Likelihood-based Estimation of Dynamic Models of Delegated Portfolio Management

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November 15, 2007

Abstract

I develop a new approach to estimate managerial ability and risk preferences in dynamic models of delegated portfolio management in complete markets. The main complication is that the dynamics of assets under management can often not be discretized exactly. In addition, the investment strategy is not known analytically so that standard discretization methods cannot be applied. I propose to use the martingale method introduced by Cox and Huang (1989) to solve the problem. This method links assets under management at each time and state to the state-price density. The state-price density can be discretized easily. This mapping is unique and invertible under mild conditions on the investor’s preferences. I combine this mapping with the Jacobian formula to construct the exact likelihood of the model, up to a univariate expectation. I use a simple model to explain the method and to illustrate its accuracy.

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This paper develops an estimation methodology for dynamic models of delegated portfolio management in complete markets. A key feature of these models is that they are of finite nature. This contrasts with the large class of dynamic models that are studied in the finance literature that are infinite-horizon or one-period models.\textsuperscript{1} So far, there exists no estimation method to study dynamic finite-horizon models empirically, which leaves the rich set of cross-equation restrictions unexplored. This paper is the first to propose a likelihood-based method that allows researcher to directly estimate a large class of dynamic models.

One approach would be to solve the model numerically by means of dynamic programming and to use the resulting optimal strategy in estimation. However, this is computationally expensive for two reasons. First, despite the recent advances in solving these models numerically,\textsuperscript{2} they all suffer from the curse of dimensionality. The optimization problem becomes high-dimensional if the number of assets increases. In addition, it requires simulations or high-dimensional quadrature methods to integrate out financial uncertainty. In contrast, the method proposed in this paper is insensitive to the number of assets and involves only one-dimensional integration, regardless of the number of assets. Second, it is often the case that the problem is inhomogeneous in assets under management (see Koijen (2007) for an example). The investment problem then features one endogenous state variable, assets under management. This implies that one needs to construct a grid in each period and solve for the optimal policies along the grid.\textsuperscript{3} This is computationally expensive, in particular when the utility index features local convexities or kinks. This requires a fine grid points to accurately solve the problem around the kink (Binsbergen and Brandt (2007)). My method also side-steps this problem and the computational effort does not depend on the homogeneity of the utility index.

The main idea is to use the martingale method developed by Cox and Huang (1989). This approach first solves for the optimal terminal level of assets, subject to a (static) budget constraint. Since markets are dynamically complete, it is always possible to replicate this claim. The replicating strategy is the optimal investment strategy that maximizes the manager’s utility. The resulting dynamics of assets under management is often complex, which impedes exact discretization. This also implies that standard stabilization techniques to mitigate the discretization bias\textsuperscript{4} cannot be applied. However, the martingale method results in an exact mapping from assets under management to

\begin{itemize}
  \item \textsuperscript{1}In the area of delegated portfolio management, Becker, Ferson, Myers, and Schill (1999) estimate a single-period (that is, myopic) market-timing model.
  \item \textsuperscript{2}See Balduzzi and Lynch (1999), Brandt, Goyal, Santa-Clara, and Stroud (2005), and Koijen, Nijman, and Werker (2007).
  \item \textsuperscript{3}Koijen, Nijman, and Werker (2007) modify the method of Brandt, Goyal, Santa-Clara, and Stroud (2005) to efficiently handle endogenous state variables, and they provide a fast way to compute the optimal portfolio in case of multiple assets. This modification has been applied to models of delegated portfolio management by Chapman, Evans, and Xu (2007). Despite these modifications, it can take minutes to solve for the optimal portfolio strategy, which renders it intractable for estimation purposes.
  \item \textsuperscript{4}See for instance Ait-Sahalia (2002), Ait-Sahalia (2007), Bakshi and Ju (2005), and DeTemple, Garcia, and Rindisbacher (2003).
\end{itemize}
the state-price density. I exploit this mapping to recover the time series of the state-price
density. By combining the martingale method with the Jacobian formula, I can construct
the likelihood of fund returns based on the time series of the state-price density. This
involves only one computational step, which is to compute a one-dimensional expectation
for which I use Gaussian quadrature. I show that the resulting likelihood and point
estimates are hardly affected for a low number of quadrature points. As such, the method
is relatively fast. I illustrate the estimation method in a simple model in which all
computations can also be done analytically. Koijen (2007) provides further empirical
results for this model and studies alternative, more complex, models for which analytical
results are unavailable.

Structural models are parametric in nature and therefore maximum likelihood is the
preferred estimation procedure. This paper offers a procedure to compute the likelihood.
The standard asymptotic theory directly applies, and I can perform tests based on the
likelihood and compute the asymptotic standard errors using the outer-product gradient
estimator. Also, it is possible to take a Bayesian approach if one has prior views on the
model parameters.

The method may have several other applications. For instance, it is possible to
recover preferences and ability of hedge fund managers (Panageas and Westerfield (2007)),
to estimate games with strategic interaction (Basak and Makarov (2007)), or to solve
dynamic corporate finance models. The extent to which this method is useful in estimating
alternative models is left for future research.

This paper summarizes the method and demonstrates its effectiveness in a simple
model in which the likelihood can also be computed analytically. I show the impact of the
number of quadrature points on the likelihood and the maximum-likelihood estimates.
Finally, I provide a list of potential extensions. These extensions include (i) multiple
assets, (ii) stochastic volatility, and (iii) more general asset price dynamics.

1 Financial market

The financial market is of the Black and Scholes type\(^5\) that is comprised of a passive
portfolio (or benchmark, \(S^P_t\)), an active portfolio (\(S^A_t\)), and a cash account (\(S^0_t\)):

\[
\frac{dS^P_t}{S^P_t} = (r + \sigma_P \lambda_P) \, dt + \sigma_P \, dZ^P_t, \quad (1)
\]

\[
\frac{dS^A_t}{S^A_t} = (r + \sigma_A \lambda_A) \, dt + \sigma_A \, dZ^A_t, \quad (2)
\]

\[
\frac{dS^0_t}{S^0_t} = r \, dt, \quad (3)
\]

\(^5\)Section 4 considers extensions of the financial market model.
with $Z_P^t$ and $Z_A^t$ standard Brownian motions that are independent, $\sigma_P$ and $\sigma_A$ the volatilities, $\lambda_P$ and $\lambda_A$ the prices of risk. The instantaneous short rate is denoted by $r$. The dynamic budget constraint for assets under management reads:

$$\frac{dA_t}{A_t} = (r + x^t_1\Sigma\Lambda)dt + x^t_1\Sigma dZ_t,$$  \hspace{1cm} (4)

where $x_t \in \mathbb{R}^2$ denotes the fractions invested in the passive and active portfolio and the remainder, $1 - x^t_1$, is invested in cash. I further define: $\Sigma \equiv \text{diag}(\sigma_P, \sigma_A)$, $\Lambda \equiv (\lambda_P, \lambda_A)'$, and $Z_t \equiv (Z_P^t, Z_A^t)'$. In this model, the dynamics of the state-price density, $\varphi_t$, is given by:

$$\frac{d\varphi_t}{\varphi_t} = -rdt - \Lambda'dZ_t.$$  \hspace{1cm} (5)

To simplify notation, I normalize $A_0 = 1$ and $\varphi_0 = 1$.

The market parameters that are common to all managers are collected in $\Theta_P \equiv \{r, \lambda_P, \sigma_P\}$ and the parameters that are manager-specific in $\Theta_A \equiv \{\lambda_A, \Gamma\}$. $\Gamma$ contains all parameters that describe the manager’s preferences. I will adopt a two-step estimation procedure. First, I estimate $\Theta_P$ by means of maximum likelihood and denote the resulting estimates by $\hat{\Theta}_P$. Second, I estimate the manager-specific parameters, $\Theta_A$, conditional on the passive parameters. The main complication is to construct the second-stage likelihood.

While a single-step estimation would enhance the efficiency of the estimates, it would require modeling the cross-sectional correlation of active portfolio returns across managers. The two-step procedure accommodates any cross-sectional dependence in active returns. It therefore requires less restrictive statistical assumptions, is not subject to misspecification of the correlation structure, and still results in consistent estimates. In addition, the two-step procedure saves substantially on computational time.

### 2 The method

The goal of this paper to compute the log-likelihood of mutual fund returns, $r_A^t \equiv \log A^t_t - \log A^t_{t-h}$, conditional on passive returns, $r_P^t \equiv \log S^t_{P,t} - \log S^t_{P,t-h}$, and first-stage estimates of the common parameters, $\hat{\Theta}_P$.\(^6\)

$$\mathcal{L} \left( r_A^T \mid r_P^T; \Theta_A, \hat{\Theta}_P \right) = \sum_{t=h}^{T/H} \ell \left( r_A^t \mid r_P^t, r_A^{t-h}; \Theta_A, \hat{\Theta}_P \right),$$  \hspace{1cm} (6)

with $y^t \equiv \{y_t, \ldots, y_t\}$.

I now show the martingale method is useful to construct the second-stage likelihood.

\(^6\)To simplify notation, I compute the likelihood for one investment period, $[0, T]$. If multiple periods are available, one can sum the likelihoods to obtain the overall likelihood of the sample.
The manager’s preferences are given by:

$$\max_{(x_s)_{s\in[0,T]}} E_0 [u(A_T)],$$

(7)

subject to the static budget constraint:

$$E_0 [\varphi_T A_T] = 1,$$

(8)

and note that $\varphi_0 A_0 = 1$ due to the before-mentioned normalizations. Cox and Huang (1989) show that one can always replace the dynamic budget constraint by a static budget constraints if the financial market is dynamically complete. In this paper, I assume that $u'(A) > 0$ and $u''(A) < 0, \forall A$. In case of local convexities, one first needs to concavify $u(\cdot)$, but the main procedure is unaffected. The first-order condition reads:

$$u'(A^*_T) = \xi \varphi_T,$$

(9)

in which $\xi$ denotes the Lagrange multiplier corresponding to the budget constraint. Since $u''(A) < 0$, $u'(A)$ is invertible. I furthermore define $I(\xi \varphi_T) \equiv \frac{\partial u^{-1}(x)}{\partial x} |_{x=\xi \varphi_T}$. It follows:

$$A^*_T = I(\xi \varphi_T).$$

(10)

Since $(\varphi_t)_{t\geq 0}$ is Markovian, it holds that:

$$A^*_t = E_t \left[ \frac{\varphi_T}{\varphi_t} A^*_T \right]$$

$$\equiv f(\varphi_t),$$

(11)

which defines the function $f(\cdot)$. I first introduce some notation before showing that $f$ is invertible: $w(\tau, \Delta Z_{t:T}) \equiv \varphi_T/\varphi_t$, $\tau \equiv T - t$, and $\Delta Z_{t:T} \equiv Z_T - Z_t$.

**Proposition 1** The function $f(\cdot)$ as defined in (11) is invertible.

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7I consider for simplicity preferences that depend only on $A_T$. Koijen (2007) considers more general preferences that depend both on assets under management and on the value of the benchmark.

8There is one subtlety though. In Proposition 1, the concavified function is concave, yet not strict as convex regions are replaced by a linear function. For any time $t < T$ this poses no problem as there is a positive probability that one ends up in the strictly concave region, rendering the function $f$ strictly concave. At time $T$, the inverse is defined from a set $D = [A_*, A^*]$ to one value of the state-price density, and there exists a unique mapping from $A_T$ to $\varphi_T$ for $A_T \notin D$. Alternatively, one can omit the last fund return. For the problem studied in Koijen (2007), the results are very similar.
Proof. Rewrite:

\[ A_t^* = E_t \left[ \frac{\phi_T A_T}{\phi_t} \right] = E_t \left[ \frac{\phi_T I (\xi \phi_T)}{\phi_t} \right] = E_t \left[ w (\tau, \Delta Z_{t:T}) I (\xi \phi_t w (\tau, \Delta Z_{t:T})) \right]. \] (12)

Hence:

\[ f' (\phi_t) = E_t \left[ \xi w (\tau, \Delta Z_{t:T})^2 I (\xi \phi_t w (\tau, \Delta Z_{t:T})) \right] = E_t \left[ \frac{1}{w'' (I (\xi \phi_t w (\tau, \Delta Z_{t:T})))} \right] < 0. \] (13)

Since \( f \) is monotonically decreasing in \( \phi_t \), it follows that \( f \) is invertible. \( \blacksquare \)

Proposition 1 implies that the mapping from \( \phi_t \) to \( A_t^* \), \( f (\phi_t) \), is unique for \( t \in [0, T] \). As such, observing \( A^T \) is equivalent to observing \( \phi^T \). I can use this to construct the likelihood of fund returns. I define \( \Delta \phi_t \equiv \log \phi_t - \log \phi_t^{t-h} \). It then follows for the log-likelihood contribution:

\[ \ell \left( r_{A,t+h} | r_P^{t+h}, r_A^t; \Theta_A, \hat{\Theta}_P \right) = \ell \left( r_{A,t+h} | r_P^{t+h}, \phi_t; \Theta_A, \hat{\Theta}_P \right) \]

(14)

\[ = \ell \left( r_{A,t+h} | \Delta Z^P_{t:t+h}, \phi_t; \Theta_A, \hat{\Theta}_P \right) \]

\[ = \ell \left( \Delta Z^A_{t:t+h} \right) + \log \left( \left( \frac{\partial \left( \log A^*_t \right)}{\partial \Delta Z^A_{t:t+h}} \right) \right) \]

\[ = \ell \left( \Delta Z^A_{t:t+h} \right) + \log \left( \left( \frac{\partial \log A^*_t}{\partial \Delta Z^A_{t:t+h}} \right) \right) \]

\[ = \ell \left( \Delta Z^A_{t:t+h} \right) + \log \left( \left( \frac{\partial f (\phi_t)}{\partial \Delta Z^A_{t:t+h}} \right) \right) + \log A^*_t, \]

where the second equality uses:

\[ r_{P,t+h} = \left( r + \sigma_P \lambda_P - \frac{1}{2} \sigma_P^2 \right) h + \sigma_P \Delta Z^P_{t:t+h}. \] (15)

The third equality is an application of the Jacobian formula and uses \( A_{t+h} = f (\phi_{t+h}) \) and:

\[ \phi_{t+h} = \phi_t \exp \left( - \left( r + \frac{1}{2} \Lambda A \right) h - \lambda_P \Delta Z^P_{t:t+h} - \lambda_A \Delta Z^A_{t:t+h} \right). \] (16)

Obviously, \( \Delta Z^A_{t:t+h} \sim N (0, h) \) and the log-likelihood contribution can therefore be computed explicitly.
The three main steps can now be summarized as follows:

1. **Computing the Lagrange parameter** Compute the Lagrange parameter, $\xi$, such that the budget constraint in (8) holds with equality. The expectation can be approximated by means of simulation or quadrature methods. Note that this is one-dimensional integration as $\log \varphi_T \sim N(\mu_{\varphi}(T), \sigma_{\varphi}(T)^2)$, with $\mu_{\varphi}(t) \equiv \log \varphi_t - r\tau - \frac{1}{2}A\Lambda \tau$ and $\sigma_{\varphi}(t) \equiv A\Lambda \tau$. Denote the simulated draws by $\varphi_j^T$, $j = 1, \ldots, J$. For each $\varphi_j^T$, compute the assets under management that maximizes the Lagrangian:

$$L^j = u(A_T) - \xi \varphi_j^T A_T,$$

which leads to $A_T^{*j}$. Then find $\xi$ so that:

$$\sum_{j=1}^J \omega_j A_T^{*j} \varphi_j^T = 1,$$

with $(\omega_j)_{j=1}^J$ the weights attached to the different draws.\(^9\)

2. **Recover the state-price density process** For $t = h, \ldots, T$, compute $\varphi_t$ so that:

$$A_t^* = f(\varphi_t) = E_t \left[ \frac{\varphi_T A_T^*}{\varphi_t} \right] \approx \sum_{j=1}^J \omega_j A_T^{*j} \frac{\varphi_j^T}{\varphi_t},$$

This implies that for any candidate $\varphi_t$, I simulate $\varphi_j^T$, and compute $A_T^{*j}$ from (17). Excellent starting values can be obtained using the passive fund innovation ($\Delta Z_{P,t-h}$) and the state-price density of the previous period ($\varphi_{t-h}$).

3. **Compute the log-likelihood contributions** Given a time series $\varphi^T$, I can construct the likelihood as in (14). This only requires one additional computation, namely to compute:

$$\frac{\partial f(\varphi_t)}{\partial Z_{t-h:t}^A} = \frac{\partial E_t \left[ \frac{\varphi_T A_T^*}{\varphi_t} \right]}{\partial Z_{t-h:t}^A} \approx \frac{\partial \left[ \sum_{j=1}^J \omega_j A_T^{*j} \frac{\varphi_j^T}{\varphi_t} \right]}{\partial Z_{t-h:t}^A},$$

in which the differentiation is performed numerically.\(^{10}\)

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\(^9\)The simulated draws can refer to either Monte Carlo integration or Gaussian quadrature.

\(^{10}\)For Monte Carlo integration, the weights simply equal $\omega_j = J^{-1}$, but they can be different for the quadrature approximation.
As I show below, the accuracy of the method increases with the number of quadrature points. Since this also increases the computational effort, it is useful to have some guidance on the number of points to use. As a general guideline, one can follow Fernández-Villaverde, Rubio-Ramírez, and Santos (2006). They suggest to compute a likelihood-ratio test for two different numbers of quadrature points. Using the tests developed in Vuong (1989), it is possible to formally compare the two approximations. One increases the number of quadrature points until it is no longer possible to statistically distinguish the models.

3 Example

In this section, I study a simple model in which the likelihood can be computed analytically. This allows me to analyze the performance of my estimation method.

3.1 Preferences

The manager’s preferences are given by:

\[
\max_{x_s \in [0,T]} E_0 [u(A_T)] = \max_{x_s \in [0,T]} E_0 \left[ \frac{1}{1 - \gamma A_T^{1-\gamma}} \right],
\]

with \( \gamma \) the coefficient of relative risk aversion.

3.2 Likelihood of one observation

In this section, I explicitly compute the likelihood of one observation to demonstrate the equivalence of observing assets under management at discrete points in time and the state-price density. It also demonstrates the use of Jacobian formula. For notational convenience, I normalize \( T = 1 \). Setting up the Lagrangian and solving the first-order conditions implies:

\[
A_1^* = (\xi \varphi_1)^{-\frac{1}{\gamma}},
\]

and by substitution in the budget constraint (8):

\[
\xi = \exp \left( - (\gamma - 1) r - \left( 1 - \frac{1}{\gamma} \right) \Lambda' \Lambda \right),
\]

\( ^{11} \)The value function in this model is simply given by:

\[
J(A_t, t) = \frac{1}{1 - \gamma} A_t^{1-\gamma} \xi^\tau,
\]

with \( \tau \equiv T - t \).
Now suppose I only observe $\ln A_1$, and I want to compute the likelihood of $\ln A_1$. For general preferences, this is non-trivial. In this case, however, we know the distribution of $A_1$ exactly, and it is log normal:

$$\ln A_1 = r + \frac{\Lambda' \Lambda}{\gamma} - \frac{\Lambda' \Lambda}{2\gamma^2} + \frac{\Lambda'}{\gamma} Z_1.$$  \hfill (24)

Direct computation delivers, with $L$ denoting the likelihood instead of the log-likelihood ($\ell$) contribution:

$$L \left( \ln A_1; \Theta_A, \hat{\Theta}_P \right) = \frac{1}{\sqrt{2\pi} \sqrt{\Lambda' \Lambda/\gamma^2}} \exp \left( -\frac{1}{2} \left( \frac{\ln A_1 - r - \frac{\Lambda' \Lambda}{\gamma} + \frac{\Lambda'}{\gamma} Z_1}{\sqrt{\Lambda' \Lambda/\gamma^2}} \right)^2 \right).$$  \hfill (25)

The mapping from $\varphi_T$ to $A_1^*$ in (22) can be exploited to construct the likelihood. After all:

$$\ln \varphi_1 = -r - \frac{\Lambda' \Lambda}{2} - \Lambda' Z_1,$$
$$\ln A_1^* = -\frac{1}{\gamma} (\ln \xi + \ln \varphi_1),$$
$$\ln \varphi_1 = -\gamma \ln A_1^* - \ln \xi,$$

and hence apply the Jacobian formula:

$$L \left( \ln A_1; \Theta_A, \hat{\Theta}_P \right) = \frac{1}{\sqrt{2\pi} \sqrt{\Lambda' \Lambda/\gamma^2}} \exp \left( -\frac{1}{2} \left( -\gamma \ln A_1^* - \ln \xi + r + \frac{\Lambda' \Lambda}{2} \right)^2 \right) \left| \frac{\partial \ln A_1}{\partial \ln \varphi_1} \right|^{-1} \left| \frac{\partial \ln \varphi_1}{\partial \ln A_1^*} \right|^{-1}$$

$$= \frac{1}{\sqrt{2\pi} \sqrt{\Lambda' \Lambda/\gamma^2}} \exp \left( -\frac{1}{2} \left( \ln A_1^* + \frac{\ln \xi}{\gamma} - \frac{r}{\gamma} - \frac{\Lambda' \Lambda}{2\gamma^2} \right)^2 \right),$$
$$= \frac{1}{\sqrt{2\pi} \sqrt{\Lambda' \Lambda/\gamma^2}} \exp \left( -\frac{1}{2} \left( \ln A_1^* - r - \frac{\Lambda' \Lambda}{\gamma} + \frac{\Lambda' \Lambda}{2\gamma^2} \right)^2 \right),$$

which uses (23) and coincides with (25). This shows that the likelihood can be computed by exploiting the mapping to from $\varphi_1$ to $A_1$.

### 3.3 Analytical derivation of the likelihood

I first show that the likelihood can be computed analytically in this model, which serves as a benchmark in the next section. The standard first-order conditions imply:

$$A_1^* = (\xi \varphi_T)^{-\frac{1}{\gamma}},$$  \hfill (30)

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12For tractability, I derive the unconditional likelihood, but the conditional likelihood follows along the same lines.
and:

\[
A_t^* = E_t \left[ \frac{\varphi_T A_T^*}{\varphi_t} \right] = \varphi_t^{-\frac{1}{2}} \xi^{-\frac{1}{2}} E_t \left[ \left( \frac{\varphi_T}{\varphi_t} \right)^{1-\frac{1}{2}} \right] = \varphi_t^{-\frac{1}{2}} g(t). \tag{31}
\]

Straightforward algebra shows that:

\[
x^* \equiv \left( \begin{array}{c} x_P^* \\ x_A^* \end{array} \right) = \frac{1}{\gamma} (\Sigma^\prime \Sigma) \gamma \Sigma, \tag{32}
\]

and \(A_t^*\) follows a geometric Brownian motion as a result. The exact likelihood of log returns, conditional on passive returns, is given by:

\[
\log A_{t+h}^* - \log A_t^* \sim N \left( \mu_t, x_t^* A_t^* \sigma^2_A h \right), \tag{33}
\]

in which:

\[
\mu_t = \left( r + x^* \Sigma \Lambda - \frac{1}{2} x^* \Sigma \Sigma^\prime x^* \right) h + x_P^* \sigma_P \Delta Z_{t+h}. \tag{34}
\]

### 3.4 Numerical results

Because the numerically approximated expectations converge to the true values if \(j \to \infty\), the relevant question is how large \(J\), the number of quadrature points, needs to be for the likelihood following from the Martingale Estimation Method, as I will refer to the method from now on, to be sufficiently close to the analytical likelihood. This section studies the accuracy of the method for different numbers of \(J\).\(^\text{14}\)

**Parameter setting** I consider the following parameter configuration:

\[
\lambda_P = .3, \ \sigma_P = .15, \ \lambda_A = .1, \ \sigma_A = .2, \ r = .05, \ \gamma \in \{2, 5, 10\}. \tag{35}
\]

**Comparison of the log-likelihoods** To gauge the accuracy of the numerical method, I compare the exact likelihood \(L^\text{Analytical}\) to the one I obtain from the Martingale Estimation Method \(L^\text{MEM}\) for (i) different levels of risk aversion and (ii) for different numbers of quadrature points. To this end, I compute the relative error in log-likelihoods

\(^{13}\)As is standard in the martingale method, the optimal portfolio is recovered by matching the diffusion coefficient on \(\frac{dA_t^*}{A_t^*}\), that is \(-\varphi_t \frac{\partial f(\varphi_t)}{\partial \varphi_t} \Lambda\), and \(x_t^* \Sigma\).

\(^{14}\)A Matlab version of the code used in this note to estimate the models is available upon request.
and the absolute relative error in log likelihoods:

\[ \varepsilon_A \equiv \frac{1}{N} \sum_{i=1}^{N} \frac{L^{MEM} - L^{Analytical}}{L^{Analytical}}, \]  

(36)

\[ \varepsilon_R \equiv \frac{1}{N} \sum_{i=1}^{N} \left| \frac{L^{MEM} - L^{Analytical}}{L^{Analytical}} \right|. \]  

(37)

The likelihood is evaluated at the true parameters. The next section analyzes the implications for the maximum likelihood estimates that are obtained by both procedures. The model is estimated using three years of monthly data for \( N = 100 \) replications. The results are presented in the Table 1.

| Table 1 about here. |

Clearly, when the number of quadrature points increases, the error in the likelihood decreases. The relative error is about \( 10^{-3} - 10^{-4} \), and can be reduced by increasing the number of quadrature points. Further extensions of this method can focus on more efficient ways to approximate the conditional expectation or its derivative, if needed at all.

Simulation exercise  In this section, I simulate a time series of three years of monthly data. I subsequently estimate the manager-specific parameters, \( \Theta = \{ \gamma, \lambda_A \} \), using the exact likelihood and the likelihood following from the Martingale Estimation Method. The results in term of bias and efficiency are presented in Table 2. The number of quadrature points equals \( J = 6 \) and \( N = 100 \).

| Table 2 about here. |

The simulation evidence indicates that the estimators are (close to) unbiased. The one following from the MEM are slightly less accurate, which is caused by approximating the expectation using quadrature methods. However, this error can be reduced easily by increasing the number of quadrature points.

4 Extensions

In this section, I briefly discuss potential extensions. These include (i) multiple assets (Section 4.1), (ii) stochastic volatility (Section 4.2), and (iii) more general dynamics of asset prices (Section 4.3).

4.1 Multiple assets

It is straightforward to allow for multiple passive portfolios. For instance, one can consider the case in which the manager can trade the benchmark, a size-oriented portfolio,
and a momentum portfolio. In case of standard dynamic programming techniques, this would significantly increase the computational burden. For the martingale method, the computational complexity is identical. To see this, note that all I need to know is the state-price density dynamics. If the manager can trade multiple passive portfolios, this simply means that I can condition on multiple shocks in (16). The main procedure is however unaffected. If the manager can trade multiple active portfolios, then it is always possible to merge this to one active portfolio because the manager will fully diversify the active portfolio. Since I observe only the asset dynamics, I cannot disentangle multiple shocks that affect different active portfolios.

4.2 Stochastic volatility

The approach so far ignores time variation in the conditional volatilities. I consider the case in which volatilities are time varying, $\Sigma_t$, but the prices of risk, $\Lambda$, remain constant. In this case, the optimal portfolios depend on the current level of volatilities, $x_t'(\Sigma_t)$, but the implied volatility matrix, $x_t'(\Sigma_t)\Sigma_t$, does not.\(^\dagger\) This is a consequence of the fact that the investor is merely interested in obtaining the right exposures to the risk factors, $Z_t$, and $\Sigma_t$ is only relevant for the implementation with the particular set of securities that are present in the asset menu. This can also be seen from the fact that in the martingale approach, $\Sigma_t$ plays no role whatsoever.

However, it is not the case that time-varying volatilities are inconsequential. The only place in which the volatility enters the estimation procedure is in backing out the passive innovation, $\Delta Z_{P,t:t+h}$, from benchmark returns. After all, suppose that $\Sigma_t$ is constant from $t$ to $t+h$, then it holds:

$$\Delta Z_{P,t:t+h} \equiv \frac{r_{t+h}^P - (r + e_1'\Sigma_t\Lambda - \frac{1}{2}e_1'\Sigma_t\Sigma_t'e_1)h}{\sqrt{he_1'\Sigma_t \Sigma_t' e_1}}. $$

Clearly, all that matters is the stochastic volatility on the passive portfolio, $\sigma_{P_t} = \sqrt{e_1'\Sigma_t\Sigma_t'e_1}$. One straightforward way to account for time-varying volatility is therefore to specify a model at a $h$-period frequency and construct the likelihood as before.

4.3 General dynamics of the financial market

The method is designed to estimate dynamic finite-horizon models in complete markets. Section 1 provides the simplest example of such a market, which is the workhorse model\(^\dagger\dagger\) It may be unclear why stochastic volatility matters in, for instance, Chacko and Viceira (2005) and Liu (2007). Chacko and Viceira (2005) assume risk premia to be constant, whereas Liu (2007) assumes that the Sharpe ratio divided by the stock volatility is constant. Under both modeling assumptions, the price of risk is time varying and a long-horizon optimally holds hedging demands if the coefficient of relative risk aversion is different from unity. In contrast, I consider the case with constant prices of risk and perturb the model to accommodate stochastic volatility, which implies that $x_t'\Sigma_t$ will be independent of $\Sigma_t$, see also DeTemple, Garcia, and Rindisbacher (2003) and Munk and Sorensen (2004).
for theoretical models of delegated portfolio management. One potential extension of the financial market is to accommodate time variation in the short rate. Suppose the short rate follows a one-factor Vasicek model:

\[ dr_t = -\kappa (r_t - \bar{r}) \, dt + \sigma \, dZ_t, \]  

\[ d\phi_t = -r_t \, dt - \Lambda \, dZ_t, \]  

in which \( Z_t \in \mathbb{R}^3 \) and I assume that the investor can trade a long-term bond (or any asset that has exposure to the third Brownian motion) to complete the financial market. It is trivial to discretize this model exactly and to compute the joint likelihood of the state-price density and the instantaneous short rate.

A more challenging problem arises when the prices of risk, \( \Lambda \), are allowed to be time varying. In this case, it may be impossible to discretize the state-price density exactly. One potential solution is to update the prices of risk only, say, annually as most models of delegated portfolio management make predictions on the dynamics of fund returns within the year. Provided the high persistence of risk premia, this approximation may have little effect. Further, there may be considerable benefits from mapping the wealth dynamics to the dynamics of the state-price density. It is well-known that the discretization bias can be reduced by stabilizing the diffusion coefficient. For the asset dynamics, the diffusion coefficient is typically unknown, which renders such stabilizations infeasible. For the state-price density, in contrast, the diffusion coefficient is known and it is therefore straightforward to stabilize the state-price density. If it is not possible to discretize the state-price density exactly to compute the likelihood, one can alternative resort to moment-based estimators of the type developed in Chacko and Viceira (2003). However, studying richer dynamics of the financial market and the economic implications for the estimates of structural dynamic models is beyond the scope of this paper and is left for further research.

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\(^{17}\)It is straightforward to consider recently introduced two-factor models in the portfolio choice literature by for instance Campbell and Viceira (2001) and Brennan and Xia (2002).


\(^{19}\)See for instance Ait-Sahalia (2002), Ait-Sahalia (2007), DeTemple, Garcia, and Rindisbacher (2003), and Bakshi and Ju (2005).
5 Conclusions

This paper develops a novel likelihood-based estimation procedure to study dynamic models of delegated portfolio management empirically. It relies on the martingale method introduced by Cox and Huang (1989) to map the dynamics of wealth, or assets under management, to the state-price density. The transition density of the latter can be constructed easily in standard models. I illustrate the method in a simple model and demonstrate the accuracy of the method. Future research can focus on more elaborate models of the financial market as to provide a more realistic description of observed mutual fund returns. Also, it is of interest to extend the method by including information in portfolio holdings if such information is available at a monthly frequency (Elton, Gruber, Krasny, and Ozelge (2006)).

References


Table 1: Comparing log-likelihoods
The table displays the relative differences in log-likelihoods by comparing the analytical log-likelihood to the one following from the Martingale Estimation Method. The two measures, $\mathcal{E}_A$ and $\mathcal{E}_R$, are defined in (36) and (37), respectively. The results are computed for three values of the coefficient of relative risk aversion, $\gamma \in \{2, 5, 10\}$, and three numbers of quadrature points, $J \in \{2, 6, 10\}$. The remaining parameters are set to: $\lambda_P = .3$, $\sigma_P = .15$, $\lambda_A = .1$, $\sigma_A = .2$, $r = .05$. The model is estimated using three years of monthly data for $N = 100$ replications.

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<th>$\mathcal{E}_A$</th>
<th>Standard error</th>
<th>RMSE</th>
<th>$\mathcal{E}_R$</th>
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<th>RMSE</th>
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Table 2: Comparing maximum-likelihood estimates
The table displays the estimates following from maximum likelihood based on the analytical log-likelihood and on the one following from the Martingale Estimation Method. The table provides the bias and efficiency, as well as the mean-squared error. The results are computed for three values of the coefficient of relative risk aversion, $\gamma \in \{2, 5, 10\}$. The number of quadrature points equals $J = 6$. The remaining parameters are set to: $\lambda_P = .3$, $\sigma_P = .15$, $\lambda_A = .1$, $\sigma_A = .2$, $r = .05$. The model is estimated using three years of monthly data for $N = 100$ replications.

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