Time-Series Restrictions for the Cross-Section of Expected Returns: Evaluating Multifactor CCAPMs

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Abstract

A number of recent papers have developed multifactor extensions of the classic consumption capital asset pricing model (CCAPM), and found that they perform remarkably well in explaining the cross-section of stock returns. While the extant literature has generally concluded that conditioning information improves the empirical performance of the CCAPM, the empirical work to date has primarily employed cross-sectional regressions that ignore theoretical restrictions on the time-series intercepts in regressions of each test asset return on the model's factors. This paper asks whether the superior empirical performance of the multifactor CCAPMs is maintained once the time-series intercept restrictions have been explicitly tested. The use of mimicking portfolios makes it straightforward to test whether such multifactor CCAPMs satisfy the time-series intercept restrictions, since in this case the single testable implication of the model is that each intercept should be zero. The empirical findings generally support the conclusion that multifactor CCAPMs can explain the cross-section of expected stock returns better than classic unconditional models such as the CAPM and CCAPM.

1 Introduction

A number of recent papers have developed multifactor extensions of the classic consumption capital asset pricing model (hereafter CCAPM), and found that they perform remarkably well in explaining the cross-section of stock returns. Multifactor models can be thought of as scaled versions of the

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standard CCAPM because the weights in the linear factor representations of these models are not fixed, but rather modeled as functions of time t information. These consumption-based models have factors which are not returns, and have been tested using cross-sectional regressions. For models in which factors are returns, the single testable implication is that intercepts from time-series regressions of test asset returns on the factors should be jointly zero. In models where factors are not returns, such time-series intercepts are not unrestricted, but they involve unknown parameters that must be estimated. This complicates testing whether the intercept restrictions hold. As a result such restrictions are typically not tested or imposed, and the performance of the model is evaluated solely on the basis of the cross-sectional fit, which does not impose or test the time-series intercept restrictions.

One way to test whether time-series intercept restrictions of multifactor CCAPMs are satisfied is to use mimicking portfolios, as proposed by Breeden (1979), and Breeden, Gibbons and Litzenberger (1989). By using mimicking portfolios in place of original factors in time-series regressions, tests of the models once again collapse to the single implication that the time-series intercepts must be jointly zero. Breeden, Gibbons and Litzenberger (1989) consider the standard CCAPM with consumption growth as a single factor, and show that the CCAPM holds with respect to a set of test assets when betas are measured relative to the mimicking portfolios obtained from the test assets. I extend these results to settings with multiple non-return factors.

In this paper I test specific multifactor CCAPM models that have been found elsewhere to explain the cross-section of expected stock returns better than the standard unconditional CCAPM. The main question is whether the superior cross-sectional performance of such models is maintained once the time-series intercept restrictions are explicitly recognized. I consider the scaled CCAPM proposed by Lettau and Ludvigson (2001b), in which the consumption-wealth ratio is used as a conditioning variable; the consumption-housing CAPM of Piazzesi, Schneider and Tuzel (2003), in which the non-housing consumption expenditure share is used as a conditioning variable; and the collateral-CCAPM of Lustig and Van Nieuwerburgh (2004), in which the housing collateral ratio is used as a conditioning variable. For comparison, we also test the standard CAPM and two other models, the Fama-French three-factor model (Fama and French (1993)), and the conditional CAPM with labor income of Santos and Veronesi (2004).

The rest of this paper is organized as follows. In section 2 I explain the specifications of unconditional and multifactor CCAPMs. Section 3 presents restrictions on the intercepts in time-series regressions which provide the basis for the cross-sectional asset pricing test. The methodology for testing time-series restrictions using mimicking portfolios in place of factors is also discussed in section 3. Section 4 describes the multifactor CCAPM models I test and discusses which variables are considered as factors in those models. Section 5 describes the data and presents the results of tests. I first compare the pricing errors across the candidate models, and then conduct statistical tests proposed by Gibbons, Ross and Shanken (1989), as well as alternative test by bootstrap. Section 6 concludes.

2 Unconditional vs. Scaled Multifactor CCAPM

I start by motivating the general multifactor extensions of the classic CCAPM. Throughout the paper, we assume that the risk-free rate R_t^f is observed. Let M_{t+1} be the stochastic discount factor. Any tradable asset with return R_{t+1} must satisfy

$$1 = E_t[M_{t+1}R_{t+1}], \tag{1}$$

where E_t denotes the expectation conditional on time t information. For the basic consumption-based model, the asset pricing equation (1) comes from the first-order condition for optimal consumption choice of a representative agent, that is,

$$1 = E_t[\delta\frac{u'(C_{t+1})}{u'(C_t)}R_{t+1}],$$

where $u(C_t)$ is instantaneous utility function, $u'(C_t)$ is marginal utility with consumption C_t, and δ is the subjective discount factor. In this model the stochastic discount factor $M_{t+1} = \delta\frac{u'(C_{t+1})}{u'(C_t)}$ is the intertemporal marginal rate of substitution.

M_{t+1} can be approximated as a linear function of consumption growth,

$$M_{t+1} = a + b\Delta c_{t+1}, \tag{2}$$

where $c_{t+1} = \log(C_t)$ and a, b are parameters. The standard CCAPM of Breeden (1979) specifies these parameters as constant, where consumption growth is the single factor. But if we derive a, b by combining equation (1), (2), and the relation for the risk-free rate which is known at time t

$$1 = E_t[M_{t+1}]R_t^f,$$

then we get

$$\begin{aligned} a &= \frac{1}{R_t^f} - bE_t[\Delta c_{t+1}] \\ b &= \frac{E_t[R_{t+1}] - R_t^f}{R_t^f Cov_t[\Delta c_{t+1}, R_{t+1}]}, \end{aligned}$$

which shows that a, b may vary over time to the extent that conditional moments vary. Based on this, conditional versions of the CCAPM can be written that allow the coefficients to be varying over time such as

$$M_{t+1} = a_t + b_t\Delta c_{t+1}. \tag{3}$$

For example, following Cochrane (1996), the coefficients may be modeled as linear functions of conditioning variables z_t known at time t:

$$\begin{aligned} M_{t+1} &= (a_0 + a_1 z_t) + (b_0 + b_1 z_t)\Delta c_{t+1} \\ &= a_0 + a_1 z_t + b_0 \Delta c_{t+1} + b_1(\Delta c_{t+1} z_t) \\ &= a_0 + A' F_{t+1}, \end{aligned} \tag{4}$$

where

$$A = [a_1 \ b_0 \ \ b_1]', \quad F_{t+1} = [z_t \ \ \Delta c_{t+1} \ \ \Delta c_{t+1} z_t]'.$$

I refer to this as the scaled multifactor CCAPM. The empirical models I consider below are of this form, and differ based on the choice of conditioning variables z_t. I call Δc_{t+1} the *fundamental* factor. In some of the models that I consider below, there are more than one fundamental factors to be scaled by conditioning variables. If we model time-variation of the coefficients explicitly as functions of known conditioning variables, then we can rewrite the conditional single-factor model as an unconditional multifactor model. These models can then be tested unconditionally as multi-factor models with consumption growth, the conditioning variable and the product term as factors. Then, using the unconditional multifactor specification of the stochastic discount factor with constant coefficients, we can derive the unconditional asset pricing equation, by taking unconditional expectation on both sides, as follows:

$$\begin{aligned} 0 &= E_t[M_{t+1} R^e_{t+1}] \quad \Rightarrow \\ 0 &= E[M_{t+1} R^e_{t+1}] \\ &= E[M_{t+1}]E[R^e_{t+1}] + Cov[M_{t+1}, R^e_{t+1}] \\ &= E[M_{t+1}]E[R^e_{t+1}] + A' Cov[F_{t+1}, R^e_{t+1}], \end{aligned} \tag{5}$$

where R^e_{t+1} is excess return over risk-free rate, $R^e_{t+1} = R_{t+1} - R^f_t$. From this unconditional asset pricing equation (5) we can derive the unconditional expected return-beta representation as

$$\begin{aligned} E[R^e_{t+1}] &= -\frac{1}{E[M_{t+1}]} A' Cov[F_{t+1}, R^e_{t+1}] \\ &= -\frac{1}{E[M_{t+1}]} A' Cov[F_{t+1}, F'_{t+1}]\beta \\ &= \beta' \lambda, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \beta &= Cov[F_{t+1}, F'_{t+1}]^{-1} Cov[F_{t+1}, R^e_{t+1}] \\ \lambda &= -\frac{1}{E[M_{t+1}]} Cov(F_{t+1}, F'_{t+1})A. \end{aligned}$$

In this paper we focus on testing the *unconditional* implications of such multifactor models. The unconditional factor model has an expected return-beta representation with constant (unconditional) betas on the multiple factors. So, when we consider the expected return-beta model to impose the restrictions on the time-series regressions, it is appropriate to "unconditionally" estimate the intercepts and the coefficients on the factors from the time-series regressions of the test assets on the multiple factors.

Again, we can think of multifactor models as *conditional* versions of the standard CCAPM. Here the "conditioning" in the conditional CCAPM refers to allowing the coefficients a_t and b_t to depend on time t information. I then go on to test the unconditional implications of the model, as has been done in the paper cited above. An alternative approach would be to test the conditional implications of the scaled multifactor models.[1] Because the models and empirical results discussed above are all based on tests of the unconditional implications of multifactor models, I do not pursue this avenue here.

3 Time-Series Restriction for Cross-Sectional Tests

3.1 Restriction on Time-series Intercept

In this section I discuss the restriction implied by the model for the time-series regressions of test asset returns on factors, and apply this restriction in our setting when the factors are not returns.

Suppose we have a vector of K factors, $f_{t+1} = [f_{1,t+1} \ \cdots \ f_{K,t+1}]'$, and N test asset returns $R^e_{i,t+1}, i = 1, \cdots, N$, excess over the risk-free rate, of which we want to explain the cross-sectional variation in unconditional average. The expected return-beta representation of the linear factor pricing model is, as stated in equation (6),

$$E[R^e_{i,t+1}] = \beta_i' \lambda, \tag{7}$$

where β_i is $K \times 1$ vector of multiple regression coefficients from a time-series regression of the return of asset i, $R^e_{i,t+1}$, on the K factors. β_i can be interpreted as the amount of exposure to the risk captured by each factor, and λ is $K \times 1$ vector of the "price" of such risk. The time-series regression of each test asset return on the factors, used to get β_i, is written

$$R^e_{i,t+1} = \alpha_i + \beta_i' f_{t+1} + \epsilon_{i,t+1}. \tag{8}$$

Taking unconditional expectations on both sides of the time-series regression, we have

$$E[R^e_{i,t+1}] = \alpha_i + \beta_i' E[f_{t+1}]. \tag{9}$$

[1] Ferson and Harvey (1999) tests the conditional implications of the Fama-French three-factor model directly by modeling time-variation in the alphas and betas.

Equating (7) and (9), we have

$$\beta_i' \lambda = \alpha_i + \beta_i' E[f_{t+1}] \Rightarrow \alpha_i = \beta_i'(\lambda - E[f_{t+1}]), \tag{10}$$

resulting in a restriction on the time series intercept of the regression (8).

When the factors f_{t+1} are not returns, the above restriction includes a vector of free parameters λ, which should be estimated. But in the special case where the factors are excess returns, we can simplify this restriction. We can apply the expected return-beta representation to the factors, since the factors are also excess returns. Doing so, the betas on the factors themselves are one and those on the other factors are all zero. From these relations we can derive $\lambda = E[f_{t+1}]$. When f_{t+1} is an excess return, this means the parameters in λ are no longer free, implying that the intercepts in the time-series regressions should be all zero for each test asset. So, the null hypotesis for the cross-sectional test, with time-series intercept restrictions, is

$$H_0 : \alpha_i = 0, \quad i = 1, \cdots, N.$$

3.2 Testing Time-Series Intercept Restrictions

As explained, in case the factors we are interested in are macroeconomic variables, not returns, we cannot directly test the null hypotesis that the intercepts from time-series regressions are jointly zero. One way to deal with this problem is to make use of mimicking portfolios, as proposed by Breeden (1979), and Breeden, Gibbons and Litzenberger (1989). This section extends their analysis to the case of multiple non-return factors. Denote the excess return space as $\overline{R^e}$. Then the mimicking portfolio for the factor f_{t+1}, denoted as f_{t+1}^*, can be represented as projection of the factor on the excess return space:

$$f_{t+1}^* = proj(f_{t+1} \mid \overline{R^e}),$$

In the asset pricing equations for excess return we have

$$E[f_{t+1} \cdot R_{t+1}^e] = E[proj(f_{t+1} \mid \overline{R^e}) \cdot R_{t+1}^e] = E[f_{t+1}^* \cdot R_{t+1}^e],$$

so f_{t+1}^* has the same pricing implication as that of f_{t+1} in the excess return space. In practice this mimicking portfolio is derived from the regression of factors on a set of *base* asset returns, and is therefore a linear combination of those base asset returns. Let's suppose that we choose a vector of M base asset returns $R_{t+1}^b = [R_{1,t+1}^b \cdots R_{M,t+1}^b]'$, where all the returns are excess over the risk-free rate. Then the mimicking portfolio regression for a factor $f_{k,t+1}$ is

$$f_{k,t+1} = \omega_{k,0} + \omega_{k,1} R_{1,t+1}^b + \cdots + \omega_{k,M} R_{M,t+1}^b + \eta_{k,t+1}, \tag{11}$$

with mimicking portfolio $f_{k,t+1}^*$ given as the fitted value from this regression:

$$f_{k,t+1}^* = \hat{\omega}_{k,1} R_{1,t+1}^b + \cdots + \hat{\omega}_{k,M} R_{M,t+1}^b,$$

where "hats" denote estimated coefficients. Here the estimated coefficients on the base assets are used as portfolio weights.

To test the time-series intercept restrictions for models with multiple non-return factors, I form the mimicking portfolio $f^*_{k,t+1}$ for each factor, and use the mimicking portfolios in place of the original factors in time-series regressions (8). To use this mimicking portfolio strategy, we need to verify that, if the expected return-beta representation holds with betas on the factors, then it also holds with betas on the mimicking portfolios.[2] Breeden, Gibbons and Litzenberger (1989) consider the standard CCAPM with consumption growth as the single factor, and choose base assets to be the same as test assets. Then they show that the consumption beta is proportional to consumption-mimicking portfolio beta, a condition that assures that if the model with consumption beta holds, then it also holds with consumption-mimicking portfolio beta, with the price of risk parameter rescaled.

The case that we consider in this paper is more general than that considered in Breeden, Gibbons and Litzenberger (1989), for two reasons. First, because we are interested in testing multifactor models, we need to derive the betas on the factors as the coefficients from the multivariate time-series regressions, so we should check if we can apply the same methodology used for testing single-factor models to our multifactor setting. Second, we may want to consider a larger set of base assets that can be applied to our test.

As to the first question, if we derive the betas on mimicking portfolios from the multivariate time-series regressions, then generally we cannot keep one-by-one proportionality of beta on each factor to beta on mimicking portfolio for that factor. But we can show that if we choose base assets properly, then an expected return-beta representation still holds with betas on mimicking portfolios instead of factors. In general, if the base assets span the test assets, then the expected return-beta representation holds with betas on mimicking portfolios when it holds with betas on factors. For example, when we want to test the model with N test assets $R^e_{t+1} = [R^e_{1,t+1} \cdots R^e_{N,t+1}]$, if we choose M base assets $R^b_{t+1} = [R^b_{1,t+1} \cdots R^b_{M,t+1}]$ such that $R^e_{t+1} = R^b_{t+1}\Gamma$, where Γ is an $M \times N (M \geq N)$ matrix, then we can use mimicking portfolios formed from these base assets, in place of factors, to test the multifactor model. In other words, if the relations

$$E[R^e_{i,t+1}] = \beta_{i1}\lambda_1 + \cdots + \beta_{iK}\lambda_K, \quad i = 1, \cdots, N,$$

hold with $\beta_{i1}, \cdots, \beta_{iK}$ measured for each of K factors, then the following relations

$$E[R^e_{i,t+1}] = \beta^*_{i1}\lambda^*_1 + \cdots + \beta^*_{iK}\lambda^*_K, \quad i = 1, \cdots, N,$$

also hold with $\beta^*_{i1}, \cdots, \beta^*_{iK}$ measured for K factor-mimicking portfolios. It follows that we can perform the cross-sectional test by testing the intercepts from the time-series regressions of the

[2] I refer to "betas on factors" as the coefficients in a multivariate regression of test assets on factors, and "betas on mimicking portfolio" as the coefficients in the multivariate regression of test assets on factor-mimicking portfolios.

test assets on the mimicking portfolios. In Appendix 1 I show that the expected return-beta representation with betas on the mimicking portfolios can be derived from the beta representation with betas on the original factors when we choose a set of base asset returns that span the test asset returns.

4 Description of the Candidate Models

4.1 Unconditional Models

For comparison with the scaled multifactor CCAPMs, we consider two types of unconditional CCAPMs. The benchmark model is the classic CCAPM of Lucas (1978) and Breeden (1979), where consumption growth is the single factor; the specification of this model is given in equation (2).

Recently this model has been augmented to deal with non-separable preference over non-housing consumption and housing consumption.[3] Piazzesi, Schneider and Tuzel (2003) argue that the composition of the consumption bundle is a new risk factor, and they show that under the assumption of CES utility the composition risk factor can be represented as growth of the ratio of non-housing consumption to overall consumption expecditure, or non-housing consumption expenditure share. The stochastic discount factor is augmented by the growth of the non-housing consumption expenditure share. If we denote C_t and H_t as non-housing and housing consumption, with p_t^C and p_t^H as prices of non-housing and housing consumption goods respectively, then the non-housing consumption expenditure share S_t is defined as

$$S_t = \frac{p_t^C C_t}{p_t^C C_t + p_t^H H_t}.$$

Then the stochastic discount factor augmented by the composition risk factor is

$$M_{t+1} = a + b\Delta c_{t+1} + d\Delta s_{t+1}, \tag{12}$$

where $s_{t+1} = \log(S_{t+1})$ and the coefficients a, b and d are considered as constant. Here d depends on the intratemporal elasticity of substitution between non-housing and housing consumption, as well as the coefficient of relative risk aversion. Following Piazzesi, Schneider and Tuzel (2003), I call this model as the (unconditional) consumption-housing CAPM, or CHCAPM.

4.2 Scaled Multifactor Models

As I already explained, I follow several recent empirical papers and capture time-variation of the coefficients in the linearized stochastic discount factor by specifying those coefficients as linear

[3] Earlier work by Dunn and Singleton (1986) and Eichenbaum and Hansen (1990) report substantial evidence against the null of separability in a representative agent model. Earlier work by Flavin (2001) also considers non-separability non-housing and housing consumption.

functions of chosen conditioning variables in the scaled version of CCAPM, as in equaiton (4). As in these empirical papers we can test the unconditional asset pricing implications of the model by interpreting it as unconditional multifactor model.

For the scaled multifactor versions of the CCAPM, I test three empirical models distinguished by different choice of conditioning variables. The first model is the scaled CCAPM proposed by Lettau and Ludvigson (2001b). In their model, a proxy for the log consumption-wealth ratio (hereafter *cay*) is used as a conditioning variable. The variable *cay* is computed as a cointegrating residual between log of consumption, log of asset wealth and log of labor income. Detailed explanation for *cay* can be found in Lettau and Ludvigson (2001a). The second model is the housing-CCAPM considered in Piazzesi, Schneider and Tuzel (2003). They use the non-housing consumption expenditure share as a conditioning variable, and consider the scaled CCAPM by scaling the coefficients of the CCAPM with their conditioning variable. The third model is the collateral-CCAPM derived from the model with housing collateral by Lustig and Van Nieuwerburgh (2004). They consider a heterogeneous agent model with endogenously incomplete market, with collateralized borrowing. The tightness of the borrowing constraint depends on the housing collateral ratio, my_t, which is the ratio of housing wealth to total wealth. They consider a model with separable preferences and a model with non-separable preferences, and I test the model with separable preferences as the scaled CCAPM.

For the scaled versions of CHCAPM, the coefficients in (12) are allowed to be time-varying, so we have

$$M_{t+1} = a_t + b_t \Delta c_{t+1} + d_t \Delta s_{t+1}.$$

Keeping the assumption that the coefficients are linear functions of the chosen conditioning variable, the scaled version of CHCAPM is

$$\begin{aligned} M_{t+1} &= (a_0 + a_1 z_t) + (b_0 + b_1 z_t)\Delta c_{t+1} + (d_0 + d_1 z_t)\Delta s_{t+1} \\ &= a_0 + a_1 z_t + b_0 \Delta c_{t+1} + b_1(\Delta c_{t+1} \cdot z_t) + d_0 \Delta s_{t+1} + d_1(\Delta s_{t+1} \cdot z_t). \end{aligned}$$

I test two models for the scaled CHCAPM. One is the scaled CHCAPM proposed by Piazzesi, Schneider and Tuzel (2003), which I call housing-CHCAPM. This model considers non-housing and housing consumption under the assumption of non-separable utility function. For this model the conditioning variable z_t is the non-housing consumption expenditure share. The other model is the collateral-CHCAPM of Lustig and Van Nieuwerburgh (2004) with non-separable preferences. The conditioning variable z_t in this model is the housing collateral ratio.

4.3 Extensions of CAPM

To further compare the cross-sectional performance of the models above with the classic models, I also test the standard CAPM and two other models which are the extensions of CAPM. The standard CAPM is the single-factor unconditional model with a market portfolio return as a factor, as

$$M_{t+1} = a + bR^M_{t+1},$$

where R^M_{t+1} is a market portfolio return. The second one is the Fama-French three-factor model, which includes the return on a portfolio long in stocks of small-size firms and short in stocks of large-size firms (SMB) and the return on a portfolio long in high book-to-market stocks and short in low book-to-market stocks (HML) as additional factors. It takes the form

$$M_{t+1} = a + bR^M_{t+1} + dSMB_{t+1} + hHML_{t+1}$$

This model is known to have particular success explaining the cross-section of stock returns, especially the size and value effects.

We also test the conditional CAPM with labor income, proposed by Santos and Veronesi (2004). They include two types of returns as factors, one for non-human wealth and the other for human wealth. The return on non-human, or financial, wealth is proxied by a market portfolio return, where the return on human wealth R^W_{t+1} is proxied by labor income growth,[4] respectively. And they use the ratio of labor income to consumption, s^w_t, as a conditioning variable. This model has the form

$$M_{t+1} = a + b_0 R^M_{t+1} + b_1 R^M_{t+1} s^w_t + d_0 R^W_{t+1} + d_1 R^W_{t+1} s^w_t$$

These CAPM-type models have factors which are all excess returns, so we can directly test them using the intercept restrictions on the time-series regressions.

All the candidate models described above are summarized in Table 1, with their specifications of the stochastic discount factors.

5 Empirical Results

In this section we first describe the data, and then present the empirical results. Our data are quarterly, and the full-sample period is 1952:Q1-2002:Q4. We will also present the results from two subsamples, 1952:Q1-1977:Q4 and 1978:Q1-2002:Q4.

5.1 Data Description

Financial Data I use the Fama-French 25 portfolios formed on firm size and book-to-market value, provided by professor Kenneth French. The 25 portfolios are the intersections of 5 portfolios sorted by firm size and 5 portfolios sorted by the ratio of book value to market value of equity.

[4] They use the wage growth as measure of return to human wealth, and measure the excess return as the log of labor income growth minus the risk-free rate. So we consider the labor income growth as return, as they do.

But, instead of taking all of the 25 portfolios as test assets, I perform the cross-sectional tests for two groups chosen out of the 25 portfolios. One group is composed of 10 portfolios chosen by size, and we call this group as "size group". This group takes the 5 portfolios from the smallest-size quintile and the 5 portfolios from the biggest-size quintile. The other group takes the 10 portfolios chosen by book-to-market, and we call this group as "book-to-market group". This group takes the 5 portfolios from the lowest-book-to-market quintile and the 5 portfolios from the highest-book-to-market quintile. The reason we choose 10 asset returns for each group is the following. As shown in Appendix 1, to form the mimicking portfolios which will be used for the cross-sectional tests, we need at least as many base assets as test assets so that the base assets can span the test assets. Recalling that these base assets are regressors in the mimicking portfolio regressions, we need to control the number of base assets given the relatively small time-series sample ($T = 204$ for full sample) in quarterly data.

Summary statistics for the two groups of test assets are presented in Table 2. The values are annualized in real terms. It is observed that on average the stock returns of small-size firms are higher than those of big-size firms by 3.07%, and the stock returns of firms with high book-to-market value are higher than those of frims with low book-to-market value by 5.7%. For each of the size quintile, we observe that the stocks with higher book-to-market value have higher returns. Also for the book-to-market quintile, the returns are higher as size is smaller, though it's not clear in the low-book-to-market quintiles.

As for comparison, Lewellen and Nagel (2004) consider two groups related to the size and book-to-market criteria.[5] One group includes the average of the stock returns in smallest quintile, the average of the stock returns in biggest quintile, and the difference of the two to capture the size-premium. The other group is composed of the average of the stock returns in highest book-to-market quintile, the average of the stock returns in lowest book-to-market quintile, and the difference of the two to capture the value-premium. Then they look at how large the average of conditional time-series intercepts is for each group. In this paper, instead of focusing on the magnitude of intercpts for size premium and value premium separately, we look at 10 time-series intercepts from the time-series regressions of test assets on the mimicking portfolios for each group, and test if the intercepts are jointly zero.

The Fama-French 25 portfolio returns sorted by size and book-to-market value, and the Fama-French three factors, value-weighted market excess returns, SMB and HML are from Kenneth French's website. For the risk-free rate, I use the three-month treasury bill rate, from Federal Reserve Board's website.

Macroeconomic Data The consumption series used in the *cay*-CCAPM of Lettau and Ludvigson

[5] Lewellen and Nagel(2004) also consider another subgroup related to the momentum portfolios. In this paper we focus on the size-related and book-to-market-related portfolios.

(2001b) is slightly different from the consumption measure used in the housing-CCAPM of Piazzesi, Schneider and Tuzel (2003) and collateral-CCAPM of Lustig and Van Nieuwerburgh (2004). That is, the series of consumption expenditure for non-durables and services excluding shoes and clothings is used to measure the consumption flow in *cay*-CCAPM, while housing-CCAPM and collateral-CCAPM use consumption expenditure for nondurables and services excluding housing services as consumption flow measure. For the standard CCAPM, I use the consumption measure used in the *cay*-CCAPM. All the consumption series are real, chain-weighted values in 2000 dollars, from Bureau of Economic Analysis (hereafter BEA) website.

All the returns used in this paper are represented as real values, computed by dividing nominal returns by the inflation rate. The inflation rate is measured from the price index for personal consumption expenditure, which is also from BEA.

Housing consumption and the price of housing and non-housing consumption goods used to compute the expenditure share of non-housing consumption are all from BEA, following the description in Piazzesi, Schneider and Tuzel (2003). The expenditure share of non-housing consumption is used as a conditioning variable for consumption-housing CAPM. The series of consumption-wealth ratio, *cay*, is from Sydney Ludvigson's website, and the series of the housing-collateral ratio, *my*, is from Stijn Van Nieuwerburgh. In Lustig and Van Nieuwerburgh (2004), they consider three different series for the measure of the housing collateral stock and computed the housing collateral ratio for each of the measures. Among these, we use the housing collateral ratio computed using the market value of residential real estate wealth.

5.2 Mimicking Portfolio Regression

For each factor, a mimicking portfolio is obtained from a time-series regression of the factor on a set of base asset returns. We have already discussed appropriate ways to choose the base assets in mimicking portfolio regressions in practice. One way is to follow Breeden, Gibbons and Litzenberger (1989) and keep the base asset returns the same as the test asset returns. In our notation, their choice corresponds to the case of $\Gamma = I$ where I is the $N \times N$ identity matrix, in the restriction $R^e_{t+1} = R^b_{t+1}\Gamma$. I use this approach here.

The results of the mimicking portfolio regressions are presented in Table 3. The first column shows (adjusted) R-squared from the regressions of the factors on the 10 returns in the size group, and the second column shows (adjusted) R-squared from the regressions of the factors on the 10 returns in the book–to-market group. We normalize the portfolio weight on each of the base asset returns so that the weights sum to one, by dividing each coefficient by sum of all coefficients.

Overall the (adjusted) R-squared are not so high, considering that the mimicking portfolio means the linear combination of the base assets to give the "maximum correlation" with the factor. Sometimes it is suggested to use dynamic mimicking portfolio, where the portfolio weights are time-varying, as a way to achieve higher R-squared, such as

$$f_{k,t+1} = \omega_{k0,t} + \omega_{k1,t} R^b_{1,t+1} + \cdots + \omega_{kM,t} R^b_{M,t+1} + \eta_{k,t+1}. \tag{13}$$

Usually the time-variation of the portfolio weights is captured by specifying the weights in the mimicking portfolio regressions as linear functions of some conditioning variables. For example, the following dynamic mimicking portfolio regression for each factor $f_{k,t+1}$, with base assets $[R^b_{1,t+1} \cdots R^b_{M,t+1}]$ and conditioning variable z_t can be run:

$$f_{k,t+1} = (\omega^0_{k0} + \omega^1_{k0} z_t) + (\omega^0_{k1} + \omega^1_{k1} z_t) R^b_{1,t+1} + \cdots + (\omega^0_{kM} + \omega^1_{kM} z_t) R^b_{M,t+1} + \eta_{k,t+1}, \tag{14}$$

implying that the dynamic mimicking portfolio is

$$f^*_{k,t+1} = (\hat{\omega}^0_{k1} + \hat{\omega}^1_{k1} z_t) R^b_{1,t+1} + \cdots + (\hat{\omega}^0_{kM} + \hat{\omega}^1_{kM} z_t) R^b_{M,t+1}.$$

It is observed that, by capturing the time-variation of portfolio weights, we can get higher R-squared from the mimicking portfolio regressions. For example, if we run a mimicking portfolio regression of the consumption growth on the 10 base assets for each group with time-varying coefficients captured using *cay* as a conditioning variable, then we have R-squared 0.24 and 0.25 (adjusted R-squared 0.15 and 0.16) for size and book-to-market group respectively in full sample, much higher than those from the constant-weight mimicking portfolio regressions.

Then, is it better to use dynamic mimicking portfolios since we can achieve higher R-squared? If one wants to test the *conditional* implications of the CCAPM models considered here, then this approach makes sense, since the portfolio weights are derived as functions of conditional covariance between factors and base asset returns. In this case if we estimate the dynamic mimicking portfolios and derive the conditional betas on the dynamic mimicking portfolios, then we can show that the conditional expected return-beta representations hold with dynamic mimicking portfolios when the conditional beta representations hold with the original factors, with a proper choice of a set of base assets. But our focus is on the test of the unconditional implications of the models, and we need to use fixed weights. Appendix 2 shows that we can use dynamic mimicking portfolios when we perform the test of the conditional implications of the models. But I also show that, if we use dynamic mimicking portfolios to test the unconditional implications, then the expected return-beta representations may not hold with betas on the dynamic mimicking portfolios even if the representations hold with betas on the factors. Based on these results, we form the constant-weight mimicking portfolios and use them for the unconditional tests of conditional CCAPMs.[6]

[6] Breeden, Gibbons and Litzenberger (1989) also comments about this point saying that "constant weights are appropriate for the empirical work focuses on unconditional moments"(p.248).

5.3 Test from the Time-Series Regressions

5.3.1 Pricing Error

As one dimension to compare the cross-sectional performance across the models, we first compare each model's pricing errors. The time-series intercepts can be interpreted as pricing errors when the factors are returns, and since we use mimicking portfolios instead of factors, our approach provides an easy way to compare different versions of CCAPMs in terms of pricing errors by looking at the magnitude of the time-series intercepts. In Table 4, we present the square root of the average squared pricing error, $\sqrt{\frac{1}{N}\sum_{i=1}^{N}\alpha_i^{*2}}$, for the N test assets and time-series intercept α_i^* from the regression of test asset i on the mimicking portfolios, for each model. The values presented in Table 4 are in quarterly percentage units.

With full sample it can be found that, for both CCAPM and CHCAPM, the scaled models produce smaller average squared pricing errors than the unconditional models, that is, their time-series intercepts are smaller. For the CCAPMs, all of the scaled models have smaller average squared pricing errors compared with the unconditional model. And for the CHCAPMs, especially the collateral-CHCAPM produce smaller pricing errors with both the size and the book-to-market groups than the unconditional model. Similar patterns can be found in the subsamples. The results show that the scaled models have smaller pricing errors than the classic unconditional models. The *cay*-CCAPM and the housing-CCAPM shows smaller average squared pricing errors than the unconditional CCAPM, and the housing-CHCAPM and collateral-CHCAPM perform as well as, or better than, the unconditional CHCAPM in lowering the magnitude of pricing errors.

Let's compare the average squared pricing errors of consumption-based models with the CAPM-type models. Among the CAPM and its extensions, the Fama-French three-factor model performs best in terms of the average squared pricing errors. This model also shows smaller magnitude of errors than the classic and scaled CCAPMs, as well as the unconditional CHCAPM. The Fama-French three-factor model is known to explain the size and value effect very well. But, we can find that the scaled CHCAPMs have smaller magnitude of pricing errors than Fama-French three-factor model in some cases. For example, in full sample and especially in the second subsample, the housing-CHCAPM shows smaller average squared pricing errors than the Fama-French three-factor model with size group. And, in the first subsample, both the housing-CHCAPM and collateral-CHCAPM outperform Fama-French three-factor model in lowering the pricing errors.

5.3.2 GRS Test

Now we perform the statistical tests of the null hypothesis that, for each model, the intercept terms from the time-series regressions of test assets on mimicking portfolios are jointly zero. Let's remind that we denote mimicking portfolio for factor $f_{k,t+1}$ as $f^*_{k,t+1}$, $k = 1, \cdots K$, and also the vector of mimicking portfolios as $f^*_{t+1} = [f^*_{1,t+1} \cdots f^*_{K,t+1}]$ where K is the number of factors in the model under evaluation. Then the time-series regressions, which provide measures of betas on the mimicking portfolios, are

$$R^e_{i,t+1} = \alpha^*_i + \beta^{*\prime}_i f^*_{t+1} + \epsilon^*_{i,t+1}, \quad i = 1, \cdots, N. \tag{15}$$

The null hypotesis for the cross-section test is

$$H_0 : \alpha^*_i = 0, \quad \forall i.$$

Gibbons, Ross and Shanken (1989) derive the appropriate finite-sample test statistics and its distribution under the null hypothesis, assuming that the regression residuals are jointly normally distributed. The GRS test statistics is given as

$$\frac{T-N-K}{N}[1 + E_T(f^*)'\hat{\Omega}^{-1}E_T(f^*)]^{-1}\hat{\alpha}^{*\prime}\hat{\Sigma}^{-1}\hat{\alpha}^* \sim F(N, T-N-K), \tag{16}$$

where

$$\begin{aligned}
\hat{\alpha}^* &= [\hat{\alpha}^*_1 \ \cdots \ \hat{\alpha}^*_N]' \\
E_T(f^*) &= \frac{1}{T}\sum_{t=1}^{T} f^*_t \\
\hat{\Omega} &= \frac{1}{T}\sum_{t=1}^{T}[f^*_t - E_T(f^*)][f^*_t - E_T(f^*)]' \\
\hat{\Sigma} &= \frac{1}{T}\sum_{t=1}^{T}\hat{\epsilon}^*_t\hat{\epsilon}^{*\prime}_t \ , \ \ \hat{\epsilon}^*_t = [\hat{\epsilon}^*_{1,t} \ \cdots \ \hat{\epsilon}^*_{N,t}]'.
\end{aligned}$$

The GRS test statistics and p-values for size and book-to-market groups in full sample are summarized in Table 5. As we can see, the null hypothesis that the time-series intercepts are jointly zero is rejected at the 5% significance level for both the unconditional and scaled CCAPMs, Santos-Veronesi model and the Fama-French three-factor model, as well as the standard CAPM. The only model that is not rejected is the collateral-CHCAPM, the scaled CHCAPM which uses the housing collateral ratio as a conditioning variable, but the unconditional CHCAPM is rejected. Compared with the unconditional CHCAPM, adding conditioning variable capturing time-variation of the coefficients in the stochastic discount factor is shown to improve the cross-sectional performance, both for size and book-to-market groups.

Now we do the GRS test for two subsamples. For each subsample we estimate mimicking portfolios again and run the time-series regressions with the new mimicking portfolios. Those results are presented in Table 6 for the first subsample and in Table 7 for the second subsample respectively. For the first subsample, the unconditional CCAPM is rejected, but the scaled versions of the CCAPM, *cay*-CCAPM, housing-CCAPM and collateral-CCAPM, are not rejected at the 5% significance level, when we perform the tests with size group. None of the unconditional and scaled CHCAPMs are rejected, and Fama-French three-factor model is not rejected either. With book-to-market group, no candidate models are rejected.

But, with the second subsample, it becomes much more difficult to explain the size and value effects with any of these models, as the results show many rejections of the candidate models in Table 7. For the size group, the unconditional and all of the scaled CCAPMs, as well as all of the classic and the extensions of CAPM including the Fama-French three-factor model, are rejected. But here we observe that the scaled CHCAPMs are not rejected while the unconditional CHCAPM is rejected at the 5% significance level. Again, the scaled versions of the CHCAPM improves the cross-sectional performance of the unconditional model, according to this test. For book-to-market group, however, all of the candidate models are rejected, which means that in the latter subsample the scaled multifactor models are not enough to explain the cross-sectional variation of the average stock returns of the firms with the highest and lowest book-to-market values.

5.3.3 Distributional Test for Residuals

The GRS test is based on the assumption that residual terms from the time-series regressions follow normal distribution. But the normality assumption has been pointed out as a problem by a number of papers (Zhou (1993), Dufour, Khalaf and Beaulieu (2003), Beaulieu, Dufour and Khakaf (2004)). These papers find that the null hypothesis that the residuals of the time-series regressions are jointly normally distributed is frequently rejected in common applications which test the classic CAPM.[7] Also they argue that if we test the CAPM based on the assumption that the residual terms follow normal distribution, but these residuals actually follow alternative fat-tail distributions, then we tend to reject the CAPM too often from the GRS test.

To address this issue, I perform the distributional goodness-of-fit test for Normality of the residuals, based on the multivariate skewness and kurtosis measures proposed by Mardia (1970). Let $X_1, \cdots, X_T$ be the observations on an $N \times 1$ random vector over the period T. The multivariate skewness and kurtosis statistics are defined as

$$\begin{aligned} SK &= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} r_{ts}^3 \\ KU &= \frac{1}{T}\sum_{t=1}^{T} r_{tt}^2, \end{aligned}$$

where $r_{ts} = (X_t - \bar{X})' S^{-1} (X_s - \bar{X})$ and $\bar{X}$ and S are the sample mean and sample covariance

[7] Zhou(1993), Dufour, Khalaf and Beaulieu(2003), Beaulieu, Dufour and Khalaf(2004) test the mean-variance efficiency of the market portfolio return under the assumption that the residual distributions follow either Student-t or mixture-of-normal distributions, using Monte-Carlo simulation.

matrices, respectively. Under the null hypothesis that $X_1, \cdots, X_T$ follow multivariate normal distribution, it is derived that SK and KU are distributed as follows:

$$\frac{T}{6} \cdot SK \sim \chi^2(\nu),$$

where ν is the degree of freedom determined as

$$\nu = \frac{N(N+1)(N+2)}{6},$$

and

$$\frac{KU - N(N+2)}{(\frac{8N(N+2)}{T})^{\frac{1}{2}}} \sim N(0,1).$$

Based on these measures of multivariate skewness and kurtosis and their distributions as defined above, we can test for the null hypothesis that the residual terms follow the multivariate normal distribution. Mardia (1970) proposes the combined skewness-kurtosis test statistic for multivariate Normality for the case when X follows a multivariate normal distribution,

$$CSK = \frac{T}{6}SK + \frac{T[KU - N(N+2)]^2}{8N(N+2)} \sim \chi^2(\frac{N(N+1)(N+2)}{6} + 1). \tag{17}$$

Table 8 reports p-values from the distributional test for normality based on the CSK statistics for the full sample and the two subsamples. Except for the standard CCAPM in the first subsample, the null hypothesis that the residuals follow the normal distribution is strongly rejected for all the models.

5.3.4 Bootstrap Test

Since we observe the strong rejection of normality for all the models, there is reason to doubt the validity of the GRS test presented above. To address this issue, I perform a bootstrap test. Again, we test the model using the time-series regressions (15), under the null hypothesis that the time-series intercepts are jointly zero. I form bootstrap test statistics using the Wald test statistic

$$W = T \cdot [1 + E_T(f^*)'\hat{\Omega}^{-1}E_T(f^*)]^{-1}\hat{\alpha}^{*\prime}\hat{\Sigma}^{-1}\hat{\alpha}^*.$$

Under the null hypothesis, the Wald test statistic is asymptotically distributed as a chi-square distribution with degree of freedom equal to the number of test assets. For this procedure, it is not required that the time-series residuals follow a normal distribution in finite sample. It can be shown that the test statistics have a well-behaved asymptotic distribution, which is a necessary condition for consistency of the bootstrap.

In practice, I perform the bootstrap tests following the suggestion in Horowitz (2003) and MacKinnon (2002). First I obtain the residuals from the time-series regressions of the test asset returns on the mimicking portfolios, using the original data. Then I generate bootstrap errors by resampling the residuals with replacement. Here I use a block bootstrap with block-length chosen following the recommendation of Horowitz (2003).[8] Since the OLS residuals $\hat{\epsilon}^*_{t+1}$ have smaller variance than the population errors, we need to rescale the OLS residuals when generating the bootstrap errors $\tilde{\epsilon}_{t+1}$:

$$\tilde{\epsilon}^*_{t+1} = \sqrt{\frac{T}{T-k}}\hat{\epsilon}^*_{t+1},$$

where k is the number of regressors including constant. Using these bootstrap errors, we create the bootstrap sample of test asset returns $\tilde{R}^e_{t+1}$, with imposing the null hypothesis, by constructing

$$\tilde{R}^e_{t+1} = 0 + \beta^{*\prime} f^*_{t+1} + \tilde{\epsilon}^*_{t+1},$$

where β^* are the multiple regression coefficients from a regression of the test asset on the multiple factors, using the original data.

Using the bootstrap sample, I re-run the time-series regressions of $\tilde{R}^e_{t+1}$ on the mimicking portfolios and compute the Wald test statistics. By iterating these procedures 1000 times, we can generate an empirical confidence interval of the test statistics for each of the candidate models. One possible caveat is that, since I use the mimicking portfolios estimated from the original data, the procedure does not take into account that the regressors are generated in a first-stage regressions. To the best of my knowledge, this problem has not been worked out in the literature. In future work, I plan to determine both how important such a correction might be to the test statistics, and how to implement it in the bootstrap.

The empirical 95% confidence intervals and the results of the bootstrap tests based on these estimated confidence intervals for each candidate model are presented in Table 9 for the full sample and in Table 10 and Table 11 for the first and second subsamples, respectively. In the full sample the bootstrap test shows basically the same results as the GRS test. That is, only the collateral-CHCAPM is not rejected; the Wald test statistics based on the original data fall within the empirical 95% confidence interval, for both size and book-to-market groups. For these models, the bootstrap test reinforces the conclusion that specific scaled multifactor models can explain the cross-section of test asset returns better than the unconditional models.

The results for the size group in two subsamples are somewhat different from those with the GRS test. In the first subsample (Table 10), no model is rejected from the bootstrap test with 95% empirical confidence interval. But in the second subsample (Table 11), we find that the housing-CCAPM and collateral-CCAPM, which are scaled multifactor extensions of the classic CCAPM, are not rejected from the test, where the unconditional CCAPM is rejected. For book-to-market

[8] It is recommended in Horowitz(2003) that the asymptotically optimal block-length is $l \sim T^{\frac{1}{4}}$ for estimating the one-sided distribution function.

group, the boostrap test shows the same results as the GRS test, that is, none of the models are rejected in the first subsample, but all the models are rejected in the second subsample.

In conclusion, I found that some scaled multifactor CCAPMs are not rejected statistically, where the unconditional models almost always are, including often the Fama-French three-factor model. However, it is inappropriate to make model comparisons based on these test statistics, since they do not tell us whether one model's pricing errors are different from another's. In particular, the Fama-French three-factor model and the housing-CHCAPM have much lower pricing errors than the other models (Table 4), even though they are statistically rejected.

6 Conclusion

This paper performs cross-sectional tests of scaled multifactor CCAPMs, by explicitly considering the theoretical restrictions on the time-series intercepts. For models whose factors are all returns, the restriction is simple: the time-series intercepts should be jointly zero. But for models in which the factors are not returns, such as the CCAPM and the multifactor extensions that have been investigated in the recent empirical literature, the models cannot be directly tested with this restriction. So, to test the CCAPMs by applying the time-series intercept restrictions, I use factor-mimicking portfolios, constructed by projecting the original factors on the proper choice of base assets. Those mimicking portfolios are used in the time-series regressions in place of the original factors.

I show that if the expected return-beta representation holds with betas on the original factors, then the beta representation also holds with betas on the mimicking portfolios, when we choose the set of base assets that spans the test assets. By using the mimicking portfolios in place of the original factors, we can transform the model into one in which the single testable implication is that the time-series intercepts be jointly zero.

This method provides an explicit way to take into account these theoretical restrictions, when evaluating the cross-sectional performance of scaled multifactor CCAPM models. Recent studies have found that the scaled CCAPM can explain the cross-sectional variation of the expected stock returns much better than the standard unconditional CCAPM, but these results are usually based on the cross-sectional regressions, ignoring the time-series intercept restrictions. The mimicking portfolio approach employed in this paper makes it possible to test the models, and to check if the superior cross-sectional performance of the scaled CCAPMs can be maintained when the time-series intercept restrictions are explicitly considered.

As candidate models, I consider several versions of unconditional and scaled CCAPMs. For unconditional models, I consider the classic single-factor CCAPM with consumption growth as the single factor, and also the consumption-housing CAPM, or CHCAPM, with composition risk factor as additional fundamental factor derived from the assumption of non-separable preferences between non-housing and housing consumption goods.

For the scaled multifactor CCAPM models, I test the models that have proven successful empirically: the *cay*-CCAPM of Lettau and Ludvigson (2001b), a three-factor model in which the consumption-wealth ratio is used as a conditioning variable, the housing-CCAPM of Piazzesi, Schneider and Tuzel (2003), in which the non-housing consumption expenditure share is used as a conditioning variable, and the collateral-CCAPM of Lustig and Van Nieuwerburgh (2004), in which the housing collateral ratio is used as a conditioning variable, respectively. Following Piazzesi, Schneider and Tuzel (2003) and Lustig and Van Nieuwerburgh (2004), I test two types of their models: three-factor models with separable preferences as the scaled CCAPM, and five-factor models with non-separable preferences as the scaled CHCAPM.

The empirical findings show that the scaled multifactor versions of CCAPM and CHCAPM can explain the cross-section of expected stock returns better than the corresponding unconditional models. In terms of the pricing errors, the scaled CCAPMs and CHCAPMs deliver a smaller magnitude of average squared pricing errors compared to the unconditional models, and in some cases the scaled versions of CHCAPM models, the housing-CHCAPM and the collateral-CHCAPM, outperform the Fama-French three-factor model in lowering the pricing error.

From a statistical perspective, we do the GRS test and also the bootstrap test of the candidate models. For the GRS test, both in the full sample and in two subsamples we observe that some candidate scaled CCAPMs and CHCAPMs are not rejected while the corresponding unconditional models are rejected. Considering the questions on the Normality assumptions for the time-series residuals for the validity of the GRS test raised by several studies, I also do a bootstrap test by estimating the empirical confidence intervals which do not depend on the Normality assumptions in finite sample. The results from the bootstrap test mainly support the results from the GRS test.

As is already explained in (10), in models where the factors are not returns, the time-series intercepts are not unrestricted, but the restrictions involve free parameters that must be estimated. In this paper, I use factor-mimicking portfolios in place of the original factors to take account of these restrictions and eliminate the free parameters. This is not the only way such restrictions can be evaluated, however. An alternative is to directly impose the time-series intercept restrictions in Generalized Method of Moments estimation of the model. Since this approach can both impose the appropriate restrictions and estimate the free parameters upon which it depends, Generalized Method of Moments can be used to test the restricted version of the model. I plan to pursue this in future work.

7 Tables

Table 1 : Summary of the Candidate Models

Abbreviation	Description
	Unconditional Models
CAPM	classic CAPM
	$M_{t+1} = a + bR^M_{t+1}$
CCAPM	classic CCAPM
	$M_{t+1} = a + b\Delta c_{t+1}$
CHCAPM	consumption-housing CAPM
	$M_{t+1} = a + b\Delta c_{t+1} + d\Delta s_{t+1}$
FF	Fama-French three-factor model
	$M_{t+1} = a + bR^M_{t+1} + dSMB_{t+1} + hHML_{t+1}$
	Scaled Multifactor Models
cay-CCAPM	Lettau-Ludvigson model, Scaled CCAPM
	$M_{t+1} = a_0 + a_1 cay_t + b_0\Delta c_{t+1} + b_1\Delta c_{t+1} cay_t$
housing-CCAPM	Piazzesi-Schneider-Tuzel model, Scaled CCAPM
	$M_{t+1} = a_0 + a_1 s_t + b_0\Delta c_{t+1} + b_1\Delta c_{t+1} s_t$
collateral-CCAPM	Lustig-Van Nieuwerburgh model, Scaled CCAPM
	$M_{t+1} = a_0 + a_1 my_t + b_0\Delta c_{t+1} + b_1\Delta c_{t+1} my_t$
housing-CHCAPM	Piazzesi-Schneider-Tuzel model, Scaled CHCAPM
	$M_{t+1} = a_0 + a_1 s_t + b_0\Delta c_{t+1} + b_1\Delta c_{t+1} s_t + d_0\Delta s_{t+1} + d_1\Delta s_{t+1} s_t$
collateral-CHCAPM	Lustig-Van Nieuwerburgh model, Scaled CHCAPM
	$M_{t+1} = a_0 + a_1 my_t + b_0\Delta c_{t+1} + b_1\Delta c_{t+1} my_t + d_0\Delta s_{t+1} + d_1\Delta s_{t+1} my_t$
SV	Santos-Veronesi model
	$M_{t+1} = a_0 + b_0 R^M_{t+1} + b_1 R^M_{t+1} s^w_t + d_0 R^W_{t+1} + d_1 R^W_{t+1} s^w_t$

R^M_{t+1}- market portfolio return, Δc_{t+1}- consumption growth

Δs_{t+1}- growth of non-housing consumption expenditure share

SMB_{t+1}- small minus big, HML_{t+1}- high minus low

cay_t- consumption-wealth ratio, my_t- housing collateral ratio

R^W_{t+1}- return on human wealth, s^w_t- ratio of labor income to consumptio

Table 2 : Summary statistics for test assets					
size group			b-m group		
	mean	std.dev		mean	std.dev
s1b1	5.39	31.04	s1b1	5.39	31.04
s1b2	11.59	26.61	s2b1	6.89	27.82
s1b3	12.02	23.31	s3b1	8.20	25.02
s1b4	14.89	22.27	s4b1	8.78	22.75
s1b5	15.86	24.19	s5b1	8.13	18.27
E[s1]	11.95		E[b1]	7.48	
s5b1	8.12	18.27	s1b5	15.87	24.19
s5b2	8.13	16.15	s2b5	14.55	21.95
s5b3	9.28	14.64	s3b5	13.34	20.65
s5b4	9.19	15.53	s4b5	12.50	20.51
s5b5	9.65	17.81	s5b5	9.65	17.81
E[s5]	8.88		E[b5]	13.18	

Notes - This table summarizes sample mean and standard deviation of the returns in size and book-to-market groups, in %. Size group includes 5 returns from the smallest size quintile and 5 returns from the biggest size quintile. Book-to-market group includes 5 returns from the lowest book-to-market quintile and 5 returns from the highest book-to-market quintile. Means are annualized by multiplying by 4 and standard deviations are multiplied by 2. All the returns are in real value, divided by inflation rate.

Table 3 : R^2 ($\bar{R}^2$) - mimicking portfolio regression

	size group	b-m group
	standard CCAPM	
$\Delta \log c_{t+1}$	0.09 (0.04)	0.05 (0.01)
	Lettau-Ludvigson model	
$\Delta \log c_{t+1}$	0.09 (0.04)	0.05 (0.01)
cay_t	0.11 (0.06)	0.12 (0.08)
$\Delta \log c_{t+1} \cdot cay_t$	0.07 (0.02)	0.08 (0.03)
	Piazzesi-Schneider-Tuzel model	
$\Delta \log c_{t+1}$	0.09 (0.05)	0.06 (0.01)
$\Delta \log c_{t+1} \cdot s_t$	0.09 (0.05)	0.06 (0.01)
s_t	0.03 (0.00)	0.06 (0.01)
$\Delta \log s_{t+1}$	0.06 (0.02)	0.07 (0.02)
$\Delta \log s_{t+1} \cdot s_t$	0.06 (0.02)	0.07 (0.02)
	Lustig-Van Nieuwerburgh model	
$\Delta \log c_{t+1}$	0.09 (0.05)	0.06 (0.01)
$\Delta \log c_{t+1} \cdot my_t$	0.05 (0.00)	0.05 (0.00)
my_t	0.06 (0.01)	0.05 (0.00)
$\Delta \log s_{t+1}$	0.06 (0.02)	0.07 (0.02)
$\Delta \log s_{t+1} \cdot my_t$	0.06 (0.02)	0.07 (0.03)

Notes - R^2 ($\bar{R}^2$) from mimicking portfolio regressions of each factor on the base asset returns. Base assets for size group are same as test assets in size group, and base assets for book-to-market group are same as test assets in book-to-market group.

Table 4 : Average squared pricing error						
$R^e_{i,t+1} = \alpha^*_i + \beta^{*\prime}_i f^*_{t+1} + \epsilon^*_{i,t+1} \quad i = 1, \cdots, N$						
	size	b-m	size	b-m	size	b-m
	1952:1-2002:4		1952:1-1977:4		1978:1-2002:4	
	Unconditional Models					
CAPM	0.920	1.098	0.723	0.893	1.198	1.391
CCAPM	0.953	0.979	0.593	0.598	1.772	2.074
CHCAPM	0.792	0.784	0.535	0.410	2.054	2.425
FF	0.596	0.640	0.489	0.488	0.859	0.931
	Scaled Multifactor Models					
cay-CCAPM	0.863	0.890	0.580	0.593	1.942	1.743
housing-CCAPM	0.707	0.874	0.546	0.565	1.276	1.585
collateral-CCAPM	0.805	0.944	0.816	0.769	2.265	2.825
housing-CHCAPM	0.561	0.850	0.451	0.404	0.350	2.895
collateral-CHCAPM	0.715	0.721	0.434	0.415	1.027	3.952
SV	0.954	1.083	1.009	0.984	1.123	1.288

Notes - This table reports the square root of average squared pricing errors across the test assets for size and book-to-market groups, which are measured by $\sqrt{\frac{1}{N}\sum_{i=1}^{N}\alpha_i^{*2}}$ in full sample and two subsamples. α^*_i are the time-series intercept, interpreted as pricing error, for each of the test assets, in quarterly percentage unit.

Table 5 : GRS statistics and p-value : 1952:Q1 - 2002:Q4				
$R^e_{i,t+1} = \alpha^*_i + \beta^{*\prime}_i f^*_{t+1} + \epsilon^*_{i,t+1} \quad i = 1, \cdots, N$				
$H_0 : \alpha^*_i = 0, \forall i$				
	size group		b-m group	
	GRS statistics	p-value	GRS statistics	p-value
	Unconditional Models			
CAPM	5.833	0.000	4.969	0.000
CCAPM	4.660	0.000	4.710	0.000
CHCAPM	4.155	0.000	4.171	0.000
FF	4.447	0.000	3.666	0.000
	Scaled Multifactor Models			
cay-CCAPM	4.376	0.000	4.424	0.000
housing-CCAPM	3.844	0.000	3.940	0.000
collateral-CCAPM	3.570	0.000	3.428	0.000
housing-CHCAPM	2.550	0.007	3.773	0.000
collateral-CHCAPM	**1.865**	**0.053**	**1.762**	**0.070**
SV	5.389	0.000	4.466	0.000

Notes - This table reports the GRS statistics and p-values from the time-series regressions of each test asset $R^e_{i,t+1}$ on the mimickong portfolios f^*_{t+1} in full sample. Bold letters correspond to the model which is not rejected at 5% significance level.

Table 6 : GRS statistics and p-value : 1952:Q1 - 1977:Q4

$$R^e_{i,t+1} = \alpha^*_i + \beta^{*\prime}_i f^*_{t+1} + \epsilon^*_{i,t+1} \quad i = 1, \cdots, N$$

$$H_0 : \alpha^*_i = 0, \forall i$$

	size group		b-m group	
	GRS statistics	*p*-value	GRS statistics	*p*-value
	Unconditional Models			
CAPM	2.601	0.008	**1.821**	**0.068**
CCAPM	1.979	0.044	**1.230**	**0.283**
CHCAPM	**1.758**	**0.080**	**0.921**	**0.518**
FF	**1.796**	**0.072**	**0.956**	**0.488**
	Scaled Multifactor Models			
cay-CCAPM	**1.922**	**0.052**	**1.203**	**0.300**
housing-CCAPM	**1.789**	**0.074**	**1.104**	**0.368**
collateral-CCAPM	**1.526**	**0.143**	**1.077**	**0.389**
housing-CHCAPM	**1.606**	**0.118**	**0.881**	**0.554**
collateral-CHCAPM	**0.590**	**0.818**	**0.534**	**0.862**
SV	3.020	0.003	**1.646**	**0.107**

Notes - This table reports the GRS statistics and p-values from the time-series regressions of each test asset on the mimicking portfolios in the first subsample. Bold letters correspond to the model which is not rejected at 5% significance level.

Table 7 : GRS statistics and p-value : 1978:Q1 - 2002:Q4

$$R^e_{i,t+1} = \alpha^*_i + \beta^{*\prime}_i f^*_{t+1} + \epsilon^*_{i,t+1} \quad i = 1, \cdots, N$$

$$H_0 : \alpha^*_i = 0, \forall i$$

	size group		b-m group	
	GRS statistics	p-value	GRS statistics	p-value
	Unconditional Models			
CAPM	5.318	0.000	5.953	0.000
CCAPM	3.875	0.000	6.288	0.000
CHCAPM	3.231	0.001	5.664	0.000
FF	4.800	0.000	5.337	0.000
	Scaled Multifactor Models			
cay-CCAPM	3.721	0.000	6.040	0.000
housing-CCAPM	2.668	0.007	4.700	0.000
collateral-CCAPM	2.367	0.016	3.875	0.000
housing-CHCAPM	**0.159**	**0.998**	3.765	0.000
collateral-CHCAPM	**1.048**	**0.412**	3.189	0.001
SV	4.598	0.000	5.051	0.000

Notes - This table reports the GRS statistics and p-values from the time-series regressions of each test asset on the mimicking portfolios in the second subsample. Bold letters correspond to the model which is not rejected at 5% significance level.

Table 8 : Distributional test : p-value						
	size	b-m	size	b-m	size	b-m
	1952:1-2002:4		1952:1-1977:4		1978:1-2002:4	
	Unconditional Models					
CAPM	0.00	0.00	0.00	0.00	0.00	0.00
CCAPM	0.00	0.00	0.01	**0.06**	0.00	0.00
CHCAPM	0.00	0.00	0.00	0.00	0.00	0.00
FF	0.00	0.00	0.01	0.00	0.00	0.00
	Scaled Multifactor Models					
cay-CCAPM	0.00	0.00	0.00	0.00	0.00	0.00
housing-CCAPM	0.00	0.00	0.00	0.00	0.00	0.00
collateral-CCAPM	0.00	0.00	0.00	0.00	0.00	0.00
housing-CHCAPM	0.00	0.00	0.00	0.00	0.00	0.00
collateral-CHCAPM	0.00	0.00	0.00	0.00	0.00	0.00
SV	0.00	0.00	0.00	0.00	0.00	0.00

Notes - This table reports the p-value for combined multivariate skewness and kurtosis test statistics from chi-square distribution. The test statistics are computed under the null hypothesis that the residuals from the time-series regressions follow multivariate normal distribution. Bold letters correspond to the model which is not rejected.

Table 9 : bootstrap results : 1952:Q1 - 2002:Q4				
	size group		b-m group	
	95% confidence interval	W_{data}	95% confidence interval	W_{data}
	Unconditional Models			
CAPM	[1.25 40.91]	61.67	[1.56 38.61]	52.53
CCAPM	[0.90 38.36]	49.27	[1.51 34.03]	49.79
CHCAPM	[0.95 35.63]	43.92	[0.66 29.03]	44.10
FF	[1.44 38.77]	47.51	[1.91 34.45]	39.17
	Scaled Multifactor Models			
cay-CCAPM	[0.33 35.38]	46.76	[0.47 33.61]	47.27
housing-CCAPM	[0.59 28.93]	41.08	[0.84 29.35]	42.09
collateral-CCAPM	[0.94 29.46]	38.15	[0.24 28.15]	36.63
housing-CHCAPM	[0.09 25.75]	27.53	[0.06 23.28]	40.74
collateral-CHCAPM	[0.22 27.51]	**20.13**	[0.14 22.60]	**19.02**
SV	[1.36 40.50]	57.88	[0.80 36.35]	47.96

Notes - This table reports the empirical 95% confidence interval from bootstrap and Wald test statistics from the original data in full sample. The empirical confidence interval from bootstrap is based on the bootstrap sample of test assets created by resampling the rescaled residuals with replacement for 1000 times. W_{data} denotes the value of Wald test statistics from original data. Bold letters correspond to the model which is not rejected.

Table 10 : bootstrap results : 1952:Q1 - 1977:Q4				
	size group		b-m group	
	95% confidence interval	W_{data}	95% confidence interval	W_{data}
	Unconditional Models			
CAPM	[0.51 46.30]	**29.12**	[1.52 49.96]	**20.39**
CCAPM	[1.03 39.64]	**22.16**	[0.36 49.05]	**13.77**
CHCAPM	[0.85 34.44]	**19.69**	[0.79 32.83]	**10.31**
FF	[1.21 35.86]	**20.56**	[1.53 52.88]	**10.94**
	Scaled Multifactor Models			
cay-CCAPM	[0.21 36.40]	**22.00**	[0.38 32.99]	**13.76**
housing-CCAPM	[0.61 33.59]	**20.47**	[0.25 53.03]	**12.63**
collateral-CCAPM	[0.87 27.67]	**17.47**	[0.40 49.21]	**12.32**
housing-CHCAPM	[0.13 31.77]	**18.80**	[0.11 25.11]	**10.31**
collateral-CHCAPM	[0.19 23.87]	**6.91**	[0.42 31.49]	**6.25**
SV	[1.66 38.78]	**34.95**	[1.94 43.69]	**19.05**

Notes - This table reports the empirical 95% confidence interval from bootstrap and Wald test statistics from the original data in the first subsample. The empirical confidence interval from bootstrap is based on the bootstrap sample of test assets created by resampling the rescaled residuals with replacement for 1000 times. W_{data} denotes the value of Wald test statistics from original data. Bold letters correspond to the model which is not rejected.

Table 11 : bootstrap results : 1978:Q1 - 2002:Q4				
	size group		b-m group	
	95% confidence interval	W_{data}	95% confidence interval	W_{data}
		Unconditional Models		
CAPM	[1.48 50.45]	59.75	[1.97 48.52]	66.88
CCAPM	[1.53 42.27]	43.53	[1.86 41.67]	70.65
CHCAPM	[1.26 43.19]	**36.31**	[0.73 30.55]	63.64
FF	[1.42 51.49]	55.17	[1.28 52.14]	61.35
		Scaled Multifactor Models		
cay-CCAPM	[0.60 29.07]	42.77	[0.81 33.71]	69.42
housing-CCAPM	[0.30 33.69]	**30.67**	[0.57 38.57]	54.03
collateral-CCAPM	[0.99 36.85]	**27.20**	[0.50 37.74]	44.54
housing-CHCAPM	[0.12 24.93]	**1.87**	[0.18 35.62]	44.30
collateral-CHCAPM	[0.31 30.38]	**12.32**	[0.16 30.60]	37.52
SV	[1.68 52.17]	53.46	[2.28 50.00]	58.73

Notes - This table reports the empirical 95% confidence interval from bootstrap and Wald test statistics from the original data in the second subsample. The empirical confidence interval from bootstrap is based on the bootstrap sample of test assets created by resampling the rescaled residuals with replacement for 1000 times. W_{data} denotes the value of Wald test statistics from original data. Bold letters correspond to the model which is not rejected.

References

[1] Beaulieu, Marie-Claude, Dufour, Jean-Marie, and Khalaf, Lynda, 2004, Multivariate tests of mean-variance efficiency with possibly non-Gaussian errors: an exact simulation-based approach, Unpublished paper, Université de Montréal.

[2] Breeden, Douglas T., 1979, An intertemporal asset pricing model with stochastic consumption and investment opportunities, *Journal of Financial Economics* 7, 265-296.

[3] Breeden, Douglas T.,Gibbons, Michael R., and Litzenberger, Robert H., 1989, Empirical test of the consumption-oriented CAPM, *Journal of Finance* 44, 231-262.

[4] Campbell, John Y., 1996, Understanding risk and return, *Journal of Political Economy* 104, 298-345.

[5] Campbell, John Y., Lo, Andrew W., and MacKinlay, A. Craig, 1997, The Econometrics of Financial Markets, Princeton University Press, Princeton, N.J.

[6] Chen, Xiaohong, and Ludvigson, Sydney, 2004, Land of addicts? An empirical investigation of habit-based asset pricing model, Unpublished paper, New York University.

[7] Cochrane, John H., 1996, A cross-sectional test of an investment-based asset pricing model, *Journal of Political Economy* 104, 572-621.

[8] Cochrane, John H., 2001, Asset Pricing, Princeton University Press, Princeton, N.J.

[9] Dufour, Jean-Marie, Khalaf, Lynda, and Beaulieu, Marie-Claude, 2003, Exact skewness-kurtosis tests for multivariate normality and goodness-of-fit in multivariate regressions with application to asset pricing models, *Oxford Bulletin of Economics and Statistics* 65, 891-906.

[10] Dunn, Kenneth, and Singleton, Kenneth, 1986, Modeling the term structure of interest rates under nonseparable utility and durability of goods, *Journal of Financial Economics* 17, 27-55.

[11] Fama, Eugene F., and French, Kenneth R., 1992, The cross-section of expected returns, *Journal of Finance* 47, 427-465.

[12] Fama, Eugene F., and French, Kenneth R., 1993, Common risk factors in the returns on stocks and bonds, *Journal of Financial Economics* 33, 3-56.

[13] Fama, Eugene F., and French, Kenneth R., 1996, Multifactor Explanations of Asset Pricing Anomalies, *Journal of Finance* 51, 55-84.

[14] Ferson, Wayne, and Harvey, Campbell, 1999, Conditioning variables and the cross section of stock returns, *Journal of Finance* 54, 1325-1360.

[15] Flavin, Marjorie, Owner-occupied housing in the presence of adjustment costs: implications for asset pricing and nondurable consumption, Unpublished paper, UCSD.

[16] Gibbons, Michael R., Ross, Stephen A., and Shanken, Jay, 1989, A test of the efficiency of a given portfolio, *Econometrica* 57, 1121-1152.

[17] Horowitz, Joel L., 2003, The bootstrap, *Handbook of Econometrics*, ed. by J. J. Heckman, and E. Leamer, vol.5, Elsevier Science B.V., North Holland.

[18] Jagannathan, Ravi, and Wang, Zhenyu, 1996, The conditional CAPM and the cross-section of expected returns, *Journal of Finance* 51, 3-53.

[19] Lamont, Owen A., 2001, Economic tracking portfolios, *Journal of Econometrics* 105, 161-184.

[20] Lettau, Martin, and Ludvigson, Sydney, 2001a, Consumption, aggregate wealth and expected stock returns, *Journal of Finance* 56, 815-849.

[21] Lettau, Martin, and Ludvigson, Sydney, 2001b, Resurrecting the (C)CAPM: a cross-sectional test when risk premia are time-varying, *Journal of Political Economy* 109, 1238-1287.

[22] Lewellen, Jonathan, and Nagel, Stefan, 2004, The conditional CAPM does not explain asset-pricing anomalies, Unpublished paper, MIT Sloan.

[23] Lustig, Hanno, and Van Nieuwerburgh, Stijn, 2004, Housing collateral, consumption insurance and risk premia: an empirical perspective, *Journal of Finance,* forthcoming.

[24] MacKinlay, A. Craig, and Richardson, Matthew, 1991, Using generalized method of moments to test mean-variance efficiency, *Journal of Finance* 46, 511-527.

[25] MacKinnon, James G., 2002, Bootstrap inference in econometrics, *Canadian Journal of Economics* 35, 615-645.

[26] Mardia, K. V., 1970, Measures of multivariate skewness and kurtosis with applications, *Biometrika* 57, 519-530.

[27] Piazzesi, Monika, Schneider, Martin, and Tuzel, Selale, 2003, Housing, consumption, and asset pricing, Unpublished paper, UCLA.

[28] Santos, Tano, and Veronasi, Pietro, 2004, Labor income and predictable stock returns, Unpublished paper, University of Chicago GSB.

[29] Zhou, Guofu, 1993, Asset-pricing tests under alternative distributions, *Journal of Finance* 48, 1927-1942.

Appendix 1

In appendix 1, I show that if a set of base asset returns can span the test asset returns, then mimicking portfolios can be used instead of factors to test the multifactor model. Suppose that I have K factor $f_{1,t+1}, \cdots, f_{K,t+1}$, and N test asset returns $R^e_{1,t+1}, \cdots, R^e_{N,t+1}$, and I choose M base asset $R^b_{1,t+1}, \cdots, R^b_{M,t+1}$ $(M \geq N)$ such that test asset returns can be generated as linear combinations of the base asset returns. To simplify, I denote f as $T \times K$ matrix of the factors, R^e as $T \times N$ matrix of the test asset returns, and R^b as $T \times M$ matrix of the base assets, with sample size of T. Then this choice of the base assets means that we have $M \times N$ matrix Γ satisfying $R^e = R^b\Gamma$.

As the first step, I derive betas on the factors from the time-series regressions of the test asset returns on the factors as follows.

$$R^e_{i,t+1} = \alpha_i + \beta_{i1} f_{1,t+1} + \cdots + \beta_{iK} f_{K,t+1} + \epsilon_{i,t+1}, \quad i = 1, \cdots, N.$$

From these multivariate regressions, a $K \times N$ matrix of betas on the factors is given as

$$\begin{aligned} \beta &= \begin{bmatrix} \beta_{11} & \cdots & \beta_{N1} \\ \vdots & \ddots & \vdots \\ \beta_{1K} & \cdots & \beta_{NK} \end{bmatrix} \\ &= Cov[f', f]^{-1} Cov[f', R^e]. \end{aligned}$$

Next I consider the mimicking portfolio regression of each factor on the base asset returns as

$$f_{j,t+1} = \omega_{j0} + \omega_{j1} R^b_{1,t+1} + \cdots + \omega_{jM} R^b_{M,t+1} + \eta_{j,t+1}, \quad j = 1, \cdots, K.$$

From these regressions we have an $M \times K$ matrix of portfolio weighs

$$\begin{aligned} \omega &= \begin{bmatrix} \omega_{11} & \cdots & \omega_{K1} \\ \vdots & \ddots & \vdots \\ \omega_{1M} & \cdots & \omega_{KM} \end{bmatrix} \\ &= Cov[R^{b\prime}, R^b]^{-1} Cov[R^{b\prime}, f], \end{aligned}$$

and a $T \times K$ matrix of mimicking portfolios

$$f^* = R^b \omega.$$

Then betas on the mimicking portfolios can be estimated from the time-series regressions of the test asset returns on the mimicking portfolios as

$$R^e_{i,t+1} = \alpha^*_i + \beta^*_{i1} f^*_{1,t+1} + \cdots + \beta^*_{iK} f^*_{K,t+1} + \epsilon^*_{i,t+1}, \quad i = 1, \cdots, N,$$

and a $K \times N$ matrix of betas on the mimicking portfolios is given as follows.

$$\begin{aligned}\beta^* &= \begin{bmatrix} \beta^*_{11} & \cdots & \beta^*_{N1} \\ \vdots & \ddots & \vdots \\ \beta^*_{1K} & \cdots & \beta^*_{NK} \end{bmatrix} \\ &= Cov[f^{*\prime}, f^*]^{-1} Cov[f^{*\prime}, R^e].\end{aligned}$$

Using the expression for f^* and ω, we can derive β^*, betas on the mimicking portfolios, with a set of base assets R^b given the test assets:

$$\begin{aligned}\beta^* &= (\omega' Cov[R^{b\prime}, R^b]\omega)^{-1}(\omega' Cov[R^{b\prime}, R^e]) \\ &= (Cov[R^{b\prime}, f]' Cov[R^{b\prime}, R^b]^{-1} Cov[R^{b\prime}, R^b] Cov[R^{b\prime}, R^b]^{-1} Cov[R^{b\prime}, f])^{-1} \cdot \\ &\quad (Cov[R^{b\prime}, f]' Cov[R^{b\prime}, R^b]^{-1} Cov[R^{b\prime}, R^e]) \\ &= (Cov[R^{b\prime}, f]' Cov[R^{b\prime}, R^b]^{-1} Cov[R^{b\prime}, f])^{-1} Cov[R^{b\prime}, f]' Cov[R^{b\prime}, R^b]^{-1} Cov[R^{b\prime}, R^e].\end{aligned}$$

Suppose that I choose a set of base asset returns which span the test asset returns, satisfying the relation $R^e = R^b\Gamma$. Then β^* can be rewritten as

$$\begin{aligned}\beta^* &= (Cov[R^{b\prime}, f]' Cov[R^{b\prime}, R^b]^{-1} Cov[R^{b\prime}, f])^{-1} Cov[R^{b\prime}, f]' Cov[R^{b\prime}, R^b]^{-1} Cov[R^{b\prime}, R^b]\Gamma \\ &= (Cov[R^{b\prime}, f]' Cov[R^{b\prime}, R^b]^{-1} Cov[R^{b\prime}, f])^{-1} Cov[f', R^b\Gamma] \\ &= (Cov[R^{b\prime}, f]' Cov[R^{b\prime}, R^b]^{-1} Cov[R^{b\prime}, f])^{-1} Cov[f', f] Cov[f', f]^{-1} Cov[f', R^e] \\ &= \Pi\beta\end{aligned}$$

where

$$\Pi = (Cov[R^{b\prime}, f]' Cov[R^{b\prime}, R^b]^{-1} Cov[R^{b\prime}, f])^{-1} Cov[f', f].$$

Here betas on the mimicking portfolios are linear combinations of betas on the factors. Generally one-by-one proportionality of beta on the mimicking portfolio to beta on the corresponding factor that Breeden, Gibbons and Litzenberger (1989) derive in the single-factor model cannot be maintained in the multifactor model.[9] If $K \times K$ matrix Π is nonsingular, we have the relation $\beta = \Pi^{-1}\beta^*$. Let's denote

$$\Pi^{-1} = \begin{bmatrix} \pi_{11} & \cdots & \pi_{1K} \\ \vdots & \ddots & \vdots \\ \pi_{K1} & \cdots & \pi_{KK} \end{bmatrix},$$

then I have the relations, for all $i = 1, \cdots, N$,

$$\begin{aligned}\beta_{i1} &= \pi_{11}\beta^*_{i1} + \cdots + \pi_{1K}\beta^*_{iK} \\ &\vdots \\ \beta_{iK} &= \pi_{K1}\beta^*_{i1} + \cdots + \pi_{KK}\beta^*_{iK}.\end{aligned}$$

[9] The special case that we can have proportionality is when Π is diagonal.

Suppose that the expected return-beta representations hold for the model, such as

$$E(R^e_{i,t+1}) = \beta_{i1}\lambda_1 + \cdots + \beta_{iK}\lambda_K, \quad i = 1, \cdots, N,$$

where λ_k is the parameter for the price of risk exposure to the factor f_k, $k = 1, \cdots, K$. Using the above relation between β and β^*, I can rewrite these representations as

$$\begin{aligned} E(R^e_{i,t+1}) &= (\pi_{11}\beta^*_{i1} + \cdots + \pi_{1K}\beta^*_{iK})\lambda_1 + \cdots + (\pi_{K1}\beta^*_{i1} + \cdots + \pi_{KK}\beta^*_{iK})\lambda_K \\ &= \beta^*_{i1}(\pi_{11}\lambda_1 + \cdots + \pi_{K1}\lambda_K) + \cdots + \beta^*_{iK}(\pi_{1K}\lambda_1 + \cdots + \pi_{KK}\lambda_K) \\ &= \beta^*_{i1}\lambda^*_1 + \cdots + \beta^*_{iK}\lambda^*_K, \quad i = 1, \cdots, N. \end{aligned}$$

So, the above results show that if the expected return-beta representations hold with β and λ, then the beta representations also hold with β^* and λ^*, when I choose a set of base asset returns which can span the test asset returns.

For comparison, suppose that I choose base assets such that test asset returns can span the base asset returns, but not vice versa. In this case, with N test assets and M base assets ($N > M$), I have the relation $R^b = R^e\Gamma$ where Γ is $N \times M$ matrix specifying the linear relations. If I apply this relation to the expression of β^*, then I get

$$\begin{aligned} \beta^* &= (Cov[R^{b\prime}, f]'Cov[R^{b\prime}, R^b]^{-1}Cov[R^{b\prime}, f])^{-1}Cov[R^{e\prime}, f]'\Gamma(\Gamma' Cov[R^{e\prime}, R^e]\Gamma)^{-1}\Gamma' Cov[R^{e\prime}, R^e] \\ &= (Cov[R^{b\prime}, f]'Cov[R^{b\prime}, R^b]^{-1}Cov[R^{b\prime}, f])^{-1}Cov[f', f]Cov[f', f]^{-1}Cov[f', R^e] \cdot \\ &\quad \Gamma(\Gamma' Cov[R^{e\prime}, R^e]\Gamma)^{-1}\Gamma' Cov[R^{e\prime}, R^e] \\ &= \Pi\beta\Psi, \end{aligned}$$

where

$$\begin{aligned} \Pi &= (Cov[R^{b\prime}, f]'Cov[R^{b\prime}, R^b]^{-1}Cov[R^{b\prime}, f])^{-1}Cov[f', f] \\ \Psi &= \Gamma(\Gamma' Cov[R^{e\prime}, R^e]\Gamma)^{-1}\Gamma' Cov[R^{e\prime}, R^e]. \end{aligned}$$

So, if $K \times K$ matrix Π and $N \times N$ matrix Ψ are nonsingular, I have $\beta = \Pi^{-1}\beta^*\Psi^{-1}$. With

$$\Pi^{-1} = \begin{bmatrix} \pi_{11} & \cdots & \pi_{1K} \\ \vdots & \ddots & \vdots \\ \pi_{K1} & \cdots & \pi_{KK} \end{bmatrix}, \ \Psi^{-1} = \begin{bmatrix} \psi_{11} & \cdots & \psi_{1N} \\ \vdots & \ddots & \vdots \\ \psi_{N1} & \cdots & \psi_{NN} \end{bmatrix},$$

we have, for all $i = 1, \cdots, N$,

$$\begin{aligned} \beta_{i1} &= (\pi_{11}\beta^*_{11} + \cdots + \pi_{1K}\beta^*_{1K})\psi_{1i} + \cdots + (\pi_{11}\beta^*_{i1} + \cdots + \pi_{1K}\beta^*_{iK})\psi_{ii} \\ &\quad \cdots + (\pi_{11}\beta^*_{N1} + \cdots + \pi_{1K}\beta^*_{NK})\psi_{Ni} \\ &\quad \vdots \\ \beta_{iK} &= (\pi_{K1}\beta^*_{11} + \cdots + \pi_{KK}\beta^*_{1K})\psi_{1i} + \cdots (\pi_{K1}\beta^*_{i1} + \cdots + \pi_{KK}\beta^*_{iK})\psi_{ii} + \\ &\quad \cdots + (\pi_{K1}\beta^*_{N1} + \cdots + \pi_{KK}\beta^*_{NK})\psi_{Ni}. \end{aligned}$$

What is the difference between this case and the previous case? In this case, if I express betas on the factors as functions of betas on the mimicking portfolios, then the betas of the test asset i on the factor k, β_{ik}, is a function not only of $\beta^*_{i1} \cdots \beta^*_{iK}$ but also of $\beta^*_{j1} \cdots \beta^*_{jK}$, $j \neq i$. As in the first case, if I look at the expected return-beta representations with betas on the mimicking portfolios β^*, then I have

$$\begin{aligned} E[R^e_{i,t+1}] &= \{(\pi_{11}\beta^*_{11} + \cdots + \pi_{1K}\beta^*_{1K})\psi_{1i} + \cdots + (\pi_{11}\beta^*_{N1} + \cdots + \pi_{1K}\beta^*_{NK})\psi_{Ni}\}\lambda_1 + \cdots \\ &\quad + \{(\pi_{K1}\beta^*_{11} + \cdots + \pi_{KK}\beta^*_{1K})\psi_{1i} + \cdots + (\pi_{K1}\beta^*_{N1} + \cdots + \pi_{KK}\beta^*_{NK})\psi_{Ni}\}\lambda_K \\ &= (\pi_{11}\psi_{1i} + \cdots + \pi_{K1}\psi_{1i})\beta^*_{11} + \cdots + (\pi_{1K}\psi_{1i} + \cdots + \pi_{KK}\psi_{1i})\beta^*_{1K} + \cdots \\ &\quad + (\pi_{11}\psi_{ii} + \cdots + \pi_{K1}\psi_{ii})\beta^*_{i1} + \cdots + (\pi_{1K}\psi_{ii} + \cdots + \pi_{KK}\psi_{ii})\beta^*_{iK} + \cdots \\ &\quad + (\pi_{11}\psi_{Ni} + \cdots + \pi_{K1}\psi_{Ni})\beta^*_{N1} + \cdots + (\pi_{1K}\psi_{Ni} + \cdots + \pi_{KK}\psi_{Ni})\beta^*_{NK}. \end{aligned}$$

From these representations it is clear that the expected return-beta representation for $E[R^e_{i,t+1}]$ includes not just $\beta^*_{i1}, \cdots, \beta^*_{iK}$, but also $\beta^*_{j1}, \cdots, \beta^*_{jK}$ ($j \neq i$). This is because, when I derive the relations between the betas on the factors and betas on the mimicking portfolios, I have post-multiplied matrix Ψ^{-1}. So even if the representation for test asset i holds with betas on the factors, I cannot transform the representation as linear combination of the asset i's betas on the mimicking portfolios, but the other assets' betas are included in the representation. Compared with this case, when the base assets can span the test assets, I can cancel out the post-multiplied matrix, and keep $\beta_{i1}, \cdots, \beta_{iK}$ as functions of $\beta^*_{i1}, \cdots, \beta^*_{iK}$. And this makes it possible to rewrite the expected return-beta representation for each test asset with betas on factors into the representation with betas on the mimicking portfolios.

Appendix 2

In appendix 2, I show that the expected return-beta representations hold conditionally with betas on dynamic mimicking portfolios if the conditional beta models hold with betas on factors, but if I derive unconditional betas on the dynamic mimicking portfolios then I may not transform the unconditional expected return-beta representations with betas on the original factors into the beta representations with betas on the dynamic mimicking portfolios.

The conditional, or time-varying, betas of the test assets R^e_{t+1} on the factors f_{t+1} can be represented as

$$\beta_t = Cov_t[f'_{t+1}, f_{t+1}]^{-1} Cov_t[f'_{t+1}, R^e_{t+1}].$$

From the mimicking portfolio regression of each factor $f_{k,t+1}$ on a set of base assets with general expression of time-varying weights,

$$f_{k,t+1} = \omega_{k0,t} + \omega_{k1,t} R^b_{1,t+1} + \cdots + \omega_{kM,t} R^b_{M,t+1} + \eta_{k,t+1},$$

an $M \times K$ matrix of portfolio weights

$$\omega_t = \begin{bmatrix} \omega_{11,t} & \cdots & \omega_{K1,t} \\ \vdots & \ddots & \vdots \\ \omega_{1M,t} & \cdots & \omega_{KM,t} \end{bmatrix},$$

(the first subscript denotes the index for each factor, and the second subscript denotes the index for each base asset) can be derived such as

$$\omega_t = Cov_t[R^{b\prime}_{t+1}, R^b_{t+1}]^{-1} Cov_t[R^{b\prime}_{t+1}, f_{t+1}],$$

from which we derive the dynamic mimicking portfolios $f^{\,*}_{t+1} = R^b_{t+1}\omega_t$. So, if I derive the conditional betas on the dynamic mimicking portfolios, then

$$\begin{aligned}
\beta^*_t &= Cov_t[f^{*\prime}_{t+1}, f^*_{t+1}]^{-1} Cov_t[f^{*\prime}_{t+1}, R^e_{t+1}] \\
&= (\omega'_t Cov_t[R^{b\prime}_{t+1}, R^b_{t+1}]\omega_t)^{-1} \omega'_t Cov_t[R^{b\prime}_{t+1}, R^e_{t+1}] \\
&= (\omega'_t Cov_t[R^{b\prime}_{t+1}, R^b_{t+1}]\omega_t)^{-1} Cov_t[R^{b\prime}_{t+1}, f_{t+1}]' Cov_t[R^{b\prime}_{t+1}, R^b_{t+1}]^{-1} Cov_t[R^{b\prime}_{t+1}, R^e_{t+1}].
\end{aligned}$$

By choosing the base asset returns which can span the test assets such as $R^e = R^b\Gamma$, β^*_t can be rewritten as

$$\begin{aligned}
\beta^*_t &= (\omega'_t Cov_t[R^{b\prime}_{t+1}, R^b_{t+1}]\omega_t)^{-1} Cov_t[R^{b\prime}_{t+1}, f_{t+1}]' Cov_t[R^{b\prime}_{t+1}, R^b_{t+1}]^{-1} Cov_t[R^{b\prime}_{t+1}, R^b_{t+1}]\Gamma \\
&= (\omega'_t Cov_t[R^{b\prime}_{t+1}, R^b_{t+1}]\omega_t)^{-1} Cov_t[f'_{t+1}, R^b_{t+1}\Gamma] \\
&= \Pi_t \beta_t
\end{aligned}$$

where

$$\Pi_t = (\omega_t' Cov_t[R_{t+1}^{b\prime}, R_{t+1}^b]\omega_t)^{-1} Cov_t[f_{t+1}', f_{t+1}].$$

So, as in Appendix 1, if Π_t is nonsingular then I have $\beta_t = \Pi_t^{-1}\beta_t^*$, and by this relation I can show that if the conditional expected return-beta representaions hold with betas on the factors, then those conditional beta representations also hold with betas on the dynamic mimicking portfolios. In other words, if

$$E_t(R_{i,t+1}^e) = \beta_{i1,t}\lambda_{1,t} + \cdots + \beta_{iK,t}\lambda_{K,t}, \quad i = 1, \cdots, N,$$

hold, then also the followings hold:

$$E_t(R_{i,t+1}^e) = \beta_{i1,t}^*\lambda_{1,t}^* + \cdots + \beta_{iK,t}^*\lambda_{K,t}^*, \quad i = 1, \cdots, N.$$

Let's consider the case that I derive the unconditional betas on the dynamic mimicking portfolios and use them for the unconditional beta representaions. Here I use the specific assumption that the portfolio weights are linear functions of a conditioning variable z_t which is known at time t and show that the unconditional expected return-beta representations with constant betas on dynamic-mimicking portfolios

$$f_{k,t+1}^* = (\omega_{k,1}^0 + \omega_{k,1}^1 z_t)R_{1,t+1}^b + \cdots + (\omega_{k,N}^0 + \omega_{k,N}^1 z_t)R_{N,t+1}^b,$$

formed from the regressions

$$f_{k,t+1} = \omega_{k,0}^0 + \omega_{k,0}^1 z_t + (\omega_{k,1}^0 + \omega_{k,1}^1 z_t)R_{1,t+1}^b + \cdots + (\omega_{k,N}^0 + \omega_{k,N}^1 z_t)R_{N,t+1}^b + \eta_{k,t+1}, \ k = 1, \cdots, K,$$

may not hold even if the representations hold with betas on factors. For simplicity, I choose a set of base asset returns same as the test asset returns, $R_{t+1}^e = R_{t+1}^b$, with specifying $\Gamma = I$. First define $T \times (2N+1)$ matrix

$$R_{t+1}^Z = [z_t \ \ R_{1,t+1}^e z_t \ \cdots \ R_{N,t+1}^e z_t \ R_{1,t+1}^e \cdots R_{N,t+1}^e]$$

then I can denote the base asset, same as test asset, as $R_{t+1}^e = [R_{1,t+1}^e \cdots R_{N,t+1}^e] = R_{t+1}^Z D_0$ with D_0 is $(2N+1) \times N$ matrix

$$D_0 = \begin{bmatrix} 0_{(N+1)\times N} \\ I_{N\times N} \end{bmatrix}.$$

Using these notations, I derive the betas on the factors

$$\begin{aligned} \beta &= \begin{bmatrix} \beta_{11} & \cdots & \beta_{N1} \\ \vdots & \ddots & \vdots \\ \beta_{1K} & \cdots & \beta_{NK} \end{bmatrix} \\ &= Cov[f_{t+1}', f_{t+1}]^{-1} Cov[f_{t+1}', R_{t+1}^e] = Cov[f_{t+1}', f_{t+1}]^{-1} Cov[f_{t+1}', R_{t+1}^Z] D_0. \end{aligned}$$

Next, I form the dynamic mimicking portfolios from the mimicking portfolio regressions specified above such as

$$\begin{aligned} f^*_{t+1} &= (\omega^0_{k,1} + \omega^1_{k,1} z_t) R^b_{1,t} + \cdots + (\omega^0_{k,N} + \omega^1_{k,N} z_t) R^b_{N,t} \\ &= R^Z_{t+1} D_1 \varpi, \end{aligned}$$

where

$$\begin{aligned} \varpi &= Cov[R^{Z\prime}_{t+1}, R^Z_{t+1}]^{-1} Cov[R^{Z\prime}_{t+1}, f_{t+1}] \\ D_1 &= \begin{bmatrix} 0 & 0_{1\times 2N} \\ 0_{2N\times 1} & I_{2N\times 2N} \end{bmatrix}. \end{aligned}$$

I have $(2N+1)\times(2N+1)$ matrix D_1 with zero in the upper left corner because R^Z contains the conditioning variable, ϖ contains the coefficient on the conditioning variable, and I just want to have a linear combination of base asset returns.

Then, I estimate betas on the dynamic mimicking portfolios from the time series regressions

$$R^e_{i,t} = \alpha^*_i + \beta^*_{i1} f^*_{1,t} + \cdots + \beta^*_{iK} f^*_{K,t} + \epsilon^*_{i,t}, \quad i = 1, \cdots, N,$$

and I get

$$\begin{aligned} \beta^* &= Cov[f^{*\prime}, f^*]^{-1} Cov[f^{*\prime}, R^e] = Cov[f^{*\prime}, f^*]^{-1} Cov[f^{*\prime}, R^Z] D_0 \\ &= (\varpi' D_1' Cov[R^{Z\prime}, R^Z] D_1 \varpi)^{-1} \varpi' D_1' Cov[R^{Z\prime}, R^Z] D_0 \\ &= (\varpi' D_1' Cov[R^{Z\prime}, R^Z] D_1 \varpi)^{-1} Cov[R^{Z\prime}, f]' Cov[R^{Z\prime}, R^Z]^{-1} D_1' Cov[R^{Z\prime}, R^Z] D_0. \end{aligned}$$

If $D_1' = I$, then I can get the relation $\beta^* = \Pi\beta$, the condition that we need to satisfy the relation between the expected return-beta representations with betas on the original factors and beta representaions with betas on the mimicking portfolios, as in Appendix 1. But with $D_1' \neq I$, I cannot derive $\beta^* = \Pi\beta$ condition. The reason is as follows. When I derive the portfolio weights conditionally, the weights are functions of conditional covariance between the base asset returns and the factors, and the covariance between the factors and conditioning variable can be ignored because the conditioning variable is known at time t. But when I run the dynamic mimicking portfolio regressions unconditionally, the covariances between the factors and conditioning variable are included in the coefficients. So I need to add D_1 to pick up only the linear combinations of the base asset returns, which prevents the relation $\beta^* = \Pi\beta$ from holding.