Pricing Policies for Perishable Products with Demand Substitution

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Abstract

This paper studies optimal pricing policies for a family of substitute perishable products with demand correlation. Potential buyers arrive according to an exogenous stochastic process. At each demand epoch, the arriving customer observes the set of substitute products for which there is still inventory available together with their corresponding prices. Based on this information, the customer either buys one of the available products at the posted price, or leaves the system without purchasing anything. We propose a simple choice model to capture buyers' purchasing behavior from which a price-sensitive demand function is derived. In this context, we study the seller's problem of optimally selecting a pricing policy that maximizes expected cumulative revenues over a finite selling horizon.

Keywords: Pricing, demand substitution, consumer choice model, retail operations, approximations.

1 Introduction

In many retail settings the demand for certain products does not only depend on their own price and stock levels, but also depends on the price and inventory of other products (substitute and complementary products). This occurs, for example, when customers are willing to substitute their favorite quality and style product for a cheaper one they can afford, or when customers prefer buying a similar (but different) product, than leaving the store making no purchase when their initial preference is out of stock. When these kind of behaviors are relevant in a product category, omitting the effects of demand substitution over inventory and pricing decisions can have significant profit implications. Some retailers have realized the importance of this issue, and have been able to compete against big discount stores by offering a one-stop shopping with a wide variety of products (*e.g.* Smith and Agrawal [20]).

This paper investigates the effects of demand substitution on optimal pricing policies for perishable products in stochastic environments. In the context of this work, we will understand by substitute products a family of products that satisfy (or are perceived to satisfy) the same customers' needs. An important aspect of substitution is the way it materializes, that is, how customers pick a particular

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product within the family of substitute products. We consider in this work two main sources of substitution in retail settings (assuming all products' attributes, except for the price, are kept fixed over time). The first is due to pricing issues. This *price-driven substitution* considers that changes in the price vector affect the way customers perceive the value of the different substitute products and therefore, the way they purchase. The second source of substitution in retail environments, referred to as *inventory-driven substitution*, is due to stock outs. If a product runs out of stock then part of the demand for that product will shift to a substitute product.

The model that we study in this paper deals with these two types of substitution effects. In our formulation, which is presented in §2 and §3, a retailer is endowed with a finite inventory of a set of substitute products that he/she can sell during a finite selling season. These substitute products differ in two attributes: quality and price. Quality is an exogenous factor that the retailer chooses when selecting the assortment of products to offer. We assume that consumers have identical "taste" for quality. That is, if only quality were considered, then all customers would have the same preferences over the set of substitute products. Price, on the other hand, is dynamically adjusted by the seller as a function of time, quality, and available inventory.

In terms of the existing literature, there are three main streams of research that are closely related to our work: (*i*) demand substitution models, (*ii*) dynamic pricing models, and (*iii*) consumer choice models. In what follows, we attempt to position our paper with respect to similar research without reviewing the vast literature in these areas.

Problems regarding stochastic demand with substitution have been addressed by different authors in several ways. However, most of this research has focused on assortment issues, omitting pricing decisions. Some of this assortment research, like van Ryzin and Mahajan [21], consider a static Multinomial Logit (MNL) demand model. In this setting, customers select the product (from the set of choices offered by the retailer) that most satisfies their needs; but if the product is out of stock, they leave the store undertaking no second choice. While this approach offers a simplified analytical setting, it is unsatisfying for many product categories where inventory-driven substitution is relevant. A second type of substitution effect over assortment policies that has been addressed is the *supplier-driven substitution* as opposed to a *customer-driven substitution* (Bassok et al. [1], Bitran and Dasu [5], Shumsky and Zhang [19]). In this case, suppliers may choose to satisfy the demand of an out of stock product with the inventory of a higher quality product. This supplier-driven substitution, however, is rarely observed in retail settings. The third type of demand substitution effect considered in previous assortment studies, is the inventory-driven substitution (or *dynamic substitution* as referred to by Mahajan and van Ryzin [15]). Smith and Agrawal [20] and Netessine and Rudi [17] include this spill-over effect but assume customers have only one substitution attempt. This last work also assumes that substitution demand is a deterministic fraction of the excess demand. Mahajan and van Ryzin [15] generalize these limitations allowing for multiple and random inventory-driven substitutions.

The second stream of literature related to our work corresponds to papers that study intertemporal pricing strategies with stochastic demand. There is abundant literature in this area, and interested readers are referred to Elmaghraby and Keskinocak [8] and Bitran and Caldentey [4] for comprehensive surveys on optimal pricing problems. Most of these works, however, concentrate on single product pricing problems. Two exceptions are the works of Bell [3] and Gallego and van Ryzin [10]. Bell [3] examines a newsvendor type model where inventory and pricing decisions for two substitute products are made simultaneously. This work is similar to ours, in the sense that it assumes inventory-driven substitution and establishes a ranking over the products (product 1 is preferred to product 2). The presence of just two products simplifies the problem substantially because only one spill-over substitution attempt must be considered. Our paper generalizes this work by allowing multiple products and substitution attempts. Gallego and van Ryzin [10] address the dynamic pricing problem of multiple products in a network revenue management context with limited inventory and price-sensitive Poisson demand. Even though the paper concentrates on airline yield management applications, the model is extensible to retail settings. Their work, however, does not explicitly model consumers' preferences, and some popular choice models, like the MNL, do not fit their framework[†]. Our work, on the contrary, is build up upon a specific consumer choice model, which allows us to obtain further insights on the relationship between customers' characteristics and optimal pricing policies.

The final stream of research related to our work is the literature on Consumer Behavior Models. Roberts and Lilien [18] present a framework for consumer behavior models, and interested readers are referred to this work for an extensive review. In this survey the authors develop a taxonomy of consumer behavior models, establishing a framework organized in five stages: *need arousal, information search, evaluation, purchase decision, post-purchase feelings.* According to this taxonomy, the demand model we present here would classify as a *purchase decision* stage model, where consumers (after evaluating the products) form a ranking of the alternatives, and develop an intention to purchase the product they like best. A distinctive feature of our choice model is that we assume that consumers have a fixed budget which limits their purchasing decisions. In general, most consumer choice model do not model this constraint. A notable exception is the work by Hauser and Urban [11] who study a budget constraint consumer model closely related to ours. However, there are some important differences between Hauser and Urban's model and ours. We postpone this discussion to section §2 where we spell out the details of our choice model.

In light of the existing literature, we summarize the main contributions of our work. First, we develop a consumer model that adequately represents purchasing decisions for substitute products taking into account budget constraints. The model captures in a parsimonious way the interplay between price and quality, which we believe is amenable to other contexts. Second, we incorporate demand substitution effects and their impact on optimal pricing policies in a retail setting with multiple products. In contrast with the previous literature, we allow for more than one spill-over event. Finally, we propose a methodology to solve the seller's problem which generates some simple and robust pricing rules as well as practical managerial insights.

[†]Their model and analysis is based on –what the authors call– a *regular demand* function, which requires, among other things, a concave revenue rate.

The remainder of this paper is organized as follows. In §2 we present a particular demand model that characterizes customers' buying decisions in a retail setting. In §3 we establish admissible pricing policies and present a dynamic programming formulation of the stochastic problem faced by the retailer. Section 4 studies the special case of unlimited supply, and shows some insights on the structure of optimal pricing policies. In §5 we develop a deterministic fluid like approximation of the limited inventory problem under consideration. Finally, in §6 we summarize our conclusions.

2 Demand Model

In this section we present the specific choice model that we use to characterize customers' purchasing behavior. For simplicity, we will discuss the choice model in the absence of inventory constraints. That is, we assume that there is infinite supply of every product. In the following sections we relax this condition and show how our proposed choice model extends to the limited supply case. Let $\mathcal{S} \triangleq \{1, 2, \ldots, N\}$ be a family of substitute products. We define for this family the cumulative demand process $D(t)$. For the purpose of our pricing model, we assume that this aggregate demand is independent of the price vector chosen by the retailer. In other words, at each moment in time there is a fixed demand intensity of potential buyers that are willing to purchase a product within the family $\mathcal{S}$.

On the other hand, the specific choice that an arriving customer makes does depend on prices. In particular, we assume that given a vector of prices $p_{\mathcal{S}}(t) = \{p_i(t) : i \in \mathcal{S}\}$[‡] a particular buyer purchases product $i \in \mathcal{S}$ with probability $q_i(p_{\mathcal{S}}(t))$. We denote by $q_0(p_{\mathcal{S}}(t))$ the non-purchase probability. Observe that these probabilities depend on the price vector but are independent of time. Hence, the demand for product i satisfies $\mathrm{d}D_i(p_{\mathcal{S}}(t)) = q_i(p_{\mathcal{S}}(t))\mathrm{d}D(t)$ for all $i \in \mathcal{S}$.

In order to fit the data to this type of model, we need to understand the nature of $D(t)$ and the probabilities $\{q_i(p_{\mathcal{S}}(t)) : i \in \mathcal{S}\}$. A variety of different approaches can be used to model total demand. For instance, the total demand can be modeled as a deterministic process using seasonality data. We can also try to fit a stochastic process such as a non-homogeneous Poisson process. A more static approach would be to consider that the demand $D(t)$ for the next T days (*e.g.* a week) is normally distributed with mean $\mu(t)$ and variance $\sigma(t)$. For the purpose of this paper however, we will model cumulative demand $D(t)$ as a time-homogeneous Poisson process with intensity λ, following the common assumptions in the revenue management literature.

To compute $\{q_i(p_{\mathcal{S}}(t)) : i \in \mathcal{S}\}$ we need to understand customer's buying behavior. As we discussed in §1, the literature on customer's choice model is extensive and has looked at the problem of modeling the $q_i(p_{\mathcal{S}}(t))$'s from various different angles. One of the most commonly used models is the MNL model (introduced by Luce [13]). The MNL assumes that every consumer will assign a certain level of utility to each product, and will select the one with the highest utility level. To capture the lack of knowledge that the seller has about the population of potential clients, and their inherent heterogeneity, the MNL

[‡]In general, quantities with subscript $\mathcal{S}$ will be used to denote the corresponding vector; for instance the price vector at time t is $p_{\mathcal{S}}(t) = (p_1(t), \ldots, p_N(t))$.

models the utility of each product as the sum of a nominal (expected) utility, plus a zero-mean random component representing the difference between an individual's actual utility and the nominal utility. When these stochastic components are modeled as i.i.d random variables with a Gumbel (or double exponential) distribution, then the probability of selecting each product is given by

$$q_i(p_{\mathcal{S}}) = \frac{\exp(u_i(p_{\mathcal{S}}))}{\sum_{j \in \mathcal{S} \cup \{0\}} \exp(u_j(p_{\mathcal{S}}))},$$

where $u_i(p_{\mathcal{S}})$ is the utility of product i given the vector of price $p_{\mathcal{S}}$. The simplicity of the MNL model makes it appealing from an analytical perspective, however, it has some restrictive properties. In particular, it does not establish a single (absolute) ranking of the products based on non-price attributes such as quality or brand prestige. Thus, it is hard to incorporate customer segmentation using the MNL framework.

Hauser and Urban [11] proposed a quite different consumer choice model that overcomes these limitations. Their model assumes each customer solves the following knapsack problem

$$\begin{aligned} \max_{g_i\, i \in \mathcal{S},\, y} \quad & u_y(y) + \sum_{i \in \mathcal{S}} u_i\, g_i \\ \text{subject to} \quad & \sum_{i \in \mathcal{S}} p_i\, g_i + y \le w \\ & y \ge 0 \;\; g_i \in \{0,1\} \quad \text{for all } i \in \mathcal{S}, \end{aligned} \qquad \text{(MP2)}$$

where w is the buyer's budget, p_i is the price of product i, and $u_y(y)$ is the marginal utility of allocating y dollars to other products.

In this paper, we introduce a variation of Hauser and Urban's MP2 model. Suppose we can rank the products in family $\mathcal{S}$ according to non-price criteria. Let u_i be the *intrinsic* utility associated to product i and let us order the products in $\mathcal{S}$ in descending order of utility, that is, $u_1 > u_2 > \cdots > u_N$, where $N \triangleq |\mathcal{S}|$ is the total number of products in $\mathcal{S}$[§]. We can think of u_i, for example, as proxy for the quality offered by product i. In the absence of price considerations, *every* customer prefers product i to product j if $i < j$. We assume that this ordering induced by the u_i's is common knowledge to the consumers and the seller (in the airline or hotel industries this differentiation is quite clear).

Suppose now that each customer has some private pair (w, u_0), where w is the buyer's budget and u_0 is the buyer's non-purchasing (or reservation) utility. Then, this customer when facing the options in $\mathcal{S}$ and prices $p_{\mathcal{S}}$ will solve the following utility maximization problem to decide his purchasing behavior.

$$\begin{aligned} \max_{x,\, x_0} \quad & u_0\, x_0 + \sum_{i \in \mathcal{S}} u_i\, x_i \\ \text{subject to} \quad & \sum_{i \in \mathcal{S}} p_i\, x_i \le w \\ & x_0 + \sum_{i \in \mathcal{S}} x_i = 1 \\ & x_0, x_i \in \{0,1\} \quad \text{for all } i \in \mathcal{S}. \end{aligned} \qquad \text{(WAL)}$$

[§]The use of strict inequality to rank the $\{u_i\}$ is at no loss of generality because if $u_i = u_j$ then from the point of view of the buyers products i and j are indistinguishable and the seller can treat them as single product.

We will refer to the previous model as *Walrasian Choice* model (WAL) because of its derivation based on a budget constraint. The interpretation of the WAL model is as follows:

1. First, there is a fixed and common knowledge ranking of the products which we represent by the sequence of utility levels $\{u_i\}$.

2. Every customer that walks into the store is characterized by two quantities (w, u_0). The non-purchasing utility u_0 defines the subset of products that a particular customer is considering as possible candidates to buy. This subset contains all those products i such that $u_i \geq u_0$.

3. Finally, the customer's budget w represents the effective amount of money that a customer is willing to expend for a product.

The WAL model is similar to Hauser and Urban's MP2 model, however, there are some differences: (i) MP2 assumes the possibility of choosing/buying multiple products, while the WAL model allows only a single product purchase; (ii) MP2 considers that the remaining budget has an associated utility, while our model includes the non-purchase option; (iii) MP2 corresponds to a Knapsack problem, for which there is no efficient (polynomial) algorithm known, while the WAL model can be easily solved with a greedy algorithm. More importantly, we think that from a modeling standpoint, WAL captures in a simpler way the trade-off between price and quality and allows further tractability in our dynamic pricing setting.

To ease the notation, we will denote sometimes a customer's type by $\theta \triangleq (w, u_0)$. We assume that this information –that characterizes consumers' purchasing preferences– is not observable by the retailer. Instead, he/she assigns it a probability distribution $F(p, u) \triangleq \mathbb{P}(w \leq p \text{ and } u_0 \leq u)$. This joint probability distribution for (w, u_0) allows us to model the correlation between w and u_0 that we expect to observe in practice. Figure 1 shows schematically the segmentation of the customers' population among the different products on the (w, u_0) space.

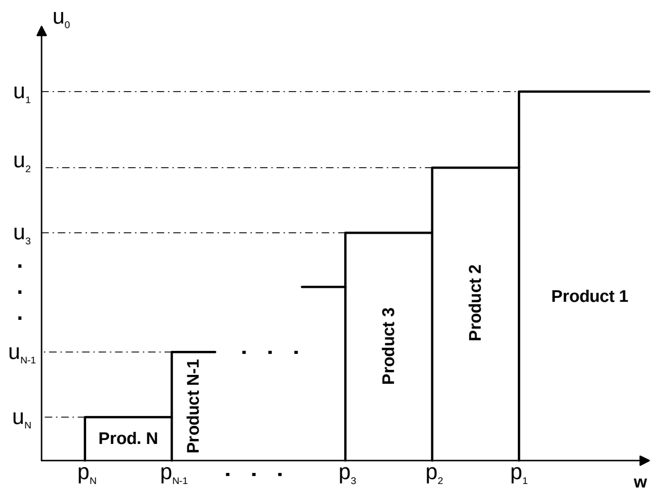

Figure 1: *Customers' segmentation in the (w, u_0) space. For example, product 2 is purchased by all those consumers that have $u_0 \leq u_2$ and $p_2 \leq w < p_1$.*

For the sake of mathematical tractability, throughout this paper we will assume that F satisfies the following assumption.

Assumption 1 *The probability distribution $F(p, u)$ is strictly increasing and twice continuously differentiable in the first argument p for every u. In addition, for fixed u, the function $F(p, u) + p\, F_p(p, u)$ is unimodal in p and converges to $F(\infty, u)$ as $p \uparrow \infty$.*

The notation $F_p(p, u)$ stands for the partial derivative of $F(p, u)$ with respect to p. The assumption about the smoothness and monotonicity of F are rather standard and they are satisfied by most common bivariate distribution functions such as bivariate normal or bivariate weibull. The assumptions on the auxiliary function, $F(p, u) + p\, F_p(p, u)$, is required to guarantee the existence of an optimal pricing policy in section §4. Again, this condition is not particularly restrictive.

The solution to the WAL problem can be obtained using a greedy algorithm. We search the list of products sequentially starting from product 1. We stop as soon as we find a product i with price $p_i \le w$ and $u_i \ge u_0$, in which case we purchase this product, or we realize that (a) for the next product in the list $u_0 > u_i$ or (b) the list is exhausted. In these two cases, (a) and (b), we do not buy any product. Given this solution, we expect that for an optimal pricing strategy $p_1 \ge p_2 \ge \cdots \ge p_N$. In section 3 we will formalize this result showing that an optimal price $p^*_{\mathcal{S}} \in \mathcal{P}_{\mathcal{S}} \triangleq \{p_{\mathcal{S}} : p_1 \ge p_2 \ge \cdots \ge p_N\}$.

Based on the WAL choice model, we can compute the probability that an arriving client chooses product i given the vector of prices $p_{\mathcal{S}}$, $q_i(p_{\mathcal{S}})$. For a given distribution F, it is straightforward to show (see Figure 1) that for any $p_{\mathcal{S}} \in \mathcal{P}_{\mathcal{S}}$

$$q_i(p_{\mathcal{S}}) = q_i(p_{i-1}, p_i) = F(p_{i-1}, u_i) - F(p_i, u_i) \quad \text{and} \quad q_0(p_{\mathcal{S}}) = 1 - \sum_{i \in \mathcal{S}} q_i(p_{i-1}, p_i), \tag{1}$$

where we set $p_0 \triangleq \infty$. From a pricing perspective, we note that the probability of purchasing product i depends exclusively on p_i and the price of the next alternative p_{i-1}. Interestingly, this model is not sensitive to equivalent alternatives, and by construction, fully incorporates the notions of product differentiation and demand segmentation. Finally, suppose that $N = 1$, *i.e.*, there is only one product, then according to (1) the probability that a customer buys the product at price p is equal to

$$q(p) = F(\infty, u) - F(p, u).$$

In the particular case when $u = \infty$ (*i.e.*, the perceived utility associated with the product is very high), the fraction of customers that buy the products is given by $q(p) = 1 - F(p)$. The distribution function F, in this single-product case, characterizes the distribution of the *reservation price*[¶] that the population of consumers has for that particular product. Thus, we view (1) as a generalization of the notion of reservation price to a multi-product setting. Notice also that the MNL model does not have this simple interpretation.

To conclude this section, let us briefly discuss how the WAL model just presented can be combined with other choice models to jointly capture customers' purchasing behavior. For the sake of exposition, let us consider the popular MNL as the alternative choice model.

[¶]The reservation price is the maximum price that a customer is willing to pay for a product in a single-product setting. See Bitran and Mondschein [6] for details about the use of reservation price distributions in pricing models.

Both the MNL and WAL models capture different aspects of the substitution phenomenon of the demand process. As a matter of fact, we can argue that in some cases (as in large retail stores) both occur simultaneously in a hierarchical way. For example, let us consider the family $\mathcal{S}$ of all shirts that are offered at a retail store. A careful look at set $\mathcal{S}$ will usually reveal that we can partition it into subset $\{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$. In group $\mathcal{S}_1$ we recognize all those shirts that are high quality high value which are usually targeted to those customer with large budgets. Next, group $\mathcal{S}_2$ corresponds to those shirts with lower quality than $\mathcal{S}_1$ but also more affordable. At the end of the spectrum we have shirts in $\mathcal{S}_n$ which are low quality but also low price.

Suppose that D is the total number of customers coming to the store looking for a shirt. Then, we can first use the partition $\{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ to determine the volume of demand for each group. Given our previous discussion this first segmentation should be based on the WAL model, as every single customer ranks the shirts in the same order (in the absence of any price consideration). Let $D_{\mathcal{S}_k}$ be the total demand for group $\mathcal{S}_k$. On the other hand, within each subfamily $\mathcal{S}_k$ there is no common ranking that can be established among the products belonging to $\mathcal{S}_k$. In this case, for each product $i \in \mathcal{S}_k$ we can now compute the individual demand D_i using the MNL model. Figure 2 shows schematically this two stage substitution model.

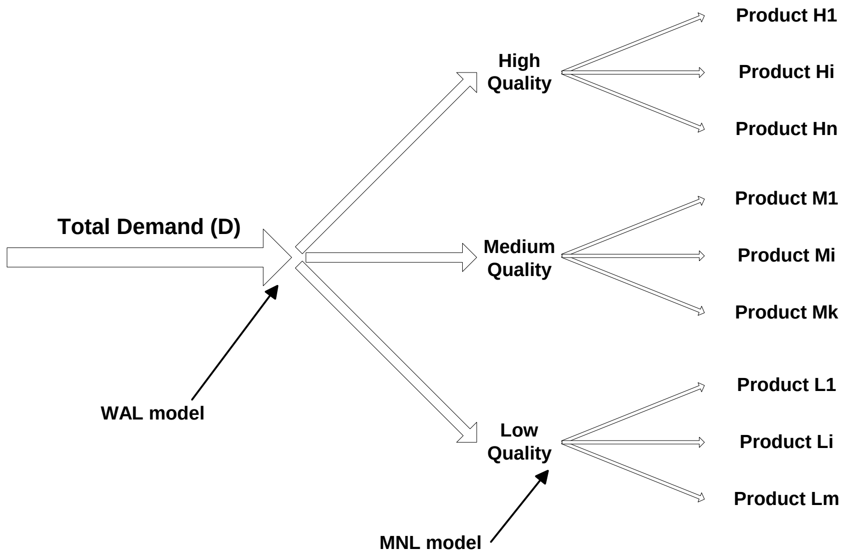

Figure 2: *Hierarchical approach for demand substitution. Segmentation at the subfamily level is represented by the WAL model and the specific choice at the product level is modeled using the MNL model.*

Within a subfamily $\mathcal{S}_k$ all the products are relatively equivalent in terms of the utility they generate and so they should sell at similar prices. More precisely, we expect that the variation of price and utility among the products of a given subfamily is negligible with respect to the variation across products of different subfamilies. For each subfamily $\mathcal{S}_k$ we associate a utility $u_{\mathcal{S}_k}$ level and price $p_{\mathcal{S}_k}$; we might view these quantities as some sort of subfamily average.

From Figure 2, the retailer pricing problem can be decomposed in two sequential pricing decisions. First, given the estimates $\{u_{\mathcal{S}_k}\}$ find an aggregate set of prices $\{p_{\mathcal{S}_k}\}$ at the subfamily level. Then, once the subfamily prices are determined, find an optimal pricing policy at the product level that is consistent with the subfamily prices.

In summary, the model of substitution that we have just described has three main elements.

1. A family $\mathcal{S}$ of products that we view as substitutes with a corresponding exogenous demand D. Note that the stream of customers introduces demand correlation among the different products in $\mathcal{S}$.

2. A partition $\{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ of subfamilies of $\mathcal{S}$. Products within each subfamily $\mathcal{S}_k$ are similar in terms of price and non-price attributes. For each subfamily $\mathcal{S}_k$ we associate a price $p_{\mathcal{S}_k}$ and a demand $D_{\mathcal{S}_k}$ using the WAL model.

3. For each product i in $\mathcal{S}_k$ we associate a price p_i that is consistent with $p_{\mathcal{S}_k}$ and we compute demand D_i using the MNL model.

In this paper we are only concerned with optimal pricing policies at a subfamily level (WAL model). The pricing problem within each subfamily has received some attention in the literature (*e.g.*, [10], [14]). The following section formalizes the problem under investigation, and presents properties of admissible pricing policies. In order to ease the notation, we will refer to each subfamily as a different *product* (understanding however, that each of these products correspond to a group of choices).

3 Admissible Pricing Policies and Problem Formulation

In this section we formulate the dynamic pricing problem for a family of substitute products $\mathcal{S}$ using the framework presented in §2. Recall that under the WAL model consumers are segmented according to their type $\theta = (w, u_0)$, were w is the budget and u_0 is the reservation utility, respectively. Each product i has an intrinsic utility u_i common to all buyers. The retailer's objective is to optimally control the price vector $p_{\mathcal{S}}(t) = (p_1(t), \ldots, p_N(t))$, over the selling horizon of length T, so as to maximize expected revenues given a vector of initial stocks $I_{\mathcal{S}}(0) = (I_1(0), \ldots, I_N(0))$.

The first step towards a mathematical formulation of the pricing problem is to extend the WAL model to incorporate the possibility of inventory stock-outs and the corresponding demand substitution. One possible way of doing this is to dynamically adjust the set of products $\mathcal{S}$ to include only those products with positive inventory. Another alternative, which is the one that we adopt here, is to keep the set $\mathcal{S}$ fixed over time but modifying the pricing policy in such a way that $q_i(p_{\mathcal{S}}(t)) = 0$ if $I_i(t) = 0$. This approach of capturing inventory-driven substitution using price-driven substitution is common in the Revenue Management literature, see for example Gallego and van Ryzin [9]-[10].

Under the choice model considered, we can "shut down" the demand for product i at any time t if we set its price equal to the next best alternative, that is, $p_i(t) = p_{i-1}(t)$ (see equation (1)). This follows from the fact that the utility of product $i-1$ is higher than the utility of product i and so no rational buyer will purchase product i instead of product $i-1$ when their prices are equal. In the case of product 1, recall that we have defined $p_0 = \infty$.

Our first result, which is rather intuitive in our setting, shows that at optimality the price of product i is nondecreasing in the utility level u_i (for a proof, see the appendix at the end).

Proposition 1 *Consider the problem of pricing N products. Suppose that customers behave according to the WAL model and let u_i be the utility associated to product i. Suppose also that the products are ordered such that $u_1 \geq u_2 \geq \cdots \geq u_N$. Then, there is an optimal price vector $p^*_{\mathcal{S}}(t)$ that belongs to $\mathcal{P}_{\mathcal{S}} \triangleq \{p_{\mathcal{S}} : p_1 \geq p_2 \geq \cdots \geq p_N\}$ for $0 \leq t \leq T$.*

According to this result, we can restrict the search of an optimal pricing policy to the set $\mathcal{P}_{\mathcal{S}}$.

Based on these observations, we say that the price process $p_{\mathcal{S}}(t)$ is *admissible* if the following two conditions are satisfied: (i) for all $t \in [0,T]$, $p_{\mathcal{S}}(t) \in \mathcal{P}_{\mathcal{S}}$ and (ii) for all $i \in \mathcal{S}$, $p_i(t) = p_{i-1}(t)$ if $I_i(t) = 0$. The set of admissible price processes will be denoted by $\mathcal{A}$. With this definition, we can write the retailer's optimization process as the following stochastic control problem.

$$\max_{p_{\mathcal{S}} \in \mathcal{A}} -\mathrm{E}\left[\int_0^T p_{\mathcal{S}}(t) \cdot \mathrm{d}I_{\mathcal{S}}(t)\right] \tag{2}$$

$$\text{subject to} \qquad I_i(t) = I_i(0) - D_i\left(\lambda \int_0^t q_i(p_{\mathcal{S}}(\tau))\, \mathrm{d}\tau\right) \quad \text{for all } i \in \mathcal{S}, \tag{3}$$

where $\{D_i(\lambda t) : i \in \mathcal{S}\}$ is a set of independent Poisson processes of rate λ.

As usual, a solution to (2)-(3) can be searched using dynamic programming. For this, let us introduce the value function $V(t, I_{\mathcal{S}})$ representing the optimal expected revenue from time $t \in [0,T]$ onwards if the inventory position at t satisfies $I_{\mathcal{S}}(t) = I_{\mathcal{S}}$. Under the (smoothness) assumption that $V(t, I_{\mathcal{S}})$ is differentiable in t, the Hamilton-Jacobi-Bellman equation is given by (*e.g.*, chapter VII in Brémaud [7])

$$-\frac{\partial V(t, I_{\mathcal{S}})}{\partial t} = \max_{p_{\mathcal{S}} \in \mathcal{A}} \left\{\lambda \sum_{i=1}^{N} q_i(p_{\mathcal{S}})\,[p_i + V(t, I_{\mathcal{S}} - e_i) - V(t, I_{\mathcal{S}})]\right\}, \tag{4}$$

where e_i is the N-dimensional canonical vector having a one in the i^{th} component and zero elsewhere. The boundary conditions are:

$$V(T, I_{\mathcal{S}}) = V(t, 0) = 0 \quad \text{for all } t \in [0,T] \;\text{ and }\; I_{\mathcal{S}} \in \mathbb{Z}^N_+. \tag{5}$$

From equation (4) it is straightforward to prove that the value function $V(t, I_{\mathcal{S}})$ is decreasing in t and increasing in each component of $I_{\mathcal{S}}$. Unfortunately, as in most of these dynamic pricing problems, a closed form derivation of an optimal solution to (2)-(3) is not available. For this reason, we will consider approximated solutions based on different types of asymptotic analysis.

From an analytical perspective, the main difficulties for solving problem (2)-(3) are driven by three main factors: (a) the inventory constraints (3), (b) the stochastic nature of the problem, and (c) the spill-over effect of stockouts (or inventory-driven substitution); if we run out of stock for product i then some demand for i will shift to product $i+1$.

Our first asymptotic approximation deals with factor (a). In section §4, we will consider the special case of unlimited supply, $I_{\mathcal{S}}(0) \uparrow \infty$. From a practical standpoint, this extreme situation may correspond to the beginning of the selling period when it is very unlikely that the inventory of any product will be depleted in the near future. As we will see in the next section, this approximation simplifies considerably the analysis as the optimal solution becomes time independent (under the assumption that current prices will not affect future demand).

Our second approximation deals with factor (b). In section §5 we consider the deterministic (also called sometimes certainty equivalent) counterpart of problem (2)-(3). We will argue that this deterministic problem can be viewed as a limiting situation in which both the vector of initial inventory $I_{\mathcal{S}}(0)$ and the demand rate λ increase proportionally large. Thus, we can think of this deterministic formulation as a good approximation for large retail operations.

4 Unlimited Supply Case

Since in the unlimited supply case the inventory constraints are not binding, the optimization problem (2)-(3) decouples in time. In this way, to solve the unlimited supply problem we simply maximize the expected revenue rate in each time instant, instead of maximizing the cumulative expected revenue. Furthermore, since the demand process is time homogeneous, the optimal pricing strategy is constant over time. Let D be the cumulative number of customers arriving during the entire horizon. Thus, conditioned on the value of D, the total expected revenue associated to a price vector $p_{\mathcal{S}} \in \mathcal{P}_{\mathcal{S}}$ can be written as $D\,W(p_{\mathcal{S}})$, where $W(p_{\mathcal{S}})$ is the expected revenue rate given by

$$W(p_{\mathcal{S}}) \triangleq \sum_{i=1}^{N} p_i\, q_i(p_{i-1}, p_i)$$

and the resulting optimization problem in this unlimited supply case reduces to

$$\max_{p_{\mathcal{S}} \in \mathcal{A}} \; W(p_{\mathcal{S}}). \tag{6}$$

Problem (6) can be written as a dynamic programming problem. Let us define the auxiliary value function $W_k(p_{k-1})$ representing the maximum expected revenue rate obtainable from products k, $k+1$,..., N given p_{k-1}, the posted price of product $k-1$. The resulting Bellman equation for $W_k(p_{k-1})$ is given by,

$$W_k(p_{k-1}) = \max_{0 \le p_k \le p_{k-1}} \{p_k\; q_k(p_{k-1}, p_k) + W_{k+1}(p_k)\} \quad \text{for all } k \in \mathcal{S}, \tag{7}$$

with boundary conditions,

$$W_k(0) = 0, \quad \text{for all } k \in \mathcal{S} \qquad \text{and} \qquad W_{N+1}(\bar{p}) = 0, \quad \text{for all } \bar{p} \ge 0.$$

Note that the solution to (6) is obtained computing $W_1(p_0)$ with $p_0 = \infty$. The following proposition is useful for characterizing the optimal solution to the dynamic program in (7).

Proposition 2 *The value function $W_k(p)$ is non-decreasing in p for all $k = 1, \ldots, N$. In addition, the optimal price in stage k*

$$p_k^*(p) \triangleq \operatorname*{argmax}_{0 \le p_k \le p} \{p_k\; q_k(p, p_k) + W_{k+1}(p_k)\}$$

is a non-decreasing function of p for all $k = 1, \ldots, N$.

Proof: See the appendix.

From a computational standpoint, instead of solving the dynamic program (7), we can derive a much simpler algorithm that relies on a line-search procedure to compute the optimal solution.
The first-order optimality conditions of problem (6) are given by

$$F(p_{i-1}, u_i) = F(p_i, u_i) + p_i\, F_p(p_i, u_i) - p_{i+1}\, F_p(p_i, u_{i+1}) \quad \text{for all } i = 1, \dots, N. \tag{8}$$

Equation (8) has an interesting intuitive interpretation. To see this, let us first multiply both sides by $\mathrm{d}p_i$, and then rearrange the terms as follows,

$$\mathrm{d}p_i\, q_i(p_{i-1}, p_i) = p_i\, \mathrm{d}p_i\, F_p(p_i, u_i) - p_{i+1}\, \mathrm{d}p_i\, F_p(p_i, u_{i+1}).$$

The left-hand side corresponds to the incremental expected revenue obtained by increasing the price of product i by $\mathrm{d}p_i$. On the other hand, the right-hand side is the associated expected cost of this price increment. The first term of the right-hand side is the lost revenue due to the fraction of customers who were willing to buy i at the initial price, but are not willing to buy it at the higher price. However, since the retailer offers a less expensive product $i+1$, some of these customers will switch and buy this less expensive product $i+1$, allowing the seller to recover part of the lost benefits. This effect is captured by the second term of the right-hand-side.

In general, (8) is a multidimensional system of nonlinear equations. Fortunately, it turns out that a simple one dimensional search can be set to solve it efficiently because of its diagonal structure. In order to see this, note that for $i = N$ (8) becomes

$$F(p_{N-1}, u_N) = F(p_N, u_N) + p_N\, F_p(p_N, u_N).$$

Therefore, fixing $p_N = \bar{p}$ we can solve for p_{N-1} as a function of $\bar{p}$. The value of p_{N-1}, as a function of $\bar{p}$, is uniquely determined by

$$p_{N-1}(\bar{p}) = F^{-1}\Big(F(\bar{p}, u_N) + \bar{p}\, F_p(\bar{p}, u_N), u_N\Big). \tag{9}$$

The function $F^{-1}(\cdot, u)$ is the inverse function of $F(p, u)$ with respect to p for a fixed u. Under the conditions in Assumption 1, this inverse function $F^{-1}(x, u)$ is well defined for $x \in [0, F(\infty, u))$. Therefore, our choice of $\bar{p}$ must by restricted so that $F(\bar{p}, u_N) + \bar{p}\, F_p(\bar{p}, u_N) < F(\infty, u_N)$. Let us define

$$\bar{p}_N^{\max} \triangleq \sup\{p \geq 0 \,:\, F(p, u_N) + p\, F_p(p, u_N) < F(\infty, u_N)\}.$$

Assumption 1 guarantees that the condition $F(p, u_N) + p\, F_p(p, u_N) < F(\infty, u_N)$ is satisfied if and only if $p < \bar{p}_N^{\max}$. Therefore, we can restrict the choice of $\bar{p}$ to the interval $[0, \bar{p}_N^{\max}]$. Note that $p_{N-1}(\bar{p})$ is increasing in $\bar{p}$ with $p_{N-1}(0) = 0$ and $p_{N-1}(\bar{p}_N^{\max}) = \infty$.

Similarly, we can sequentially (backward on the index i) solve for all p_i as a function of $\bar{p}$ using (8), that is,

$$p_{i-1}(\bar{p}) = F^{-1}\Big(F(p_i(\bar{p}), u_i) + p_i(\bar{p})\, F_p(p_i(\bar{p}), u_i) - p_{i+1}(\bar{p})\, F_p(p_i(\bar{p}), u_{i+1}), u_i\Big) \quad \text{for all } i = N-1, \dots, 2. \tag{10}$$

For each i, we need to guarantee that the argument of F^{-1} is bounded from above by $F(\infty, u_i)$. In other words, we have to restrict the choice of $\bar{p}$ such that

$$F(p_i(\bar{p}), u_i) + p_i(\bar{p})\, F_p(p_i(\bar{p}), u_i) - p_{i+1}(\bar{p})\, F_p(p_i(\bar{p}), u_{i+1}) < F(\infty, u_i), \qquad \text{for all } i = N-1, \dots, 2.$$

For an arbitrary distribution F, the left-hand side can be a complicated function of $\bar{p}$ and so imposing this inequality condition is not straightforward. Let us suppose for a moment that the left-hand side is a unimodal function of $\bar{p}$. Then, as before, we can show that $\bar{p}$ must be restricted to a closed interval of the form $[0, \bar{p}_i^{\max}]$ where the sequence of upper bounds $\bar{p}_i^{\max}$ is increasing in the index i. Furthermore, the solution $p_i(\bar{p})$ is increasing in $\bar{p}$ with $p_{i-1}(0) = 0$ and $p_{i-1}(\bar{p}_i^{\max}) = \infty$.

In general, we have not been able to prove this unimodal property under Assumption 1. However, all the computational experiments that we have performed using bivariate distributions such as normal, weibull, and exponential have shown this property. In what follows, we will assume that this condition is in fact satisfied.

Finally, the condition for $i = 1$ is used for checking optimality. That is, if

$$F(p_0, u_1) = F(p_1(\bar{p}), u_1) + p_1(\bar{p})\, F_p(p_1(\bar{p}), u_1) - p_2(\bar{p})\, F_p(p_1(\bar{p}), u_2) \tag{11}$$

holds then the solution $p_{\mathcal{S}}(\bar{p}) \triangleq (p_1(\bar{p}), p_2(\bar{p}), \dots, p_N(\bar{p}))$ satisfies the optimality condition (8), if not we change $\bar{p}$ and iterate. The following algorithm formalizes this procedure.

<u>UNLIMITED INVENTORY ALGORITHM:</u>

Step 1: Set $p_{N+1} = 0$, $p_N = \bar{p}$, and $p_0 = \infty$ for some $\bar{p} \le \bar{p}_2^{\max}$.

Step 2: Solve recursively the system

$$F(p_{i-1}, u_i) = F(p_i, u_i) + p_i\, F_p(p_i, u_i) - p_{i+1}\, F_p(p_i, u_{i+1})$$

to compute p_i, $i = N-1, \dots 1$.

Step 3: Compute

$$\eta = F(p_0, u_1) - F(p_1, u_1) - p_1\, F_p(p_1, u_1) + p_2\, F_p(p_1, u_2).$$

If $|\eta| \le \epsilon$ then stop the solution $(p_1, \dots, p_N)$ is an ϵ-solution. If $\eta > \epsilon$ then $p_N \leftarrow p_N + \delta$, otherwise $p_N \leftarrow p_N - \delta$. Goto 2 and iterate. □

It is straightforward to show that for every $\bar{p}$ the solution $p_{\mathcal{S}}(\bar{p})$ belongs to $\mathcal{P}_{\mathcal{S}}$[†] which is consistent with the optimality condition identified in Proposition 1.

To ensure that the previous algorithm is well defined, we need to address the problem of existence of a solution to the first-order optimality conditions in (8). The following result identifies *necessary* and *sufficient* conditions for the existence of a solution as well as a set of bounds for this solution. Let us define two auxiliary functions

$$L(p, u, \hat{u}) \triangleq F(p, u) + p \left(F_p(p, u) - F_p(p, \hat{u}) \right) \;\text{ and }\; U(p, u) \triangleq F(p, u) + p\, F_p(p, u).$$

[†]This conclusion follows using induction over $i = N, N-1, \dots, 1$ and the monotonicity of $F(p, u)$ with respect to p.

Proposition 3 *A sufficient condition for the existence of a solution to (8) is that there exists a price $\hat{p}$ that solves $F(p_0, u_1) = L(\hat{p}, u_1, u_2)$. On the other hand, a necessary condition for the existence of a solution is that there exists a $\tilde{p}$ such that $U(\tilde{p}, u_1) = F(p_0, u_1)$. In addition, every solution $(p_1^*, \ldots, p_N^*)$ to (8) satisfies $p_i^{min} \leq p_i^* \leq p_i^{max}$, where the sequence of lower and upper bounds is computed recursively, for $i = 1, 2, \ldots, N$, as follows: $p_i^{min} = \text{argmin}\{p : F(p_{i-1}^{min}, u_i) = U(p, u_i)\}$ and $p_i^{max} = \text{argmax}\{p : F(p_{i-1}^{max}, u_i) = L(p, u_i, u_{i+1})\}$ with boundary conditions $p_0^{max} = p_0^{min} = p_0 = \infty$.*

Proof: See the appendix.

The sufficient condition identified in Proposition 3 is not particularly restrictive and most of the distributions commonly used in practice satisfy it. One important case in this group is the bivariate normal distribution. Figure 3 plots the functions $L(p, u_1, u_2)$ and $U(p, u_1)$ and shows how to identify the lower and upper bounds, $p_1^{\min}$ and $p_1^{\max}$, respectively, for the case in which $F(p, u)$ is a bivariate normal distribution.

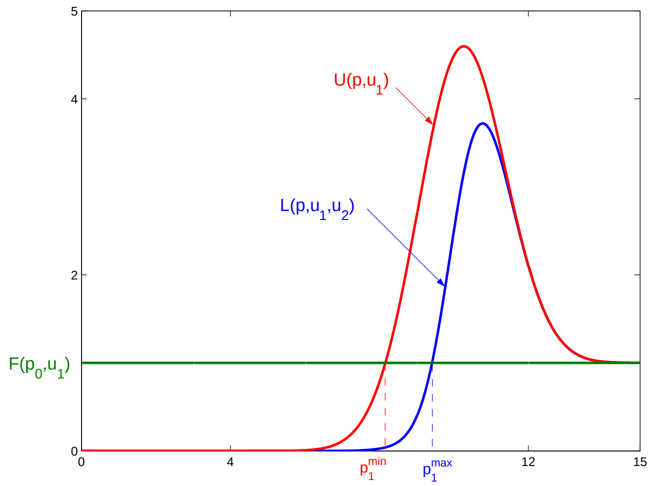

Figure 3: *Shape of $L(p_1, u_1, u_2)$ and $U(p, u_1)$ with $u_1 = 15$, $u_2 = 12$ for the case when (w, u_0) has a bivariate normal distribution with mean $(10, 10)$, variance $(2.0, 1.0)$, and coefficient of correlation $\rho = 0.8$.*

We next present a set of numerical experiments to show the behavior of optimal prices and revenues under different settings. To model customers' type, we consider a bivariate normal distribution with mean $(\mu_w, \mu_{u_0}) = (1, 1)$ and variance $(\sigma_w^2, \sigma_{u_0}^2) = (0.5, 0.4)$. The quality of products is assumed to be evenly distributed over $(0.5, 3)$.

Our first analysis studies the effect of correlation between customers' budget (w) and their non-purchasing utility (u_0) over pricing policies. Figure 4 (a) shows optimal pricing policies for a family of 20 substitute products for a set of four different ρ's ($\rho = 0, 0.5, 0.7, 0.9$). As presented in this plot, optimal prices raise with the magnitude of the coefficient of correlation.

When the coefficient of correlation grows, the seller increases his/her ability to segment customers according to their budget (disposition to pay). Under high correlation settings, the quality of products serves the seller as a proxy for customers' budget, in the same way as time is used as a proxy for customers' disposition to pay in the airline industry. In the limit, when $\rho \uparrow 1$, the seller knows with

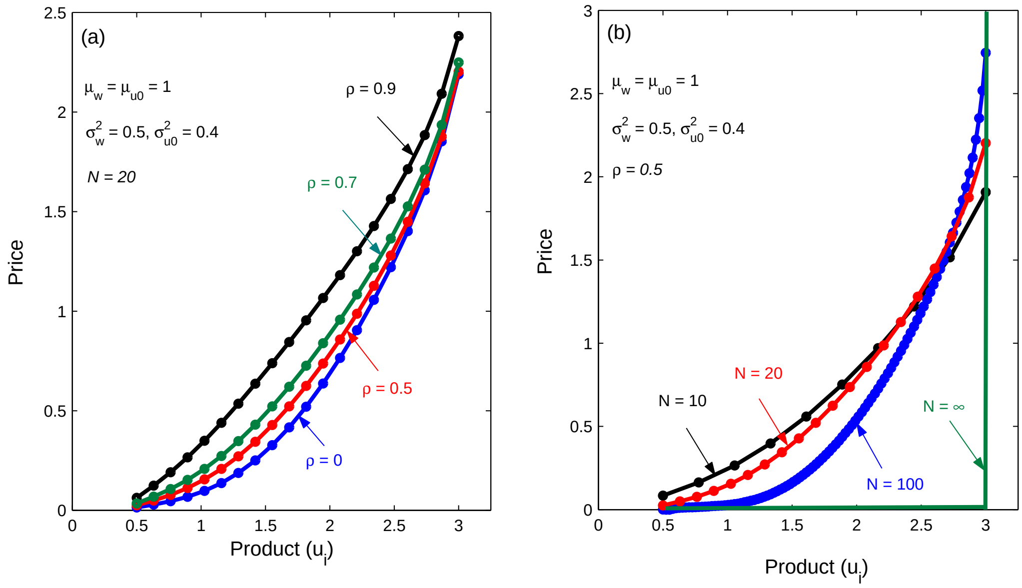

Figure 4: *(a) Pricing policies for different correlation levels; (b) Pricing policies for different number of choices.*

certainty that low quality products will be demanded only by low budget customers, whereas high budget customers will exclusively demand high quality products. This segmentation ability allows the seller to discriminate customers according to their budget, making it possible to increase prices so as to charge each customer type as much as he/she can pay.

Figure 4 (b) presents the effect of the number of choices over optimal prices. When the number of different products increases, the prices for low quality choices get reduced while for the high quality choices the opposite occurs. As the number of choices grows very large, all the products, except for those with quality near u_{max}, will have prices close to zero (observe in the figure the case when $n = \infty$). Let us analyze the asymptotic pricing problem (*i.e.* $n \uparrow \infty$). Note that in this limit situation, prices should be posted in the whole range of possible prices (*i.e.* from 0 to p_0). Not doing so would imply loosing sales from very low budget customers, since they cannot afford even the cheapest product, and loosing revenues from very high budget customers, since they pay less than what they can afford. In this setting, the demand for a product of quality u is $D\,F_p(p(u), u)\,\mathrm{d}p(u)$. Assuming $p(u)$ can be inverted, total revenues will be given by $\int_0^{p_0} p\,D\,F_p(p, u(p))\,\mathrm{d}p$. Since $F_p(p, u(p)) \leq F_p(p, u_{max})$ (where $u_{max} = 3$ in our example), an optimal price assignment is $u(p) = u_{max}$, for $0 < p \leq p_0$, and $p(u) = 0$, for $u_{min} \leq u < u_{max}$.

Variations in the coefficient of correlation and the number of different products available do not only influence pricing policies, but also have an important impact over total revenues. Our next two computational experiments study these issues. We consider a fixed selling horizon in which the retailer faces an average arrival of 100 customers during the whole period ($D = 100$).

Figure 5(a) presents the influence of the correlation coefficient over total revenues. When correlation increases, it is possible to increment prices without reducing the number of non-purchasing customers.

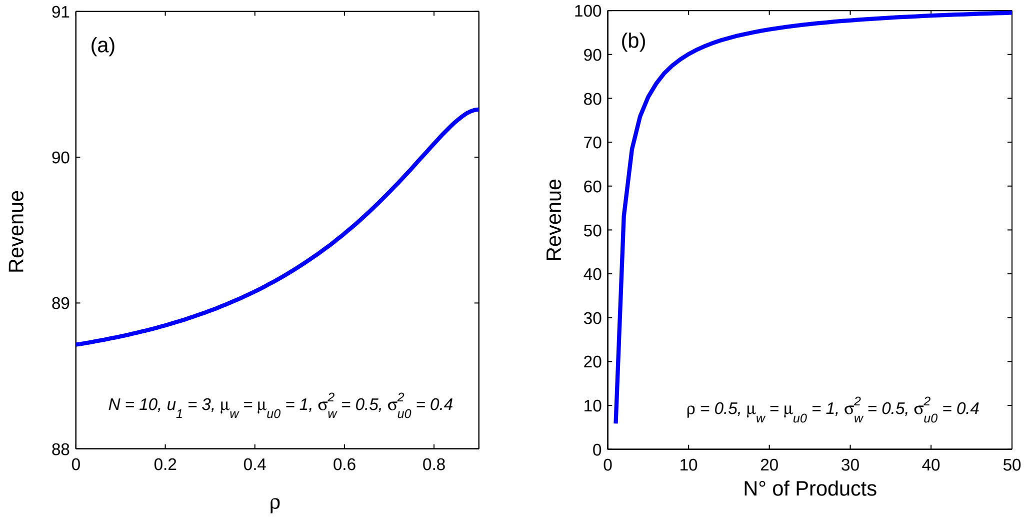

Figure 5: *(a) Optimal revenues as a function of coefficient of correlation ρ; (b) Optimal revenues as a function of the number of substitute products.*

This explains the increasing effect of ρ over revenues presented in the plot. It is interesting to note however, that the level of correlation has only a limited influence over total revenues. In this particular example, the impact is less than 2%.

The effect of the number of choices over total revenues is shown in Figure 5 (b). We emphasize two interesting issues: (i) revenues reach an asymptotical limit as the number of products grows infinitely large, and (ii) this limit is approached quite rapidly. It is possible to obtain an upper bound for this asymptotic revenue. Observe that there are no costs associated with the number of choices available, so the maximum revenue is reached when the number of products grow infinitely large. Also recall that the optimal revenue in an infinite choice setting is given by $\int_0^{p_0} p\, D\, F_p(p, u_{max})\, \mathrm{d}p$, so when u_{max} is large, $F_p(p, u_1) \approx G_p(p)$ (where $G_p(\cdot)$ is the marginal density of w), and $\int_0^{p_0} p\, D\, F_p(p, u_{max})\, \mathrm{d}p \approx D \int_0^{p_0} p\, G_p(p)\, \mathrm{d}p = D\, \mathrm{E}[w]$. In the example under consideration, no more than 10 products are required to generate around 90% of the maximum potential income ($D \cdot \mathrm{E}[w] = 100 \cdot 1$). This result mimics actual retailing practices, where it is quite uncommon to observe more than 10 substitute brands compete simultaneously in a certain category of products.

4.1 First-Order Taylor Approximation

To get further insights about the structure of the optimal pricing policy, let us use a first-order Taylor approximation in (8). Recall from Assumption 1 that $F^{-1}(p, u)$ denotes the inverse function of $F(p, u)$ with respect to p for a fixed u. Then, based on equation (8) we get that

$$\begin{aligned} p_{i-1} &= F^{-1}\big(F(p_i, u_i) + p_i\, F_p(p_i, u_i) - p_{i+1}\, F_p(p_i, u_{i+1}), u_i\big) \\ &\approx F^{-1}\big(F(p_i, u_i), u_i\big) + F_p^{-1}\big(F(p_i, u_i), u_i\big)\, \big(p_i\, F_p(p_i, u_i) - p_{i+1}\, F_p(p_i, u_{i+1})\big) \\ &= 2\, p_i - p_{i+1}\, \frac{F_p(p_i, u_{i+1})}{F_p(p_i, u_i)}, \end{aligned} \tag{12}$$

where the approximation follows from the first-order expansion of $F^{-1}(x, u_i)$ around $x = F(p_i, u_i)$ with $F_p^{-1}(p, u)$ denoting the partial derivative of $F^{-1}(p, u)$ with respect to p. The second equality follows from the identity $F_p^{-1}(F(p, u), u)\, F_p(p, u) = 1$. We note that the approximation is exact for the case in which the budget w (or reservation price) is uniformly distributed and independent of the reservation utility u_0 (see Example 1 below).

Since $F(p, u)$ is increasing in both arguments and $u_i \geq u_{i+1}$, it follows that $\frac{F_p(p_i, u_{i+1})}{F_p(p_i, u_i)} \in [0, 1]$. Therefore, if we assume that the Taylor expansion is accurate –as we do for the rest of this section– we obtain

$$2\, p_i - p_{i+1} \leq p_{i-1} \leq 2\, p_i \quad (i = 1, \ldots, N). \tag{13}$$

In words, the inequality on the right implies that it is never optimal to mark-up more than twice the price of a product with respect to the next "lower quality" product. On the other hand, the inequality on the left implies that $p_i - p_{i+1} \leq p_{i-1} - p_i$, that is, the price differential between two consecutive products increases with the level of quality.

Let us define $\beta_i \triangleq \frac{p_i}{p_{i+1}}$ for $i = 1, \ldots, N-1$, which represents the relative mark-up of the price of product i with respect to price of the next "lower quality" product $i + 1$[†]. From proposition 1 and equation (13), we know that $1 \leq \beta_i \leq 2$. On the other hand, equation (13) implies that

$$2 - \frac{1}{\beta_i} \leq \beta_{i-1} \leq 2 \quad (i = 2, \ldots, N-1).$$

Using these two inequalities iteratively, we get a lower bound for β_{i-1} with the following continued-fraction representation

$$2 \geq \beta_{i-1} \geq 2 - \frac{1}{\beta_i} \geq 2 - \frac{1}{2 - \frac{1}{\beta_{i+1}}} \geq 2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{\beta_{i+2}}}} \geq \cdots \geq 2 - \frac{1}{2 - \frac{1}{2 - \cdots \frac{1}{2 - \frac{1}{\beta_{N-1}}}}}.$$

For $i = N$, equation (13) implies that $\beta_{N-1} = 2$ and so we can explicitly compute the continued-fraction above. After some straightforward manipulations, we get that

$$1 + \frac{1}{N + 1 - i} \leq \beta_{i-1} \leq 2 \quad (i = 2, \ldots, N-1).$$

The following proposition summarizes the previous discussion under the first-order Taylor expansion.

Proposition 4 *Suppose there is unlimited inventory, and the first-order Taylor expansion for $F^{-1}(p, u)$ in (12) holds, then: (i) the relative mark-up of the price of product i with respect to the price of product $i+1$, $\beta_i = \frac{p_i}{p_{i+1}}$, is bounded by $1 + \frac{1}{N-i} \leq \beta_i \leq 2$, for $i = 1, \ldots, N-1$; (ii) the absolute price differential, $p_i - p_{i+1}$, is decreasing in i, for $i = 1, \ldots, N-1$.*

Example 1: Independent Budget and Reservation Utility

A particular case for which the assumptions of proposition 4 hold trivially occurs when the budget w and reservation utility u_0 are independent random variables and w is uniformly distributed. In this situation, there are two distribution functions G and H such that $F(p, u) = G(p)\, H(u)$ and $G(p_{i-1}) = G(p_i) + G_p(p_i)\,(p_{i-1} - p_i)$[‡]. Under this condition, the results

[†] Note that by definition, $p_{N+1} = 0$ and so β_N is not well defined.

[‡] This linear approximation also hold if the optimal prices for the different products are relatively close to each other.

in proposition 4 hold directly. Furthermore, in this situation (8) implies that $p_i = A_i\,\bar{p}$ for $i = 1, \ldots, N$, where the coefficients $\{A_i\}$ satisfy the recursion

$$A_{i-1} = 2\,A_i - A_{i+1}\,\frac{H(u_{i+1})}{H(u_i)} \quad \text{for all } i = 1, \ldots, N,$$

and boundary conditions $A_{N+1} = 0$ and $A_N = 1$.
The sequence $\{A_i : i = 0, \ldots, N\}$ is nonincreasing in i. This follows directly from the fact that $H(u_{i+1}) \le H(u_i)$ since our ordering of the products satisfies $u_{i+1} \le u_i$. Moreover, using induction it is straightforward to show that

$$2^{N-i}\left[1 - \frac{1}{4}\sum_{k=i+1}^{N-1}\frac{H(u_{k+1})}{H(u_k)}\right] \le A_i \le 2^{N-i}.$$

Finally, $\bar{p}$ solves the fixed-point condition

$$\bar{p} = \frac{1 - G(A_1\,\bar{p})}{(A_0 - A_1)G_p(A_1\,\bar{p})} \tag{14}$$

In the special case where $G(p)$ is uniformly distributed in $[p_{\mathsf{min}}, p_{\mathsf{max}}]$ then (14) implies that the optimal price strategy is given by

$$p_i = \frac{A_i}{A_0}\,p_{\mathsf{max}} \quad \text{for all } i = 1, \ldots, N.$$

As we have already mentioned our WAL model can be viewed as a generalization of the simple *reservation price* formulation for single product. Similarly, condition (14) generalizes condition (2) in Bitran and Mondschein [6]. □

The simplicity of Proposition 4 is very attractive for a managerial implementation. For instance, the bounds on the relative mark-ups are distribution-free which make them particularly appealing in those cases were there is a little or non information about the demand distribution.

In Table 1 we present a family of 10 substitute products under two different customer segmentation schemes: a *bivariate weibull distribution*§ and a *bivariate normal distribution*. The first two columns of the table characterize the product according to its quality (utility). The following ten columns present the optimal price (p_i^*), the lower and upper bound for the optimal price (p_i^{min} and p_i^{max}), the optimal price difference between two consecutive products ($p_i^* - p_{i+1}^*$), and the relative mark-ups (β_i^*) for each of the ten products under both segmentation settings.

These results show that under both distributions, optimal prices comply quite well with proposition 4 (*i.e.* price differentials, $p_i^* - p_{i+1}^*$, are decreasing in i, and relative mark-ups move within the established bounds). However, in order to implement the results in proposition 4 some additional work is required. In particular, we need to be able to translate the suggested bounds on the relative mark-ups on actual price recommendations. For this, we first get an approximation on the relative mark-up for product i

§ $\mathbb{P}[X > x, Y > y] = exp\left\{-\left[\left(\frac{x}{\theta_x}\right)^{\gamma_x/\delta} + \left(\frac{y}{\theta_y}\right)^{\gamma_y/\delta}\right]^{\delta}\right\}$

Product		Bivariate Weibull					Bivariate Normal						
i	u_i	p_i^*	p_i^{min}	p_i^{max}	$p_i^* - p_{i+1}^*$	β_i^*	p_i^*	p_i^{min}	p_i^{max}	$p_i^* - p_{i+1}^*$	β_i^*	β_i^{min}	β_i^{max}
1	1.50	1.90	0.73	6.80	0.65	1.52	1.42	0.86	2.01	0.37	1.35	1.11	2.00
2	1.39	1.25	0.32	5.21	0.36	1.40	1.05	0.45	1.68	0.25	1.31	1.13	2.00
3	1.28	0.89	0.15	4.11	0.23	1.36	0.80	0.24	1.43	0.19	1.31	1.14	2.00
4	1.17	0.65	0.07	3.27	0.17	1.34	0.61	0.13	1.22	0.15	1.32	1.17	2.00
5	1.06	0.49	0.04	2.59	0.12	1.34	0.47	0.06	1.05	0.12	1.34	1.20	2.00
6	0.94	0.36	0.02	2.04	0.10	1.37	0.35	0.03	0.90	0.09	1.37	1.25	2.00
7	0.83	0.27	0.01	1.58	0.08	1.43	0.25	0.02	0.76	0.08	1.43	1.33	2.00
8	0.72	0.19	0.00	1.20	0.07	1.58	0.18	0.01	0.64	0.06	1.55	1.50	2.00
9	0.61	0.12	0.00	0.89	0.06	2.08	0.11	0.00	0.54	0.05	1.97	2.00	2.00
10	0.50	0.06	0.00	0.29	-	-	0.06	0.00	0.32	-	-	-	-

Table 1: *Numerical Optimization of 10 substitute products. Two distributions are considered: i) bivariate weibull distribution with scale param. $\theta_w = \theta_{u_0} = 1$, shape param. $\gamma_w = \gamma_{u_0} = 1$, and correlation param. $\delta = 0.5$; ii) bivariate normal distribution with mean $\mu_w = \mu_{u_0} = 1$, variance $\sigma_w^2 = 0.5$ and $\sigma_{u_0}^2 = 0.4$, and coefficient of correlation $\rho = 0.5$.*

using a convex combination of the bounds computed in proposition 4. That is, for a fixed $\alpha \in [0, 1]$, we define the approximated relative mark-up for product i as

$$\tilde{\beta}_i(\alpha) \triangleq \alpha \left(1 + \frac{1}{N - i}\right) + (1 - \alpha)\, 2.$$

From Table 1, we see that the lower bound on β_i is more accurate than the upper bound. Hence, we expect α to be closer to one. In our computation experiments below, we choose $\alpha = 1$ and $\alpha = 0.7$. We can think of more sophisticated rules to choose α (*e.g.*, making it a function of i) but we do not investigate this issue here.

The next step is to get an approximation for the price of product 1, which we denote by $\tilde{p}_1$¶. One possible approach, that we use in our computation experiments, is to consider the solution using a particular demand distribution such as the uniform (see Example 1). Alternatively, the seller might have some prior estimate of the value of p_1 based on past experiences or based on the prices set by competitors. Once $\tilde{p}_1$ has been determined, we can compute the prices of products $2, 3, \ldots, N$ as follows

$$\tilde{p}_i = \frac{\tilde{p}_{i-1}}{\tilde{\beta}_{i-1}(\alpha)} = \frac{\tilde{p}_1}{\tilde{\beta}_1(\alpha)\,\tilde{\beta}_2(\alpha)\cdots\tilde{\beta}_{i-1}(\alpha)}, \qquad i = 2, \ldots, N.$$

When selecting the value of $\tilde{p}_1$ (and therefore the price of all the products), the seller should consider other constraints which are not captured by our model, such as price bounds based on costs and competition.

In Table 2 we compare optimal revenues with those generated applying the pricing strategy $\tilde{p}_S$ derived above. To do so, we define $\mathcal{R}$ as the ratio between the revenue obtained by using $\tilde{p}_S$ and the optimal revenue. In these numerical experiments, customers are characterized by a bivariate normal distribution with $\mu_w = \mu_{u_0} = 1$ and $\sigma_w = \sigma_{u_0} = 1$. Three different values of ρ are considered. We analyze a setting of 10 substitute products with their quality randomly distributed over $[\mu_{u_0} - \sigma_{u_0}\,,\, \mu_{u_0} + \sigma_{u_0}]$. For each of the cases studied, a set of 100 random instances of product quality were generated to compute the mean and standard deviation of $\mathcal{R}$ ($\mathcal{R}_{mean}$ and $\mathcal{R}_{std}$, respectively).

¶In fact, we only need an approximation for the price of one of the N products; for ease of exposition we consider product 1.

We perform the analysis using two values of α (1.0 and 0.7). The value of $\tilde{p}_1$ is obtained by using a bivariate uniform distribution approximation ($\tilde{p}_1 = p_1^{unif}$). This uniform distribution is given by

$$\mathbb{P}[w \leq p, u_0 \leq u] = \left(\frac{p - (\mu_w - \sigma_w)}{2\sigma_w}\right) \left(\frac{u - (\mu_{u_0} - \sigma_{u_0})}{2\sigma_{u_0}}\right).$$

Note that p_1^{unif} can be computed easily using the results in Example 1.

	$\tilde{p}_1 = p_1^{unif}$			
	$\alpha = 1.0$		$\alpha = 0.7$	
ρ	$\mathcal{R}_{mean}$	$\mathcal{R}_{std}$	$\mathcal{R}_{mean}$	$\mathcal{R}_{std}$
0.1	.9217	.0164	.9599	.0059
0.5	.9283	.0226	.9835	.0072
0.9	.8516	.0745	.9806	.0202

Table 2: *Revenues applying the approximation $\tilde{p}_{\mathcal{S}}$ versus optimal revenues.*

Table 2 shows some interesting issues. In the first place, it is important to highlight the fact that all of the mean values of $\mathcal{R}$ are above 0.85, and in most cases, $\mathcal{R}_{mean}$ is above 0.90. It is also possible to observe that the value of α plays an important role in $\mathcal{R}_{mean}$. Finally, note that the results when using $\tilde{p}_1 = p_1^{unif}$ are quite good, specially when $\alpha = 0.7$. This last observation implies that prices can be set without knowing $F(p, u)$, only the first two moments μ_{u_0}, μ_w, σ_{u_0} and σ_w are required. Based on these results, we believe Proposition 4 provides an efficient and robust (distribution-free) methodology to establish prices in a setting of substitute products.

5 Deterministic Approximation to the Finite Inventory Case

In this section we propose a deterministic approximation for problem (2)-(3), where demand is modeled in a fluid-like (continuous) and time homogeneous way. This approximation, as we will see, simplifies the path-dependent nature of the pricing problem, allowing a more tractable analytical formulation. We will also see that this deterministic continuous approximation is asymptotically optimal as the volume of expected sales and initial inventory grow proportionally large.

Consider a sequence of instances of problem (2)-(3) parameterized by $n \in \mathbb{Z}_+$. For the n^{th} instance, let us denote by $I^n_{\mathcal{S}}(0)$ and λ^n the vector of initial inventory and demand rate, respectively. All other parameters are kept fixed independent of n. In the limiting regime that we consider, we let both $I^n_{\mathcal{S}}(0)$ and λ^n grow proportionally large. In other words, we consider those regimes that approximate the operations of a large retailer. Specifically, we define

$$I^n_{\mathcal{S}}(0) = n\, I_{\mathcal{S}}(0) \qquad \text{and} \qquad \lambda^n = n\, \lambda, \tag{15}$$

where $I_{\mathcal{S}}(0)$ and λ are constants†. For the n^{th} instance, the retailer's optimization problem (2)-(3)

†A more general definition of our asymptotic regime given by $\lim_{n\to\infty} \frac{I^n_{\mathcal{S}}(0)}{n} = I_{\mathcal{S}}(0)$ and $\lim_{n\to\infty} \frac{\lambda^n}{n} = \lambda$ is possible, but for ease of exposition we restrict ourselves to the special case in (15).

becomes

$$V^n \triangleq \max_{p_{\mathcal{S}}\in\mathcal{A}} -\mathrm{E}\left[\int_0^T p_{\mathcal{S}}(t)\cdot \mathrm{d}I^n_{\mathcal{S}}(t)\right]$$

$$\text{subject to} \qquad I_i^n(t) = I_i^n(0) - D_i\left(\lambda^n \int_0^t q_i(p_{\mathcal{S}}(\tau))\,\mathrm{d}\tau\right), \quad \text{for all } i\in\mathcal{S}.$$

We note that the set $\mathcal{A}$ of admissible pricing policies remains independent of n. Our next step is to consider a normalized version of the optimization problem above. To this end, let us introduce the following scaled quantities:

$$\bar{V}^n \triangleq \frac{V^n}{n} \qquad \text{and} \qquad \bar{I}_i^n(t) \triangleq \frac{I_i^n(t)}{n}, \quad \text{for all } i\in\mathcal{S}.$$

Combining these definitions and the asymptotic regime given by condition (15) we obtain the following equivalent formulation

$$\bar{V}^n \triangleq \max_{p_{\mathcal{S}}\in\mathcal{A}} -\mathrm{E}\left[\int_0^T p_{\mathcal{S}}(t)\cdot \mathrm{d}\bar{I}^n_{\mathcal{S}}(t)\right] \tag{16}$$

$$\text{subject to} \qquad \bar{I}_i^n(t) = I_i(0) - \frac{1}{n} D_i\left(n\,\lambda \int_0^t q_i(p_{\mathcal{S}}(\tau))\,\mathrm{d}\tau\right), \quad \text{for all } i\in\mathcal{S}. \tag{17}$$

For any pricing policy $p_{\mathcal{S}}(t)\in\mathcal{A}$ and any product $i\in\mathcal{S}$, the demand intensity process

$$\lambda \int_0^t q_i(p_{\mathcal{S}}(\tau))\,\mathrm{d}\tau$$

is continuous and uniformly bounded in $[0,T]$. Therefore, in the limit as $n\uparrow\infty$ the scaled inventory process $\bar{I}_i^n(t)$ converges (almost surely and uniformly over a compact set) to a process $I_i(t)$ such that

$$\lim_{n\to\infty} \bar{I}_i^n(t) \overset{\text{a.s.}}{=} I_i(t) \ \text{ u.o.c.}, \quad \text{where } I_i(t) = I_i(0) - \lambda\int_0^t q_i(p_{\mathcal{S}}(\tau))\,\mathrm{d}\tau.$$

We do not attempt a formal proof of this convergence as it goes beyond the scope of this paper. For further details on this type of convergence and limiting regimes, the interested reader is referred to Kurtz [12], Mandelbaum and Pats [16], and reference therein.

Under this asymptotic regime, the retailer's pricing problem (2)-(3) reduces to the following deterministic continuous time control problem.

$$V^{\det}(I_{\mathcal{S}}(0)) \triangleq \max_{p_{\mathcal{S}}\in\mathcal{A}} \lambda \int_0^T p_{\mathcal{S}}(t)\cdot q_{\mathcal{S}}(p_{\mathcal{S}}(t))\,\mathrm{d}t \tag{18}$$

$$\text{subject to} \qquad I_i(t) = I_i(0) - \lambda\int_0^t q_i(p_{\mathcal{S}}(\tau))\,\mathrm{d}\tau \quad \text{for all } i\in\mathcal{S}.$$

Note that the optimization problem is autonomous in the sense that the demand rate is constant and the set $\mathcal{A}$ and the functions $\{q_i(p_{\mathcal{S}}) \,:\, i=1,\ldots,N\}$ are independent of the calendar time t. Therefore, we can search for an optimal policy within the family of pricing policies that are constant over time, that is, solving the finite dimensional optimization problem

$$V^{\det}(I_{\mathcal{S}}(0)) \triangleq \max_{p_{\mathcal{S}}\in\mathcal{A}} \lambda T \sum_{i=1}^N p_i\, q_i(p_{\mathcal{S}}) \tag{19}$$

$$\text{subject to} \qquad \lambda T\, q_i(p_{\mathcal{S}}) \le I_i(0) \quad \text{for } i=1,\ldots,N.$$

Similarly to the unlimited supply case, this problem can be re-formulated as a dynamic programming problem. In this limited supply case, the DP recursion is given by,

$$\begin{aligned} V_k^{\text{det}}(p_{k-1}) &= \max_{0 \leq p_k \leq p_{k-1}} \left\{ \lambda T\, p_k\, q_k(p_{k-1}, p_k) + V_{k+1}^{\text{det}}(p_k) \right\} \quad \text{for all } k \in \mathcal{S} \\ \text{subject to} \quad & \lambda T\, q_k(p_{k-1}, p_k) \leq I_k(0), \end{aligned} \tag{20}$$

with boundary conditions,

$$V_k^{\text{det}}(0) = 0, \quad \text{for all } k \in \mathcal{S} \qquad \text{and} \qquad V_{N+1}^{\text{det}}(\bar{p}) = 0, \quad \text{for all } \bar{p} \geq 0.$$

Again, rather than solving the dynamic program, we can use a much simpler line-search algorithm to compute the optimal solution. In fact, the Karush-Kuhn-Tucker (KKT) optimality conditions for problem (19) are (*e.g.*, chapter 4 in Bazaraa et al. [2])

$$\begin{aligned} 0 &= F(p_{i-1}, u_i) - F(p_i, u_i) - (p_i - \nu_i)\, F_p(p_i, u_i) + (p_{i+1} - \nu_{i+1})\, F_p(p_i, u_{i+1}) \quad \text{for all } i = 1, \ldots, N \\ 0 &\leq I_i(0) - \lambda T \left[F(p_{i-1}, u_i) - F(p_i, u_i) \right], \quad \text{for all } i = 1, \ldots, N \\ 0 &= \nu_i \Big(I_i(0) - \lambda T \left[F(p_{i-1}, u_i) - F(p_i, u_i) \right] \Big), \quad \text{for all } i = 1, \ldots, N \\ 0 &\leq \nu_i \quad \text{for all } i = 1, \ldots, N \end{aligned} \tag{21}$$

where ν_i is the lagrangian multiplier for the i^{th} product inventory constraint. The following algorithm characterizes a line-search procedure that simultaneously computes a vector of prices and multipliers that solve the KKT conditions above.

<u>LIMITED INVENTORY ALGORITHM</u>:

Step 1: Set $p_{N+1} = \nu_{N+1} = 0$ and fix $p_N = \bar{p}$.

Step 2: For $i = N, \ldots, 2$ compute p_{i-1} as a function of $\bar{p}$ as follows. Given p_i, p_{i+1} and ν_{i+1}, compute

$$\begin{aligned} \zeta_i &\triangleq \min \left\{ \frac{I_i(0)}{\lambda T} \,,\; p_i\, F_p(p_i, u_i) - (p_{i+1} - \nu_{i+1})\, F_p(p_i, u_{i+1}) \right\} \quad \text{and} \\ \tilde{p} &\triangleq F^{-1}(F(p_i, u_i) + p_i\, F_p(p_i, u_i) - (p_{i+1} - \nu_{i+1})\, F_p(p_i, u_{i+1}) \,,\; u_i) \end{aligned}$$

Then,

$$p_{i-1} = F^{-1}(F(p_i, u_i) + \zeta_i \,,\; u_i) \qquad \text{and} \qquad \nu_i = \frac{F(\tilde{p}, u_i) - F(p_{i-1}, u_i)}{F_p(p_i, u_i)}.$$

Step 3: Optimality check: if there exists an $\nu_1 \geq 0$ such that

$$F(p_0, u_1) - F(p_1, u_1) - (p_1 - \nu_1)\, F_p(p_1, u_1) + (p_2 - \nu_2)\, F_p(p_1, u_2) = 0,$$

$$\lambda\, T[F(p_0, u_1) - F(p_1, u_1)] \leq I_1(0), \quad \text{and} \quad \nu_1 \Big(\lambda\, T[F(p_0, u_1) - F(p_1, u_1)] - I_1(0) \Big) = 0,$$

then the sequences $\{p_1 \ldots, p_N\}$ and $\{\nu_1, \ldots, \nu_N\}$ jointly satisfy the KKT conditions and stop. Otherwise, go to Step 1, change the value of $\bar{p}$ and iterate. □

A couple of observations about this algorithm are in order. First of all, by construction in Step 2 $\tilde{p} \geq p_{i-1}$ which guarantees that $\nu_i \geq 0$. Also from Step 2 note that

$$F(p_{i-1}, u_i) - F(p_i, u_i) = \zeta \geq \frac{I_i(0)}{\lambda T}$$

which guarantees that the inventory constrain for product $i = 2, \ldots, N$ is satisfied. In addition, if the inequality is strict it follows that $\tilde{p} = p_{i-1}$, that is, $\nu_i = 0$ and so the complementary slackness condition is also satisfied for $i = 2, \ldots, N$. From Step 2 and the definitions of $\tilde{p}$, ζ_i, p_{i-1} and ν_i the reader can easily verify that the first KKT optimality condition is also satisfied for $i = 2, \ldots, N$. Finally, the optimality check in Step 3 guarantees that the KKT conditions are also satisfied for $i = 1$. In summary, if the algorithm is able to find a solution, then this solution satisfies the KKT conditions in (21).

The question now is whether there exists a solution to the KKT conditions. The following result provides necessary and sufficient conditions for this to happen. We recall from section 4 the definitions of L and U.

$$L(p, u_1, u_2) \triangleq F(p, u_1) + p\Big(F_p(p, u_1) - F_p(p, u_2)\Big) \text{ and } U(p, u_1) \triangleq F(p, u_1) + p\,F_p(p, u_1).$$

Proposition 5 *A necessary condition for the existence of a solution is that there exists a $\tilde{p}$ such that $U(\tilde{p}, u_1) = F(p_0, u_1)$. On the other hand, suppose that $L(p, u_1, u_2)$ is unimodal. Then, a sufficient condition for the existence of a solution to the KKT conditions in (21) is that there exists a price $\hat{p}$ that solves $F(p_0, u_1) = L(\hat{p}, u_1, u_2)$.*

Proof: See the appendix.

We can use the previous algorithm to get some insights about the effects that a limited inventory has on an optimal pricing strategy. From step 2, we can see that as the inventory of product i decreases the prices of product i and $i-1$ get closer. In the limit, as $I_i(0)$ goes to zero, the price of product i converges to the price of product $i-1$ ($p_i \uparrow p_{i-1}$). The intuition is simple, products with small inventory have a high chance of stocking out during the selling season. In order to mitigate this effect, the seller raises the price of these products close to the next best alternative reducing demand.

We next present a few numerical examples that highlight this and other effects of the inventory constraints. For this purpose, just as we did in §4, we consider an average arrival of 100 customers over the selling horizon ($D = 100$). Customers' type is represented by a bivariate normal distribution with mean $(\mu_w, \mu_{u_0}) = (1, 1)$, variance $(\sigma_w^2, \sigma_{u_0}^2) = (0.5, 0.4)$, and coefficient of correlation $\rho = 0.5$. The quality of products is assumed to be evenly distributed between 0.5 and 3.

Figure 6 presents optimal revenues as a function of the initial stock of a certain product. In this numerical exercise we consider a family of 5 substitute products, where all products, with the exception of product 3, have unlimited supply.

The curve that appears in this figure has the expected form. It presents an increasing monotonicity with decreasing marginal increments. The maximum revenue is limited by the optimal non-limited

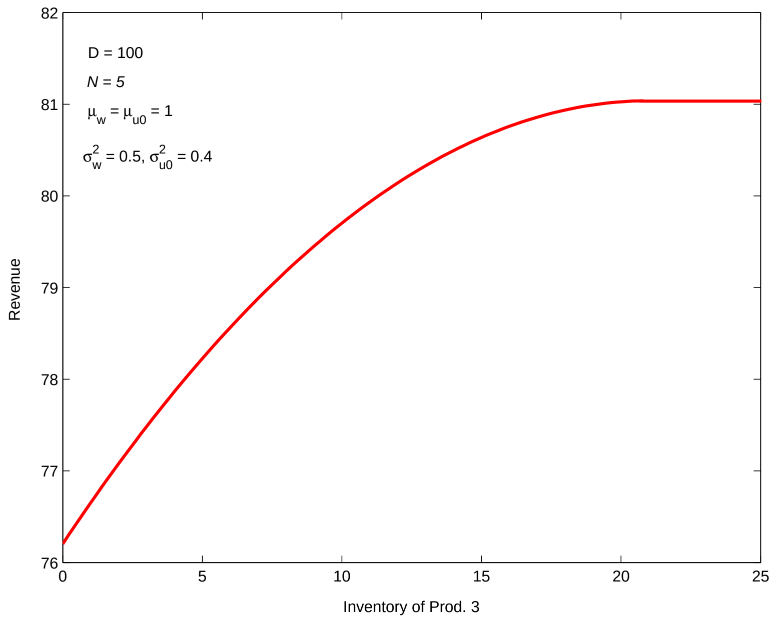

Figure 6: *Revenues as a function of the starting inventory.*

selling amount (in this particular case around 20 units). It is interesting to observe that the first 15 units are responsible for more than 90% of the potential revenue attributable to product 3.

Figure 7 studies the effect over pricing policies, and selling amounts, generated by the presence of certain products with limited inventory. We consider a family of 10 substitute products, where products 3,6 and 9 have a limited stock with only one unit of inventory (the rest of the products have unlimited supply).

As shown in Figure 7(a), the limited inventory price curve follows a stepwise form. Products with limited inventory present price increments so as to match the demand with the scarce available supply. Products with immediately higher quality than the limited ones, have prices slightly above those of their restricted neighbors, so as to satisfy part of the unsatisfied demand due to stock-outs.

Figure 7(b) compares the optimal selling amounts of the limited and unlimited supply settings. This plot shows an increment on the selling quantities of all the products, except for those with limited inventory. This behavior is rather intuitive; when a certain product is limited, part of the customers' demand for this product will be absorbed by higher or lower quality products.

Similarly to §4.1, let us conclude this section discussing a first-order approximation for problem 19.

5.1 First-Order Approximation

In this section, we use a first-order Taylor expansion to approximate p_i. From step 2 in the *Limited Inventory* algorithm it follows that $p_{i-1} = F^{-1}(F(p_i, u_i) + \zeta_i, u_i)$. Using a first-order approximation of $F^{-1}(x, u_i)$ around $x = F(p_i, u_i)$ and the definition of ζ_i we get

$$p_{i-1} \approx p_i + \frac{\zeta_i}{F_p(p_i, u_i)} = p_i + \min\left\{ \frac{I_i(0)}{\lambda T\, F_p(p_i, u_i)}\ ,\ p_i - (p_{i+1} - \nu_{i+1})\, \frac{F_p(p_i, u_{i+1})}{F_p(p_i, u_i)} \right\}.$$

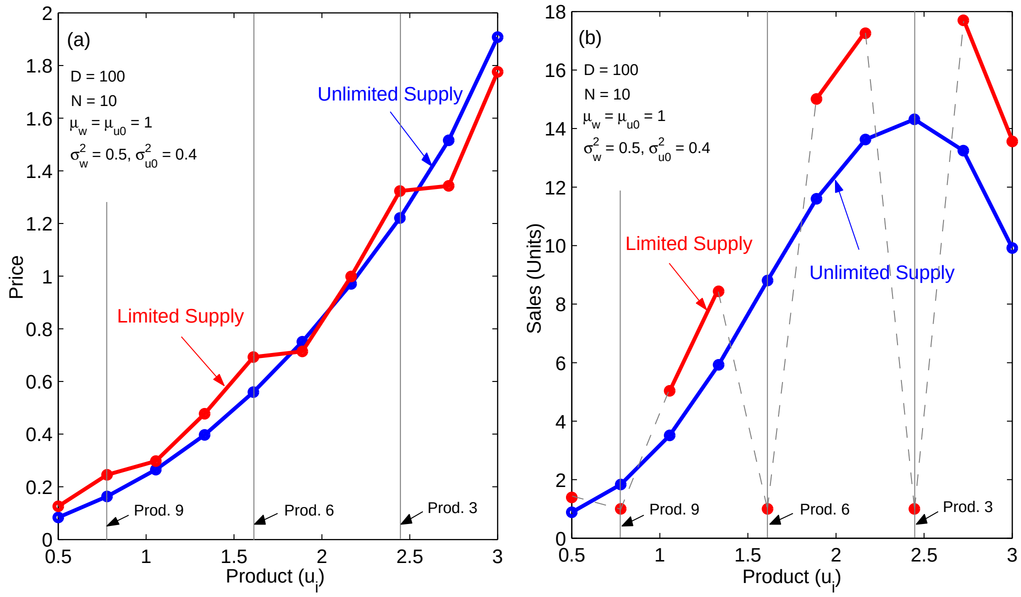

Figure 7: *Effect of limited inventory for certain products ($I_3(0) = I_6(0) = I_9(0) = 1$, $I_1(0) = I_2(0) = I_4(0) = I_5(0) = I_7(0) = I_8(0) = I_{10}(0) = \infty$) over (a) pricing policies, and (b) selling amounts.*

Like §4.1, we use the fact that $\frac{F_p(p_i,u_{i+1})}{F_p(p_i,u_i)} \in [0,1]$ to get the following bounds for p_{i-1}.

$$p_i + \min\{\frac{I_i(0)}{\lambda T\, F_p(p_i,u_i)}\ ,\ p_i - p_{i+1}\} \leq p_{i-1} \leq p_i + \min\{\frac{I_i(0)}{\lambda T\, F_p(p_i,u_i)}\ ,\ p_i\}. \tag{22}$$

A couple of observation about the bounds in (22) are in order. First note that if the inventory of product i is zero then $p_i = p_{i-1}$, that is, the bounds are tight. On the other hand, if the inventory of product i is large enough then the bounds in (22) coincide with those obtained in section 4.1.

In what follows, we present some numerical experiments that show the quality of this approximation, and its adequacy in the implementation of a simple pricing methodology. In these experiments we consider a family of 10 substitute products, and assume all of these have unlimited inventory levels, except for products 3, 6 and 9, which have a single unit only. For the ease of notation, we define $\underline{p}_i = p_{i+1} + \min\{\gamma_{i+1}, p_{i+1} - p_{i+2}\}$ and $\overline{p}_i = p_{i+1} + \min\{\gamma_{i+1}, p_{i+1}\}$, where $\gamma_i = \frac{I_i(0)}{\lambda T\, F_p(p_i,u_i)}$. Note that γ_i can be considered as the *clearing price* of product i.

Table 3 analyzes the approximation quality of the bounds of optimal prices under the same two customer segmentation schemes presented in §4.1: a *bivariate weibull distribution* and a *bivariate normal distribution*. The first three columns of the table characterize the product according to its quality (utility) and inventory level. The following eight columns present the optimal selling level, the optimal price (p_i^*), and the lower and upper bounds ($\underline{p}_i$ and $\overline{p}_i$), for each of the ten products under both segmentation settings.

The results shown in this table indicate that the quality of the approximation presented in (22) is satisfying under both studied distributions. Except for products 1 and 9 in the bivariate normal case, optimal prices are contained within the approximated bounds. This approximation will be useful, however, in the way it can be adequately implemented in a pricing methodology. To analyze this

Product			Bivariate Weibull				Bivariate Normal			
i	u_i	$I_i(0)$	$Sales_i^*$	p_i^*	$\underline{p}_i$	$\overline{p}_i$	$Sales_i^*$	p_i^*	$\underline{p}_i$	$\overline{p}_i$
1	3.00	∞	7.88	2.28	1.51	2.41	13.56	1.78	1.36	1.74
2	2.72	∞	11.32	1.46	1.46	1.46	17.70	1.34	1.34	1.34
3	2.44	1	1.00	1.41	1.29	1.55	1.00	1.32	1.29	1.36
4	2.17	∞	11.94	0.97	0.67	1.07	17.26	1.00	0.73	1.11
5	1.89	∞	12.61	0.65	0.65	0.65	15.01	0.71	0.71	0.71
6	1.61	1	1.00	0.63	0.58	0.75	1.00	0.69	0.66	1.03
7	1.33	∞	11.02	0.41	0.25	0.52	8.44	0.48	0.35	1.07
8	1.06	∞	10.54	0.24	0.24	0.24	5.04	0.30	0.30	0.30
9	0.78	1	1.00	0.23	0.20	0.36	1.00	0.25	0.26	1.93
10	0.50	∞	8.14	0.10	-	-	1.40	0.13	-	-

Table 3: *Numerical Optimization of 10 substitute products. Two distributions are considered: i) bivariate weibull distribution with scale param. $\theta_w = \theta_{u_0} = 1$, shape param. $\gamma_w = \gamma_{u_0} = 1$, and correlation param. $\delta = 0.5$; ii) bivariate normal distribution with mean $\mu_w = \mu_{u_0} = 1$, variance $\sigma_w^2 = 0.5$ and $\sigma_{u_0}^2 = 0.4$, and coefficient of correlation $\rho = 0.5$.*

issue, we proceed in a similar way as we did in §4.1. First, let us define the approximated price for product i as $\tilde{p}_i(\alpha) \triangleq \alpha\,\underline{p}_i + (1-\alpha)\,\overline{p}_i$, where $\alpha \in [0,1]$ is a fixed constant.

To obtain the set of approximated prices, $\tilde{p}_i$, we require a fixed value for $\tilde{p}_N$. To do so, a possible approach is to consider the solution of the unlimited case with a uniform distribution. Once $\tilde{p}_N$ has been established, it is possible to compute $\tilde{p}_i$ for $i = N-1, ..., 1$. In Table 4 we compare optimal revenues with those obtained applying the pricing strategy just described. Let $\mathcal{R}$ be the ratio between the revenues generated by using $\tilde{p}_S$ and the optimal revenue.

Just as we did in the experiments of the previous section, we assume customers are characterized by a bivariate normal distribution with $\mu_w = \mu_{u_0} = 1$ and $\sigma_w = \sigma_{u_0} = 1$. We analyze three different levels of correlation ($\rho = 0.1, 0.5, 0.9$) for a family of 10 substitute products, whose quality (utility) is randomly distributed over $[\mu_{u_0} - \sigma_{u_0}\,,\, \mu_{u_0} + \sigma_{u_0}]$. For each of these three values of ρ, we consider two different α's, $\alpha = 1.0$ and $\alpha = 0.7$. For each of the studied combinations, a set of 100 random instances of product quality were generated to compute the mean and standard deviation of $\mathcal{R}$ ($\mathcal{R}_{mean}$ and $\mathcal{R}_{std}$, respectively).

As mentioned earlier, the price of product N has to be assigned exogenously. To do so, we use an approximation which is calculated using the unlimited case with the following bivariate uniform distribution

$$\mathbb{P}[w \le p, u_0 \le u] = \left(\frac{p - (\mu_w - \sigma_w)}{2\sigma_w}\right)\left(\frac{u - (\mu_{u_0} - \sigma_{u_0})}{2\sigma_{u_0}}\right).$$

The price of product N generated in this way is denominated p_N^{unif}.

	$\tilde{p}_N = p_N^{unif}$			
	$\alpha = 1.0$		$\alpha = 0.7$	
ρ	$\mathcal{R}_{mean}$	$\mathcal{R}_{std}$	$\mathcal{R}_{mean}$	$\mathcal{R}_{std}$
0.1	.7667	.0214	.8766	.0283
0.5	.8030	.0249	.8479	.0427
0.9	.8418	.0318	.7618	.0754

Table 4: *Revenues applying the approximation $\tilde{p}_S$ versus optimal revenues in the limited inventory case (only products 3,6 and 9 are limited with a single unit).*

The results shown in Table 4, though not as good as those presented in the previous section, are still quite promising (the lowest $\mathcal{R}_{mean}$ is above 0.75, while the highest is almost 0.88). It is also possible to observe, that in this limited case, the value of α is not as important as in the unlimited case (at least, under the studied conditions). This last result is quite reasonable since, as shown in Table 3, neither of both bounds is clearly more accurate than the other. Considering these results, we believe the methodology presented here constitutes a very efficient and robust way to establish prices under a limited inventory setting, specially when the demand distribution is not known.

6 Conclusions

This paper has studied the problem of optimal pricing perishable products with demand substitution. We provided an original demand model, denominated WAL model, that allows for an adequate characterization of customers' purchasing decisions under a price and inventory driven substitution environment. Some of the properties presented by this demand model, such as the ability of establishing a ranking among products, and its greedy resolution nature, overcome most of the limitations inherent to other commonly used models (e.g. MNL). Based on this demand model, we formulated the retailer's pricing problem as a stochastic control problem, and identified three main factors that account for the complexity of its resolution: (i) the inventory constraints, (ii) the stochastic nature of the problem, and (iii) the presence of inventory driven substitution. To deal with these difficulties, we considered two different asymptotic approximations that permitted us the attainment of further insights.

Our first asymptotic approximation, the unlimited supply case, overcomes the presence of inventory constraints and spill-over effects. In this infinite inventory setting, we proposed an efficient line-search procedure to obtain optimal prices, and derived properties of optimal prices through a first-order Taylor approximation of $F^{-1}(p, u)$. These properties do not only establish a characterization of optimal prices, but can be used to define pricing policies in an efficient and distribution-free way, which constitutes an attractive methodology to support managerial decisions. The adequacy of these properties were tested through a set of computational experiments. The second asymptotic approximation consisted in modeling demand in a deterministic fluid-like way, which overcomes the effects of stock-outs and the stochastic nature of the problem. We showed that this deterministic continuous approximation results as the limit when the volume of expected sales and initial inventories grow proportionally large. We presented a line-search procedure that allows for an efficient resolution of the resulting KKT optimality conditions. Finally, as in the unlimited case, we developed a distribution-free pricing methodology, and tested its adequacy with a set of computational experiments.

An interesting direction for future research is to study the problem of jointly optimizing pricing and assortment policies. The optimal assortment problem with substitute products, as indicated in §1, has received plenty of attention (e.g., van Ryzin and Mahajan [21], Smith and Agrawal [20], and Mahajan and van Ryzin [15]). However, all these works have excluded pricing decisions in their analysis, assuming prices are determined exogenously. In this way, the problem of optimizing pricing and assortment decisions together under the WAL model, is a natural extension of our work that could be explored.‡

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Appendix

Proof of Proposition 1:

Consider an arbitrary pricing policy $p_{\mathcal{S}}(t)$ together with its corresponding inventory process $I_{\mathcal{S}}(t)$. Based on $p_{\mathcal{S}}(t)$ and $I_{\mathcal{S}}(t)$ we will define a new price process $\tilde{p}_{\mathcal{S}}(t) \in \mathcal{P}_{\mathcal{S}}$ with an associated inventory process $\tilde{I}_{\mathcal{S}}(t)$ such that

$$\int_0^T p_{\mathcal{S}}(t) \cdot \mathrm{d}I_{\mathcal{S}}(t) = \int_0^T \tilde{p}_{\mathcal{S}}(t) \cdot \mathrm{d}\tilde{I}_{\mathcal{S}}(t). \tag{23}$$

This result ensures that we can restrict the search for an optimal pricing policy to the set $\mathcal{P}_{\mathcal{S}}$. Let us fix a time $t \in [0, T]$ and define $\tilde{p}_0(t) \triangleq \max\{p_j(t) : j \in \mathcal{S}\}$. We construct the price vector $\tilde{p}_{\mathcal{S}}(t)$ using the following algorithm:

1. Set $i = 1$.
2. If $I_i(t) = 0$ then set $\tilde{p}_i(t) = \tilde{p}_{i-1}(t)$. Otherwise, if $I_i(t) > 0$ then $\tilde{p}_i(t) = \min\{\tilde{p}_{i-1}(t), p_i(t)\}$.
3. Set $i = i + 1$. If $i < N + 1$ goto 2, otherwise stop.

It follows from step 2 that the resulting price process $\tilde{p}_{\mathcal{S}}(t)$ belongs to $\mathcal{P}_{\mathcal{S}}$. In order to prove that condition (23) holds, we will show that for every realization of the demand process, $I_{\mathcal{S}}(t) = \tilde{I}_{\mathcal{S}}(t)$ and $p_{\mathcal{S}}(t) \cdot \mathrm{d}I_{\mathcal{S}}(t) = \tilde{p}_{\mathcal{S}}(t) \cdot \mathrm{d}\tilde{I}_{\mathcal{S}}(t)$, for every t.

Let us first suppose by contradiction that $I_{\mathcal{S}}(t) \neq \tilde{I}_{\mathcal{S}}(t)$. Then there exists a time $\tau \in (0, T]$ such that $\tau = \inf\{t > 0 \,:\, \mathrm{d}I_{\mathcal{S}}(t) \neq \mathrm{d}\tilde{I}_{\mathcal{S}}(t)\}$. At this time τ there is a customer arriving and buying some product i. By construction $\tilde{p}_{\mathcal{S}}(\tau) \leq p_{\mathcal{S}}(\tau)$ and so we must have $\mathrm{d}I_{\mathcal{S}}(\tau) = 0$ and $\mathrm{d}\tilde{I}_{\mathcal{S}}(\tau) = -1$. In other words, at time $\tau-$ the inventory levels under $p_{\mathcal{S}}(t)$ and $\tilde{p}_{\mathcal{S}}(t)$ are the same and at time τ they differ. This means that the customer who arrives at time τ is willing to purchase a product i under $\tilde{p}_{\mathcal{S}}(\tau)$ but he/she is not willing to purchase i under $p_{\mathcal{S}}(\tau)$. For this event to happen, we must have that $I_i(\tau-) > 0$ and $\tilde{p}_i(\tau) < p_i(\tau)$. So by step 2 of the algorithm we must have that $\tilde{p}_i(\tau) = \tilde{p}_{i-1}(\tau) < p_i(\tau)$. But if this is the case, it must be that $\tilde{I}_{i-1}(\tau-) = 0$, otherwise the arriving buyer would prefer to get product $i-1$ instead of product i since they are selling at the same price $\tilde{p}_i(\tau) = \tilde{p}_{i-1}(\tau)$. But by the definition of τ, $\tilde{I}_{i-1}(\tau-) = I_{i-1}(\tau-)$ and so by step 2 it follows that $\tilde{p}_{i-1}(\tau) = \tilde{p}_{i-2}(\tau)$. Again, this condition implies that $\tilde{I}_{i-2}(\tau-) = 0$, otherwise the arriving buyer would have purchased product $i-2$ instead of i. Applying this reasoning recursively, we can show that if the arriving customer purchases product i then $\tilde{I}_j(\tau-) = I_j(\tau-) = 0$ for all $j = 1, \ldots, i-1$. But if this is the case, step 2 in the algorithm implies that $\tilde{p}_j(\tau) = \tilde{p}_0(\tau)$ for $j = 1, \ldots, i$. But this contradicts our previous conclusion that $\tilde{p}_i(t) < p_i(t)$ since $\tilde{p}_i(\tau) \geq p_i(\tau)$. Hence, we conclude that $I_{\mathcal{S}}(t) = \tilde{I}_{\mathcal{S}}(t)$ for all t.

Finally, we prove that $p_{\mathcal{S}}(t) \cdot \mathrm{d}I_{\mathcal{S}}(t) = \tilde{p}_{\mathcal{S}}(t) \cdot \mathrm{d}\tilde{I}_{\mathcal{S}}(t)$. The condition holds trivially if $\mathrm{d}\tilde{I}_{\mathcal{S}}(t) = \mathrm{d}I_{\mathcal{S}}(t) = 0$. Now suppose that at time t there is a sale such that $\mathrm{d}\tilde{I}_i(t) = \mathrm{d}I_i(t) = -1$ and $p_i(t) > \tilde{p}_i(t)$. This condition requires that $I_i(t-) > 0$ and by step 2 of the algorithm it also implies $\tilde{p}_i(t) = \tilde{p}_{i-1}(t)$. Once again, we can use the same argument of the previous paragraph to conclude that we must have $\tilde{p}_j(t) = \tilde{p}_0(t)$ for all $j = 1, \ldots, i$, which contradicts $p_i(t) > \tilde{p}_i(t)$. Hence, it follows that $p_{\mathcal{S}}(t) \cdot \mathrm{d}I_{\mathcal{S}}(t) = \tilde{p}_{\mathcal{S}}(t) \cdot \mathrm{d}\tilde{I}_{\mathcal{S}}(t)$. $\square$

Proof of Proposition 2:

To prove that $W_k(p_{k-1})$ is non-decreasing in p_{k-1} we simply use the definition and backward induction over k. To prove the monotonicity of $p_k^*(p_{k-1})$, let us introduce the change of variable $z_{k-1} = F(p_{k-1}, u_k)$ and the auxiliary function

$$G_k(p) = W_{k+1}(p) - p\, F(p, u_k).$$

Using a slight abuse of notation, we define the value function $W_k(z_{k-1})$ and optimal price $p_k^*(z_{k-1})$ as follows

$$W_k(z_{k-1}) = \max\left\{G_k(p) + p\, z_{k-1} \;:\; 0 \leq p \leq F^{-1}(z_{k-1}, u_k)\right\}$$

and

$$p_k^*(z_{k-1}) = \operatorname{argmax}\left\{G_k(p) + p\, z_{k-1} \;:\; 0 \leq p \leq F^{-1}(z_{k-1}, u_k)\right\}.$$

Take $z_1 \geq z_2$, let us prove that $p_k^*(z_1) \geq p_k^*(z_2)$. In fact, for all $0 \leq p \leq F^{-1}(z_2, u_k)$

$$G_k(p) + p\, z_2 \leq G_k(p_k^*(z_2)) + p_k^*(z_2)\, z_2 \leq G_k(p_k^*(z_2)) + p_k^*(z_2)\, z_1.$$

The first inequality follows from the optimality of $p_k^*(z_2)$. The second inequality follows the non-negativity of $p_k^*(z_2)$ and the assumption $z_1 \geq z_2$. Therefore, $p_k^*(z_2)$ maximizes $G_k(p) + p\, z_1$ in the range

$0 \leq p \leq F^{-1}(z_2, u_k)$. Since $F^{-1}(z_2, u_k) \leq F^{-1}(z_1, u_k)$ it follows that the optimizer of $G_k(p) + p\, z_1$ in the range $0 \leq p \leq F^{-1}(z_1, u_k)$ is greater than or equal to $p_k^*(z_2)$. In other words, $p_k^*(z_2) \leq p_k^*(z_1)$. Finally, since $z_{k-1} = F(p_{k-1}, u_k)$ is increasing in p_{k-1}, we conclude that $p_k^*(p_{k-1})$ is non-decreasing in p_{k-1}. $\square$

Proof of Proposition 3:
We first note that for $\bar{p} = 0$ we have that $p_i(\bar{p}) = 0$ and for $\bar{p} \to \infty$ we have that $p_i(\bar{p}) \to \infty$, for all $i = 1, \ldots, N$. Thus, by the continuity of the distribution function $F(p, u)$, the range of the mapping $p_i(\bar{p})$ is the entire $\mathbb{R}_+$. This observation implies that the issue of existence reduces to find a price $\bar{p} \in \mathbb{R}_+$ such that condition (11) is satisfied.

Note that $p_{\mathcal{S}}(\bar{p}) \in \mathcal{P}_{\mathcal{S}}$ and so we can bound the right-hand side in (11) as follows.

$$L(p_1(\bar{p}), u_1, u_2) \leq F(p_1(\bar{p}), u_1) + p_1(\bar{p})\, F_p(p_1(\bar{p}), u_1) - p_2(\bar{p})\, F_p(p_1(\bar{p}), u_2) \leq U(p_1(\bar{p}), u_1). \tag{24}$$

Since the range of $p_1(\bar{p})$ is $\mathbb{R}_+$, a sufficient condition for the existence of a solution to condition (11) is that there exists a price $\hat{p} \in \mathbb{R}_+$ such that $L(\hat{p}, u_1, u_2) = F(p_0, u_1)$. On the other hand, a necessary condition for the existence of a solution is that there exists a $\tilde{p} \in \mathbb{R}_+$ such that $U(\tilde{p}, u_1) = F(p_0, u_1)$.

Finally, the prove that the sequences $\{p_i^{\min}\}$ and $\{p_i^{\max}\}$ are effectively lower and upper bounds on the solutions to condition (8) follows by induction over $i = 1, 2, \ldots, N$ together with the inequalities

$$\begin{aligned} F(p_{i-1}^*, u_i) &\geq F(p_{i-1}^{\min}, u_i) = U(p_i^{\min}, u_i) > U(p_i, u_i) \\ &\geq F(p_i, u_i) + p_i\, F_p(p_i, u_i) - p_{i+1}\, F_p(p_i, u_{i+1}), \quad \text{for all } p_{i+1} \leq p_i < p_i^{\min} \end{aligned}$$

and

$$\begin{aligned} F(p_{i-1}^*, u_i) &\leq F(p_{i-1}^{\max}, u_i) = L(p_i^{\max}, u_i, u_{i+1}) < L(p_i, u_i, u_{i+1}) \\ &\leq F(p_i, u_i) + p_i\, F_p(p_i, u_i) - p_{i+1}\, F_p(p_i, u_{i+1}), \quad \text{for all } p_{i+1} \leq p_i \text{ and } p_i^{\max} < p_i. \ \square \end{aligned}$$

Proof of Proposition 5:
As in the proof of proposition 3, we note that the prices p_i and lagrangian multipliers ν_i satisfy

- For $\bar{p} = 0$, $p_i(\bar{p}) = 0$ for all $i = 1, \ldots, N$ and $\nu_i(\bar{p}) = 0$ for all $i = 2, \ldots, N$.
- As $\bar{p} \uparrow \infty$, $p_i(\bar{p}) \uparrow \infty$ for all $i = 1, \ldots, N$ and $\nu_i(\bar{p}) \downarrow 0$ for all $i = 2, \ldots, N$.

Therefore, by the continuity of the distribution function $F(p, u)$, we conclude that the range of the mapping $p_i(\bar{p})$ $(i = i, \ldots, N)$ is the entire $\mathbb{R}_+$. This observation implies that the problem of existence reduces to find a price $\bar{p} \in \mathbb{R}_+$ such that the conditions in Step 3 of the algorithm are satisfied.
First of all, to ensure feasibility we need to select $\bar{p}$ in such a way that

$$\lambda\, T\, [F(p_0, u_1) - F(p_1(\bar{p}), u_1)] \leq I_1(0) \quad \Longleftrightarrow \quad p_1(\bar{p}) \geq F^{-1}\left(F(p_0, u_1) - \frac{I_1(0)}{\lambda T}\, u_1 \right).$$

Since the right-hand side of the inequality is a constant, we can always select a $\bar{p}$ that will satisfy this feasibility condition. Let us define $\bar{p}^{\min}$ to be the price $\bar{p}$ for which

$$p_1^{\min} \triangleq p_1(\bar{p}^{\min}) = F^{-1}\left(F(p_0, u_1) - \frac{I_1(0)}{\lambda T}\, u_1\right).$$

Suppose that $\hat{p} \le p_1^{\min}$ then because of the unimodal assumption on $L(p, u_1, u_2)$, it is not hard to see that

$$F(p_0, u_1) - L(p_1^{\min}, u_1, u_2) = F(p_0, u_1) - F(p_1^{\min}, u_1) - p_1^{\min}\, F_p(p_1^{\min}, u_1) + p_1^{\min}\, F_p(p_1^{\min}, u_2) \le 0.$$

In addition, since $p_1^{\min} \ge p_2(\bar{p}^{\min})$ and $\nu_2(\bar{p}^{\min}) \ge 0$ we conclude that

$$F(p_0, u_1) - F(p_1^{\min}, u_1) - p_1^{\min}\, F_p(p_1^{\min}, u_1) + (p_2(\bar{p}^{\min}) - \nu_2(\bar{p}^{\min}))\, F_p(p_1^{\min}, u_2) \le 0.$$

Therefore, we can find an $\nu_1 \ge 0$ such that

$$F(p_0, u_1) - F(p_1^{\min}, u_1) - (p_1^{\min} - \nu_1)\, F_p(p_1^{\min}, u_1) + (p_2(\bar{p}^{\min}) - \nu_2(\bar{p}^{\min}))\, F_p(p_1^{\min}, u_2) = 0$$

and so by choosing $\bar{p} = \bar{p}^{\min}$ the algorithm generates a solution to the KKT condition.
Suppose, on the other hand, that $\hat{p} \ge p_1^{\min}$ then we can repeat the previous argument but replacing $p_1^{\min}$ by $\hat{p}$.
To prove the necessity part, we simply notice that

$$F(p_0, u_1) - U(p_1, u_1) \le F(p_0, u_1) - F(p_1, u_1) - (p_1 - \nu_1)\, F_p(p_1, u_1) + (p_2 - \nu_2)\, F_p(p_1, u_2)$$

for all $p_1 \ge p_2$ and $\nu_1 \ge 0$ and $p_2 \ge \nu_2 \ge 0$. The fact that $p_2 \ge \nu_2$ follows from Step 2 in the algorithm. Therefore, if there is a solution to the KKT conditions then the right-hand side of the inequality would be equal to zero for this solution. This implies that the left-hand side would less than or equal to zero for this solution. But for $p_1 = 0$ the left hand side is strictly positive. Thus by continuity we conclude that there has to be a $\tilde{p}$ such that $F(p_0, u_1) = U(\tilde{p}, u_1)$. $\square$