading rate is uniformly bounded. This construction allows us to introduce technical constraints in the strategy space of the insider that capture the natural limits of what is possible in a discrete time model, while at the same time preserving existence of equilibrium.

Similar to the discrete-time model, we define the intrinsic value V_t to be the expected discounted value of the fundamental at time τ given the insider's information at time t. That is,

$$V_t = \mathbb{E}[e^{-\delta(\tau-t)}\,\bar{V}_\tau \mid \mathcal{F}^I_t, t > \tau] = \frac{\theta}{\theta+\delta}\,\bar{V}_t.$$

We also define $\sigma_v = \bar{\sigma}_v\,\theta/(\theta+\delta)$ so that V_t is a driftless Brownian motion with dynamics

$$\mathrm{d}V_t = \sigma_v\,\mathrm{d}B^v_t.$$

A strategy profile is a pair of processes (X, P), where $X_t \in \mathcal{F}^I_t$ is the insider's cumulative trading volume up to time t, and $P_t \in \mathcal{F}^M_t$ is the price set by the market maker at time t. For a given profile (X, P), the insider's expected discounted payoff, $\Pi(P, X)$, is defined as follows

$$\Pi(P, X) = \mathbb{E}\left[e^{-\delta\,\tau}\,\bar{V}_\tau\,X_\tau - \int_0^\tau e^{-\delta\,t} P_t\,\mathrm{d}X_t - \int_0^\tau e^{-\delta\,t}\mathrm{d}[X, P]_t\right],$$

where $[X, P]_t$ is the quadratic covariation between X_t and P_t.[3]

Given the space $\mathcal{C}$ of continuous processes adapted to the insider's information $\mathcal{F}^I_t$, a continuous-time equilibrium is a profile (X, P) with the following properties: (i) given P, $\Pi(X, P)$ is bounded and $X \in \mathcal{C}$ maximizes[4] $\Pi(X, P)$, and (ii) the price process P satisfies the equilibrium condition

$$P_t = \mathbb{E}\left[V_t \,\middle|\, \mathcal{F}^M_t, X\right] \qquad 0 \le t < \tau,$$

given the insider's trading strategy X.

For the analysis that follows, we find convenient to rewrite the insider's payoff using the following identity

$$e^{-\delta\,\tau}\,\bar{V}_\tau\,X_\tau = \int_0^\tau e^{-\delta\,\tau}\,\bar{V}_t\,\mathrm{d}X_t + \int_0^\tau e^{-\delta\,\tau}\,X_t\,\mathrm{d}\bar{V}_t + \int_0^\tau e^{-\delta\,\tau}\,\mathrm{d}[X, \bar{V}]_t,$$

where $[X, \bar{V}]_t$ is the quadratic covariation between X_t and $\bar{V}_t$. Plugging back this identity in Π, taking expectation and canceling the stochastic integral with respect to the martingale $\bar{V}_t$, we get

$$\begin{aligned}\Pi(P, X) &= \mathbb{E}\left[\int_0^\tau \left(e^{-\delta\,\tau}\,\bar{V}_t - e^{-\delta\,t}\,P_t\right)\mathrm{d}X_t + \int_0^\tau e^{-\delta\,t}\mathrm{d}[X, V]_t - \int_0^\tau e^{-\delta\,t}\mathrm{d}[X, P]_t\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-\mu\,t}(V_t - P_t)\,\mathrm{d}X_t + \int_0^\infty e^{-\mu\,t}\mathrm{d}[X, V]_t - \int_0^\infty e^{-\mu\,t}\mathrm{d}[X, P]_t\right],\end{aligned}$$

[3]Intuitively, this term arises because the price paid by the insider is computed 'at the end of the period', and therefore it includes the effect of the insider's 'last trade' $\mathrm{d}X_t$. For a formal derivation, see equation (11) in Back (1992).

[4]We rule out discontinuities in X because they would immediately inform the market marker that he is mispricing the asset.

where the second equality is based on the fact that τ is exponentially distributed with rate θ and is independent of $\mathcal{F}_t^I$. Recall also the definition $\mu = \delta + \theta$.

Our construction of a continuous-time limit equilibrium (P, X) builds on the features that we heuristically derived in the previous section. That is, the limit equilibrium has two phases: an absolutely continuous phase in the interval $[0, T)$ in which X has bounded variation and a singular phase in the interval $[T, \infty)$ in which X has unbounded variation, for some switching time $T > 0$. In the absolutely continuous phase, the insider trades at a rate $\beta_t (V_t - P_t)$ and the market maker adjusts prices at a rate λ_t, where $\beta, \lambda : [0, T) \to \mathbb{R}_+$, so that

$$\mathrm{d}X_t = \beta_t (V_t - P_t) \,\mathrm{d}t \qquad \text{and} \qquad \mathrm{d}P_t = \lambda_t \,\mathrm{d}Z_t, \qquad t < T.$$

In the interval $[0, T)$ the variance Σ_t decreases from Σ_0 to 0. In the singular phase $[T, \infty)$, $\Sigma_t \equiv 0$, the market maker adjusts the price at a constant rate λ_T and the insider buys/sells at an infinite rate driving the gap between the price and the valuation instantaneously to 0. That is,

$$\mathrm{d}X_t = \frac{\mathrm{d}V_t}{\lambda_T} - \mathrm{d}Y_t \qquad \text{and} \qquad \mathrm{d}P_t = \lambda_T \,\mathrm{d}Z_t = \mathrm{d}V_t, \qquad t \geq T.$$

4.1 Continuous-Time Equilibrium with Restricted Trading

Since the continuous-time extension of a discrete-time trading strategy (as we defined it in Section 3) is always of bounded variation, it would be natural to let $\mathcal{C}$ be the set of continuous processes of *bounded variation* adapted to $\mathcal{F}_t^I$. However, exactly because the insider would like to use (in equilibrium) a strategy of unbounded variation, a continuous-time model with such a space $\mathcal{C}$ would not have an equilibrium. Our purpose is to approximate the discrete-time equilibrium for Δ small by a continuous-time strategy profile. We construct such a strategy profile as the limit of a sequence of continuous-time equilibria in which the insider trading rate is uniformly bounded. More specifically, for each $\bar{\beta} > 0$, we consider the restricted strategy space $\mathcal{C}(\bar{\beta})$ for the insider of all processes $X \in \mathcal{C}$ such that

$$\mathrm{d}X_t = \beta_t (V_t - P_t) \,\mathrm{d}t$$

for some process β_t adapted to $\mathcal{F}_t^I$ with $|\beta_t| \leq \bar{\beta}$ for all $t \geq 0$. The continuous-time model with this strategy space for the insider does have an equilibrium. We obtain our continuous-time approximation of the discrete-time equilibrium (for Δ small) by taking the limit of a continuous-time sequence of equilibria as $\bar{\beta} \to \infty$.

With a bounded trading rate, the two phases of the equilibrium are characterized by (*i*) $\beta_t < \bar{\beta}$ in the 'absolutely continuous' phase $[0, T)$, and (*ii*) $\beta_t = \bar{\beta}$ in the 'singular phase' $[T, \infty)$.

When X is of bounded variation, the quadratic covariations $[X, V]_t$ and $[X, P]_t$ are both zero, and accordingly the last two terms of $\Pi(P, X)$ drop out and

$$\Pi(X, P) = \mathbb{E}\left[\int_0^\infty e^{-\mu t} \beta_t (V_t - P_t)^2 \,\mathrm{d}t\right].$$

Let us suppose that the market maker uses the pricing rule

$$\mathrm{d}P_t = \lambda_t \,\mathrm{d}Z_t, \qquad t \geq 0, \tag{13}$$

for some nonnegative process λ_t. Hence, under the restriction $X \in \mathcal{C}(\bar{\beta})$, the evolution of P_t is governed by the following SDE

$$\mathrm{d}P_t = \lambda_t \left[\beta_t (V_t - P_t)\mathrm{d}t + \sigma_y \, \mathrm{d}B_t^y\right].$$

Note that both the insider's payoff function and the dynamics of P_t depend on P_t and V_t only through their difference. Thus, we find it convenient to define the price-gap process $M_t = V_t - P_t$ with dynamics

$$\mathrm{d}M_t = -\lambda_t \, \beta_t \, M_t \, \mathrm{d}t + \sigma_v \, \mathrm{d}B_t^v - \lambda_t \, \sigma_y \, \mathrm{d}B_t^y, \qquad t \geq 0.$$

The process $\sigma_v \, B_t^v - \lambda_t \, \sigma_y \, B_t^y$ is a driftless Gaussian process with variance $\sigma_t^2 = \sigma_v^2 + \lambda_t^2 \, \sigma_y^2$. Therefore,

$$\mathrm{d}M_t = -\lambda_t \, \beta_t \, M_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}B_t \qquad t \geq 0,$$

where B_t is a standard Brownian motion. In equilibrium both λ_t and β_t are nonnegative and so the process M_t reverts towards 0. We define the value function

$$\begin{aligned} \Pi(t, M) \quad &= \quad \sup_{|\beta_t| \leq \bar{\beta}} \; \mathbb{E}\left[\int_t^\infty e^{-\mu\,(s-t)} \, \beta_s \, M_s^2 \, \mathrm{d}s\right] \\ &\quad\; \text{s.t.} \quad \mathrm{d}M_s = -\lambda_s \, \beta_s \, M_s \, \mathrm{d}t + \sigma_s \, \mathrm{d}B_s \quad \text{for } s \geq t, \quad \text{and} \quad M_t = M. \end{aligned} \tag{14}$$

The process $\Pi(t, M)$ is the insider's optimal expected profit-to-go starting at time t with an initial price-value gap $M_t = M$. We note that M_t and Π depend on both the pricing policy λ and $\bar{\beta}$. When we wish to emphasize this dependence we will include λ and/or $\bar{\beta}$ as part of their arguments, for example, we will write $M_t(\bar{\beta})$ or $\Pi(t, M, \lambda, \bar{\beta})$.

The dynamic programming HJB equation for $\Pi(t, M)$ is

$$0 = \max_{|\beta| \leq \bar{\beta}} \left\{ -\lambda_t \, \beta \, M \, \Pi_M + \frac{1}{2} \, (\sigma_v^2 + \lambda_t^2 \, \sigma_y^2) \, \Pi_{MM} + \Pi_t - \mu \, \Pi + M^2 \, \beta \right\}, \qquad t \in [0, \infty)$$

where Π_M (Π_{MM}) and Π_t are the first (second order) partial derivative of Π with respect to M and t, respectively. We will show that the profit-to-go is a quadratic function $\Pi(t, M_t) = \alpha_t \, M_t^2 + \gamma_t$, where α_t and γ_t are two deterministic functions of t. Then, the HJB reduces to

$$0 = \max_{|\beta| \leq \bar{\beta}} \left\{ \left[\beta \, (1 - 2\lambda_t \, \alpha_t) + \dot{\alpha}_t - \mu \, \alpha_t\right] M^2 + \alpha_t(\sigma_v^2 + \lambda_t^2 \sigma_y^2) + \dot{\gamma}_t - \mu \gamma_t \right\}. \tag{15}$$

The market maker's equilibrium condition is characterized in the following proposition.

Proposition 2 *Suppose the insider selects a deterministic trading rate β_t, and the market maker chooses the pricing rule (13). Then, the market maker's equilibrium condition $P_t = \mathbb{E}[V_t | \mathcal{F}_t^M, X]$ is satisfied if λ_t is a deterministic function such that*

$$\Sigma_t \, \beta_t = \sigma_y^2 \, \lambda_t \qquad \text{and} \qquad \dot{\Sigma}_t = \sigma_v^2 - \sigma_y^2 \, \lambda_t^2, \tag{16}$$

where $\Sigma_t = \mathbb{E}[(V_t - P_t)^2 | \mathcal{F}_t^M, X]$ and $\dot{\Sigma}_t$ represents its first derivative with respect to t.

The equilibrium that we construct below is defined by a pair of deterministic nonnegative processes (λ_t, β_t) that satisfy (16) and solve (14) for $t = 0$.

Note that if $\lambda_t < 0$, choosing $\beta_t = \bar{\beta} > 0$ is optimal because it maximizes current payoffs and *increases* the gap between V_t and P_t. For example, if $V_t - P_t > 0$ the insider maximizes the flow of profits when $\beta_t = \bar{\beta}$, while at the same time she *decreases* the price, and hence *increases* the profits she makes in futures purchases. However, by Proposition 2 and the fact that $\Sigma_t \geq 0$, λ_t and β_t must have the same sign. Since $\bar{\beta} > 0$, λ_t should also be positive, contradicting our assumption. Thus, in equilibrium, $\lambda_t \geq 0$ for all t.

Theorem 2 *Let $L = 2\sigma_v \bar{\beta}/\sigma_y$. An equilibrium of the game with upper bound $\bar{\beta}$ is given by*

$$\lambda_t = \begin{cases} \lambda_0 \, e^{-\mu t} & \text{if } \; t < T \\ \dfrac{\sigma_v}{\sigma_y} \left[\dfrac{e^{Lt} + k\, e^{LT}}{e^{Lt} - k\, e^{LT}} \right] & \text{if } \; t \geq T \end{cases} \qquad \text{and} \qquad \beta_t = \begin{cases} \dfrac{\sigma_y^2 \lambda_t}{\Sigma_t} & \text{if } \; t < T \\ \bar{\beta} & \text{if } \; t \geq T, \end{cases} \tag{17}$$

where

$$\Sigma_t = \begin{cases} \Sigma_0 + \sigma_v^2 t - \dfrac{\lambda_0^2 \sigma_y^2}{2\mu} \left[1 - e^{-2\mu t} \right] & \text{if } \; t < T \\ \dfrac{\sigma_v \, \sigma_y}{\bar{\beta}} \left[\dfrac{e^{Lt} + k\, e^{LT}}{e^{Lt} - k\, e^{LT}} \right] & \text{if } \; t \geq T. \end{cases} \tag{18}$$

Moreover, in equilibrium, $\Pi(t, M) = \alpha_t \, M^2 + \gamma_t$, where

$$\alpha_t = \begin{cases} \dfrac{e^{\mu t}}{2\,\lambda_0} & \text{if } \; t < T \\ \bar{\beta} \, e^{-(L-\mu)t} \left[e^{Lt} - k\, e^{LT} \right]^2 \displaystyle\int_t^\infty e^{(L-\mu)s} \left[e^{Ls} - k\, e^{LT} \right]^{-2} \mathrm{d}s & \text{if } \; t \geq T \end{cases} \tag{19}$$

$$\gamma_t = \begin{cases} \left[\gamma_0 - \dfrac{\lambda_0 \sigma_y^2}{4\mu} - \dfrac{\sigma_v^2}{2\lambda_0} t \right] e^{\mu t} + \dfrac{\lambda_0 \sigma_y^2}{4\mu} e^{-\mu t} & \text{if } \; t < T \\ \bar{\beta} e^{\mu t} \displaystyle\int_t^\infty e^{(L-\mu)s} \left[e^{Ls} - k\, e^{LT} \right]^{-2} \int_t^s \sigma_u^2 e^{-Lu} \left[e^{Lu} - k\, e^{LT} \right]^2 \mathrm{d}u \, \mathrm{d}s & \text{if } \; t \geq T. \end{cases} \tag{20}$$

Let $\lambda_T^- = \lim_{t \uparrow T} \lambda_t = \lambda_0 \, e^{-\mu T}$, and define Σ_T^-, α_T^- and γ_T^- similarly. Then the constants $(k, \lambda_0, \gamma_0, T)$ are determined by the value-matching conditions

$$\lambda_T^- = \lambda_T, \qquad \Sigma_T^- = \Sigma_T, \qquad \alpha_T^- = \alpha_T, \quad \text{and} \quad \gamma_T^- = \gamma_T. \tag{21}$$

The constants k and T satisfies $0 < k < \mu/L$. Though the value $\Pi(t, M)$ is defined piecewise, it is continuously differentiable in t and twice continuously differentiable in M.

Tedious computations show that β_t is increasing in $t \in [0, T]$. Since $\beta_T = \bar{\beta}$, the inside's strategy satisfies $|\beta_t| \leq \bar{\beta}$ for all t.

Proposition 3 *For $t \in [T, \infty)$, the price differential M_t has mean reverting dynamics and satisfies*

$$M_t = M_T \, e^{-\bar{\beta} \int_T^t \lambda_s \, \mathrm{d}s} + \int_T^t \sigma_s \, e^{-\bar{\beta} \int_s^t \lambda_u \, \mathrm{d}u} \, \mathrm{d}B_s.$$

4.2 Limiting Solution as $\bar{\beta} \to \infty$

To obtain the continuous-time limit equilibrium, we let $\bar{\beta}$ go to infinity. We have to be careful, however, with the interpretation of this limit because of the nature of the insider trading strategy. According to Theorem 2, as $\bar{\beta}$ grows large so does the insider trading rate in $[T, \infty)$. In the limit as $\bar{\beta} \to \infty$, the insider wants to trade at an infinite rate. In the language of optimal control, the insider is exerting *singular* control, that is, she is using a trading strategy that is not absolutely continuous with respect to time. Nevertheless, the following Theorem guarantees that both $\Pi(t, M, \bar{\beta})$ and $M_t(\bar{\beta})$ admit a well-defined limit as $\bar{\beta} \to \infty$.

Theorem 3 *For each $\bar{\beta} > 0$, let $(k(\bar{\beta}), \lambda_0(\bar{\beta}), \gamma_0(\bar{\beta}), T(\bar{\beta}))$ be the equilibrium associated with $\bar{\beta}$ specified in Theorem 2. Then $(k(\bar{\beta}), \lambda_0(\bar{\beta}), \gamma_0(\bar{\beta}), T(\bar{\beta})) \to (0, \lambda_0, \gamma_0, T)$, where $\lambda_0 = \frac{\sigma_v}{\sigma_y} e^{\mu T}$ and T is the unique nonnegative root of the equation*

$$\Sigma_0 + \sigma_v^2 T = \sigma_v^2 \left[\frac{e^{2\mu T} - 1}{2\mu} \right].$$

The continuous-time limit equilibrium has two phases separated by the switching time T.

- Absolutely Continuous Phase in $[0, T)$: *In this phase, the insider's trading strategy and market maker's pricing rule are given by*

$$\mathrm{d}X_t = \beta_t (V_t - P_t)\, \mathrm{d}t \qquad \textit{and} \qquad \mathrm{d}P_t = \lambda_t\, \mathrm{d}Z_t \qquad t < T,$$

where β_t and λ_t are the two deterministic functions

$$\beta_t = \frac{\sigma_v\, \sigma_y\, e^{\mu(T-t)}}{\Sigma_t} \qquad \textit{and} \qquad \lambda_t = \frac{\sigma_v}{\sigma_y} e^{\mu(T-t)}, \quad t < T.$$

The variance of the market maker's estimate of V_t is given by

$$\Sigma_t = \Sigma_0 + \sigma_v^2 t - \sigma_v^2 e^{2\mu T} \left[\frac{1 - e^{-2\mu t}}{2\mu} \right] \qquad t < T,$$

which decreases monotonically to 0 in $[0, T)$.

- Singular Phase in $[T, \infty)$: *In this phase, the (inverse) market depth is constant: $\lambda_t = \sigma_v / \sigma_y$, and the limiting market maker's pricing rule satisfies*

$$\mathrm{d}P_t = \frac{\sigma_v}{\sigma_y} \mathrm{d}Z_t, \qquad t \geq T.$$

In addition, the price differential $M_t(\bar{\beta})$ converges weakly to 0 over compacts in $[T, \infty)$. As a result, the insider's trading strategy converges weakly to $X_t = X_T + \sigma_y\, [(B_t^v - B_T^v) - (B_t^y - B_T^y)]$ and $\Sigma_t = 0$ for all $t \geq T$.

- Insider's Payoff: *Let* $(T-t)^+ = \max\{0, T-t\}$. *As* $\bar{\beta} \to \infty$, *the insider's value function* $\Pi(t, M_t, \bar{\beta})$ *converges to the quadratic function*

$$\Pi(t, M_t) = \alpha_t\, M_t^2 + \gamma_t \qquad \textit{where} \qquad \alpha_t = \frac{\sigma_y}{2\,\sigma_v}\, e^{-\mu\,(T-t)^+} \qquad \textit{and}$$

$$\gamma_t = \frac{\sigma_y\,\sigma_v}{4\,\mu}\left[3\,e^{-\mu\,(T-t)^+} + e^{\mu\,(T-t)^+}\right] + \frac{\sigma_y\,\sigma_v}{2}\,(T-t)^+\,e^{-\mu\,(T-t)^+}.$$

The previous result summarizes a number of important features of the limit equilibrium. One of the remarkable property of this equilibrium is the existence a finite time T, endogenously determined, at which market efficiency is reached and preserved thereafter. Indeed, the fact that $M_t \Rightarrow 0$ for $t \geq T$ means that after T, the insider is cashing out her private information instantaneously. Despite this market efficiency, the insider is able to collect positive rents in $[T, \infty)$ since $\Pi(t, 0) > 0$. The source of these rents is the continuous inflow of new information that the insider gets from privately observing the evolution of V_t. From the market maker's perspective, the strategy profile in $[T, \infty)$ validates his work in a rather strong sense. In fact, the market maker is concerned with setting prices so that $P_t = \mathbb{E}[V_t|\mathcal{F}_t^M]$. Theorem 3 implies that P_t converges uniformly on compact sets to V_t in $[T, \infty)$.[5] Hence, in the limit as $\bar{\beta} \to \infty$, the market maker knows exactly the intrinsic value of the asset and the price reflects this value at all times $t \geq T$. As a result, the insider trading volume X_t behaves as a martingale after T. It is also interesting to note that $X_t - X_T$ is independent of σ_v.

Intuitively, the existence of a finite threshold time T is the result of two forces that influence the insider trading strategy in opposite directions. On one hand, the fact that $\Pi(T, 0) > 0$ implies that the insider is able to collect positive rents after T, despite the fact that market prices are efficient after this time. Hence, given the unpredictability of the information leakage, the insider is anxious to collect these rents as quickly as possible pushing the value of T towards 0. On the other hand, the market maker's choice of a decreasing (inverse) market depth, λ_t, gives the insider incentives to slow down her trading activity pushing T away from 0. In equilibrium, the choice of λ_t is such that these two forces compensate each other and the insider gradually reveals her private information resulting in a finite time T bounded away from 0.

The informational rents after time T are due to the continuous inflow of new information that the insider gets by privately tracking the evolution of V_t. In the absence of these rents, either because V_t is constant or because the insider looses her capacity to track V_t, the insider would have no incentive to speed up her trading and market efficiency would only be reached asymptotically ($T = \infty$).

Proposition 4 *The insider's ex-ante (at time 0, before observing any signals) expected payoff-to-go is*

$$\mathbb{E}[\Pi(t, M_t)] = \frac{\sigma_v\,\sigma_y}{\mu}\,\cosh(\mu(T-t)^+).$$

[5] This follows from the Skorohod Representation Theorem and the fact that $M_t = V_t - P_t$ converges weakly to (the continuous process) 0.

$\mathbb{E}[\Pi(t, M_t)]$ is also the market's best estimate of the insider's expected payoff-to-go from time t on. Hence, from the market point of view the insider's expected payoff decreases monotonically with time in $[0, T)$ and stays constant after T.

Theorem 4 *When $\sigma_v = 0$, there exists a continuous-time linear Markovian equilibrium where*

$$\Sigma_t = \Sigma_0\, e^{-2\,\mu\,t}, \qquad \beta_t = \sqrt{\frac{2\,\mu\,\sigma_y^2}{\Sigma_0}}\, e^{\mu\,t}, \qquad \lambda_t = \sqrt{\frac{2\,\mu\,\Sigma_0}{\sigma_y^2}}\, e^{-\mu\,t}, \tag{22}$$

$$\alpha_t = \frac{e^{\mu\,t}}{2}\sqrt{\frac{\sigma_y^2}{2\,\mu\,\Sigma_0}} \qquad \textit{and} \qquad \gamma_t = \frac{e^{-\mu\,t}}{2}\sqrt{\frac{\Sigma_0\,\sigma_y^2}{2\,\mu}}. \tag{23}$$

PROOF: using the results in section 2.1, and replacing $n = t/\Delta$, we get the following equilibrium as a function of t and Δ.

$$\Sigma_t = \frac{\Sigma_0}{(1+S)^{\frac{t}{\Delta}}}, \qquad \beta_t = \frac{1}{\Delta}\sqrt{\frac{\sigma_y^2\,\Delta\,S}{\Sigma_0}}\,(1+S)^{\frac{t}{2\Delta}}, \qquad \lambda_t = \sqrt{\frac{\Sigma_0\,S}{\sigma_y^2\,\Delta}}\,(1+S)^{-(1+\frac{t}{2\Delta})},$$

$$\alpha_t = \frac{e^{-\mu\,\Delta}}{2}\sqrt{\frac{\sigma_y^2\,\Delta}{\Sigma_0\,S}}\,(1+S)^{\frac{t}{2\Delta}} \qquad \text{and} \qquad \gamma_t = \frac{e^{-2\,\mu\,\Delta}}{2}\,\frac{\sqrt{\Sigma_0\,\sigma_y^2\,\Delta\,S}}{\sqrt{1+S} - e^{-\mu\,\Delta}}\,(1+S)^{-(2+\frac{t}{2\Delta})},$$

where S is the unique root in $[0,1]$ of the equation $(1+S)\,(1-S)^2 = e^{-2\,\mu\,\Delta}$ and $\Sigma_0 = \Sigma_{-\Delta}\,(1+S)^{-1}$. For Δ small, it follows that $S = 2\,\mu\,\Delta + O(\Delta)$. Hence, in the limit as $\Delta \downarrow 0$, we get (22) and (23). ∎

Note that when $\sigma_v = 0$, in equilibrium $\Sigma_t \to 0$ as $t \to \infty$, but $\Sigma_t > 0$ for all $t \geq 0$. Also, the trading rate β_t remains bounded for all $t \geq 0$. Let us conclude this section discussing other remarks about the limiting solution in Theorem 3.

Remarks:

- The switching time T is independent of σ_y. On the other hand, T decreases with both σ_v and μ and increases with Σ_0. Furthermore, as $\sigma_v \downarrow 0$, this switching time diverges to $+\infty$ and the resulting profile coincides with the equilibrium derived in Theorem 4.

- One can show that the limit equilibrium in Theorem 3 satisfies the smooth-pasting condition

  $$\lim_{t \uparrow T} \dot{\Sigma}_t = 0.$$

  This is in contrast to the equilibria obtained in models that assume a fixed announcement date (*e.g.*, Kyle (1985)), where Σ_t does not approach 0 smoothly.

5 Discussion

5.1 The Effect of Noisy Information

A key difference between our model and those in the existing literature on strategic trading is that our insider continuously updates her knowledge of the fundamental value of the asset. The volatility coefficient σ_v determines the amount of information asymmetry between the insider and the rest of the market. The higher is σ_v the larger is the advantage of the insider.

As noted above, the switching time T decreases with σ_v, that is, the insider is willing to reveal her private information faster as the fundamental value becomes more volatile. The following result shows that this effect holds in a strong sense.

Proposition 5 *The value of Σ_t weakly decreases with σ_v for all $t \geq 0$. On the other hand, the insider's ex-ante expected payoff $\mathbb{E}[\widehat{\Pi}(t, M_t)] = \alpha_t \, \Sigma_t + \gamma_t$ is weakly increasing in σ_v for all $t \geq 0$.*

In other words, the more volatile is the fundamental value, the faster the price adjusts to the current intrinsic value. However, this efficiency come at a cost. Indeed, the insider is willing to trade away her private information faster because the market maker is willing to compensate her for doing so. Hence, we expect market prices to be more informative when the volatility of the fundamental value is higher. For example, in the special case in which there is no volatility ($\sigma_v = 0$), market efficiency ($\Sigma_t = 0$) is only reached asymptotically as $t \to \infty$ and the insider's ex-ante payoff is minimized.

5.2 Market Efficiency and Insider's Expected Payoff

The solution in Theorem 3 reveals that despite the fact that the market reaches full informational efficiency –that is, market price perfectly tracks the value of the asset– after time T, the insider still makes positive rents. In what follows, we show that this limiting solution cannot be an equilibrium of the game in continuous time.

Recall that the insider's expected payoff-to-go after time T can be written as

$$\Pi(T, M_T, P, X) = \mathbb{E}\left[\int_T^\infty e^{-\mu\,(t-T)} M_t \,\mathrm{d}X_t + \int_T^\infty e^{-\mu\,(t-T)} \,\mathrm{d}[X,V]_t - \int_T^\infty e^{-\mu\,(t-T)} \,\mathrm{d}[X,P]_t\right].$$

Let $(P^{\bar{\beta}}, X^{\bar{\beta}})$ be the equilibrium constructed in Theorem 2 and (P, X) be the limit equilibrium derived in Theorem 3. After time T, the market maker's pricing strategy P is given by $\mathrm{d}P_t = \lambda_T \mathrm{d}Z_t$, where $\lambda_T = \sigma_v/\sigma_y$, and the insider's cumulative volume of trade is a martingale process such that $\mathrm{d}X_t = \sigma_y \, [\mathrm{d}\, B_t^v - \mathrm{d}B_t^y]$. Thus, when $M_T = 0$, the first stochastic integral with respect to X_t has 0 expectation and the quadratic covariations between X_t and V_t and between X_t and P_t satisfy $\mathrm{d}[X,V]_t = \sigma_y \, \sigma_v \, \mathrm{d}t$ and $\mathrm{d}[X,P]_t = \lambda_t \, \sigma_y^2 \mathrm{d}t = \sigma_y \, \sigma_v \, \mathrm{d}t$ respectively. It follows that

$$\Pi(T, 0, P, X) = 0 \neq \lim_{\bar{\beta}\to\infty} \Pi(T, 0, P^{\bar{\beta}}, X^{\bar{\beta}}) = \frac{\sigma_v \sigma_y}{\mu}.$$

That is, Π has a discontinuity at (P, X) as X is approached by strategies of bounded variation.

5.3 On the Fitness of the Continuous Time Approximation

In this section, we discuss the relationship between the discrete-time equilibrium of Theorem 1 and the continuous-time limit equilibrium of Theorem 3. Specifically, we show that this continuous-time equilibrium is indeed a good approximation for the discrete-time equilibrium as the period length $\Delta \downarrow 0$.

Given $\Delta > 0$, let us denote by $\{(\Sigma_{n-1}^{\Delta}, \beta_n^{\Delta})\}_{n\geq 0}$ the corresponding discrete-time profile generated by the recursions in (3) and (10) with initial conditions $(\Sigma_{-1}^{\Delta}, \beta_0^{\Delta})$. In the analyis that follows, the value of Σ_{-1}^{Δ} is always set equal to the same initial value Σ_0, independent of Δ. Using the notation introduced in Section 3, we can extend $\{(\Sigma_{n-1}^{\Delta}, \beta_n^{\Delta})\}_{n\geq 0}$ to a continuous-time profile $\{(\Sigma_t^{\Delta}, \beta_t^{\Delta})\}_{t\geq 0}$ for all calendar time $t \geq 0$ defined (with a small abuse of notation) as follows: $\Sigma_t^{\Delta} = \Sigma_{\lfloor t-\Delta \rfloor}^{\Delta}$ and $\beta_t^{\Delta} = \beta_{\lfloor t \rfloor}^{\Delta}/\Delta$, where $\lfloor t \rfloor$ denotes the largest integer such that $n\Delta \leq t$.

The profile $(\Sigma_t^{\Delta}, \beta_t^{\Delta})$ is consistent with the equilibrium conditions if it is nonnegative and satisfies the recursion in (11), that is,

$$\Sigma_{t+\Delta}^{\Delta} = \sigma_v^2 \Delta + \frac{\sigma_y^2\, \Sigma_t^{\Delta}}{(\beta_t^{\Delta})^2 \Sigma_t^{\Delta}\, \Delta + \sigma_y^2} \quad \text{and} \quad \beta_{t+\Delta}^{\Delta} \Sigma_{t+\Delta}^{\Delta} = \frac{e^{-\mu\,\Delta}\, \beta_t^{\Delta}\, \sigma_y^4\, \Sigma_t^{\Delta}}{\sigma_y^4 - (\beta_t^{\Delta})^4\, (\Sigma_t^{\Delta})^2\, \Delta^2}, \quad t \geq 0. \tag{24}$$

Given Σ_0, the profile $(\Sigma_t^{\Delta}, \beta_t^{\Delta})$ is uniquely specified by β_0^{Δ} and (24). The pair of difference equations in (24) can be rewritten as

$$\frac{\Sigma_{t+\Delta}^{\Delta} - \Sigma_t^{\Delta}}{\Delta} = \sigma_v^2 - \frac{(\beta_t^{\Delta}\, \Sigma_t^{\Delta})^2}{\sigma_y^2 + (\beta_t^{\Delta})^2\, \Sigma_t^{\Delta}\, \Delta} \quad \text{and} \tag{25}$$

$$\frac{\beta_{t+\Delta}^{\Delta}\, \Sigma_{t+\Delta}^{\Delta} - \beta_t^{\Delta}\, \Sigma_t^{\Delta}}{\Delta} = \left[\frac{\sigma_y^4\, (e^{-\mu\,\Delta} - 1)/\Delta + (\beta_t^{\Delta})^4\, (\Sigma_t^{\Delta})^2\, \Delta}{\sigma_y^4 - (\beta_t^{\Delta})^4\, (\Sigma_t^{\Delta})^2\, \Delta^2}\right] \beta_t^{\Delta}\, \Sigma_t^{\Delta}. \tag{26}$$

This alternative representation suggests that these equations can be approximated by a pair of differential equations for Δ sufficiently small. For a given t, suppose that $\limsup_{\Delta\downarrow 0} (\beta_t^{\Delta})^2\, \Sigma_t^{\Delta} < \infty$. Then, $(\beta_t^{\Delta})^2\, \Sigma_t^{\Delta}\, \Delta$ is negligible for Δ sufficiently small and, as $\Delta \downarrow 0$, (25)-(26) converge to

$$\frac{\mathrm{d}\Sigma_t}{\mathrm{d}t} = \sigma_v^2 - \frac{(\Sigma_t\, \beta_t)^2}{\sigma_y^2} \qquad \text{and} \qquad \frac{\mathrm{d}(\Sigma_t\, \beta_t)}{\mathrm{d}t} = -\mu\, \Sigma_t\, \beta_t. \tag{27}$$

Since the condition $\limsup_{\Delta\downarrow 0} (\beta_t^{\Delta})^2\, \Sigma_t^{\Delta} < \infty$ is trivially satisfied at $t = 0$, we expect (by continuity) that the convergence above holds for all $t \in [0, \tau)$ for some positive $\tau > 0$. Then, integrating (27) in this range we obtain a continuous profile (Σ_t^0, β_t^0) given by

$$\Sigma_t^0 = \Sigma_0 + \sigma_v^2\, t - \frac{(\beta_0\, \Sigma_0)^2}{2\,\mu\,\sigma_y^2}\,(1 - e^{-2\,\mu\,t}) \quad \text{and} \quad \beta_t^0 = \frac{\beta_0\, \Sigma_0\, e^{-\mu\,t}}{\Sigma_t^0}, \qquad t < \tau. \tag{28}$$

Note that for this continuous-time solution the condition $\limsup_{\Delta\downarrow 0} (\beta_t^{\Delta})^2\, \Sigma_t^{\Delta} = (\beta_t^0)^2\, \Sigma_t^0 < \infty$ reduces to $\Sigma_t^0 > 0$. Hence, τ is uniquely determined as the smallest (positive) solution of the equation $\Sigma_\tau^0 = 0$, that is,

$$0 = \Sigma_0 + \sigma_v^2\, \tau - \frac{(\beta_0\, \Sigma_0)^2}{2\,\mu\,\sigma_y^2}\,(1 - e^{-2\,\mu\,\tau}).$$

We will denote this solution by $\tau(\beta_0)$ to emphasize its dependence on the initial condition β_0. If such a solution does not exist then we set $\tau(\beta_0) = \infty$ and the continuous-time approximation holds for all $t \geq 0$.

According to the previous discussion, the discrete-time equilibrium $(\Sigma_t^\Delta, \beta_t^\Delta)$ converges to (Σ_t^0, β_t^0) for all $t \in [0, \tau(\beta_0))$. However, this analysis, based on the limiting behavior of the difference equations (25)-(26), is not sufficient to pinpoint the right value of β_0 to initialize the continuous-time solution (Σ_t^0, β_t^0). For this, we need to impose the conditions that characterize the unique discrete-time equilibrium of Theorem 1.

From Lemma 3 (and the discussion that follows it), the initial value β_0^Δ for the discrete-time equilibrium with period length Δ can be found as the supremum of the set $\mathcal{B}^\Delta(\Sigma_0)$. This is the set of all those β_0^Δ for which the corresponding profile $(\Sigma_t^\Delta, \beta_t^\Delta)$ generated by (24) is nonnegative for all t. Hence, we need to characterize the limit of the set $\mathcal{B}^\Delta(\Sigma_0)$ as $\Delta \downarrow 0$. Let us denote this limiting set by $\mathcal{B}^0(\Sigma_0) = \liminf_{\Delta \downarrow 0} \mathcal{B}^\Delta(\Sigma_0)$. That is, $\beta_0 \in \mathcal{B}^0(\Sigma_0)$ if and only if there exists $\hat{\Delta} > 0$ such that $\beta_0 \in \mathcal{B}^\Delta(\Sigma_0)$ for all $\Delta \leq \hat{\Delta}$.

Proposition 6 *Let η be the unique nonnegative root of the equation*

$$0 = \Sigma_0 + \frac{\sigma_v^2}{2\,\mu}\,[\ln{(\eta - 1)} - \eta].$$

Then, $\mathcal{B}^0(\Sigma_0) = [0, \bar{\beta}_0)$ where $\bar{\beta}_0 = \frac{\sigma_v\,\sigma_y}{\Sigma_0}\sqrt{\eta - 1}$. Furthermore, $\tau(\bar{\beta}_0) = \frac{1}{\mu}\ln\left(\frac{\Sigma_0\,\bar{\beta}_0}{\sigma_v\,\sigma_y}\right)$.

According to this result, as $\Delta \downarrow 0$, the initial condition β_0^Δ that characterized the discrete-time equilibrium converges to $\bar{\beta}_0$. It follows then from our previous discussion that for all $t < \tau(\bar{\beta}_0)$,

$$\lim_{\Delta\downarrow 0}\ \Sigma_t^\Delta = \Sigma_t^0 = \Sigma_0 + \sigma_v^2\,t - \frac{(\Sigma_0\,\bar{\beta}_0)^2}{2\,\mu\,\sigma_y^2}(1 - e^{-2\,\mu\,t}) \quad \text{and} \quad \lim_{\Delta\downarrow 0}\ \beta_t^\Delta = \beta_t^0 = \frac{\Sigma_0\,\bar{\beta}_0}{\Sigma_t^0}\,e^{-\mu\,t}.$$

The proof of Proposition 6 reveals that at $t = \tau(\bar{\beta}_0)$, both $\Sigma_t^0 = 0$ and $\dot{\Sigma}_t^0 = 0$. That is, the limit equilibrium satisfies a smooth pasting condition.

One can also show that $\tau(\bar{\beta}_0)$ is equal to T in Theorem 3 and that the limiting solution (Σ_t^0, β_t^0) above coincides with the continuous-time solution (Σ_t, β_t) derived in Theorem 3 for $t < T$. In other words, for Δ small, the discrete-time equilibrium is well approximated by the *absolutely continuous phase* of the solution in Theorem 3.

As for what happens after time T, we note that there is no real discrete-time counterpart for the *singular phase*. In fact, for the continuous-time model, T is the time at which the system reaches stationarity. In discrete-time, however, stationary is only reached asymptotically (unless $\Sigma_0 = 0$). Nevertheless, the continuous-time variance process Σ_t in Theorem 3 is a good approximation of the discrete-time variance Σ_t^Δ for all $t \geq 0$. Indeed, for Δ sufficiently small, the discrete-time variance Σ_t^Δ is nonincreasing in t and, since $\Sigma_t = 0$ for all $t \geq T$, it follows that $|\Sigma_t^\Delta - \Sigma_t| \leq |\Sigma_T^\Delta - \Sigma_T|$ for all $t \geq T$. Since $\Sigma_T^\Delta \to \Sigma_T = 0$ as $\Delta \downarrow 0$ we conclude that

$$\lim_{\Delta\downarrow 0}\ \sup_{t\geq 0}\ |\Sigma_t^\Delta - \Sigma_t| = 0.$$

Appendix

Proof of Lemma 1.

For each $n \geq 0$ and each $k \geq 0$, consider the finite horizon problem for the insider where the fundamental value is made public for sure at the end of period $n+k$ if it has not been publicly revealed before. Let $\Pi_{k,n}(p,V)$ be the insider's optimal discounted value from period n onward in this problem, when the price and fundamental value in period $n-1$ are (p,V). Obviously, $\Pi_{k,n}(p,V) \leq \hat{\Pi}_n(p,V)$ (because the insider can always choose $x_s = 0$ for all $s > n+k$) and $\lim_{k\to\infty} \Pi_{k,n}(p,V) = \hat{\Pi}_n(p,V)$ for all $n \geq 0$ and all $(p,V) \in \mathbb{R}^2$.

We first show inductively in k that for each n, either

$$\Pi_{k,n}(p,V) = \frac{a_{k,n}}{\lambda_n}(V-p)^2 + \frac{b_{k,n}}{\lambda_n}\Sigma_v + c_{k,n}\lambda_n\Sigma_y \tag{29}$$

for some constants $(a_{k,n}, b_{k,n}, c_{k,n})$, or $\Pi_{k,n} \equiv \infty$. When $k = 0$,

$$\Pi_{0,n}(p,V) = \max\ (V - p - \lambda_n x)x = \frac{(V-p)^2}{4\lambda_n},$$

so $\Pi_{0,n}$ satisfies (29) with $a_{0,n} = 1/4$ and $b_{0,n} = c_{0,n} = 0$ for all $n \geq 1$. By induction, assume first that $\Pi_{k,n+1}$ satisfies (29) for a given (k,n). We then show that either $\Pi_{k+1,n}$ also satisfies (29) or $\Pi_{k+1,n} \equiv \infty$. We have that

$$\Pi_{k+1,n}(p,V) = \max_{x\in\mathbb{R}}\ (V-p-\lambda_n x)x + \rho\mathbb{E}[\Pi_{k,n+1}(V+W_n, p+\lambda_n(x+Y_n))]$$

$$= \max_{x\in\mathbb{R}}\ (V-p-\lambda_n x)x + \rho\left[\frac{a_{k,n+1}}{\lambda_{n+1}}[(V-p-\lambda_n x)^2 + \Sigma_v + \lambda_n^2\Sigma_y] + \frac{b_{k,n+1}}{\lambda_{n+1}}\Sigma_v + c_{k,n+1}\lambda_{n+1}\Sigma_y\right].$$

When $\rho\lambda_n a_{k,n+1}/\lambda_{n+1} < 1$, the quadratic objective function is concave and

$$\Pi_{k+1,n}(p,V) = \frac{\lambda_{n+1}(V-p)^2}{4\lambda_n[\lambda_{n+1} - a_{k,n+1}\rho\lambda_n]} + \rho\left[\frac{\Sigma_v}{\lambda_{n+1}}[a_{k,n+1} + b_{k,n+1}] + \Sigma_y\left[a_{k,n+1}\frac{\lambda_n^2}{\lambda_{n+1}} + c_{k,n+1}\lambda_{n+1}\right]\right].$$

Hence $\Pi_{k+1,n}$ satisfies (29) with

$$a_{k+1,n} = \frac{1}{4}\left[1 - a_{k,n+1}\frac{\rho\lambda_n}{\lambda_{n+1}}\right]^{-1} \tag{30}$$

$$b_{k+1,n} = \frac{\rho\lambda_n}{\lambda_{n+1}}[a_{k,n+1} + b_{k,n+1}] \quad \text{and} \quad c_{k+1,n} = \rho\left[a_{k,n+1}\frac{\lambda_n}{\lambda_{n+1}} + c_{k,n+1}\frac{\lambda_{n+1}}{\lambda_n}\right]. \tag{31}$$

When $a_{k,n+1}\rho\lambda_n/\lambda_{n+1} \geq 1$, the quadratic objective function is convex, and $\Pi_{k+1,n} \equiv \infty$. By induction, now assume instead that $\Pi_{k,n+1} \equiv \infty$. Then $\Pi_{k+n+1-s,s} \equiv \infty$ for all $s = 0, \ldots, n$. This concludes our proof by induction.

Let us now assume that $\sum \rho^n/\lambda_n = \infty$. In this case we will show that $\Pi_{k,n}(p,v) \to \infty$ as $k \to \infty$, for all n and (p,v).

Special Case: When $\lambda_{n+1}/\lambda_n = \rho$ for all $n \geq 0$, the sequences $\{a_{k,n}\}$, $\{b_{k,n}\}$ and $\{c_{k,n}\}$ are independent of n and

$$a_{k+1} = \frac{1}{4(1-a_k)}, \quad b_{k+1} = b_k + a_k \quad \text{and} \quad c_{k+1} = \rho^2 c_k + a_k.$$

These difference equations have the solutions

$$a_k = \frac{k+1}{2(k+2)}\,, \quad b_k = a_0 + \cdots + a_{k-1} \quad \text{and} \quad c_k = a_{k-1} + a_{k-2}\rho^2 + \cdots + a_0\rho^{2(k-1)}.$$

Since $1/4 \le a_k < a_{k+1} < 1/2$ for all k and $a_k \to 1/2$, we have that $b_k \to \infty$ and $c_k \to [2(1-\rho^2)]^{-1}$. Therefore, as $\Pi_{k,n}(p,v) \ge b_k/\lambda_n$, $\Pi_{k,n}(p,v) \to \infty$ for all n and (p,v).

The situation $\lambda_{n+1}/\lambda_n = \rho$ for all $n \ge 1$ represents a limit case. If the sequence $\{\lambda_n\}$ goes to 0 faster, then for any n there exists k such $\Pi_{k,n} \equiv \infty$. Suppose for example that $\lambda_{n+1}/\lambda_n \le \rho 3/4$ for all n. The function $f(a,d) = [4(1-ad)]^{-1}$ is increasing in a and d (when $ad < 1$). Therefore, $a_{k,n} \ge \hat{a}_k$ for all (k,n), where $\hat{a}_0 = 1/4$ and $\hat{a}_{k+1} = f(\hat{a}_k, 4/3)$ for all $k \ge 0$. Since $\hat{a}_1 = 3/8$, $\hat{a}_2 = 1/2$, and $\hat{a}_3 = 3/4$, we have that $1 \le a_{3,n+1}\rho\lambda_n/\lambda_{n+1}$, and $\Pi_{4,n} \equiv \infty$ for all n.

In the general case, since $a_{0,n} = 1/4$ and $\rho\lambda_n/\lambda_{n+1} > 0$ for all $n \ge 1$, it is easy to see (by induction) that (30) implies $a_{k,n} > 1/4$ for all $k \ge 1$ and $n \ge 0$. Fix $n \ge 0$. For any $k \ge 1$, if there exists $j \in \{1,\dots,k\}$ such that $1 \le a_{k-j.t+j}\rho\lambda_{n+j}/\lambda_{n+j+1}$, then $\Pi_{k-j+1,n+j-1} \equiv \infty$, which implies that $\Pi_{k,n} \equiv \infty$. If $1 > a_{k-j.t+j}\rho\lambda_{n+j}/\lambda_{n+j+1}$ for all $j \in \{1,\dots,k\}$, then (31) implies that

$$b_{k,n} \ge \frac{\rho\lambda_n}{\lambda_{n+1}}\left[\frac{1}{4} + b_{k-1,n+1}\right] \ge \frac{\rho\lambda_n}{\lambda_{n+1}}\left[\frac{1}{4} + \frac{\rho\lambda_{n+1}}{\lambda_{n+2}}\left[\frac{1}{4} + b_{k-2,n+2}\right]\right] \cdots \ge \frac{\lambda_n}{4}\left[\frac{\rho}{\lambda_{n+1}} + \cdots + \frac{\rho^k}{\lambda_{n+k}}\right].$$

Note that

$$\sum_{j=1}^{\infty} \frac{\rho^j}{\lambda_{n+j}} = \frac{1}{\rho^n}\sum_{j=t+1}^{\infty}\frac{\rho^j}{\lambda_j} = \frac{1}{\rho^n}\left[\sum_{j=1}^{\infty}\frac{\rho^j}{\lambda_j} - \sum_{j=1}^{n}\frac{\rho^j}{\lambda_j}\right] = \infty.$$

Therefore, either $\Pi_{k,n} \equiv \infty$ for some $k \ge 1$, which implies that $\hat{\Pi}_n \equiv \infty$, or for all $k \ge 1$,

$$\hat{\Pi}_n(p,v) \ge \Pi_{k,n}(p,v) \ge \frac{\Sigma_v}{\lambda_n} b_{k,n} \ge \frac{\Sigma_v}{4}\sum_{j=1}^{k}\frac{\rho^j}{\lambda_{n+j}}.$$

Since the right-hand side of the last inequality converges to ∞ as $k \to \infty$, $\hat{\Pi}_n \equiv \infty$ in this case as well.

Finally, assume that $\sum \rho^n/\lambda_n < \infty$ and $\rho\lambda_n/\lambda_{n+1} \le 1$ for all $n \ge 1$ In this case we will show that each $\hat{\Pi}_n(p,V)$ is a quadratic function of $(V-p)$ and that $\hat{\Pi} = B(\hat{\Pi})$.

Since $a_{0,n} = 1/4$, it is easy to show by induction that $1/4 < a_{k,n} < 1/2$ for all $k \ge 1$ and $n \ge 0$. Recall that $f(a,d) = [4(1-ad)]^{-1}$ is increasing in a and d. Since $a_{1,n+1} > 1/4 = a_{0,n+1}$ for all $n \ge 0$, $a_{2,n} = f(a_{1,n+1}, d_n) > f(a_{0,n+1}, d_n) = a_{1,n}$ for all $n \ge 0$. Repeating this argument forward, we conclude that $\{a_{k,n}\}_{k=1}^{\infty}$ is an increasing sequence and it must converge. Let $\alpha_n = \lim_{k\to\infty} \rho a_{k,n}/\lambda_n$. Since $\rho\lambda_n/\lambda_{n+1} \le 1$ and $a_{k,n+1} < 1/2$ for all k, $\lambda_n\alpha_{n+1} \le 1/2$.

Since $a_{k,n} < 1/2$ for all $k \ge 0$ and $n \ge 0$,

$$b_{k,n} \le \frac{\rho\lambda_n}{\lambda_{n+1}}\left[\frac{1}{2} + b_{k-1,n+1}\right] \le \cdots \le \frac{\lambda_n}{2}\left[\frac{\rho}{\lambda_{n+1}} + \cdots + \frac{\rho^k}{\lambda_{n+k}}\right] < \frac{\lambda_n}{2\rho^n}\sum_{j=t+1}^{\infty}\frac{\rho^j}{\lambda_j} < \infty.$$

By induction in k, we now show that $b_{k,n} < b_{k+1,n}$ for all $k \ge 0$ and $n \ge 1$. Clearly $b_{0,n} = 0 < b_{1,n}$ for all $n \ge 0$. Since $a_{k,n+1} < a_{k+1,n+1}$, if the inequality holds for (k,n), then

$$b_{k+1,n} = d_n[a_{k,n+1} + b_{k,n+1}] < d_n[a_{k+1,n+1} + b_{k+1,n+1}] = b_{k+2,n}.$$

That is, for each $n \geq 0$, the sequence $\{b_{k,n}\}_{k=0}^{\infty}$ is increasing and hence it must converge. Solving (31) we obtain

$$c_{k,n} = \frac{1}{\lambda_n} \sum_{j=1}^{k} \rho^j \frac{\lambda_{n+j-1}^2}{\lambda_{n+j}} a_{k-j,n+j}.$$

One can show that for each $n \geq 0$, the sequence $\{c_{k,n}\}_{k=1}^{\infty}$ is increasing, and since $\lambda_s \leq M$ and $a_{j,s} < 1/2$ for all j and s,

$$c_{k,n} \leq \frac{M^2}{2\lambda_n} \sum_{j=1}^{k} \frac{\rho^j}{\lambda_{n+j}} < \frac{M^2}{2\lambda_n \rho^n} \sum_{j=t+1}^{\infty} \frac{\rho^j}{\lambda_j} < \infty,$$

the sequence must converge. Let $\gamma_n = \lim_{k\to\infty} \rho[b_{k,n}\Sigma_v/\lambda_n + c_{k,n}\lambda_n\Sigma_y]$ and define $\hat{\Pi}_n$ by $\rho\hat{\Pi}_n(p,v) = \alpha_n(v-p)^2 + \gamma_n$.

Let $\mathbb{Q}$ be the set of $f \in \mathbb{B}$ such that for some $a, b \in \mathbb{R}$, $f(p,v) = a(v-p)^2 + b$, and define the norm $||f|| = \max\{|a|, |b|\}$. Let $\mathbb{Q}_n = \{a(v-p)^2 + b \mid a\rho\lambda_n < 1\}$. Then, $b_n : \mathbb{Q}_n \to \mathbb{Q}_n$ is continuous. Therefore, for each $n \geq 0$,

$$\hat{\Pi}_n = \lim_{k\to\infty} \Pi_{k,n} = \lim_{k\to\infty} b_n(\Pi_{k,n+1}) = b_n(\hat{\Pi}_{n+1}).$$

That is, $\hat{\Pi} = B(\hat{\Pi})$. ∎

Proof of Lemma 3.

When $\beta_0 < \Psi(\Sigma_{-1})$, the sequence $\{(A_n, B_n)\}$ remains feasible forever. Moreover, for some finite N, $(A_n, B_n) \in R_3$ for all $n \geq N$. Therefore $A_n < A_{n+1}$ for all $n \geq N$ and $A_n \to \infty$. Recall that the graphs of G_1 and G_2 intersect at $(\hat{A}, \hat{B})$, and that $(A, B) \in R_3$ and $A \geq \hat{A}$ imply that $B \leq G_1(A)$. The function $h(A) = (A-1)^2/[A(A-2)]$ is decreasing for all $A > 2$, and $h(A) \to 1$ as $A \to \infty$. Let $\omega \in (\rho, 1)$. Without loss of generality, assume that N is such that $A_N \geq \hat{A}$ and $h(A_N) \leq \omega/\rho$. Then, $B_n \leq G_1(A_n)$ for all $n \geq N$, and therefore for all $n \geq N$,

$$B_{n+1} = F_B(A_n, B_n) = \rho\left[\frac{A_n^2 B_n}{A_n^2 - B_n^4}\right] \leq \rho\left[\frac{A_n^2 B_n}{A_n^2 - [G_1(A_n)]^4}\right] = \rho h(A_n) B_n \leq \omega B_n. \tag{32}$$

Since $B_N \leq \hat{B}$, this implies that $B_n \leq \hat{B}\omega^{n-N}$ for all $n \geq N$. From (4),

$$\lambda_n = \frac{\beta_n \Sigma_{n-1}}{\beta^2\Sigma_{n-1} + \Sigma_y} = \frac{A_n B_n}{A_n + B_n^2}\sqrt{\frac{\Sigma_v}{\Sigma_y}} < B_n\sqrt{\frac{\Sigma_v}{\Sigma_y}}.$$

Since we would like to show that $\sum \rho^n/\lambda_n = \infty$, we need a tighter upper bound on B_n. Note, however, that

$$B_{n+1} = F_B(A_n, B_n) = \rho\left[\frac{A_n^2 B_n}{A_n^2 - B_n^4}\right] \geq \rho B_n \quad \text{for all } n \geq 0,$$

so there is not a lot of slack in the previous upper bound (32) for B_{n+1}.

For any $\epsilon > 0$, let $N^* > N$ be such that $\hat{B}\omega^{N^*-N} < \epsilon$. Then, for all $n \geq N^*$,

$$A_{n+1} = F_A(A_n, B_n) = 1 + \frac{A_n^2}{A_n + B_n^2} \geq 1 + \frac{A_n}{1 + \epsilon^2/A_n} \geq 1 + A_n\left[1 - \frac{\epsilon^2}{A_n}\right] = A_n + (1 - \epsilon^2).$$

Let $e = 1 - \epsilon^2$. Then $A_{N^*+n} \geq A_{N^*} + ne > ne$ for all $n \geq N^*$. Feeding this bound back into (32), we obtain that

$$B_{N^*+n+3} \leq \rho h((n+2)e) B_{N^*+2+n} \leq \cdots \leq \rho^n h((n+2)e) h((n+1)e) \cdots h(3e) B_{N^*+3}.$$

Choose $\epsilon < 1/4$ so that $\epsilon^2 < 1/16$. Then, for all $k \geq 3$,

$$\begin{aligned} h(ke) &= \frac{[k-1-k\epsilon^2]^2}{[k-k\epsilon^2][k-2-k\epsilon^2]} = 1 + \frac{1}{k(k-2) - 2k(k-1)\epsilon^2 + k^2\epsilon^4} \\ &< 1 + \frac{1}{k[k-2-2(k-1)\epsilon^2]} < 1 + \frac{8}{k[7k-15]} \leq 1 + \frac{4}{k^2}\,. \end{aligned}$$

Let

$$H_n = \left[1 + \frac{4}{1^2}\right]\left[1 + \frac{4}{2^2}\right] \cdots \left[1 + \frac{4}{n^2}\right] \quad \text{and} \quad a_n = \frac{1}{H_n} = \left[\frac{1^2}{1^2+4}\right] \cdots \left[\frac{n^2}{n^2+4}\right].$$

Note that $[1+4/1^2][1+4/2^2] = 10$. Hence, $B_{N^*+n+3} < \rho^n B_{N^*+3} H_n/10$. Therefore,

$$\sqrt{\frac{\Sigma_v}{\Sigma_y}} \sum_{n\geq 1} \frac{\rho^n}{\lambda_n} > \sum_{n \geq 1} \frac{\rho^n}{B_n} > \sum_{n\geq 1} \frac{10\rho^{N^*+3+n}}{\rho^n H_n B_{N^*+3}} = \frac{10}{B_{N^*+3}} \rho^{N^*+3} \sum_{n \geq 1} a_n.$$

Gauss's test (see, for example, Knopp 1990) states that if

$$\frac{a_{n+1}}{a_n} = 1 - \frac{c}{n} - \frac{g_n}{n^\epsilon}$$

where $\epsilon > 1$ and $\{g_n\}$ is bounded, then $\sum a_n$ converges when $c > 1$ and diverges when $c \leq 1$. In our case

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(n+1)^2+4} = 1 - \left[\frac{4n^2}{(n+1)^2+4}\right]\frac{1}{n^2}\,,$$

so $c = 0$ and $\epsilon = 2$. Therefore $\sum a_n = \infty$ and $\sum \rho^n/\lambda_n = \infty$. ∎

Proof of Proposition 1.

Consider equations (3) and (10) with $\Sigma_v = 0$. Let us introduce the change of variable $S_n = \beta_{n+1}^2\,\Sigma_n/\Sigma_y$. Replacing β_n by S_n, equations (3) and (10) become

$$\Sigma_n = \frac{\Sigma_{n-1}}{S_{n-1}+1} \qquad \text{and} \qquad S_n = \frac{\rho^2\, S_{n-1}}{(1+S_{n-1})\,(1-S_{n-1})^2}.$$

With a partial substitution, equation (10) can also be written as $\beta_{n+1}\Sigma_n = \rho\beta_n\Sigma_{n-1}/(1-S_{n-1}^2)$. Therefore, for $\{S_n\}$ to be compatible with equilibrium it must be that $0 < S_n \leq 1$ for all n. Since the dynamics of S_n on the right equation are independent of Σ_n we can solve this equation independently of the first equation. A solution for this equation is $S_n = S$ for all $n \geq 0$, where S is a root of the equation

$$(1+S)\,(1-S)^2 = \rho^2.$$

The function $f(x) = (1+x)\,(1-x)^2$ is monotonically decreasing in $(0,1)$ with $f(1) = 0 < \rho < 1 = f(0)$ and we conclude that there is a unique root $S \in (0,1)$. If we set $S_{-1} = S$ then $S_n = S$ for all $n \geq 0$. In this case, the evolution of Σ_n is given by

$$\Sigma_n = \frac{\Sigma_{-1}}{(1+S)^{n+1}} \qquad \text{for all } n \geq 0. \tag{33}$$

Thus, the variance decreases geometrically over time. The evolution of β_n follows directly from the definition of S_n. Similarly, the value of α_{n+1} can be computed from equation (9). Finally, to get the value of γ_n we iterate equation (8) to get

$$\begin{aligned}\gamma_n &= \rho\gamma_{n+1} + \rho\alpha_{n+1}\,\lambda_n^2\,\Sigma_y = \rho^2\gamma_{n+2} + \rho^2\alpha_{n+2}\,\lambda_{n+1}^2\,\Sigma_y + \rho\alpha_{n+1}\,\lambda_n^2\,\Sigma_y \\ &= \sum_{k=1}^{\infty}\rho^k\alpha_{n+k}\,\lambda_{n+k-1}^2\,\Sigma_y + \lim_{k\to\infty}\rho^k\,\gamma_{n+k}.\end{aligned}$$

Replacing α_n and λ_n one can show that the summation converges to the value of γ_n stated in the proposition, which also shows that the limit converges to 0 since $0 < \rho < 1$.

Let $g(x) = \rho^2 x/f(x)$, so that $S_n = g(S_{n-1})$. The function $g : [0,1) \to \mathbb{R}$ is convex, $g(0) = 0$, $\lim_{x\to 1} g(x) = \infty$, and $g(S) = S$. If we set $S_{-1} > S$, then the sequence generated by $S_n = g(S_{n-1})$ increases monotonically until $S_n > 1$ for some n. That is, the sequence becomes infeasible. If we set $S_{-1} < S$, then the sequence generated by $S_n = g(S_{n-1})$ decreases monotonically to 0, and the corresponding sequence $\{\lambda_n\}$ converges to 0 'too fast', making $\sum \rho^n/\lambda_n = \infty$. Therefore, only the choice $S_{-1} = S$ is consistent with equilibrium. ∎

Proof of Proposition 2.

The condition $P_t = \mathbb{E}[V_t|\mathcal{F}_t^M]$ implies that P_t is the orthogonal projection V_t on $\mathcal{F}_t^M$ in L^2. Hence, we can interpret the market maker's equilibrium condition as the solution to a classical Kalman-Bucy filtering problem. Let the signal process be the value of the fundamental V_t, with dynamics

$$\mathrm{d}V_t = \sigma_v\,\mathrm{d}B_t^v,$$

and the observation process be the price process P_t, with dynamics

$$\mathrm{d}P_t = \lambda_t\,\mathrm{d}Z_t = \beta_t\,\lambda_t\,(V_t - P_t)\,\mathrm{d}t + \sigma_y\,\lambda_t\,\mathrm{d}B_t^y.$$

Let v_t be the corresponding optimal (in mean square sense) filtering estimate of V_t and Σ_t be the filtering error. Then, the equilibrium condition is $P_t = v_t$.

The generalized Kalman filter conditions for the pair (V_t, P_t) are given by

$$\mathrm{d}v_t = \frac{\Sigma_t\,\beta_t}{\lambda_t\,\sigma_y^2}\,[\mathrm{d}P_t - \lambda_t\,\beta_t\,(v_t - P_t)\,\mathrm{d}t] \qquad \text{and} \qquad \dot{\Sigma}_t = \sigma_v^2 - \frac{(\Sigma_t\,\beta_t)^2}{\sigma_y^2}.$$

To recover the identity $P_t = v_t$ we need to impose that

$$\frac{\Sigma_t\,\beta_t}{\lambda_t\,\sigma_y^2} = 1 \qquad \text{or equivalently} \qquad \Sigma_t\,\beta_t = \lambda_t\,\sigma_y^2.$$

This equality together with the border condition $v_0 = P_0$ imply that $v_t = P_t$ for all $t > 0$. This equality also implies that $(\Sigma_t\,\beta_t)^2 = \lambda_t^2\,\sigma_y^4$. Therefore, the second filtering condition leads to the differential equation

$$\dot{\Sigma}_t = \sigma_v^2 - \sigma_y^2\,\lambda_t^2,$$

which completes the proof of the Lemma. ∎

Proof of Theorem 2.

To prove this theorem we establish that (i) λ_t satisfies the filtering condition (16) for the market-maker; and (ii) given λ_t, $(\beta_t, \Pi(t, M))$ solves the HJB equation (15).

FILTERING CONDITONS: In $[0, T)$, (17) and (18) imply that $\Sigma_t\, \beta_t = \sigma_y^2\, \lambda_t = \sigma_y^2\, \lambda_0\, e^{-\mu\, t}$, and the second filtering condition of (16) is satisfied if and only if

$$\dot{\Sigma}_t = \sigma_v^2 - \sigma_y^2\, \lambda_t^2 = \sigma_v^2 - \sigma_y^2\, \lambda_0^2\, e^{-2\,\mu\, t}.$$

The solution of this differential equation is

$$\Sigma_0 + \sigma_v^2 t - \frac{\lambda_0^2\, \sigma_y^2}{2\mu}\big[1 - e^{-2\,\mu\, t}\big].$$

In (T, ∞), $\beta_t \equiv \bar{\beta}$. In this case, the filtering conditions (16) become

$$\Sigma_t\, \bar{\beta} = \lambda_t\, \sigma_y^2 \qquad \text{and} \qquad \dot{\Sigma}_t = \sigma_v^2 - \sigma_y^2\, \lambda_t^2.$$

The second filtering equation leads to the differential equation

$$\frac{\sigma_y^2}{\bar{\beta}}\dot{\lambda}_t = \sigma_v^2 - \sigma_y^2\, \lambda_t^2.$$

Therefore, the filtering conditions are satisfied if and only if

$$\lambda_t = \frac{\sigma_v}{\sigma_y}\left[\frac{e^{L\,t} + k\, e^{L\,T}}{e^{L\,t} - k\, e^{L\,T}}\right] \qquad \text{and} \qquad \Sigma_t = \frac{\sigma_v\, \sigma_y}{\bar{\beta}}\left[\frac{e^{L\,t} + k\, e^{L\,T}}{e^{L\,t} - k\, e^{L\,T}}\right],$$

for some constant of integration k. Note that λ_t and Σ_t are decreasing in t if and only if $k \geq 0$, a fact we establish below.

OPTIMALITY CONDITONS: We guess that $\Pi(t, M) = \alpha_t M^2 + \gamma_t$, with the coefficients α_t and γ_t defined by (19) and (20). Then, the HJB equation becomes

$$0 = \max_{|\beta| \leq \bar{\beta}} \left\{ \big[\beta\,(1 - 2\lambda_t\, \alpha_t) + \dot{\alpha}_t - \mu\, \alpha_t\big]\, M^2 + \alpha_t(\sigma_v^2 + \lambda_t^2 \sigma_y^2) + \dot{\gamma}_t - \mu\gamma_t \right\}.$$

Note that the right-hand side is linear in β, and recall that $\sigma_t^2 = \sigma_v^2 + \lambda_t^2 \sigma_y^2$.

In $(0, T)$, $0 < \beta_t < \bar{\beta}$. Thus, the HJB equation is satisfied if and only if $\dot{\alpha}_t - \mu\, \alpha_t = 0$, $\dot{\alpha}_t - \mu\, \alpha_t = 0$, and $\alpha_t(\sigma_v^2 + \lambda_t^2 \sigma_y^2) + \dot{\gamma}_t - \mu\gamma_t = 0$. The first two conditions are met if and only if $\lambda_t = \lambda_0\, e^{-\mu t}$ and $\alpha_t = e^{\mu t}/[2\lambda_0]$, for some constant $\lambda_0 > 0$. Replacing these two functions, the solution of the last differential equation is

$$\gamma_t = \left[\gamma_0 - \frac{\lambda_0 \sigma_y^2}{4\mu} - \alpha_0 \sigma_v^2 t\right] e^{\mu t} + \frac{\lambda_0 \sigma_y^2}{4\mu} e^{-\mu t},$$

for some constant γ_0.

In (T, ∞), $\beta_t \equiv \bar{\beta}$, so the HJB equation is satisfied if and only if $1 - 2\lambda_t \alpha_t \geq 0$,

$$\bar{\beta}(1 - 2\lambda_t \alpha_t) + \dot{\alpha}_t - \mu\, \alpha_t = 0 \qquad \text{and} \qquad \alpha_t\, \sigma_t^2 + \dot{\gamma}_t - \mu\gamma_t = 0.$$

We check the last two conditions first. The coefficient α_t defined by (19) satisfies

$$\dot{\alpha}_t = -(L-\mu)\,\alpha_t + \frac{2\,L\,e^{L\,t}}{e^{L\,t} - k\,e^{L\,T}}\,\alpha_t - \bar{\beta} = \mu\,\alpha_t + \bar{\beta}(2\,\lambda_t\,\alpha_t - 1). \tag{34}$$

The coefficient γ_t defined by (20) satisfies

$$\dot{\gamma}_t = \mu\,\gamma_t - \bar{\beta}\,e^{\mu\,t}\int_t^\infty e^{(L-\mu)\,s}[e^{L\,s} - k\,e^{L\,T}]^{-2}\sigma_t^2\,e^{-L\,t}[e^{L\,t} - k\,e^{L\,T}]^2 \mathrm{d}s = \mu\,\gamma_t - \alpha_t\,\sigma_t^2. \tag{35}$$

Finally we show that $2\,\lambda_t\,\alpha_t < 1$ for all $t \in (T,\infty)$. Since $2\,\lambda_t\,\alpha_t = 1$ for all $t \in (0,T)$, we have that $2\,\lambda_T^-\,\alpha_T^- = 1$. The value-matching conditions (21) also require that $\lambda_T^- = \lambda_T$ and $\alpha_T^- = \alpha_T$. Therefore $2\,\lambda_T\,\alpha_T = 1$. To ensure the nonnegativity of λ_t for $t \geq T$, we must require that $|k| \leq 1$. Using the definitions of λ_t and α_t for $t \geq T$, and with the change of variable $u = L\,s$, it follows that

$$2\,\alpha_t\,\lambda_t = L\int_0^\infty e^{-(L+\mu)\,s}\,\frac{1-[k\,e^{-L(t-T)}]^2}{[1-k\,e^{-L(s+t-T)}]^2}\,\mathrm{d}s = \int_0^\infty e^{-(1+\mu/L)\,u}\,\frac{1-[k\,e^{-L(t-T)}]^2}{[1-k\,e^{-L(t-T)}\,e^{-u}]^2}\,\mathrm{d}u.$$

For notational convenience, let us define the constant $a = 1+\mu/L$ and the auxiliary function

$$h(\ell) = \int_0^\infty e^{-a\,u}\,\frac{1-\ell^2}{[1-\ell\,e^{-u}]^2}\,\mathrm{d}u,$$

so that $2\,\alpha_t\,\lambda_t = h\big(k\,e^{-L(t-T)}\big)$ for $t \geq T$. Note that the border condition $2\,\alpha_T\,\lambda_T = 1$ reduces to $h(k) = 1$ (a condition that is independent of T!).

In the argument that follows we will show that for $\bar{\beta}$ sufficiently large there exists a unique $k \in [-1,1]$ that solves $h(k) = 1$. Furthermore, we will show that $k > 0$ and that $h(\ell)$ is increasing in ℓ for $\ell \in [0,k]$. As a result, $2\,\alpha_t\,\lambda_t = h\big(k\,e^{-L(t-T)}\big) \leq h(k) = 1$ for all $t \geq T$, as required. Although the proof of these steps is tedious, the intuition is rather straightforward if we note that $h(\ell) \approx (1+\ell)/a$ for $\bar{\beta}$ sufficiently large (or equivalently, for a close to one).

To prove that $k > 0$ note that

$$h(0) = \frac{1}{a} < 1 \qquad \text{and} \qquad h'(\ell) = \int_0^\infty e^{-a\,u}\,\frac{2\,(e^{-u}-\ell)}{[1-\ell\,e^{-u}]^3}\,\mathrm{d}u,$$

so $h'(\ell) \geq 0$ for all $\ell \leq 0$. To prove the existence, we compute a lower bound for $h(\ell)$:

$$h(\ell) \geq \int_0^\infty e^{-a\,u}\,\frac{1-\ell^2}{[1-\ell\,e^{-a\,u}]^2}\,\mathrm{d}u = \frac{1+\ell}{a} \qquad \text{for all } \ell \in [0,1),$$

where the inequality follows from the fact that $a \geq 1$ and $\ell \geq 0$. This lower bound is equal to 1 at $\ell = a-1 = \mu/L$. Hence, for $\bar{\beta}$ sufficiently large $\mu/L < 1$ and $h(\mu/L) > 1$. Since $h(\ell)$ is continuous and $h(0) = 1/a < 1$, there exists $k \in (0,\mu/L)$ such that $h(k) = 1$.

Assume that $\bar{\beta}$ is sufficiently large so that $\mu/L < 1$. We now conclude the proof by showing that $h(k)$ is increasing in $\ell \in [0,\mu/L]$ which shows, by virtue of the lower bound above, that k is unique and that $h(\ell) \leq h(k)$ for all $\ell \in [0,k)$, as needed. For this, note that for all $\ell \in [0,\mu/L]$

$$\begin{aligned} h'(\ell) &= \int_0^\infty e^{-a\,u}\,\frac{2\,(e^{-u}-\ell)}{(1-\ell\,e^{-u})^3}\,\mathrm{d}u \geq 2\int_0^\infty \frac{e^{-(a+1)\,u}}{(1-\ell\,e^{-\frac{a+1}{2}\,u})^3}\,\mathrm{d}u - 2\,\ell\int_0^\infty \frac{e^{-u}}{(1-\ell\,e^{-u})^3}\,\mathrm{d}u \\ &= \frac{1}{(1-\ell)^2}\left[\frac{2}{a+1} - (2-\ell)\,k\right] \geq \frac{1}{(1-\ell)^2}\left[\frac{2}{a+1} - (3-a)\,(a-1)\right] > \frac{5-3\,a^2}{(1-\ell)^2(a+1)}, \end{aligned}$$

where the second inequality follows from the fact that $(2-\ell)\ell$ is increasing in $\ell \in [0,1]$. Therefore, $h'(\ell) > 0$ for all $\bar\beta$ sufficiently large so that $a \le \sqrt{5/3}$

To show that $\Pi(t,M) = \alpha_t M^2 + \gamma_t$ is continuously differentiable once in t and twice in M, note that the value matching conditions (21) imply that λ_t, α_t and γ_t are continuous functions of t. $\Pi(t,M)$ is clearly twice continuously differentiable in M. Hence, we only need to show that α_t and γ_t are continuously differentiable at $t = T$. Equations (19), (34), (20) and (35) imply that

$$\lim_{t\uparrow T} \dot\alpha_t = \mu\,\alpha_T = \lim_{t\downarrow T} \dot\alpha_t \qquad \text{and} \qquad \lim_{t\uparrow T} \dot\gamma_t = \mu\,\gamma_T - \alpha_T\,\sigma_T^2 = \lim_{t\downarrow T} \dot\gamma_t. \qquad \blacksquare$$

Proof of Proposition 3.

Since for all $t \ge T$, $\beta_t \equiv \bar\beta$, it follows that M_t has mean reverting dynamics

$$\mathrm{d}M_t = -\lambda_t\,\bar\beta\,M_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t.$$

Let $\Lambda_t = \int_0^t \lambda_s\,\mathrm{d}s$. We can integrate this equation using the integrating factor $\exp(\bar\beta\,\Lambda_t)$. Indeed, from a straightforward application of Itô's lemma we get

$$\mathrm{d}\big(e^{\bar\beta\,\Lambda_t}\,M_t\big) = \sigma_t\,e^{\bar\beta\,\Lambda_t}\,\mathrm{d}B_t,$$

and integrating this equation between T and t we get

$$M_t = M_T\,e^{-\bar\beta\,(\Lambda_t-\Lambda_T)} + \int_T^t \sigma_s\,e^{-\bar\beta\,(\Lambda_t-\Lambda_s)}\,\mathrm{d}B_s, \qquad t \ge T. \qquad \blacksquare$$

Proof of Theorem 3.

Assume that $\bar\beta$ is sufficiently large so that $\mu/L < 1/2$. By Theorem 2, $0 < k(\bar\beta) < \mu/L$. Hence

$$1 \le \left[\frac{1+k(\bar\beta)}{1-k(\bar\beta)}\right] = \frac{1+\mu/L}{1-\mu/L} \le 1 + 2\frac{\mu}{L} + 4\left[\frac{\mu}{L}\right]^2.$$

Since $L = 2\sigma_v\bar\beta/\sigma_y$, it follows that $k(\bar\beta) \to 0$ and so for all $t \ge T$,

$$\lim_{\bar\beta\to\infty} \lambda_t = \lim_{\bar\beta\to\infty} \frac{\sigma_v}{\sigma_y}\left[\frac{1+k(\bar\beta)}{1-k(\bar\beta)}\right] = \frac{\sigma_v}{\sigma_y} \qquad \text{and} \qquad \lim_{\bar\beta\to\infty} \Sigma_t = \lim_{\bar\beta\to\infty} \frac{\sigma_y^2\lambda_t}{\bar\beta} = 0.$$

Also, equation (19) implies that

$$\alpha_T = \lim_{\bar\beta\to\infty} \bar\beta\,e^{(L+\mu)\,T}\int_T^\infty e^{-(L+\mu)\,s}\,\mathrm{d}s = \lim_{\bar\beta\to\infty} \frac{\bar\beta}{L+\mu} = \frac{\sigma_y}{2\sigma_v}.$$

Therefore $\lambda_0 = e^{\mu T}/[2\alpha_T] = e^{\mu T}\sigma_v/\sigma_y$. In the limit, as $\bar\beta \to \infty$, the threshold time T solves

$$\Sigma_T = \Sigma_0 + \sigma_v^2\,T - \sigma_v^2\left[\frac{e^{2\,\mu\,T}-1}{2\,\mu}\right] = 0.$$

One can check that this equation has a unique solution and that $\dot\Sigma_T = 0$.

It only remains to prove the weak convergence of $M_t(\bar\beta)$ to 0 as $\bar\beta \to \infty$. For this we will invoke Theorem 2.1 in Prokhorov (1956) and prove the convergence of the finite-dimensional distributions of $\{M_t(\bar\beta)\}$ to 0 together with the compactness of the sequence $\{M_t(\bar\beta)\}$ (see also Billingsley 1999, Chapter 2). Let $\mathcal{T} = [T_1, T_2]$ with $T < T_1 < T_2$. In what follows, we define $\Lambda(t,s) = \int_t^s \lambda_u \,\mathrm{d}u$, $\underline{\lambda}_{\mathcal{T}} = \min\{\lambda_t : t \in \mathcal{T}\}$, $\bar\lambda_{\mathcal{T}} = \max\{\lambda_t : t \in \mathcal{T}\}$ and $\bar\sigma_{\mathcal{T}} = \max\{\sigma_t : t \in \mathcal{T}\}$.

Let $\{t_1, t_2, \dots, t_n\} \in \mathcal{T}$. For each $t \in \mathcal{T}$, $M_t(\bar\beta)$ satisfies

$$M_t(\bar\beta) = M_T\, e^{-\bar\beta\,\Lambda(T,t)} + \int_T^t \sigma_s\, e^{-\bar\beta\,\Lambda(s,t)}\,\mathrm{d}B_s.$$

Therefore, the random vector $(M_{t_1}(\bar\beta), M_{t_2}(\bar\beta), \dots, M_{t_n}(\bar\beta))$ has a Gaussian distribution. We now show that this distribution converges to the distribution of the constant vector $(0, \dots, 0)$. Let us denote by $\mu^M(\bar\beta)$ and $\Sigma^M(\bar\beta)$ its mean vector and variance-covariance matrix, respectively. It follows that the i^{th} component of $\mu^M(\bar\beta)$ is given by

$$\mu_i^M(\bar\beta) = \mathbb{E}[M_{t_i}(\bar\beta)] = \mathbb{E}[M_T]\, e^{-\bar\beta\,\Lambda(T,t_i)}, \qquad i = 1, \dots, n.$$

Similarly, the covariance between the i^{th} and j^{th} components in $\Sigma^M(\bar\beta)$ is given by (assume $t_i \le t_j$)

$$\begin{aligned}
\Sigma_{ij}^M(\bar\beta) &= \mathbb{E}[\big(M_{t_i}(\bar\beta) - \mu_i^M(\bar\beta)\big)\big(M_{t_j}(\bar\beta) - \mu_j^M(\bar\beta)\big)] \\
&= \mathbb{E}\left[\left(\int_T^{t_i} \sigma_s\, e^{-\bar\beta\,\Lambda(s,t_i)}\,\mathrm{d}B_s\right)\left(\int_T^{t_j} \sigma_s\, e^{-\bar\beta\,\Lambda(s,t_j)}\,\mathrm{d}B_s\right)\right] \\
&= \mathbb{E}\left[\int_T^{t_i} \sigma_s\, e^{-\bar\beta\,\Lambda(s,t_i)}\,\mathrm{d}B_s \left(e^{-\bar\beta\,\Lambda(t_i,t_j)} \int_T^{t_i} \sigma_s\, e^{-\bar\beta\,\Lambda(s,t_i)}\,\mathrm{d}B_s + \int_{t_i}^{t_j} \sigma_s\, e^{-\bar\beta\,\Lambda(s,t_j)}\,\mathrm{d}B_s\right)\right] \\
&= e^{-\bar\beta\,\Lambda(t_i,t_j)}\, \mathbb{E}\left[\left(\int_T^{t_i} \sigma_s\, e^{-\bar\beta\,\Lambda(s,t_i)}\,\mathrm{d}B_s\right)^2\right] = e^{-\bar\beta\,\Lambda(t_i,t_j)} \left(\int_T^{t_i} \sigma_s\, e^{-2\,\bar\beta\,\Lambda(s,t_i)}\,\mathrm{d}s\right).
\end{aligned}$$

The fourth equality uses the fact that B_t has independent increment so that two stochastic integrals with non-overlapping ranges are uncorrelated. The fifth equality uses Itô's isometry. Therefore, as $\bar\beta$ goes to infinity we get

$$\lim_{\bar\beta\to\infty} \mu_i^M(\bar\beta) = 0 \qquad \text{and} \qquad \lim_{\bar\beta\to\infty} \Sigma_{ij}^M(\bar\beta) = 0, \qquad \text{for all } i, j = 1, \dots, n.$$

We conclude that the distribution of $(M_{t_1}(\bar\beta), M_{t_2}(\bar\beta), \dots, M_{t_n}(\bar\beta))$ converges to the distribution of the constant $(0, \dots, 0)$.

We now prove that $\{M_t(\bar\beta) : \bar\beta > 0\}$ is tight. For this we show that there exists a constant R independent of $\bar\beta$ such that

$$D := \mathbb{E}\left[\big(M_{t_2}(\bar\beta) - M_{t_1}(\bar\beta)\big)^2\right] \le R\,|t_2 - t_1|, \qquad \text{for all } t_1, t_2 \in \mathcal{T}.$$

Indeed, by the definition of $M_t(\bar\beta)$ and Itô's isometry it follows for $t_1 \le t_2$ that

$$D = \left(M_T^2\, e^{-2\bar\beta\Lambda(T,t_1)} + \int_T^{t_1} \sigma_s^2 e^{-2\bar\beta\Lambda(s,t_1)}\mathrm{d}s\right)\left(1 - e^{-\bar\beta\Lambda(t_1,t_2)}\right)^2 + \int_{t_1}^{t_2} \sigma_s^2 e^{-2\bar\beta\Lambda(s,t_2)}\mathrm{d}s.$$

Since $(1-\exp(-x))^2 \le 2\,x$ for all $x \ge 0$, it follows that

$$\left(1-e^{-\bar\beta\Lambda(t_1,t_2)}\right)^2 \le 2\,\bar\beta\Lambda(t_1,t_2) \le 2\,\bar\beta\,\bar\lambda_T\,(t_2-t_1).$$

As a result, we have that

$$D \le \left[2\,\bar\lambda_T\left(M_T^2\,\bar\beta\,e^{-2\bar\beta\Lambda(T,t_1)} + \int_T^{t_1}\sigma_s^2\,\bar\beta\,e^{-2\bar\beta\Lambda(s,t_1)}\mathrm{d}s\right) + \bar\sigma_T^2\right](t_2-t_1).$$

Since $t_1 \ge T_1 > T$, it is not hard to show that

$$\bar\beta\,e^{-2\bar\beta\Lambda(T,t_1)} \le \frac{e^{-1}}{2\,\Lambda(T,t_1)} \le \frac{e^{-1}}{2\,\Lambda(T,T_1)}.$$

In addition,

$$\int_T^{t_1}\sigma_s^2\,\bar\beta\,e^{-2\bar\beta\Lambda(s,t_1)}\mathrm{d}s \le \bar\sigma_T^2\int_T^{t_1}\bar\beta\,e^{-2\bar\beta\,\underline{\lambda}_T\,(t_1-s)}\mathrm{d}s = \frac{\bar\sigma_T^2}{2\,\underline{\lambda}_T}.$$

Hence, we can choose the constant R to be equal to

$$R = \bar\lambda_T\left(\frac{M_T^2\,e^{-1}}{\Lambda(T,T_1)} + \frac{\bar\sigma_T^2}{\underline{\lambda}_T}\right) + \bar\sigma_T^2,$$

which is independent of $\bar\beta$. Hence, $\{M_t(\bar\beta)\}$ is tight. If $M_T = 0$ a.s. then we can repeat the previous steps with $T_1 = T$.

Finally, we prove the weak convergence of the insider's trading strategy. Let us denote by $X_t(\bar\beta)$ the insider's trading strategy when $\beta_t = \bar\beta$. It follows from Proposition 3 that $X_t(\bar\beta)$ has the following dynamics

$$\mathrm{d}X_t(\bar\beta) = \bar\beta\,M_t(\bar\beta)\,\mathrm{d}t = \frac{1}{\lambda_t}\left[\sigma_v\,\mathrm{d}B_t^v - \lambda_t\,\sigma_y\,\mathrm{d}B_t^y - \mathrm{d}M_t(\bar\beta)\right], \qquad t \ge T.$$

Integrating from T to t we get $X_t(\bar\beta) = X_T + \sigma_y\,[(B_t^v - B_T^v) - (B_t^y - B_T^y)] - \frac{\sigma_y}{\sigma_v}\,(M_t(\bar\beta) - M_T)$. Since $M_T = 0$, $\lambda_t \to \sigma_v/\sigma_y$ and $M_t(\bar\beta)$ converges weakly to 0 as $\bar\beta \to \infty$ for all $t > T$, it follows that $X_t(\bar\beta) \overset{\bar\beta\to\infty}{\Longrightarrow} X_t = X_T + \sigma_y\,[(B_t^v - B_T^v) - (B_t^y - B_T^y)]$ for all $t \ge T$. ∎

Proof of Proposition 4. The insider's ex-ante expected rent is

$$\mathbb{E}[\Pi(t,M_t)] = \alpha_t\,\mathbb{E}[(V_t-p_t)^2] + \gamma_t = \alpha_t\,\Sigma_t + \gamma_t.$$

For $t \in [T,\infty)$, $\Sigma_t = 0$ and $\mathbb{E}[\Pi(t,M_t)] = \gamma_T$. For $t \in (0,T)$, $\dot\alpha_t = \mu\,\alpha_t$, $2\,\alpha_t\,\lambda_t = 1$, and

$$\dot\Sigma_t = \frac{\mathrm{d}\Sigma_t}{\mathrm{d}\alpha_t}\,\dot\alpha_t = \sigma_v^2 - \lambda_t^2\,\sigma_y^2 = \sigma_v^2 - \frac{\sigma_y^2}{4\,\alpha_t^2} \qquad \text{so} \qquad \frac{\mathrm{d}\Sigma_t}{\mathrm{d}\alpha_t} = \frac{\sigma_v^2}{\mu\,\alpha_t} - \frac{\sigma_y^2}{4\,\mu\,\alpha_t^3}\,.$$

Therefore,

$$\Sigma_t = \frac{\sigma_v^2}{\mu}\,\log(\alpha_t) + \frac{\sigma_y^2}{8\,\mu\,\alpha_t^2} + c_1$$

for some constant of integration c_1. Similarly

$$\dot\gamma_t = \frac{\mathrm{d}\gamma_t}{\mathrm{d}\alpha_t}\,\dot\alpha_t = \mu\,\gamma_t - \alpha_t(\sigma_v^2 + \lambda_t^2\sigma_y^2) \qquad \text{so} \qquad \frac{\mathrm{d}\gamma_t}{\mathrm{d}\alpha_t} = \frac{\gamma_t}{\alpha_t} - \frac{1}{\mu}\left[\sigma_v^2 + \frac{\sigma_y^2}{4\,\alpha_t^2}\right].$$

Hence

$$\gamma_t = -\frac{\sigma_v^2}{\mu}\,\alpha_t\,\log(\alpha_t) + \frac{\sigma_y^2}{8\,\mu\,\alpha_t} + c_2\alpha_t,$$

for some constant c_2. Thus

$$\alpha_t\,\Sigma_t + \gamma_t = \frac{\sigma_y^2}{4\,\mu\,\alpha_t} + (c_1 + c_2)\alpha_t.$$

Since $\alpha_T = \sigma_y/[2\sigma_v]$, $\Sigma_T = 0$ and $\gamma_T = \sigma_v\,\sigma_y/\mu$, we have that

$$\frac{\sigma_v\,\sigma_y}{\mu} = \alpha_T\,\Sigma_T + \gamma_T = \frac{\sigma_y^2}{4\,\mu\alpha_T} + (c_c + c_2)\alpha_T,$$

which implies that $c_1 + c_2 = \sigma_v^2/\mu$. Therefore

$$\alpha_t\,\Sigma_t + \gamma_t = \frac{\sigma_y^2}{4\,\mu\,\alpha_t} + \frac{\sigma_v^2}{\mu}\,\alpha_t = \frac{\sigma_y\,\sigma_v}{2\,\mu}\,e^{\mu(T-t)} + \frac{\sigma_y\,\sigma_v}{2\,\mu}\,e^{-\mu(T-t)} = \frac{\sigma_y\,\sigma_v}{\mu}\,\cosh(\mu(T-t)). \qquad \blacksquare$$

Proof of Proposition 5. Recall from Theorem 3 that Σ_t satisfies

$$\Sigma_t = \Sigma_0 + \sigma_v^2\,t - \sigma_v^2\,e^{2\mu T}\left[\frac{1-e^{-2\,\mu\,t}}{2\mu}\right] \qquad t < T,$$

and $\Sigma_T = 0$ for all $t \geq T$, where $T \geq 0$ is the unique solution to

$$\Sigma_0 + \sigma_v^2\,T = \sigma_v^2\left[\frac{e^{2\,\mu\,T}-1}{2\,\mu}\right].$$

Since T decreases with σ_v, it suffices to prove that Σ_t decreases with σ_v for $t < T$.

In what follows, and without lost of generality, we will normalize the value of μ such that $2\,\mu = 1$ (this is equivalent to re-scaling time). With this normalization, the derivative of Σ_t ($t < T$) with respect to σ_v^2 is equal to

$$\frac{\partial \Sigma_t}{\partial \sigma_v^2} = t - e^T\,(1-e^{-t}) - \sigma_v^2\,e^T\,(1-e^{-t})\,\frac{\partial T}{\partial \sigma_v^2}, \qquad t < T.$$

In addition, from the definition of T it follows that

$$\frac{\partial T}{\partial \sigma_v^2} = \frac{1}{\sigma_v^2}\left[\frac{1+T-e^T}{e^T-1}\right].$$

Plugging back this value on $\frac{\partial \Sigma_t}{\partial \sigma_v^2}$ we get that for $t < T$

$$\frac{\partial \Sigma_t}{\partial \sigma_v^2} = t - (1-e^{-t})\left[\frac{T}{1-e^{-T}}\right] \leq 0.$$

The inequality follows from the fact that $t/(1-e^{-t})$ is an increasing function of t.

Let us now prove the monotonicity of the insider's ex-ante expected payoff. Given the normalization $2\,\mu = 1$, this payoff is given by

$$\mathbb{E}[\Pi(t, M_t)] = 2\,\sigma_y\,\sigma_v\,\cosh\left(\frac{1}{2}\,(T-t)^+\right) \qquad t \geq 0.$$

Note that to prove the monotonicity of $\mathbb{E}[\Pi(t, M_t)]$ with respect to σ_v it is enough to focus on the case $t \leq T$. The derivative with respect to σ_v is given by

$$\begin{aligned}\frac{\partial \mathbb{E}[\Pi(t, M_t)]}{\partial \sigma_v} &= 2\,\sigma_y \cosh\left(\frac{1}{2}(T-t)\right) + \sigma_y\,\sigma_v \sinh\left(\frac{1}{2}(T-t)\right)\frac{\partial T}{\partial \sigma_v} \\ &= 2\,\sigma_y \cosh\left(\frac{1}{2}(T-t)\right) + 2\,\sigma_y \sinh\left(\frac{1}{2}(T-t)\right)\left[\frac{1+T-e^T}{e^T-1}\right] \\ &= 2\,\sigma_y \sinh\left(\frac{1}{2}(T-t)\right)\left[\frac{T}{e^T-1}\right] + 2\,\sigma_y \exp\left(\frac{T-t}{2}\right) \geq 0. \quad \blacksquare\end{aligned}$$

Proof of Proposition 6. Let β_0 be such that $\tau(\beta_0) = \infty$. In this case, the corresponding continuous-time solution (Σ_t^0, β_t^0) in (28) satisfies $\Sigma_t^0 > 0$ for all $t \geq 0$ and it follows, trivially, that $(\beta_t^0)^2\,\Sigma_t^0 < \infty$ for all $t \geq 0$. As a result,

$$\lim_{\Delta \downarrow 0}(\Sigma_t^\Delta, \beta_t^\Delta) = (\Sigma_t^0, \beta_t^0) > 0, \qquad \text{for all } t \geq 0$$

and we conclude that $\beta_0 \in \mathcal{B}^0(\Sigma_0)$. Now, the condition $\tau(\beta_0) = \infty$ holds for all those initial conditions β_0 that generate variance processes Σ_t^0 that satisfies $\min_{t\geq 0}\Sigma_t^0 > 0$. Let $\bar\beta_0$ be such that the corresponding Σ_t^0 satisfies $\min_{t\geq 0}\Sigma_t^0 = 0$. Using the definition of Σ_t^0 in (28) one can show that $\bar\beta_0 = \frac{\sigma_v\,\sigma_y}{\Sigma_0}\sqrt{\eta - 1}$, where η is the unique nonnegative root of the equation

$$0 = \Sigma_0 + \frac{\sigma_v^2}{2\,\mu}\left[\ln(\eta - 1) - \eta\right].$$

We conclude that $[0, \bar\beta_0) \subseteq \mathcal{B}^0(\Sigma_0)$.

In what follows, we will show (by contradiction) that there is no $\beta_0 > \bar\beta_0$ such that $\beta_0 \in \mathcal{B}^0(\Sigma_0)$. For this, we will show that for any $\hat\Delta > 0$ there exits a $\Delta \leq \hat\Delta$ such that the profile $(\Sigma_t^\Delta, \beta_t^\Delta)$ is infeasible. To prove this, we will use the following sufficient condition for infeasibility discussed in Section 2 after Figure 1: if for some $\tilde t$ we have that $\Sigma_{\tilde t+\Delta}^\Delta\,\beta_{\tilde t+\Delta}^\Delta > \Sigma_{\tilde t}^\Delta\,\beta_{\tilde t}^\Delta$ then the profile $(\Sigma_t^\Delta, \beta_t^\Delta)$ is infeasible (*i.e.*, will eventually become negative).

Since $\beta_0 > \bar\beta_0$, the continuous-time limit $(\Sigma_t^0, \beta_t^0) < 0$ for some $t > \tau(\beta_0)$. This implies that we cannot have convergence of the difference equations in (25)-(26) to the differential equations in (27) at $t = \tau(\beta_0)$. In other words, there must exist $\epsilon > 0$ such that

$$\limsup_{\Delta \downarrow 0}\,(\beta_{t^\Delta}^\Delta)^2\,\Sigma_{t^\Delta}^\Delta\,\Delta \geq \epsilon,$$

for a sequence of times $\{t^\Delta\}$ such that $\lim_{\Delta\downarrow 0} t^\Delta = \tau(\beta_0)$. Hence, for any $\hat\Delta > 0$ there exists $\Delta \leq \hat\Delta$ such that $(\beta_{t^\Delta}^\Delta)^2\,\Sigma_{t^\Delta}^\Delta\,\Delta \geq \epsilon$ and so for this Δ equation (24) implies

$$\beta_{t^\Delta+\Delta}^\Delta\,\Sigma_{t^\Delta+\Delta}^\Delta = \frac{e^{-\mu\,\Delta}\,\sigma_y^4\,\beta_{t^\Delta}^\Delta\,\Sigma_{t^\Delta}^\Delta}{\sigma_y^4 - (\beta_{t^\Delta}^\Delta)^4\,(\Sigma_{t^\Delta}^\Delta)^2\,\Delta^2} \geq \frac{e^{-\mu\,\Delta}\,\sigma_y^4\,\beta_{t^\Delta}^\Delta\,\Sigma_{t^\Delta}^\Delta}{\sigma_y^4 - \epsilon^2} > \beta_{t^\Delta}^\Delta\,\Sigma_{t^\Delta}^\Delta.$$

The last inequality follows since we can take Δ arbitrarily small so that $e^{-\mu\,\Delta}\,\sigma_y^4 > \sigma_y^4 - \epsilon^2$. But this implies that the sequence $(\beta_t^\Delta\,\Sigma_t^\Delta)$ is increasing at $t = t^\Delta$ and so the resulting profile $(\Sigma_t^\Delta, \beta_t^\Delta)$ is infeasible. Hence, if $\beta_0 > \bar\beta_0$ then $\beta_0 \notin \mathcal{B}^0(\Sigma_0)$ which completes the proof. $\blacksquare$

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