Supply Contracts with Financial Hedging

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Motivation

Financial Market

Procurement Suppliers' Management
Inventory/Production Capacity/Processes Product Design
Distribution Pricing Policies Demand Forecast

Corporation

Problem:

Select an *operational strategy* and a *financial strategy* that optimize the value of the corporation.
Motivation

- A Methodology that effectively integrates Operational and Financial activities by
  - Maximizing the Economic Value of the corporation.
  - Modeling the risk preferences of decision makers.
  - Using financial markets for hedging purposes.
  - Recognizing the “incompleteness” of the problem.
  - Taking advantage of all available information.

\[
\max \{\text{Economic Value}\} \\
\text{subject to} \quad \text{problem dynamics} \\
\text{and subject to} \quad \text{risk management constraint(s)}.
\]

- Operational risk in customers’ taste, machine breakdowns, unreliable suppliers, etc...

- Financial risk in exchange rates, commodity prices, interest rates, etc...
Outline of Talk

◦ The Modelling Framework
  - Motivation for maximizing economic value of operating profits.
  - Financial market & informational structure

◦ Supply Contracts with Financial Hedging
  - The Wholesale Price Contract with Budget Constraint

◦ Conclusions & Further Research
Modelling Framework

Financial Market & Information
Motivating the Modelling Framework

- Consider two series cash-flows arising from two operating policies $I_1$ and $I_2$:

$$C^{(1)}(I_1) = (c^{(1)}_1(I_1), c^{(1)}_2(I_1), \ldots, c^{(1)}_n(I_1))$$

$$C^{(2)}(I_2) = (c^{(2)}_1(I_2), c^{(2)}_2(I_2), \ldots, c^{(2)}_n(I_2))$$

- If probability distribution of $C^{(1)}$ and $C^{(2)}$ identical then most operations models are indifferent between the two.

- However, if $C^{(1)}$ positively correlated with financial market for example, and $C^{(2)}$ negatively correlated with financial market then economic value of $C^{(2)}$ greater than economic value of $C^{(1)}$.
  - models should take this into account as they get ever more complex and include additional sources of uncertainty

- Do so by using an equivalent martingale measure, $Q$, or stochastic discount factor
  - consistent with only a hedging motivation for financial trading
The Modelling Framework

- $H(I)$ the time $T$ cumulative payoff from some operating policy, $I$
  - The operating policy may be static or dynamic.
- $G(\theta)$ the time $T$ gain from a self-financing trading strategy, $\theta$
- Operational and financial hedging problem is
  \[
  \min_{I,G(\theta)} \mathbb{E}^Q [H(I)]
  \]
  subject to problem dynamics
  and subject to risk management constraint(s) \(\mathcal{R}(H(I), G(\theta)) \leq 0 \text{ a.s.}\)
  \(\mathbb{E}^P [\mathcal{R}(H(I), G(\theta))] \leq 0.\)

- consistent with only a hedging motivation for financial trading
  - $G(\theta)$ a martingale under $\mathbb{Q}$ so $\mathbb{E}^Q[G(\theta)] = 0$
  - assuming initial capital assigned to hedging strategy is zero
Financial Market Model

- Probability space \((\Omega, \mathcal{F}, \mathbb{P})\).
- \((W_{1t}, W_{2t})\) a standard Brownian motion.
- A single risky stock with price dynamics
  
  \[ dX_t = \mu X_t \, dt + \sigma X_t \, dW_{1t}. \]

- Cash account available with \(r \equiv 0\).
- \(\mathcal{F}_t\) is the filtration generated by \((W_{1t}, W_{2t})\).
- \(\mathcal{F}_t^X\) is the filtration generated by \(X_t\).
- Operating profits, \(H(I) \in \mathcal{F}_T\), a function of \((W_{1t}, W_{2t})\).
- The gain process, \(G(\theta)_t\) of a self-financing trading strategy \(\theta_t\) is
  
  \[ G(\theta)_t = \int_0^t \theta_s \, dX_s. \]

- An equivalent martingale measure, \(\mathbb{Q}\), under which \(X_t\) and \(G(\theta)_t\) are martingales.
Information

In general, operating profits are function of \((X_t, W_{2t})\), that is, \(H(I) \in \mathcal{F}_T\).

- **Incomplete Information:**
  - Demand, product quality, customer tastes not always observable.
  - Only information related to \(X_t\) is available.

  \[ \Rightarrow \] Trading strategies must satisfy: \(\theta_t \in \mathcal{F}_t^{X}, \quad t \in [0, T].\)

- **Complete Information:**
  - The evolution of \(X_t\) and \(B_{2t}\) are both available.

  \[ \Rightarrow \] Trading strategies satisfy: \(\theta_t \in \mathcal{F}_t, \quad t \in [0, T].\)
Supply Contracts with Financial Hedging

The Wholesale Price Contract with Budget Constraint
The Wholesale Price Contract

Non-Cooperative Operation:

- Solution Concept: Stackelberg game.
  1. At time $t = 0$, Manufacturer offers a wholesale price $w$ to the Retailer.
  2. At time $t = 0$, Retailer orders a quantity $q$.
  3. At time $t = T$, a random clearance price $P(q)$ is realized: $P(q) = A - q$.

- The random price $A$ is correlated to the financial market $X_t$.

- Retailer operates under a budget constraint: $w q \leq B$. 

Supply Contracts with Financial Hedging

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Types of Contracts

**Simple Contract**

- \((w, q)\) contract is decided
- Production takes place at \(t = 0\)
- Clearance Price \(P(q)\) is realized
- Payoffs are determined at \(t = T\)

**Flexible Contract**

- \((w_\tau, q_\tau)\) contract is decided at \(t = 0\)
- Market Signal \(X_\tau\) is observed at \(t = \tau\)
- Production takes place at \(t = \tau\)
- Clearance Price \(P(q)\) is realized
- Payoffs are determined at \(t = T\)

**Flexible Contract with Financial Hedging**

- \((w_\tau, q_\tau)\) contract is decided at \(t = 0\)
- Market Signal \(X_\tau\) is observed at \(t = \tau\)
- Trading Gain \(G_\tau\) is observed
- Financial hedging takes place at \(t = \tau\)
- Production takes place at \(t = \tau\)
- Clearance Price \(P(q)\) is realized
- Payoffs are determined at \(t = T\)

**Remark:** The time \(\tau\) can be a decision variable: a deterministic time or a \(\mathcal{F}_{\tau}^X\)-stopping time.
Some Notation

* ) \((\Omega, \mathcal{F}, \mathbb{Q})\) probability space.

* ) \(X_t\): Time \(t\) value of a \(\mathbb{Q}\)-martingale tradable security \((t \in [0, T])\).

* ) \(\mathcal{F}_t = \sigma(X_s ; 0 \leq s \leq t)\) filtration generated by \(X_t\) with \(\mathcal{F}_T \subseteq \mathcal{F}\).

* ) \(E_\tau^\mathbb{Q}[E] := E_\tau^\mathbb{Q}[E|\mathcal{F}_\tau]\) for all \(E \in \mathcal{F}\).

* ) \(P(q) = A - q\): retail price at time \(T\) with \(A \in \mathcal{F}\).

* ) \(\bar{A}_\tau = E_\tau^\mathbb{Q}[A|\mathcal{F}_\tau]\) for any \(\mathcal{F}_t\)-stopping time \(\tau \leq T\).

* ) \(c_\tau\): per unit manufacturing cost at time \(\tau \in [0, T]\).

* ) \(w_\tau \in \mathcal{F}_\tau\): manufacturer’s wholesale price menu.

* ) \(q_\tau \in \mathcal{F}_\tau\): retailer’s ordering quantity menu.

* ) \(G_\tau \in \mathcal{F}_\tau\): retailer’s trading gains (or losses) with \(E_\tau^\mathbb{Q}[G_\tau] = 0\).

* ) Budget constraint: \(w_\tau q_\tau \leq B_\tau := B + G_\tau\).

Assumption.  For all \(\tau \in [0, T]\), \(\bar{A}_\tau \geq c_\tau\).
Consider a fixed $\tau \in [0, T]$.

**Step 1:** At $t = 0$, the manufacturer offers a wholesale menu $w_\tau \in \mathcal{F}_\tau$.

**Step 2:** A retailer, with no access to the financial market, decides the ordering level $q_\tau \in \mathcal{F}_\tau$ solving

$$\Pi^F_{R}(w_\tau) = \mathbb{E}_0^Q \left[ \max_{q_\tau \geq 0} \left\{ \mathbb{E}_\tau^Q [(A - q_\tau - w_\tau) q_\tau] \right\} \right]$$

subject to $w_\tau q_\tau \leq B$, for all $\omega \in \Omega$.

The optimal solution (retailer’s reaction) is

$$q(w_\tau) = \min \left\{ \left( \frac{\bar{A}_\tau - w_\tau}{2} \right)^+, \frac{B}{w_\tau} \right\}.$$

**Step 3:** The manufacturer problem is (Stackelberg leader)

$$\Pi^F_{M} = \mathbb{E}_0^Q \left[ \max_{w_\tau \geq c_\tau} \left\{ (w_\tau - c_\tau) q_\tau(w_\tau) \right\} \right].$$
Flexible Contract

**Proposition.** (Flexible Contract Solution)

*Under Assumption, the equilibrium solution for the flexible contract is*

\[
\begin{align*}
w^F_\tau &= \frac{\bar{A}_\tau + \delta^F_\tau}{2} \quad \text{and} \quad q^F_\tau = \frac{\bar{A}_\tau - \delta^F_\tau}{4},
\end{align*}
\]

where

\[
\delta^F_\tau := \max \left\{ c_\tau , \sqrt{(\bar{A}^2_\tau - 8B)_+} \right\}.
\]

*The equilibrium expected payoffs of the players are then given by*

\[
\begin{align*}
\Pi^F_{M|\tau} &= \frac{(\bar{A}_\tau + \delta^F_\tau - 2c_\tau)(\bar{A}_\tau - \delta^F_\tau)}{8} \quad \text{and} \quad \Pi^F_{R|\tau} = \frac{(\bar{A}_\tau - \delta^F_\tau)^2}{16}.
\end{align*}
\]

**Remarks:**

- \( \delta^F_\tau \) is a modified production cost \( \geq c_\tau \) that is induced by a limited budget \( B \).

- \( w^F_\tau \) decreases in \( B \) and \( q^F_\tau, \Pi^F_{M|\tau} \) and \( \Pi^F_{R|\tau} \) increase in \( B \).
Define \( w^F := E_0^Q[w^F_\tau] \), \( q^F := E_0^Q[q^F_\tau] \), \( \Pi^F_M := E_0^Q[\Pi^F_M|\tau] \) and \( \Pi^F_R := E_0^Q[\Pi^F_R|\tau] \).

**Proposition.** Suppose that \( B \leq \frac{\bar{A}_\tau^2-c_\tau^2}{8} \) almost surely. Then

\[
\begin{align*}
w^F &\leq w^S, \\
q^F &\geq q^S, \\
\Pi^F_M &\leq \Pi^S_M \quad \text{and} \quad \Pi^F_R \geq \Pi^S_R.
\end{align*}
\]

However, if \( B \geq \max \left\{ \frac{\bar{A}_\tau^2-c_\tau^2}{8}, \frac{\bar{A}_0^2-c_\tau^2}{8} \right\} \) almost surely then

\[
\begin{align*}
w^F &= w^S + \frac{c_\tau - c_0}{2} \quad \text{and} \quad q^F = q^S - \frac{c_\tau - c_0}{4}
\end{align*}
\]

and \( \Pi^F_M \geq \Pi^S_M \) and \( \Pi^F_R \geq \Pi^S_R \) if and only if \( E_0^Q[(\bar{A}_\tau - c_\tau)^2] \geq (\bar{A} - c_0)^2 \).
Flexible Contract vs. Simple Contract

\[ \bar{A}_\tau \sim \text{Uniform}[1, 3], \quad c_0 = 0.3, \quad c_\tau = 0.35 \text{ (case 1)} \quad \text{and} \quad c_\tau = 0.7 \text{ (case 2)}. \]
Flexible Contract: Efficiency

\[ \bar{A}_\tau = 2, \quad c_\tau = 0.6 \text{ (top)} \quad \text{and} \quad c_\tau = 1.2 \text{ (bottom)}. \]
Flexible Contract with Financial Hedging

\[ (w_\tau, q_\tau) \text{ contract is decided} \]
\[ \text{Market Signal } X_\tau \text{ is observed} \]
\[ \text{Trading Gain } G_\tau \text{ is observed} \]
\[ \text{Clearance Price } P(q) \text{ is realized} \]
\[ \text{Payoffs are determined} \]

\[ \text{Financial hedging takes place} \]
\[ \text{Production takes place} \]
\[ \text{time } t = 0 \quad t = \tau \quad t = T \]

**Flexible Contract with Financial Hedging**

- **Step 1:** At \( t = 0 \), and for a fixed \( \tau \leq T \), the manufacturer offers a price menu \( w_\tau \in \mathcal{F}_\tau^X \).

- **Step 2:** In response, at \( t = 0 \), the retailer selects an optimal ordering menu \( q^*_\tau(w_\tau) \in \mathcal{F}_\tau^X \) solving

  \[
  \Pi^H_R(w_\tau) = \max_{q_\tau \geq 0, G_\tau} \mathbb{E}^Q [(A - q_\tau) q_\tau - w_\tau q_\tau]
  \]

  subject to
  \[
  w_\tau q_\tau \leq B + G_\tau, \quad \text{for all } \omega \in \Omega
  \]
  \[
  \mathbb{E}^Q[G_\tau] = 0.
  \]

- **Step 3:** The manufacturer selects the optimal wholesale price menu \( w^*_\tau \) solving

  \[
  \Pi^H_M(w_\tau) = \max_{w_\tau} \mathbb{E}^Q [w_\tau q^*_\tau(w_\tau) - c_\tau q^*_\tau(w_\tau)].
  \]
Flexible Contract with Financial Hedging

**Proposition.** (Retailer’s Optimal Strategy)

Let $Q_\tau$, $\mathcal{X}$ and $\mathcal{X}^c$ be defined as follows

$$Q_\tau \triangleq \left( \frac{\bar{A}_\tau - w_\tau}{2} \right)^+,$$

$$\mathcal{X} \triangleq \{ \omega \in \Omega : B \geq Q_\tau w_\tau \}, \quad \text{and} \quad \mathcal{X}^c \triangleq \Omega - \mathcal{X}.$$

**Case 1:** Suppose that $\mathbb{E}^Q [Q_\tau w_\tau] \leq B$. Then $q^*_\tau(w_\tau) = Q_\tau$ and there are infinitely many choices of the optimal claim, $G_\tau$. One natural choice is to take

$$G_\tau = [Q_\tau w_\tau - B] \cdot \begin{cases} 
\delta & \text{if } \omega \in \mathcal{X} \\
1 & \text{if } \omega \in \mathcal{X}^c
\end{cases}$$

$$\delta \triangleq \frac{\int_{\mathcal{X}^c} [Q_\tau w_\tau - B] \, dQ}{\int_{\mathcal{X}} [B - Q_\tau w_\tau] \, dQ}.$$

**Remark:** In this case, it is possible to completely eliminate the budget constraint by trading in the financial market.
Flexible Contract with Financial Hedging

Proposition. (Continuation)

Case 2: Suppose that $B < \mathbb{E}^Q [Q_\tau w_\tau]$. Then

$$q_\tau(w_\tau) = \left( \frac{\bar{A}_\tau - w_\tau (1 + \lambda)}{2} \right)^+ \text{ where } \lambda \geq 0 \text{ solves } \mathbb{E}^Q \left[ w_\tau \left( \frac{\bar{A}_\tau - w_\tau (1 + \lambda)}{2} \right)^+ \right] = B.$$ 

Proposition. (Producer’s Optimal Strategy and the Stackelberg Solution)

Let $\phi^* \triangleq \inf \left\{ \phi \geq 1 \text{ such that } \mathbb{E}^Q \left[ \left( \frac{\bar{A}_\tau^2 - (\phi c_\tau)^2}{8} \right)^+ \right] \leq B \right\}.$

Then, $w^*_\tau = \frac{\bar{A}_\tau + \phi^* c_\tau}{2}$ and $q^*_\tau = \left( \frac{\bar{A}_\tau - \phi^* c_\tau}{4} \right)^+$ and the players’ expected payoffs satisfy

$$\Pi_{H|\tau}^M = \frac{(\bar{A}_\tau + \phi^* c_\tau - 2c_\tau)(\bar{A}_\tau - \phi^* c_\tau)^+}{8} \text{ and } \Pi_{H|\tau}^R = \frac{((\bar{A}_\tau - \phi^* c_\tau)^+)^2}{16}.$$ 

Remark: When $q^*_\tau = 0$, the manufacturer decides to overcharge the retailer making the supply chain non-operative. This is never the case if the retailer does not have not access to the financial market.
Flexible Contract with Financial Hedging

**Proposition.** *The manufacturer always prefers the H-contract to the F-Contract. On the other hand, the retailer’s preferences are*

<table>
<thead>
<tr>
<th>Small Budget</th>
<th>Large Budget</th>
</tr>
</thead>
<tbody>
<tr>
<td>H-Contract</td>
<td>H-Contract</td>
</tr>
<tr>
<td>H-Contract or F-Contract</td>
<td>H-Contract = F-Contract</td>
</tr>
</tbody>
</table>

**Retailer's Preferences**

- **Wholesale Price**
  - H-Contract
  - F-Contract

- **Ordering Level**
  - H-Contract
  - F-Contract

- **Producer’s Payoff**
  - H-Contract
  - F-Contract

- **Retailer’s Payoff**
  - H-Contract
  - F-Contract
Flexible Contract with Financial Hedging
Efficiency

○ On path-by-path basis, the Centralized system is not necessarily more efficient than
the Decentralized Supply Chain!

\[ \exists \omega \in \Omega \text{ such that } q^H_{C|\tau} = 0 \text{ and } q^H_\tau > 0. \]

Remark: This is never the case under a Flexible Contract without Hedging.

○ On average, the Centralized solution is more efficient than the Decentralized
solution.

\[ \mathbb{E}_0^Q[q^H_{C|\tau}] \geq \mathbb{E}_0^Q[q^H_\tau]. \]
Optimal Production Postponement

The manufacturer can choose the optimal time to execute the contract solving

$$\Pi^H_P = \max_\tau \mathbb{E}_0^Q \left[ \frac{(\bar{A}_\tau + \phi c_\tau - 2c_\tau)(\bar{A}_\tau - \phi c_\tau)^+}{8 \xi} \right]$$

subject to

$$\phi = \inf \left\{ \psi \geq 1 : \mathbb{E}_0^Q \left[ \frac{(\bar{A}_\tau^2 - \psi^2 c_\tau^2)^+}{8 \xi} \right] \leq B \right\}.$$

Two possibilities:
- **Open Loop**: $\tau$ is a deterministic time selected at time $t = 0$.
- **Closed Loop**: $\tau$ is an $\mathcal{F}_t$-stopping time.

**Modeling Assumptions:**

**Financial Market**

$X_t$ is a diffusion process

$$dX_t = \sigma(X_t) dW_t.$$  

**Operations**

Additive: $A = F(X_T) + \varepsilon$, $\mathbb{E}_0^Q[\varepsilon] = 0$, or

Multiplicative: $A = \varepsilon F(X_T)$, $\varepsilon \geq 0$ and $\mathbb{E}_0^Q[\varepsilon] = 1$

Production: $c_\tau = c_0 + \alpha \tau^\kappa$, for all $\tau \in [0, 1]$. 

Supply Contracts with Financial Hedging
Optimal open-loop production postponement for four different production cost functions.

\[ X_0 = 1, \sigma(X) = X, F(X) = 2 + X \text{ and } c_\tau = 0.3 + 0.7 \tau^\kappa. \]
Optimal Close-Loop Production Postponement

Optimal continuation region for four different manufacturing cost functions.

\[ X_0 = 1, \sigma(X) = X, F(X) = 2 + X \text{ and } c_\tau = 0.3 + 0.7 \tau^\kappa. \]
### Optimal Close-Loop vs. Optimal Open-Loop Production Postponement

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>Open-Loop Payoff</th>
<th>Closed-Loop Payoff</th>
<th>% Increase</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>7.29</td>
<td>7.29</td>
<td>0.0%</td>
</tr>
<tr>
<td>1</td>
<td>7.29</td>
<td>7.305</td>
<td>0.2%</td>
</tr>
<tr>
<td>4</td>
<td>7.71</td>
<td>7.99</td>
<td>3.7%</td>
</tr>
<tr>
<td>8</td>
<td>8.09</td>
<td>8.33</td>
<td>3.8%</td>
</tr>
</tbody>
</table>

Optimal continuation region for four different manufacturing cost functions.

$$X_0 = 1, \sigma(X) = X, F(X) = 2 + X \text{ and } c_\tau = 0.3 + 0.7 \tau^\kappa.$$
Summary

- Simple extension to the traditional wholesale contract.
- The proposed procurement contracts uses the Financial Market as:
  - A source of **public information** upon which contracts can be written.
  - A means for **financial hedging** to mitigate the impact of the budget constraint.
- Consistent with the notions of production postponement and demand forecast.
- Managerial Insights:
  - Manufacturer and Retailer incentives are not always aligned as a function of $B$.
  - Manufacturer prefers retailers that have access to the financial market.
  - With hedging, the supply chain might not operate in some states $\omega \in \Omega$.
  - In some cases, financial hedging **eliminates** the budget constraint.
  - Optimal time $\tau$ of the contract balances $\Var(\bar{A}_\tau)$ and $c_\tau$.
- Extensions:
  - Other types of contracts: quantity discount, buy-back, etc.
  - Include other sources of uncertainty: exchange rates, interest rates, credit risk.