Coordinating Inventory Control and Pricing Strategies with Random Demand and Fixed Ordering Cost: the Infinite Horizon Case[1]

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Abstract

We analyze an infinite horizon, single product, periodic review model in which pricing and production/inventory decisions are made simultaneously. Demands in different periods are identically distributed random variables that are independent of each other and their distributions depend on the product price. Pricing and ordering decisions are made at the beginning of each period and all shortages are backlogged. Ordering cost includes both a fixed cost and a variable cost proportional to the amount ordered. The objective is to maximize expected discounted, or expected average profit over the infinite planning horizon. We show that a stationary (s, S, p) policy is optimal for both the discounted and average profit models with general demand functions. In such a policy, the period inventory is managed based on the classical (s, S) policy and price is determined based on the inventory position at the beginning of each period.

1 Introduction

In recent years, scores of retail and manufacturing companies have started exploring innovative pricing strategies in an effort to improve their operations and ultimately the bottom line. Firms are employing methods such as dynamically adjusting price over time based on inventory levels or production schedules as well as segmenting customers based on their sensitivity to price and lead time.

For instance, no company underscores the impact of the Internet on product pricing strategies more than Dell Computers. The exact same product is sold at different prices on Dell's Web site, depending on whether the purchase is made by a private consumer, a small, medium or large business, the federal government, an education or health care provider. A more careful review of Dell's strategy, see [1], suggests that even the price of the same product for the same industry is not fixed; it may change significantly over time.

Dell is not alone in its use of a sophisticated pricing strategy. Consider:

- Boise Cascade Office Products sells many products on-line. Boise Cascade states that prices for the 12,000 items ordered most frequently on-line might change as often as daily. [11].
- Ford Motor Co. uses pricing strategies to match supply and demand and target particular customer segments. Ford executives credit the effort with \$3 billion in growth between 1995 and 1999. [12].

These developments call for models that integrate production decisions, inventory control and pricing strategies. Such models and strategies have the potential to radically improve supply chain

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efficiencies in much the same way as revenue management has changed the airline industry, see Belobaba [2] or McGill and van Ryzin [13]. Indeed, in the airline industry, revenue management provided growth and increased revenue by 5%, see Belobaba. In fact, if it were not for the combined contributions of revenue management and airline schedule planning systems, American Airlines (Cook [5]) would have been profitable only one year in the decade beginning in 1990. In the retail industry, to name another example, dynamically pricing commodities can provide significant improvements in profitability, as shown by Gallego and van Ryzin [8].

The coordination of replenishment strategies and pricing policies has been the focus of many papers, starting with the work of Whitin [18] who analyzed the celebrated newsvendor problem with price dependent demand. For a review, the reader is referred to Eliashberg and Steinberg [6], Petruzzi and Dada [14], Federgruen and Heching [7] or Chan, Simchi-Levi and Swann [3].

Recently, Chen and Simchi-Levi [4] considered a finite horizon, periodic review, single product model with stochastic demand. Demands in different periods are independent of each other and their distributions depend on the product price. Pricing and ordering decisions are made at the beginning of each period, and all shortages are backlogged. The ordering cost includes both a fixed cost and a variable cost proportional to the amount ordered. Inventory holding and shortage costs are convex functions of the inventory level carried over from one period to the next. The objective is to find an inventory policy and pricing strategy maximizing expected profit over the finite horizon.

Chen and Simchi-Levi proved that when the demand process is additive, i.e., the demand process has two components, a deterministic part which is a function of the price and an additive random perturbation, an (s, S, p) policy is optimal. In such a policy the inventory strategy is an (s, S) policy: If the inventory level at the beginning of period t is below the reorder point, s_t, an order is placed to raise the inventory level to the order-up-to level, S_t. Otherwise, no order is placed. Price depends on the initial inventory level at the beginning of the period. Unfortunately, for general demand models, including multiplicative demand processes, Chen and Simchi-Levi showed that the (s, S, p) policy is not necessarily optimal. To characterize the optimal policy in this case, Chen and Simchi-Levi developed a new concept, the symmetric k-convexity, and employed it to prove that for general demand processes, an (s, S, A, p) policy is optimal. In such a policy, the optimal inventory strategy at period t is characterized by two parameters (s_t, S_t) and a set $A_t \in [s_t, (s_t + S_t)/2]$, possibly empty depending on the problem instance. When the inventory level x_t at the beginning of period t is less than s_t or $x_t \in A_t$, an order of size $S_t - x_t$ is made. Otherwise, no order is placed. Price depends on the initial inventory level at the beginning of the period.

In this paper we analyze the corresponding infinite horizon models under both the discounted and average profit criteria. We make similar assumptions as in Chen and Simchi-Levi [4] except that here all input parameters, i.e., demand processes, costs and revenue functions, are assumed to be time independent. Surprisingly, by employing the symmetric k-convexity concept developed in Chen and Simchi-Levi [4], we establish that a stationary (s, S, p) policy is optimal for both additive demand and general demand processes under the discounted and average profit criteria. Our approach is motivated by the classic papers by Iglehart [9, 10], Veinott [19] and Zheng [20].

The paper is organized as follows. In Section 2 we review the main assumptions of our model and the concepts of k-convexity and symmetric k-convexity. We start in Section 3 by identifying properties of the best (s, S) inventory policy for both the discounted and average profit cases. These properties, together with the concept of symmetric k-convexity, enable us to construct solutions for the optimality equations of the discounted and average profit problems. In Section 4, we prove some useful bounds on the reorder level and order-up-to level for a corresponding finite horizon problem. In

Section 5 and Section 6, we apply these bounds and the optimality equations to prove the optimality of a stationary (s, S, p) policy for the infinite horizon problems with the discounted and average profit criteria, respectively. Finally, in Section 7 we provide concluding remarks.

2 The Model

Consider a firm that has to make production and pricing decisions over an infinite time horizon with stationary demand process, costs and revenue functions. For each period t, let

d_t = demand in period t
p_t = selling price in period t
$\underline{p}, \bar{p}$ are the common lower and upper bounds on p_t, respectively.

Throughout this paper, we concentrate on demand functions similar to those considered in Chen and Simchi-Levi [4]. These demand functions are of the following form:

Assumption 1 *For any t, the demand function satisfies*

$$d_t = D_t(p, \epsilon_t) := \alpha_t D(p_t) + \beta_t, \tag{1}$$

where $\epsilon_t = (\alpha_t, \beta_t)$, and α_t, β_t are two random variables with $\alpha_t \geq 0$, $E\{\alpha_t\} = 1$ and $E\{\beta_t\} = 0$. The random perturbations, ϵ_t, are identically distributed with the same distribution as $\epsilon = (\alpha, \beta)$ and are independent across time. Furthermore, the inverse function of D, denoted by D^{-1}, is continuous and strictly decreasing.

As observe in [4], by scaling and shifting, the assumptions $E\{\alpha_t\} = 1$ and $E\{\beta_t\} = 0$ can be made without loss of generality. A special case of this demand function is the **additive** demand function, where the demand function is of the form $d_t = D(p) + \beta_t$. This implies that only β_t is a random variable while $\alpha_t = 1$. Another special case is a model with the **multiplicative** demand function. In this case, the demand function is of the form $d_t = \alpha_t D(p)$, where α_t is a random variable.

Let x_t be the inventory level at the beginning of period t, just before placing an order. Similarly, y_t is the inventory level at the beginning of period t after placing an order. Lead time is assumed to be zero and hence an order placed at the beginning of period t arrives immediately before demand for the period is realized. The ordering cost function includes both a fixed cost and a variable cost and is calculated for every t, $t = 1, 2, \ldots$, as

$$k\delta(y_t - x_t) + c(y_t - x_t),$$

where

$$\delta(u) := \begin{cases} 1, & \text{if } u > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Unsatisfied demand is backlogged. Let x be the inventory level carried over from period t to the next period. Since we allow backlogging, x may be positive or negative. A cost $h(x)$ is incurred at the end of period t which represents inventory holding cost when $x > 0$ and shortage cost if $x < 0$.

Given a discount factor γ with $0 < \gamma \leq 1$, an initial inventory level, $x_1 = x$, and a pricing and replenishment policy, let

$$V_T^\gamma(x) = E\{\sum_{t=1}^{T} \gamma^{t-1}(-k\delta(y_t - x_t) - c(y_t - x_t) - h(x_{t+1}) + p_t D_t(p_t, \epsilon_t))\}, \tag{2}$$

be the T-period total expected discounted profit, where $x_{t+1} = y_t - D_t(p_t, \epsilon_t)$.

In the infinite horizon expected discounted profit model the objective is to decide on ordering and pricing policies so as to maximize

$$\limsup_{T\to\infty} V_T^\gamma(x),$$

for $0 < \gamma < 1$ and any initial inventory level x. Similarly, in the infinite horizon expected average profit model the objective is to maximize

$$\limsup_{T\to\infty} \frac{1}{T} V_T^\gamma(x),$$

for $\gamma = 1$ and any initial inventory level x.

To find the optimal strategy that maximizes (2), let $v_t^\gamma(x)$ be the maximum total expected discounted profit over a t-period planning horizon when we start with an initial inventory level x. A natural dynamic program that can be applied to find the policy maximizing (2) is as follows. For $t = 1, 2, \ldots, T$,

$$v_t^\gamma(x) = cx + \max_{y \geq x, \bar{p}_t \geq p \geq \underline{p}_t} -k\delta(y - x) + f_t^\gamma(y, p) \tag{3}$$

with $v_0^\gamma(x) = 0$ for any x, where

$$f_t^\gamma(y, p) := -cy + E\{pD_t(p, \epsilon_t) - h(y - D_t(p, \epsilon_t)) + \gamma v_{t-1}^\gamma(y - D_t(p, \epsilon_t))\}.$$

For the general demand functions (1), we can present the formulation (3) only with respect to expected demand rather than with respect to price. Note that there is a one-to-one correspondence between the selling price $p_t \in [\underline{p}, \bar{p}]$ and the expected demand $D(p_t) \in [\underline{d}, \bar{d}]$, where

$$\underline{d} = D(\bar{p}) \text{ and } \bar{d} = D(\underline{p}).$$

We denote the expected demand at period t by $d = D(p)$. Also let

$$\phi_t^\gamma(x) = v_t^\gamma(x) - cx, h^\gamma(y) = h(y) + (1 - \gamma)cy, \text{ and } \hat{R}(d) = R(d) - cd,$$

where R is the expected revenue function with

$$R(d) = dD^{-1}(d),$$

which is a function of expected demand d. These functions, $\phi_t^\gamma(x), h^\gamma(y)$ and $\hat{R}(d)$, allow us to transform the original problem to a problem with zero variable ordering cost.

Specifically, the dynamic program (3) can be written as

$$\phi_t^\gamma(x) = \max_{y \geq x} -k\delta(y - x) + g_t^\gamma(y, d_t^\gamma(y)) \tag{4}$$

with $\phi_0^\gamma(x) = -cx$ for any x, where

$$g_t^\gamma(y,d) = H^\gamma(y,d) + \gamma E\{\phi_{t-1}^\gamma(y - \alpha_t d - \beta_t)\}, \tag{5}$$

$$H^\gamma(y,d) := -E\{h^\gamma(y - \alpha d - \beta)\} + \hat{R}(d),$$

and

$$d_t^\gamma(y) \in \text{argmax}_{\bar{d} \geq d \geq \underline{d}} g_t^\gamma(y,d). \tag{6}$$

Thus, most of our focus is on the transformed problem (4) which has a similar structure to problem (3). In this transformed problem one can think of h^γ as being the holding and shortage cost function, $\hat{R}$ as being the revenue function, the variable ordering cost is equal to zero, and $\phi_t^\gamma(x)$ is the maximum total expected discounted profit over a t-period planning horizon when starting with an initial inventory level x.

Define

$$Q^\gamma(x) := \max_{\bar{d} \geq d \geq \underline{d}} H^\gamma(x,d). \tag{7}$$

For technical reasons, we need the following assumptions on the revenue function and the holding and shortage cost function.

Assumption 2 *R and $-h$ are concave. The function $Q^\gamma(x)$ is finite for any x. As a consequence $Q^\gamma(x)$ is concave. Furthermore, we assume that,*

$$\lim_{|x| \to \infty} Q^\gamma(x) = \lim_{|x| \to \infty} Q^0(x) = -\infty.$$

The following two concepts, k-convexity and symmetric k-convexity, are important in the analysis of our model. Of course, k-convexity is not a new concept; it was introduced and applied by Scarf [17] for the finite horizon, single product stochastic inventory problem. Here we use the definition of k-convexity, introduced[2] in Chen and Simchi-Levi [4], which is shown to be equivalent to the traditional definition given in [17].

Definition 2.1 *A real-valued function f is called k-convex for $k \geq 0$, if for any $x_0 \leq x_1$ and $\lambda \in [0,1]$,*

$$f((1-\lambda)x_0 + \lambda x_1) \leq (1-\lambda)f(x_0) + \lambda f(x_1) + \lambda k. \tag{8}$$

A function f is called k-concave if $-f$ is k-convex.

The symmetric k-convexity is a new concept introduced in Chen and Simchi-Levi [4].

Definition 2.2 *A real-valued function f is called sym-k-convex for $k \geq 0$, if for any x_0, x_1 and $\lambda \in [0,1]$,*

$$f((1-\lambda)x_0 + \lambda x_1) \leq (1-\lambda)f(x_0) + \lambda f(x_1) + \max\{\lambda, 1-\lambda\}k. \tag{9}$$

A function f is called sym-k-concave if $-f$ is sym-k-convex.

Observe that k-convexity is a special case of symmetric k-convexity. The following lemma describes properties of symmetric k-convex functions, which are introduced and proved in [4].

[2]While completing this paper, Professor Paul Zipkin pointed out to us that this equivalent characterization of k-convexity has appeared in Porteus [15].

Lemma 1 *(a) A real-valued convex function is also sym-0-convex and hence sym-k-convex for all $k \geq 0$. A sym-k_1-convex function is also a sym-k_2-convex function for $k_1 \leq k_2$.*

(b) If $g_1(y)$ and $g_2(y)$ are sym-k_1-convex and sym-k_2-convex respectively, then for $\alpha, \beta \geq 0$, $\alpha g_1(y) + \beta g_2(y)$ is sym-$(\alpha k_1 + \beta k_2)$-convex.

(c) If $g(y)$ is sym-k-convex and w is a random variable, then $E\{g(y-w)\}$ is also sym-k-convex, provided $E\{|g(y-w)|\} < \infty$ for all y.

(d) Assume that g is a continuous sym-k-convex function and $g(y) \to \infty$ as $|y| \to \infty$. Let S be a global minimizer of g and s be any element from the set

$$X := \{x | x \leq S, g(x) = g(S) + k \text{ and } g(x') \geq g(x) \text{ for any } x' \leq x\}.$$

Then we have the following results.

(i) $g(s) = g(S) + k$ and $g(y) \geq g(s)$ for all $y \leq s$.

(ii) $g(y) \leq g(z) + k$ for all y, z with $(s+S)/2 \leq y \leq z$.

3 Preliminaries

Consider a stationary (s, S, p) policy defined by the reorder point s, the order-up-to level S and a price $p(x)$ which is a function of the inventory level x. As pointed out earlier, there is a one-to-one correspondence between price and expected demand through the mapping $d = D(p)$. Hence, from now on we use $(s, S, \mathbf{d})$ and (s, S, p) interchangeably.

Given the stationary $(s, S, \mathbf{d})$ policy chosen above, let $I^\gamma(s, x, \mathbf{d})$ be the expected γ-discounted profit incurred during a horizon that starts with initial inventory level x and ends, at this period or a later period, with an inventory level no more than s. Let $M^\gamma(s, x, \mathbf{d})$ be the expected γ-discounted time to drop from initial inventory level x to or below s. Observe that whenever $x \leq s$, we have $I^\gamma(s, x, \mathbf{d}) = 0$ and $M^\gamma(s, x, \mathbf{d}) = 0$. On the other hand when $x > s$ we have

$$I^\gamma(s, x, \mathbf{d}) = H^\gamma(x, \mathbf{d}(x)) + \gamma E\{I^\gamma(s, x - \alpha\mathbf{d}(x) - \beta, \mathbf{d})\}, \tag{10}$$

and

$$M^\gamma(s, x, \mathbf{d}) = 1 + \gamma E\{M^\gamma(s, x - \alpha\mathbf{d}(x) - \beta, \mathbf{d})\}. \tag{11}$$

Let

$$c^\gamma(s, S, \mathbf{d}) = \frac{-k + I^\gamma(s, S, \mathbf{d})}{M^\gamma(s, S, \mathbf{d})}. \tag{12}$$

The definitions of $I^\gamma(s, x, \mathbf{d}), M^\gamma(s, x, \mathbf{d})$ and $c^\gamma(s, S, \mathbf{d})$ imply the following properties.

Lemma 2 *Given an $(s, S, \mathbf{d})$ policy,*

(i) for $\gamma = 1$ $c^\gamma(s, S, \mathbf{d})$ is the long-run average profit;

(ii) for $0 < \gamma < 1$ the function

$$c^\gamma(s, S, \mathbf{d})/(1-\gamma) + I^\gamma(s, x, \mathbf{d}) - c^\gamma(s, S, \mathbf{d}) M^\gamma(s, x, \mathbf{d})$$

is the infinite horizon expected discounted profit starting with an initial inventory level x.

Proof. Part (i) follows directly from the elementary renewal reward theory (see Ross [16]), and so does the case $x \leq s$ for part (ii). In order to prove part (ii) for $x > s$, define $\tau(s, x, \mathbf{d})$ to be the number of periods it takes to drop the inventory level from x to or below s. Therefore, we have $\tau(s, x, \mathbf{d}) = 0$ for $x \leq s$ and

$$\tau(s, x, \mathbf{d}) = 1 + \tau(s, x - \alpha\mathbf{d}(x) - \beta, \mathbf{d}), \text{ for } x > s.$$

The infinite horizon expected discounted profit starting with initial inventory level x is

$$I^\gamma(s, x, \mathbf{d}) + E\{\gamma^{\tau(s,x,\mathbf{d})}\}c^\gamma(s, S, \mathbf{d})/(1-\gamma),$$

which implies that it suffices to argue that

$$M^\gamma(s, x, \mathbf{d}) = (1 - E\{\gamma^{\tau(s,x,\mathbf{d})}\})/(1-\gamma). \tag{13}$$

For this purpose observe that from the recursion for $\tau(s, x, \mathbf{d})$,

$$(1 - E\{\gamma^{\tau(s,x,\mathbf{d})}\})/(1-\gamma) = \begin{cases} 0, \text{ for } x \leq s, \\ 1 + \gamma(1 - E\{\gamma^{\tau(s,x-\alpha\mathbf{d}(x)-\beta,\mathbf{d})}\})/(1-\gamma), \text{ for } x > s, \end{cases}$$

which is exactly the same recursion for $M^\gamma(s, x, \mathbf{d})$ (11). Therefore, (13) holds and hence part (ii) is true. ∎

To provide intuition about (ii) observe that $c^\gamma(s, S, \mathbf{d})$ is the expected discounted profit per period for the infinite horizon expected discounted profit problem starting with an initial inventory level no more than s. Therefore, $c^\gamma(s, S, \mathbf{d})/(1-\gamma)$ is the infinite horizon expected discounted profit if we start with an initial inventory level, x, no more than s and this implies that (ii) holds since in this case both $I^\gamma(s, x, \mathbf{d})$ and $M^\gamma(s, x, \mathbf{d})$ are equal to zero. For $x \geq s$, observe that $c^\gamma(s, S, \mathbf{d})M^\gamma(s, x, \mathbf{d})$ is the expected discounted profit incurred during the expected discounted time $M^\gamma(s, x, \mathbf{d})$ if we start with an initial inventory level no more than s. Thus, the difference between the infinite horizon expected discounted profit starting with an initial inventory level no more than s and the infinite horizon expected discounted profit starting with the initial inventory level x equals

$$I^\gamma(s, x, \mathbf{d}) - c^\gamma(s, S, \mathbf{d})M^\gamma(s, x, \mathbf{d}).$$

Hence (ii) follows.

We continue by assuming that the period demand is positive. Formally, this assumption says that for any realization of the random variables $\epsilon = (\alpha, \beta)$, $\alpha d + \beta \geq \alpha\underline{d} + \beta \geq \eta > 0$ for some η and any $d \in [\underline{d}, \bar{d}]$. This assumption will be relaxed by perturbing $\underline{d} = D^{-1}(\bar{p})$ and α and analyzing the limiting behavior of the best (s, S) inventory policy.

For any given (s, S), let $c^\gamma(s, S)$ be the optimal value of problem

$$\max_{\mathbf{d}:\bar{d} \geq \mathbf{d}(x) \geq \underline{d}} c^\gamma(s, S, \mathbf{d}). \tag{14}$$

Define

$$\phi^\gamma(x, s, S, s') = \begin{cases} 0, \text{ for } x \leq s', \\ \max_{\bar{d} \geq d \geq \underline{d}} g^\gamma(x, s, S, s', d), \text{ for } x > s', \end{cases} \tag{15}$$

where

$$g^\gamma(x,s,S,s',d) = H^\gamma(x,d) - c^\gamma(s,S) + \gamma E\{\phi^\gamma(x-\alpha d-\beta,s,S,s')\}.$$

Let $\phi^\gamma(x,s,S)=\phi^\gamma(x,s,S,s)$. For any feasible expected demand function $\mathbf{d}$, let

$$\psi^\gamma(x,s,S,\mathbf{d}) = I^\gamma(s,x,\mathbf{d}) - c^\gamma(s,S)M^\gamma(s,x,\mathbf{d}). \tag{16}$$

Then from the recursions for I^γ (10) and M^γ (11), we have that

$$\psi^\gamma(x,s,S,\mathbf{d}) = \begin{cases} 0, \text{ for } x\le s, \\ H^\gamma(x,\mathbf{d}(x)) - c^\gamma(s,S) + \gamma E\{\psi^\gamma(x-\alpha\mathbf{d}(x)-\beta,s,S,\mathbf{d})\}, \text{ for } x>s. \end{cases} \tag{17}$$

Lemma 3 *For any x,*

$$\limsup_{\mathbf{d}:\ \bar{d}\ge\mathbf{d}(x)\ge\underline{d}} \psi^\gamma(x,s,S,\mathbf{d}) = \phi^\gamma(x,s,S).$$

In particular, $\phi^\gamma(S,s,S)=k$.

Proof. We argue by induction that $\psi^\gamma(x,s,S,\mathbf{d})\le\phi^\gamma(x,s,S)$ for any feasible function $\mathbf{d}$ and any x. It is clearly true for $x\le s$ since in this case both functions equal zero. Assume that it is true for any x with $x\le y$ for some y. We prove that it is also true for $x\le y+\eta$. In fact, for $x>s$,

$$\begin{array}{rcl} \psi^\gamma(x,s,S,\mathbf{d}) & = & H^\gamma(x,\mathbf{d}(x)) - c^\gamma(s,S) + \gamma E\{\psi^\gamma(x-\alpha\mathbf{d}(x)-\beta,s,S,\mathbf{d})\} \\ & \le & H^\gamma(x,\mathbf{d}(x)) - c^\gamma(s,S) + \gamma E\{\phi^\gamma(x-\alpha\mathbf{d}(x)-\beta,s,S)\} \\ & \le & \max_{\bar{d}\ge d\ge\underline{d}} H^\gamma(x,d) - c^\gamma(s,S) + \gamma E\{\phi^\gamma(x-\alpha d-\beta,s,S)\} \\ & = & \phi^\gamma(x,s,S), \end{array}$$

where the first inequality is justified by the induction assumption. On the other hand, for any given $\varepsilon>0$, choose a function $\mathbf{d}_\varepsilon$ such that for any $x>s$

$$g^\gamma(x,s,S,s,\mathbf{d}_\varepsilon(x)) \ge \phi^\gamma(x,s,S)-\varepsilon.$$

We have that $\psi^\gamma(x,s,S,\mathbf{d}_\varepsilon)$ converges to $\phi^\gamma(x,s,S)$ uniformly over any bounded set as $\varepsilon\downarrow 0$. Thus for any x,

$$\limsup_{\mathbf{d}:\ \bar{d}\ge\mathbf{d}(x)\ge\underline{d}} \psi^\gamma(x,s,S,\mathbf{d}) = \phi^\gamma(x,s,S).$$

From the definitions of $c^\gamma(s,S,\mathbf{d})$ and $c^\gamma(s,S)$, we have that for any $\mathbf{d}$,

$$\psi^\gamma(S,s,S,\mathbf{d})\le k \text{ and } \limsup_{\mathbf{d}} \psi^\gamma(S,s,S,\mathbf{d}) = k,$$

where for the equality, we use the fact that $M^\gamma(S,s,\mathbf{d})$ is bounded since $\alpha d+\beta\ge\eta$ for any feasible d. Therefore, $\phi^\gamma(S,s,S)=k$. ■

Let c^γ be the optimal value of problem

$$\max_{(s,S)} c^\gamma(s,S). \tag{18}$$

Define

$$F^\gamma := \{(s,S)|c^\gamma(s,S)\ge \max Q^\gamma(x)-k, Q^\gamma(s)=c^\gamma(s,S) \text{ and } Q^\gamma(S)\ge c^\gamma(s,S)\}.$$

Proposition 1 $c^\gamma = \max_{(s,S)\in F^\gamma} c^\gamma(s,S)$.

Proof. In order to prove this result, we make the following observations.

(i) $c^\gamma \geq \max Q^\gamma(x) - k$. In fact, let x^γ be any maximum point of $Q^\gamma(x)$. Then $c^\gamma(x^\gamma - \eta, x^\gamma) = Q^\gamma(x^\gamma) - k$, since $I^\gamma(x^\gamma - \eta, x^\gamma, d) = H^\gamma(x, \mathbf{d}(x))$ and $M^\gamma(x^\gamma - \eta, x^\gamma, d) = 1$ for any expected demand function $\mathbf{d}$. Hence,

$$c^\gamma \geq c^\gamma(x^\gamma - \eta, x^\gamma) = \max Q^\gamma(x) - k.$$

(ii) (a) If $Q^\gamma(s) < c^\gamma(s,S)$, let s_1 be the smallest element in the set

$$\{x | x > s, Q^\gamma(x) = c^\gamma(s,S)\}.$$

It is easy to see that the set is nonempty and $s_1 < S$ since $\phi^\gamma(S,s,S) = k \geq 0$. From the recursive definition of $\phi^\gamma(x,s,S,s_1)$ we have that for any x,

$$\phi^\gamma(x,s,S,s_1) \geq \phi^\gamma(x,s,S),$$

since $\phi^\gamma(x,s,S) \leq 0$ for $x \in [s,s_1]$. In particular, $\phi^\gamma(S,s,S,s_1) \geq k$. We claim $c^\gamma(s_1,S) \geq c^\gamma(s,S)$. In fact, for any given $\varepsilon > 0$, choose a function $\mathbf{d}_\varepsilon(x)$ such that for any $x > s_1$,

$$g^\gamma(x,s,S,s_1,\mathbf{d}_\varepsilon(x)) \geq \phi^\gamma(x,s,S,s_1) - \varepsilon.$$

One can see that for any x,

$$\limsup_{\epsilon\downarrow 0} \psi^\gamma(x,s,S,\mathbf{d}_\varepsilon,s_1) \geq \phi^\gamma(x,s,S,s_1).$$

The above inequality, together with (16) and the fact that $\phi^\gamma(S,s,S,s_1) \geq k$, implies that

$$c^\gamma(s_1,S) \geq \limsup_{\epsilon\downarrow 0} c^\gamma(s_1,S,\mathbf{d}_\varepsilon) \geq c^\gamma(s,S) = Q^\gamma(s_1).$$

If $c^\gamma(s_1,S) > Q^\gamma(s_1)$, we repeat this process and end up with a sequence $s_1 < s_2 < \ldots < S$ with $c^\gamma(s,S) = Q^\gamma(s_1) < c^\gamma(s_1,S) = Q^\gamma(s_2) < \ldots$. If the process stops in finite steps, say n steps, then $c^\gamma(s,S) \leq c^\gamma(s_n,S) = Q^\gamma(s_n)$. Otherwise, let s^* be the limit of this sequence $\{s_n, n = 1,2,\ldots\}$ and $\tilde{c}^\gamma(s^*,S)$ be the limit of $c^\gamma(s_n,S)$. From the continuity of Q^γ as implied by its concavity, we have that $Q^\gamma(s^*) = \tilde{c}^\gamma(s^*,S)$. We argue that $\tilde{c}^\gamma(s^*,S) = c^\gamma(s^*,S)$. Define

$$\tilde{\phi}^\gamma(x,s^*,S) = \begin{cases} 0, \text{ for } x \leq s^*, \\ \max_{\bar{d}\geq d\geq \underline{d}} H^\gamma(x,d) - \tilde{c}^\gamma(s^*,S) + \gamma E\{\tilde{\phi}^\gamma(x - \alpha d - \beta, s^*, S)\}, \text{ for } x > s^*. \end{cases}$$

One can see that $\phi^\gamma(x,s_n,S)$ converges to $\tilde{\phi}^\gamma(x,s^*,S)$ uniformly for x over any bounded set. Furthermore, we have that $\tilde{\phi}^\gamma(S,s^*,S) = k$ since $\phi^\gamma(S,s_n,S) = k$. Hence, from the definition (15) of $\phi^\gamma(x,s^*,S)$ and the fact that $\phi^\gamma(S,s^*,S) = k$, we have that $c^\gamma(s^*,S) = \tilde{c}^\gamma(s^*,S)$ and $\tilde{\phi}^\gamma(x,s^*,S)$ is identical to $\phi^\gamma(x,s^*,S)$. Therefore, $Q^\gamma(s^*) = c^\gamma(s^*,S) \geq c^\gamma(s,S)$.

(b) If $Q^\gamma(s) > c^\gamma(s,S)$, let s_1 be the largest element in the set

$$\{x | x < s, Q^\gamma(x) = c^\gamma(s,S)\}.$$

Then from the recursions of I^γ (10) and M^γ (11), we have that for any x,

$$\phi^\gamma(x,s,S,s_1) \geq \phi^\gamma(x,s,S),$$

since $\phi^\gamma(x,s,S,s_1) \geq 0$ for $x \in [s_1, s]$. Following a similar argument to part (a), we can show that there exists a point s^* such that $Q^\gamma(s^*) = c^\gamma(s^*,S) \geq c^\gamma(s,S)$.

(iii) If $Q^\gamma(S) < c^\gamma(s,S)$, then from the recursive definition of ϕ^γ (15) we have that

$$k = \phi^\gamma(S,s,S) < \max_{\bar{d} \geq d \geq \underline{d}} \gamma E\{\phi^\gamma(S - \alpha d - \beta, s, S)\} \leq \max_{x \leq S-\eta} \phi^\gamma(x,s,S) = \phi^\gamma(S_1,s,S),$$

where S_1 is a maximum point of $\phi^\gamma(x,s,S)$ for $x \leq S - \eta$. From (16), we have $c^\gamma(s,S_1) \geq c^\gamma(s,S_1,d^\gamma_{(s,S)}) > c^\gamma(s,S)$. If $Q^\gamma(S_1) < c^\gamma(s,S_1)$ we can repeat the argument and find $S_{i+1} \leq S_i - \eta$, $i = 1,2,\ldots$, such that $c^\gamma(s,S_{i+1}) > c^\gamma(s,S_i)$ for $i = 1,2,\ldots$. This process has to be finite since we have $S_{i+1} \leq S_i - \eta$. Assume we end up with S_n. Then $Q^\gamma(S_n) \geq c^\gamma(s,S_n) \geq c^\gamma(s,S)$.

Observations (i)-(iii) imply that, for the maximization problem (18), it suffices to restrict the feasible set of (s,S) policies to the set F^γ. ■

For any $(s,S) \in F^\gamma$, since $Q^\gamma(s) = c^\gamma(s,S)$, one can show that $\phi^\gamma(x,s,S)$ is continuous in x and

$$\phi^\gamma(x,s,S) = \begin{cases} 0, & \text{for } x \leq s, \\ \max_{\bar{d} \geq d \geq \underline{d}} g^\gamma(x,s,S,s,d), & \text{for } x \geq s, \end{cases}$$

Furthermore, for $x \geq s$, the following function

$$d^\gamma_{(s,S)}(x) \in \operatorname{argmax}_{\bar{d} \geq d \geq \underline{d}} g^\gamma(x,s,S,s,d),$$

is well-defined and by (16), (17) and Lemma 3 solves problem (14).

In the following lemma, we characterize the properties of the best (s,S) inventory policy. This lemma is key to our analysis of the discounted and average profit problems.

Lemma 4 *There exists an optimal solution (s^γ, S^γ) to problem (18) such that the functions $\phi^\gamma(x) := \phi^\gamma(x, s^\gamma, S^\gamma)$ and $Q^\gamma(x)$ (see (7) for the definition of this function), satisfy the following properties.*

(a) $\phi^\gamma(x) \leq k$ *for any* x *and* $\phi^\gamma(S^\gamma) = k$.

(b) $Q^\gamma(s^\gamma) = c^\gamma$.

(c) $Q^\gamma(x) \geq c^\gamma$ *for* $x \in [s^\gamma, S^\gamma]$.

(d) $\phi^\gamma(x) \geq 0$ *for any* $x \leq S^\gamma$.

(e) $s^\gamma \leq x^\gamma$ *for any maximum point* x^γ *of* $Q^\gamma(x)$.

(f) $y^\gamma \leq S^\gamma$ *for any minimum point* y^γ *of* $h^\gamma(y)$.

Proof. Proposition 1 implies that for problem (18), we can focus on (s, S) in the set F^γ. Observe that F^γ is a bounded set. We now prove that it is also closed and hence compact. For this purpose assume (s, S) is the limit of a sequence $(s_n, S_n) \in F^\gamma$. We claim that $c^\gamma(s_n, S_n)$ converges to $c^\gamma(s, S)$. In fact, let $\tilde{c}^\gamma(s, S)$ be the limit of a subsequence $c^\gamma(s_{n_i}, S_{n_i})$. Then from the continuity of Q^γ, $Q^\gamma(S) \geq Q^\gamma(s) = \tilde{c}^\gamma(s, S)$. Define

$$\tilde{\phi}^\gamma(x, s, S) = \begin{cases} 0, \text{ for } x \leq s, \\ \max_{\bar{d} \geq d \geq \underline{d}} H^\gamma(x, d) - \tilde{c}^\gamma(s, S) + \gamma E\{\tilde{\phi}^\gamma(x - \alpha d - \beta, s, S)\}, \text{ for } x \geq s. \end{cases}$$

One can see that $\phi^\gamma(x, s_{n_i}, S_{n_i})$ converges to $\tilde{\phi}^\gamma(x, s, S)$ uniformly for x over any bounded set. Furthermore, we have that $\tilde{\phi}^\gamma(S, s, S) = k$ since $\phi^\gamma(S_{n_i}, s_{n_i}, S_{n_i}) = k$. Hence, from the definition (15) of $\phi^\gamma(x, s, S)$ and the fact that $\phi^\gamma(S, s, S) = k$, we have that $c^\gamma(s, S) = \tilde{c}^\gamma(s, S)$ and $\tilde{\phi}^\gamma(x, s, S)$ is identical to $\phi^\gamma(x, s, S)$. Therefore, $c^\gamma(s_n, S_n)$ converges to $c^\gamma(s, S)$ and as a consequence, F^γ is closed and hence compact.

We are ready to prove the existence of the best $(s, S, \mathbf{d})$ policy. Assume that c^γ is the limit of $c^\gamma(s_n, S_n)$ for a sequence $(s_n, S_n) \in F^\gamma$. From the compactness of F^γ there is a subsequence (s_{n_i}, S_{n_i}), such that

$$\lim_{i \to \infty} (s_{n_i}, S_{n_i}) = (s^\gamma, S^\gamma)$$

for some $(s^\gamma, S^\gamma) \in F^\gamma$. As proved in the previous paragraph, we have

$$c^\gamma(s^\gamma, S^\gamma) = \lim_{i \to \infty} c^\gamma(s_{n_i}, S_{n_i}) = c^\gamma,$$

and thus (s^γ, S^γ) is the best (s, S) inventory policy.

Hence,

- Part (a) follows from (16) and the fact that (s^γ, S^γ) solves problem (18).
- Part (b) and (c) hold since $(s^\gamma, S^\gamma) \in F^\gamma$ and Q^γ is concave.
- Part (d) follows from part (c) and the recursive definition of ϕ^γ in (15).
- From the argument of Observation (ii) in the proof of Proposition 1, it is easy to see that s^γ can be chosen as the smallest element in the set $\{x | Q^\gamma(x) = c^\gamma\}$. Therefore part (c) implies that $s^\gamma \leq x^\gamma$ for any maximum point x^γ of $Q^\gamma(x)$ and hence part (e) holds.

We now prove part (f). For any minimum point y^γ of $h^\gamma(x)$, we prove by induction that $\phi^\gamma(x)$ is non-decreasing for $x \leq y^\gamma$ and consequently we can choose S^γ such that $y^\gamma \leq S^\gamma$. Without loss of generality, assume that $s^\gamma \leq y^\gamma$. First, $\phi^\gamma(x)$ is non-decreasing for $x \leq s^\gamma$. Now assume it is true for any x with $x \leq y$ for some $y \leq y^\gamma$. Then for x and x' such that $s^\gamma \leq x \leq x' \leq \min\{y + \eta, y^\gamma\}$, we have

$$\begin{aligned} \phi^\gamma(x) &= \max_{\bar{d} \geq d \geq \underline{d}} H^\gamma(x, d) - c^\gamma + \lambda E\{\phi^\gamma(x - \alpha d - \beta)\} \\ &\leq \max_{\bar{d} \geq d \geq \underline{d}} H^\gamma(x', d) - c^\gamma + \lambda E\{\phi^\gamma(x' - \alpha d - \beta)\} \\ &= \phi^\gamma(x'), \end{aligned}$$

where the inequality holds since $x \leq x' \leq y^\gamma$, $h^\gamma(x)$ is convex and $\phi^\gamma(x)$ is non-decreasing for $x \leq y$ by induction assumption. Therefore $\phi^\gamma(x)$ is non-decreasing for $x \leq y^\gamma$. Thus part (f) follows. ∎

To provide some intuition, we point out that $Q^\gamma(x)$ is the single period maximum expected profit when we start with an inventory level x; $c^\gamma(s,S)$ can be viewed as the average discounted profit per period for a given (s,S) policy and its associated best price strategy. Thus, if (b) does not hold, one can change the reorder point, s^γ, and improve the average discounted profit per period. If (c) does not hold, one can decrease S^γ and increase average discounted profit per period.

Lemma 4 allows us to show that ϕ^γ is symmetric k-concave.

Lemma 5 *ϕ^γ is symmetric k-concave for the general demand model.*

Proof. We prove, by induction, that ϕ^γ satisfies

$$\phi^\gamma(x_\lambda) \geq (1-\lambda)\phi^\gamma(x_0) + \lambda\phi^\gamma(x_1) - \max\{\lambda, 1-\lambda\}k, \tag{19}$$

for any $x_0 < x_1$ and $\lambda \in [0,1]$, where $x_\lambda = (1-\lambda)x_0 + \lambda x_1$.

Since $\phi^\gamma(x) = 0$ for $x \leq s^\gamma$, it is obvious that (19) holds for $x_1 \leq s^\gamma$. Now assume that (19) holds for any x_0 and x_1 with $x_0 < x_1 \leq y$ for some y. We show that (19) also holds for any x_0 and x_1 with $x_0 < x_1 \leq y + \eta$. We distinguish between three cases.

Case 1: $x_0 > s^\gamma$. Letting $d_\lambda = (1-\lambda)d^\gamma_{(s^\gamma,S^\gamma)}(x_0) + \lambda d^\gamma_{(s^\gamma,S^\gamma)}(x_1)$, we have that

$$\begin{aligned}
\phi^\gamma(x_\lambda) &\geq H^\gamma(x_\lambda, d_\lambda) - c^\gamma + \gamma E\{\phi^\gamma(x_\lambda - \alpha d_\lambda - \beta)\} \\
&\geq (1-\lambda)(H^\gamma(x_0, d^\gamma_{(s^\gamma,S^\gamma)}(x_0)) - c^\gamma + \gamma E\{\phi^\gamma(x_0 - \alpha d^\gamma_{(s^\gamma,S^\gamma)}(x_0) - \beta)\}) \\
&+ \lambda(H^\gamma(x_1, d^\gamma_{(s^\gamma,S^\gamma)}(x_1)) - c^\gamma + \gamma E\{\phi^\gamma(x_1 - \alpha d^\gamma_{(s^\gamma,S^\gamma)}(x_1) - \beta)\}) - \gamma\max\{\lambda, 1-\lambda\}k \\
&\geq (1-\lambda)\phi^\gamma(x_0) + \lambda\phi^\gamma(x_1) - \max\{\lambda, 1-\lambda\}k,
\end{aligned}$$

where the second inequality follows from the concavity of H^γ, the fact that for any feasible d, $x_0 - \alpha d - \beta \leq x_1 - \alpha d - \beta \leq y$ and the induction assumption.

Case 2: $x_0 \leq s^\gamma$ and $x_\lambda \leq S^\gamma$. (19) holds since, by Lemma 4 parts (a) and (d), $\phi^\gamma(x_1) \leq k$ and $\phi^\gamma(x_\lambda) \geq 0$.

Case 3: $x_0 \leq s^\gamma \leq S^\gamma \leq x_\lambda$.

$$\begin{aligned}
\phi^\gamma(x_\lambda) &\geq (1-\mu)\phi^\gamma(S^\gamma) + \mu\phi^\gamma(x_1) - \max\{\mu, 1-\mu\}k \\
&\geq \mu(\phi^\gamma(x_1) - k) \\
&\geq \lambda(\phi^\gamma(x_1) - k) \\
&\geq (1-\lambda)\phi^\gamma(x_0) + \lambda\phi^\gamma(x) - \max\{\lambda, 1-\lambda\}k,
\end{aligned}$$

where μ is chosen such that $x_\lambda = (1-\mu)S^\gamma + \mu x_1$ with $0 \leq \mu \leq \lambda$. The first inequality follows from Case 1, the second inequality holds since $\phi^\gamma(S^\gamma) = k$ by Lemma 4 part (a), the third inequality holds since $0 \leq \mu \leq \lambda$ and, by Lemma 4 part (a), $\phi^\gamma(x_1) \leq k$, and the last inequality follows from the fact that $\phi^\gamma(x_0) = 0$ since $x_0 \leq s^\gamma$.

Therefore, by induction ϕ^γ is symmetric k-concave. ∎

In the special case of additive demand functions, we show that ϕ^γ is k-concave. Define

$$g^\gamma(x,d) := H^\gamma(x,d) - c^\gamma + \gamma E\{\phi^\gamma(x - d - \beta)\}, \text{ for } x \geq s^\gamma.$$

We need the following result, which basically implies that the higher the inventory level at the beginning of one period after placing an order, the higher the expected inventory level at the end of this period. A similar result was proven in [4] for the finite horizon case.

Lemma 6 *For the model with additive demand processes, there exists an optimal solution $d^\gamma(x)$ for problem $\max_{\bar{d}\geq d\geq \underline{d}} g^\gamma(x,d)$ such that $x - d^\gamma(x)$ is non-decreasing for $x \geq s^\gamma$.*

Proof. For $x \geq s^\gamma$, let
$$d^\gamma(x) = \max\left\{\text{argmax}_{\bar{d}\geq d\geq \underline{d}} g^\gamma(x,d)\right\}.$$
We claim that $x - d^\gamma(x)$ is non-decreasing for $x \geq s^\gamma$. If not, there exists x and x' such that $s^\gamma \leq x < x'$ and $x - d^\gamma(x) > x' - d^\gamma(x')$. Then by letting

$$d := d^\gamma(x') - (x'-x) > d^\gamma(x) \text{ and } d' = d^\gamma(x) + (x'-x) < d^\gamma(x'),$$

we have
$$g^\gamma(x, d^\gamma(x)) > g^\gamma(x,d), \text{ and } g^\gamma(x', d^\gamma(x')) \geq g^\gamma(x', d').$$
Adding the above two inequalities together, we have that

$$\hat{R}(d^\gamma(x)) + \hat{R}(d^\gamma(x')) > \hat{R}(d) + \hat{R}(d'),$$

which cannot be true since $\hat{R}$ is assumed to be concave. Therefore, $x - d^\gamma(x)$ is non-decreasing for $x \geq s^\gamma$. ■

Lemma 7 *ϕ^γ is k-concave for the additive demand model.*

Proof. We show, by induction, that ϕ^γ satisfies

$$\phi^\gamma(x_\lambda) \geq (1-\lambda)\phi^\gamma(x_0) + \lambda\phi^\gamma(x_1) - \lambda k, \tag{20}$$

for any $x_0 < x_1$ and $\lambda \in [0,1]$, where $x_\lambda = (1-\lambda)x_0 + \lambda x_1$.

Since $\phi^\gamma(x) = 0$ for $x \leq s^\gamma$, it is obvious that (20) holds for $x_1 \leq s^\gamma$. Now assume that (20) holds for any x_0 and x_1 with $x_0 < x_1 \leq y$ for some y. We show that (20) also holds for any x_0 and x_1 with $x_0 < x_1 \leq y + \eta$. We distinguish between three cases.
<u>Case 1:</u> $x_0 \geq s^\gamma$. In fact, letting $d_\lambda = (1-\lambda)d^\gamma_{(s^\gamma,S^\gamma)}(x_0) + \lambda d^\gamma_{(s^\gamma,S^\gamma)}(x_1)$, we have that

$$\begin{array}{rcl}
\phi^\gamma(x_\lambda) & \geq & H^\gamma(x_\lambda, d_\lambda) - c^\gamma + \gamma E\{\phi^\gamma(x_\lambda - d_\lambda - \beta)\} \\
& \geq & (1-\lambda)(H^\gamma(x_0, d^\gamma_{(s^\gamma,S^\gamma)}(x_0)) - c^\gamma + \gamma E\{\phi^\gamma(x_0 - d^\gamma_{(s^\gamma,S^\gamma)}(x_0) - \beta)\}) \\
& + & \lambda(H^\gamma(x_1, d^\gamma_{(s^\gamma,S^\gamma)}(x_1)) - c^\gamma + \gamma E\{\phi^\gamma(x_1 - d^\gamma_{(s^\gamma,S^\gamma)}(x_1) - \beta)\}) - \gamma\lambda k \\
& \geq & (1-\lambda)\phi^\gamma(x_0) + \lambda\phi^\gamma(x_1) - \lambda k,
\end{array}$$

where the second inequality follows from the concavity of H^γ, Lemma 4 part (b), the fact that by Lemma 6, $x_0 - d^\gamma_{(s^\gamma,S^\gamma)}(x_0) - \beta \leq x_1 - d^\gamma_{(s^\gamma,S^\gamma)}(x_1) - \beta \leq y$ and the induction assumption.
<u>Case 2:</u> $x_\lambda \leq s^\gamma$. (20) holds for x_0, x_1 and any $\lambda \in [0,1]$, since $\phi^\gamma(x) \leq k$ for any x by Lemma 4 part (a).
<u>Case 3:</u> $x_0 \leq s^\gamma \leq x_\lambda$. Select μ, $0 \leq \mu \leq \lambda$ such that $x_\lambda = (1-\mu)s^\gamma + \mu x_1$. We have that

$$\begin{array}{rcl}
\phi^\gamma(x_\lambda) & \geq & (1-\mu)\phi^\gamma(s^\gamma) + \mu\phi^\gamma(x_1) - \mu k \\
& = & \mu(\phi^\gamma(x_1) - k) \\
& \geq & \lambda(\phi^\gamma(x_1) - k) \\
& = & (1-\lambda)\phi^\gamma(x_0) + \lambda\phi^\gamma(x_1) - \lambda k,
\end{array}$$

where the first inequality follows from Case 1 and Lemma 4 part (b), the second inequality from Lemma 4 part (a) which states that $\phi^\gamma(x) \leq k$ for any x, and the last equality from $\phi^\gamma(x_0) = 0$ since $x_0 \leq s^\gamma$.

Therefore, by induction ϕ^γ is k-concave. ■

We are ready to prove that (ϕ^γ, c^γ) satisfies the equation:

$$\phi^\gamma(x) + c^\gamma = \max_{y \geq x} \Big\{ \max_{\bar{d} \geq d \geq \underline{d}} -k\delta(y-x) + H^\gamma(y,d) + \gamma E\{\phi^\gamma(y - \alpha d - \beta)\} \Big\} \tag{21}$$

and that (s^γ, S^γ) is the policy that attains the first maximization in equation (21).

Notice that when $\gamma = 1$, (21) is the optimality equation for the average profit problem. On the other hand, when $0 < \gamma < 1$, define

$$\hat{\phi}^\gamma(x) = c^\gamma/(1-\gamma) + \phi^\gamma(x).$$

Then (21) implies that

$$\hat{\phi}^\gamma(x) = \max_{y \geq x, \bar{d} \geq d \geq \underline{d}} -k\delta(y-x) + H^\gamma(y,d) + \gamma E\{\hat{\phi}^\gamma(y - \alpha d - \beta)\},$$

which is the optimality equation for the γ-discounted profit problem for $0 < \gamma < 1$, i.e. problem (4).

Theorem 3.1 *(ϕ^γ, c^γ) satisfies equation (21) and (s^γ, S^γ) attains the first maximization in equation (21).*

Proof. For any x, define

$$O^\gamma(x) := \max_{\bar{d} \geq d \geq \underline{d}} H^\gamma(x,d) - c^\gamma + \gamma E\{\phi^\gamma(x - \alpha d - \beta)\}.$$

From (15) and Lemma 4 part (b), one can see that $O^\gamma(x) = Q^\gamma(x) - c^\gamma$ for $x \leq s^\gamma$ and $O^\gamma(x) = \phi^\gamma(x)$ for $x \geq s^\gamma$. We have the following observations.

(a) $O^\gamma(x) \leq O^\gamma(s^\gamma) = 0$ for $x \leq s^\gamma$. This follows from Lemma 4 parts (b) and (e), the concavity of Q^γ and the fact that $O^\gamma(x) = Q^\gamma(x) - c^\gamma$ for $x \leq s^\gamma$.

(b) $O^\gamma(x) \leq O^\gamma(S^\gamma) = k$ for any x. This result follows from part (a) and Lemma 4 part (a) since $O^\gamma(x) = \phi^\gamma(x)$ for $x \geq s^\gamma$.

(c) $O^\gamma(y) \geq O^\gamma(z) - k$, for any y, z with $s^\gamma \leq y \leq z$. Since $O^\gamma(x) = \phi^\gamma(x)$ for $x \geq s^\gamma$, we only need to show that $\phi^\gamma(y) \geq \phi^\gamma(z) - k$. For $y \leq S^\gamma$, we have

$$\phi^\gamma(y) \geq 0 \geq \phi^\gamma(z) - k$$

by Lemma 4 parts (a) and (d). For $y \geq S^\gamma$, $\phi^\gamma(y) \geq \phi^\gamma(z) - k$ follows from Lemma 4 part (a), Lemma 5 and Lemma 1 part (d).

Observations (a), (b) and (c) imply that the optimal y in equation (21) follows the (s^γ, S^γ) policy: if $x \le s^\gamma$ then $y = S^\gamma$, otherwise $y = x$. Thus, (ϕ^γ, c^γ) satisfies (21). ■

The above results are proven under the assumption that $\alpha\underline{d} + \beta \ge \eta > 0$. Now we relax this assumption and prove that all the results in this section hold even when the assumption is not satisfied.

To do that we are going to construct a sequence of random variables α_η such that

(R_a) $E\{\alpha_\eta\} = 1$.

(R_b) α_η is bounded below by a positive constant.

(R_c) α_η converges to α in distribution as $\eta \downarrow 0$.

Let $F(x)$ be the cumulative probability distribution of α. For any $\eta < 1$, let

$$q_\eta = \frac{1 - F(\eta)}{\int_\eta^\infty (x - \eta) dF(x)} \frac{\int_0^\eta (\eta - x) dF(x)}{F(\eta)},$$

and

$$p_\eta = \frac{q_\eta F(\eta)}{1 - F(\eta)}.$$

Without loss of generality, assume that $\int_1^\infty x dF(x) > 0$. We have $q_\eta = O(\eta)$ and $p_\eta = O(\eta)$. Furthermore,

$$F(\eta)(1 + q_\eta) + (1 - p_\eta)(1 - F(\eta)) = 1, \tag{22}$$

and

$$\eta F(\eta)(1 + q_\eta) + (1 - p_\eta) \int_\eta^\infty x dF(x) = \int_0^\infty x dF(x) = 1. \tag{23}$$

Define a function F_η such that

$$F_\eta(x) = \begin{cases} 0, \text{ for } x < \eta \\ (1 + q_\eta)F(\eta) + (1 - p_\eta)(F(x) - F(\eta)), \text{ for } x \ge \eta. \end{cases}$$

Equation (22) implies that F_η is a distribution function. Let α_η be a random variable with distribution F_η. Then by (23) $E\{\alpha_\eta\} = 1$ and the requirements (R_a), (R_b) and (R_c) are satisfied.

We are ready to relax the assumption that $\alpha\underline{d} + \beta$ is bounded below by a positive constant. For this purpose, consider a similar model with α and $\underline{d}$ replaced by α_η and $\underline{d} + \eta$ respectively for $\bar{d} - \underline{d} \ge \bar{\eta} \ge \eta > 0$. We refer to this model as the *modified problem*. Notice that if $\bar{d} = \underline{d}$, it is well known that an (s, S) policy is optimal for both the average and discounted profit models (see [9, 10]).

In the modified problem, $\alpha_\eta(\underline{d} + \eta) + \beta$ is bounded below by a positive constant. Let $c_\eta^\gamma(s, S)$ be the average discounted profit per period for the stationary (s, S) policy associated with the best price under this modified model. Define

$$F_\eta^\gamma := \{(s, S) | c_\eta^\gamma(s, S) \ge -k + \max Q_\eta^\gamma(x), Q_\eta^\gamma(s) = c_\eta^\gamma(s, S) \text{ and } Q_\eta^\gamma(S) \ge c_\eta^\gamma(s, S)\},$$

where $Q_\eta^\gamma(x) = \max_{\bar{d} \ge d \ge \underline{d} + \eta} H_\eta^\gamma(x, d)$ and $H_\eta^\gamma(x, d) = \hat{R}(d) - E\{h^\gamma(x - \alpha_\eta d - \beta)\}$. From the construction of α_η, one can see that Q_η^γ converges to Q^γ uniformly over any bounded set. Therefore, by Assumption 2, F_η^γ is uniformly bounded for $0 < \eta \le \bar{\eta}$.

Let $(s^\gamma_\eta, S^\gamma_\eta)$ be the best (s, S) policy under the modified model with parameter η, and let $c^\gamma_\eta = c^\gamma_\eta(s^\gamma_\eta, S^\gamma_\eta)$. Define

$$\phi^\gamma_\eta(x) = \begin{cases} 0, \text{ for } x \leq s^\gamma_\eta, \\ \max_{\bar{d} \geq d \geq \underline{d}+\eta} H^\gamma_\eta(x, d) - c^\gamma_\eta + \gamma E\{\phi^\gamma_\eta(x - \alpha_\eta d - \beta)\}, \text{ for } x \geq s^\gamma_\eta. \end{cases} \tag{24}$$

Since F^γ_η is uniformly bounded for $0 < \eta \leq \bar{\eta}$, there exists a limit point for some subsequence $(s^\gamma_{\eta_i}, S^\gamma_{\eta_i})$, where $\eta_i \to 0$ as $i \to \infty$. Let (s^γ_0, S^γ_0) be this limit point.

Lemma 4 part (b), together with the fact that Q^γ_η converges to Q^γ uniformly over a bounded set, implies that $c^\gamma_{\eta_i}$ converges to a point c^γ_0. Hence $\phi^\gamma_{\eta_i}$ converges to a function ϕ^γ_0 uniformly over any bounded set as $i \to \infty$. Furthermore, this convergence property implies that ϕ^γ_0 satisfies the recursion (24) with $\eta = 0$, where $H^\gamma_0(x, d) = H^\gamma(x, d)$ and $\alpha_0 = \alpha$. Since $\phi^\gamma_\eta(S^\gamma_\eta) = k$, $\phi^\gamma_0(S^\gamma_0) = k$ and hence $c^\gamma_0 = c^\gamma(s^\gamma_0, S^\gamma_0)$. Therefore Lemma 4 and Lemma 5 hold and $(\phi^\gamma_0, c^\gamma_0)$ satisfies (21). Thus, all the results in this section hold even if $\alpha\underline{d} + \beta \geq 0$.

4 Bounds

The convergence results proven in Section 5 and Section 6 for the discounted and average profit cases, respectively, require bounds on some of the parameters of the optimal policy for the finite horizon model. Our approach in this section is motivated by the classical work of Veinott, [19].

Consider the dynamic program (4). A straight-forward extension of the analysis in [4] shows that an (s, S, A, p) policy is optimal for this problem.

For every t, $t = 1, \ldots$, let $(s^\gamma_t, S^\gamma_t, A^\gamma_t, p^\gamma_t)$ be the parameters of the optimal policy. We show that s^γ_t and S^γ_t are uniformly bounded. Specifically, define

$$\underline{S}^\gamma = \min_{\bar{d} \geq d \geq \underline{d}} \Big\{ \text{argmax}_x H^\gamma(x, d), \text{argmax}_x H^0(x, d) \Big\}, \text{ and } \bar{s}^\gamma = \max_{\bar{d} \geq d \geq \underline{d}} \Big\{ \text{argmax}_x H^\gamma(x, d) \Big\},$$

$$\underline{s}^\gamma = \max\{x | x \leq \underline{S}^\gamma, H^\mu(\underline{S}^\gamma, d) \geq H^\mu(x, d) + k, \text{ for } \mu = 0, \gamma \text{ and all feasible } d\},$$

and

$$\bar{S}^\gamma = \min\{x | x \geq \bar{s}^\gamma, H^\gamma(\bar{s}^\gamma, d) \geq H^\gamma(x, d) + k, \text{ for all feasible } d\}.$$

The existence of $\underline{s}^\gamma$ and $\bar{S}^\gamma$ follows from Assumption 2.

Lemma 8 *For $t \geq 1$,*

$$\phi^\gamma_t(x) \geq \phi^\gamma_t(x') - k, \textit{ for } x \leq x', \tag{25}$$

$$g^\gamma_t(y', d) - g^\gamma_t(y, d) \leq H^\gamma(y', d) - H^\gamma(y, d) + k, \textit{ for } y \leq y', \tag{26}$$

and

$$g^\gamma_t(y', d^\gamma_t(y')) \leq g^\gamma_t(y, d^\gamma_t(y)) + k, \textit{ for } y' \geq y \geq \bar{s}^\gamma. \tag{27}$$

Proof. By induction. For $t = 0$, $\phi^\gamma_t(x) = -cx$ is non-increasing since the variable ordering cost $c \geq 0$. Hence (25) holds for $t = 0$. For $t \geq 1$, (25) follows directly from (4). (26) follows from (5) and (25) for period $t - 1$. (27) follows from (26), the definition of $\bar{s}^\gamma$ and the concavity of H^γ. ■

Lemma 9

$$g_1^\gamma(y',d) - g_1^\gamma(y,d) = H^0(y',d) - H^0(y,d) \geq 0,\ \textit{for } y \leq y' \leq \underline{S}^\gamma, \tag{28}$$

$$g_t^\gamma(y',d) - g_t^\gamma(y,d) \geq H^\gamma(y',d) - H^\gamma(y,d) \geq 0,\ \textit{for } y \leq y' \leq \underline{S}^\gamma \textit{ and } t > 1, \tag{29}$$

$$g_t^\gamma(y', d_t^\gamma(y')) \geq g_t^\gamma(y, d_t^\gamma(y)),\ \textit{for } y \leq y' \leq \underline{S}^\gamma \textit{ and } t \geq 1, \tag{30}$$

and

$$\phi_t^\gamma(x') \geq \phi_t^\gamma(x),\ \textit{for } x \leq x' \leq \underline{S}^\gamma \textit{ and } t \geq 1. \tag{31}$$

Proof. (28) follows from the definition of $\underline{S}^\gamma$ and the fact that $g_1^\gamma(y,d) = H^0(y,d)$. We prove the remaining three inequalities by induction. Assume that (29) holds for some $t > 1$. (30) follows directly from (28) (if $t = 1$) or (29) (if $t > 1$). Furthermore, for any $x \leq x' \leq \underline{S}^\gamma$,

$$\begin{aligned} \phi_t^\gamma(x') &= \max\{g_t^\gamma(x', d_t^\gamma(x')), -k + \max_{y>x'} g_t^\gamma(y, d_t^\gamma(y))\} \\ &\geq \max\{g_t^\gamma(x, d_t^\gamma(x)), -k + \max_{y>x} g_t^\gamma(y, d_t^\gamma(y))\} \\ &= \phi_t^\gamma(x), \end{aligned}$$

where the inequality follows from (30). This proves inequality (31). Finally, (5), (31), and the definition of $\underline{S}^\gamma$, imply that (29) holds for $t+1$ and any feasible d, since H^γ is concave. ■

We are ready to present our bounds on s_t^γ and S_t^γ.

Lemma 10 *For every t, $s_t^\gamma \in [\underline{s}^\gamma, \bar{s}^\gamma]$ and $S_t^\gamma \in [\underline{S}^\gamma, \bar{S}^\gamma]$.*

Proof. We first show that for every t and $y \leq \underline{s}^\gamma$,

$$g_t^\gamma(y, d_t^\gamma(y)) \leq -k + g_t^\gamma(\underline{S}^\gamma, d_t^\gamma(\underline{S}^\gamma))$$

which implies that an order is placed for this level of inventory, y, and hence $s_t^\gamma \geq \underline{s}^\gamma$.

For $t > 1$, we have that for $y \leq \underline{s}^\gamma$,

$$\begin{aligned} g_t^\gamma(y, d_t^\gamma(y)) &= H^\gamma(y, d_t^\gamma(y)) + \gamma E\{\phi_{t-1}^\gamma(y - \alpha_t d_t^\gamma(y) - \beta_t)\} \\ &\leq -k + H^\gamma(\underline{S}^\gamma, d_t^\gamma(y)) + \gamma E\{\phi_{t-1}^\gamma(\underline{S}^\gamma - \alpha_t d_t^\gamma(y) - \beta_t)\} \\ &\leq -k + H^\gamma(\underline{S}^\gamma, d_t^\gamma(\underline{S}^\gamma)) + \gamma E\{\phi_{t-1}^\gamma(\underline{S}^\gamma - \alpha_t d_t^\gamma(\underline{S}^\gamma) - \beta_t)\} \\ &= -k + g_t^\gamma(\underline{S}^\gamma, d_t^\gamma(\underline{S}^\gamma)), \end{aligned}$$

where the first inequality follows from the definition of $\underline{s}^\gamma$ and (31), and the second inequality from the definition of d_t^γ.

Consider now $t = 1$. Using the fact that $g_t^\gamma(y,d) = H^0(y,d)$ and the definition of $d_t^\gamma(x)$, $\underline{s}^\gamma$ and $\underline{S}^\gamma$, we have $g_t^\gamma(y, d_t^\gamma(y)) \leq -k + g_t^\gamma(\underline{S}_t^\gamma, d_t(\underline{S}_t^\gamma))$ for $y \leq \underline{s}^\gamma$.

To show that $s_t^\gamma \leq \bar{s}^\gamma$, we apply inequality (27) which implies that no order is placed when $y \geq \bar{s}^\gamma$. Hence, $s_t^\gamma \in [\underline{s}^\gamma, \bar{s}^\gamma]$.

To show that $S_t^\gamma \leq \bar{S}^\gamma$, it suffices to show that for $y \geq \bar{S}^\gamma$ we have

$$g_t^\gamma(\bar{s}^\gamma, d_t^\gamma(\bar{s}^\gamma)) \geq g_t^\gamma(y, d_t^\gamma(y)).$$

In fact, for $y \geq \bar{S}^\gamma$,

$$\begin{aligned}
g_t^\gamma(\bar{s}^\gamma, d_t^\gamma(\bar{s}^\gamma)) &= H^\gamma(\bar{s}^\gamma, d_t^\gamma(\bar{s}^\gamma)) + \gamma E\{\phi_{t-1}^\gamma(\bar{s}^\gamma - \alpha_t d_t^\gamma(\bar{s}^\gamma) - \beta_t)\} \\
&\geq H^\gamma(\bar{s}^\gamma, d_t^\gamma(y)) + \gamma E\{\phi_{t-1}^\gamma(\bar{s}^\gamma - \alpha_t d_t^\gamma(y) - \beta_t)\} \\
&\geq k + H^\gamma(y, d_t^\gamma(y)) + \gamma E\{\phi_{t-1}^\gamma(y - \alpha_t d_t^\gamma(y) - \beta_t)\} - \gamma k \\
&\geq g_t^\gamma(y, d_t^\gamma(y)),
\end{aligned}$$

where the first inequality follows from the definition of d_t^γ, the second inequality from the definition of $\bar{S}^\gamma$ and (25) and the last inequality from definition (5).

Finally, inequality (30) implies that the function $g_t^\gamma(y, d_t^\gamma(y))$ is non-decreasing for $y \leq \underline{S}^\gamma$. Hence, $S_t^\gamma \geq \underline{S}^\gamma$ and as a result $S_t^\gamma \in [\underline{S}^\gamma, \bar{S}^\gamma]$. ∎

5 Discounted Profit Case

Consider the discounted profit case with a discount factor $0 < \gamma < 1$ and recall the definition of $\hat{\phi}^\gamma(x)$. Lemma 2 tells us that $\hat{\phi}^\gamma(x)$ is the infinite horizon expected discounted profit for the stationary $(s^\gamma, S^\gamma, d_{(s^\gamma, S^\gamma)}^\gamma)$ policy when starting with an initial inventory level x.

The following convergence result relates the t-period maximum total expected discounted profit function, $\phi_t^\gamma(x)$, and $\hat{\phi}^\gamma(x)$.

Theorem 5.1 *For any $M \geq \max\{\bar{S}^\gamma, S^\gamma\}$ and any $t \geq 1$, we have that*

$$\max_{x \leq M} |\phi_t^\gamma(x) - \hat{\phi}^\gamma(x)| \leq \gamma^{t-1} \max_{x \leq M} |\phi_1^\gamma(x) - \hat{\phi}^\gamma(x)|. \tag{32}$$

Proof. By induction. For $t = 1$ inequality (32) holds as equality. Consider $t > 1$. From (4) and (21), we have that for any $x \leq M$,

$$\begin{aligned}
\phi_t^\gamma(x) - \hat{\phi}^\gamma(x) &= \max\nolimits_{M \geq y \geq x, \bar{d} \geq d \geq \underline{d}} -k\delta(y-x) + H^\gamma(y,d) + \gamma E\{\phi_{t-1}^\gamma(y - \alpha d - \beta)\} \\
&- \max\nolimits_{M \geq y \geq x, \bar{d} \geq d \geq \underline{d}} -k\delta(y-x) + H^\gamma(y,d) + \gamma E\{\hat{\phi}^\gamma(y - \alpha d - \beta)\} \\
&\leq \max\nolimits_{M \geq y \geq x} -k\delta(y-x) + H^\gamma(y, d_t^\gamma(x)) + \gamma E\{\phi_{t-1}^\gamma(y - \alpha d_t^\gamma(x) - \beta)\} \\
&- (-k\delta(y-x) + H^\gamma(y, d_t^\gamma(x)) + \gamma E\{\hat{\phi}^\gamma(y - \alpha d_t^\gamma(x) - \beta)\}) \\
&= \gamma \max\nolimits_{M \geq y \geq x} E\{\phi_{t-1}^\gamma(y - \alpha d_t^\gamma(x) - \beta) - \hat{\phi}^\gamma(y - \alpha d_t^\gamma(x) - \beta)\} \\
&\leq \gamma^{t-1} \max\nolimits_{x \leq M} |\phi_1^\gamma(x) - \hat{\phi}^\gamma(x)|,
\end{aligned}$$

where the first equation follows from Theorem 3.1, Lemma 10 and the assumption that $M \geq \max\{\bar{S}^\gamma, S^\gamma\}$, the first inequality from the definition of d_t^γ (see (6)), and the last inequality from the induction assumption.

By employing a similar approach, we can prove that for $x \leq M$,

$$\hat{\phi}^\gamma(x) - \phi_t^\gamma(x) \leq \gamma^{t-1} \max_{x \leq M} |\phi_1^\gamma(x) - \hat{\phi}^\gamma(x)|.$$

Hence (32) holds for all t. ∎

The theorem thus implies that the t-period maximum total expected discounted profit function, $\phi_t^\gamma(x)$, converges to the infinite horizon expected discounted profit function, $\hat{\phi}^\gamma(x)$, associated with the stationary $(s^\gamma, S^\gamma, d_{(s^\gamma, S^\gamma)}^\gamma)$ policy and as a consequence, this policy is optimal for the infinite horizon expected discounted profit problem.

6 Average Profit Case

In this section we analyze the average profit case and hence assume that $\gamma = 1$. To prove that a stationary $(s, S, \mathbf{d})$ policy is optimal for the average profit case, we apply a similar approach to the one used by Iglehart [10] for the traditional stochastic inventory model. Specifically, we show that the long-run average profit of the best $(s, S, \mathbf{d})$ policy, c^1, is the limit of the maximum average profit per period over a t-period planning horizon.

Theorem 6.1 *For any x,*

$$\phi_t^1(x)/t - c^1 \to 0, \text{ as } t \to \infty.$$

Proof. We prove by induction that for any given $M \geq \max\{\bar{S}^1, S^1\}$, there exist r and R such that

$$tc^1 + \phi^1(x) + r \leq \phi_t^1(x) \leq tc^1 + \phi^1(x) + R, \text{ for } x \leq M \text{ and any } t. \tag{33}$$

First, for $x \leq \min\{\underline{s}^1, s^1\}$, $\phi^1(x)$ and $\phi_t^1(x)$ are constants. Hence, for $t = 1$, there exist two parameters r and R such that (33) holds for $x \leq M$.

Second, assume (33) is true for $t - 1$. Since $S_t^1 \leq \bar{S}^1 \leq M$, for $x \leq M$ we have

$$\phi_t^1(x) = \max_{M \geq y \geq x, \bar{d} \geq d \geq \underline{d}} -k\delta(y - x) + H^1(y, d) + E\{\phi_{t-1}^1(y - \alpha d - \beta)\},$$

and hence

$$\begin{array}{rcl} \phi_t^1(x) & \leq & \max_{M \geq y \geq x, \bar{d} \geq d \geq \underline{d}} -k\delta(y - x) + H^1(y, d) + E\{\phi^1(y - \alpha d - \beta)\} + (t-1)c^1 + R \\ & \leq & \max_{y \geq x, \bar{d} \geq d \geq \underline{d}} -k\delta(y - x) + H^1(y, d) - c^1 + E\{\phi^1(y - \alpha d - \beta)\} + tc^1 + R \\ & = & \phi^1(x) + tc^1 + R, \end{array}$$

where the first inequality follows from the induction assumption (33), the second inequality holds since we removed the constraint $M \geq y$ and the equality follows from the optimality equation, (21).

The left hand side inequality (i.e., the lower bound) of (33) can be established in a similar fashion.

By choosing M arbitrarily large, (33) implies that

$$\phi_t^1(x)/t - c^1 \to 0, \text{ as } t \to \infty,$$

for any x. ■

The theorem thus suggests that starting with any initial inventory level, the maximum average profit per period over a t-period planning horizon converges to a constant c^1, the long-run average profit of the best $(s, S, \mathbf{d})$ policy. Therefore, the best $(s, S, \mathbf{d})$ policy, the stationary $(s^\gamma, S^\gamma, d^\gamma_{(s^\gamma, S^\gamma)})$ policy, is optimal for the infinite horizon average profit problem.

7 Concluding Remarks

In this section we summarize our main results. Recall that for the finite horizon case Chen and Simchi-Levi [4] proved that an (s, S, p) policy is not necessarily optimal for general demand processes. Indeed by developing and employing the concept of symmetric k-convex functions, Chen and Simchi-Levi showed that in this case an (s, S, A, p) policy is optimal.

Surprisingly, in the current paper we show, using the concept of symmetric k-convexity, that a stationary (s, S, p) policy is optimal in the infinite horizon case for both the discounted and average profit criteria. This result holds for the general demand process defined by Assumption 1 which includes additive and multiplicative demand functions; both are common in the economics literature.

References

[1] Agrawal, V. and A. Kambil. 2000. Dynamic Pricing Strategies in Electronic Commerce. working paper, Stern Business School, New York University.

[2] Belobaba, P. P. 1987. Airline Yield Management: An Overview of Seat Inventory Control. *Transportation Science* **21**, pp. 63–23.

[3] Chan, L. M. A. D. Simchi-Levi and J. Swann. 2001. Effective Dynamic Pricing Strategies with Stochastic Demand. Massachusetts Institute of Technology.

[4] Chen, X. and D. Simchi-Levi. 2002. Coordinating Inventory Control and Pricing Strategies with Random Demand and Fixed Ordering Cost: The Finite Horizon Case. Massachusetts Institute of Technology.

[5] Cook, T. (2000). Creating Competitive Advantage in the Airline Industry. Seminar sponsored by the MIT Global Airline Industry Program and the MIT Operations Research Center.

[6] Eliashberg, J. and R. Steinberg. 1991. Marketing-production joint decision making. J. Eliashberg, J. D. Lilien, eds. *Management Science in Marketing*, Volume 5 of *Handbooks in Operations Research and Management Science*, North Holland, Amsterdam.

[7] Federgruen, A. and A. Heching. 1999. Combined pricing and inventory control under uncertainty. *Operations Research*, **47**, No. 3, pp. 454-475.

[8] Gallego, G. and G. van Ryzin. 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science*, **40**, pp. 999-1020.

[9] Iglehart, D. 1963. Optimality of (s, S) policies in the infinite-horizon dynamic inventory problem. *Management Science*, **9**, pp. 259-267.

[10] Iglehart, D. 1963. Dynamic programming and the analysis of inventory problems. Chapter 1 in *Multistage Inventory Models and Techniques*, ed. H. Scarf, D. Gilford and M. Shelly. Stanford University Press.

[11] Kay, E. 1998. Flexed pricing. *Datamation*, **44**, No. 2, pp. 58-62.

[12] Leibs, S. 2000. Ford Heads the Profits. *CFO The Magazine*, **16**, 9, pp. 33–35.

[13] McGill, J. I. and G. J. Van Ryzin. 1999. Revenue Management: Research Overview and Prospects. *Transportation Science* **33**, pp. 233–256.

[14] Petruzzi, N. C. and M. Dada. 1999. Pricing and the newsvendor model: a review with extensions. *Operations Research*, **47**, pp. 183-194.

[15] Porteus, E. 1971. On the optimality of the generalized (s, S) policies. *Management Science*, **17**, pp. 411-426.

[16] Ross, S. 1970. *Applied Probability Models with Optimization Applications*. Holden-Day, San-Francisco.

[17] Scarf, H. 1960. The optimality of (s, S) policies for the dynamic inventory problem. *Proceedings of the 1st Stanford Symposium on Mathematical Methods in the Social Sciences*, Stanford University Press, Stanford, CA.

[18] Whitin, T. M. 1955. Inventory control and price theory. *Management Science*, **2**. pp. 61-80.

[19] Veinott, A. F. 1966. On the optimality of (s, S) inventory policies: New conditions and a new proof. *SIAM Journal on Applied Mathematics*, **14**, No. 5, pp. 1067-1083.

[20] Zheng, Y. S. 1992. A simple proof for optimality of (s, S) policies in infinite-horizon inventory systems. *J. App. Pro.*, **28**, pp. 802-810.