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Viscous demand

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Abstract

In many markets, demand adjusts slowly to changes in prices, i.e., demand is "viscous". This viscosity gives each firm some monopoly power, since it can raise its price above that of its competitors without immediately losing all of its customers. The resulting equilibrium pricing behavior and market outcomes can differ significantly from what one would predict in the absence of demand viscosity. In particular, the model explains the importance of market share as an investment, as well as "kinked demand curves". It also explains how apparently "competitive" pricing behavior can lead to outcomes that mimic those of collusion.

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1. Introduction

In many markets, demand adjusts slowly to changes in prices, i.e., demand is "viscous". For such a market, the time path of a firm's prices acquires added significance, compared with the case of instantaneous demand response. In this paper I explore some problems in strategic dynamic pricing of a service, in the presence of viscous demand, for a monopoly and a duopoly. In particular, the viscosity of demand confers on each firm a kind of monopoly power, since it can raise its price above that of its competitors without immediately losing all of its customers. As we shall see, this phenomenon can lead to equilibrium pricing behavior and market outcomes that differ significantly from what one would predict

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in the absence of demand viscosity. In particular, it provides a rationale for the importance of market share as an investment, and it provides an explanation for the "kinked demand curve". It also explains how apparently "competitive" pricing behavior can lead to outcomes that mimic those of collusion.

There are many reasons for the viscosity of demand. In the case of a service (which will be the focus of this paper), such as a subscription to a magazine, newspaper, or long-distance telephone carrier, the viscosity of demand is probably best explained by an "attention budget". The (potential) consumer cannot be thinking every hour, or even every week, about which long-distance carrier to use. Rather, the consumer rethinks such decisions from time to time, regularly or at some random intervals, perhaps triggered by some events. We may think of the consumer as a "server" for a queue of decision problems, which are served according to some system of priorities. If the priority system is not fully optimized, we may think of the consumer as displaying a form of "bounded rationality". The time it takes for a decision problem to be "served" will depend not only on the duration of the service time, but also on the pattern of arrivals of decision problems at the queue. Decisions about a service that belongs to a higher priority class will be served more quickly, and so the demand for that service will display less viscosity. For example, it has been verified empirically that, on the average, consumers who spend more on long-distance telephone service also exhibit less viscosity of demand. (For more details, see Section 2.)

In the case of a durable good, a person with a piece of equipment that has been recently acquired will typically not switch to another model as soon as a supplier lowers its price, but will wait until the equipment has suffered some wear and tear, or has otherwise depreciated. The analysis of a market for durable goods requires a quite different model, and is beyond the scope of this paper.

Viscosity of demand is to be distinguished from "stickiness of demand" due to the cost of switching suppliers. In a pure switching-cost model, a consumer will switch suppliers as soon as the gross saving from switching exceeds the cost. By contrast, in the viscosity model, if two suppliers offer identical services at different prices, eventually all customers will switch to the lower-price supplier.

In the model explored in this paper, at every instant of time (time is continuous), each consumer purchases a service at a rate equal to 0 or 1. If there is more than one supplier, then a consumer who buys the service must also choose the supplier. I shall consider two cases: (1) monopoly, and (2) duopoly; in the latter case I shall assume that the services provided by the two suppliers are identical. With the exception of one result, I also assume that all consumers are identical in their two demand parameters: (1) the long-run willingness-to-pay (WTP) for the service, and (2) the "viscosity coefficient" (see below). One way to think about the demand model is that there is a continuum of consumers, and consumers flow smoothly from one category to another according to a given differential equation. This, however, is only a metaphor for a situtation in which there are many consumers, and each consumer's demand behavior is governed by a (continuous time) Markov process. Using a suitable law of large numbers for Markov processes, one can show that, as the number of consumers increases, in the limit the fractions of consumers in each category obey the postulated differential equation (see Section 2 and Appendix A). In what follows, I shall use the more convenient language of a continuum of consumers, normalized to have mass unity.

I first summarize the results for the case of a monopoly (Section 3). At every time t, let $X(t)$ denote the mass of consumers who are actually purchasing the service at that time (the "customers"), and let $P(t)$ denote the price of the service, per unit time. One usually calls $X(t)$ the market penetration at time t. Let w denote the long-run willingness-to-pay for the service (the same for all consumers). Finally, suppose that the price of the service is exogenously constrained to be between zero and some positive number m (see below for a discussion of this assumption). The market penetration evolves according to the differential equation,

$$X'(t) = \begin{cases} k[w - P(t)][1 - X(t)] & \text{if } 0 \leqslant P(t) \leqslant w, \\ -k[P(t) - w]X(t) & \text{if } w \leqslant P(t) \leqslant m. \end{cases}$$

Here the strictly positive parameter k is the reciprocal of the "viscosity coefficient"; *smaller values of k correspond to a higher viscosity*. According to the first line of the equation, if the price is less than the WTP, then noncustomers become customers, at a rate proportional to the mass of noncustomers. On the other hand, according to the second line, if the price exceeds the WTP, then customers stop buying the service, at a rate proportional to the mass of customers. If the price remains constant, say p, and p is less than w, then the market penetration will approach unity (in the limit), whereas if p exceeds w, then the market penetration will approach zero. The market penetration will remain unchanged from its initial value if p just equals w. (For a more complete motivation of the specific form of the model, see Section 2.)

Since customers do not disappear immediately when the price is raised above the WTP, increasing the market penetration by lowering the price represents a kind of *investment*.

One interpretation of the upper bound, m, is that $X'(t)$ equals minus infinity (i.e., the market penetration falls immediately to zero) if the price $P(t)$ ever exceeds m. This should be regarded as an approximation to the situation in which the market penetration declines very rapidly as the price approaches some boundary value. The lower bound could have any value, and is taken here to be zero for simplicity of exposition. Other plausible laws of motion for the market penetration, which are qualitatively similar to the one described here, are discussed in Section 2.

For this simple model, I characterize the dynamic price policy that maximizes the monopolist's total discounted profit, for the special case in which the monopolist's cost is proportional to the market penetration, i.e., there is a constant marginal cost, say c, and zero fixed cost. There are two cases to be distinguished. In Case 1, which is the focus of this paper, the discount rate, say r, is not too large relative to the other parameters of the model, and the maximum price, m, is sufficiently close to w (these conditions can be made precise). In this case, the optimal policy for the monopolist is a *target penetration policy*, namely there is an optimal target market penetration, such that the monopolist sets the price equal to zero if the current market penetration is strictly less than the target, and sets the price equal to w, the WTP, if the current penetration is at least as large as the target. The optimal target, s, is given by the simple formula

$$s = k(w - c)/[r + k(w - c)]. \tag{1.1}$$

Note that in this formula the optimal target, s, is a decreasing function of the marginal cost, c, but the target penetration is not complete even when the marginal cost is zero. Also, s is increasing in k, the inverse of the viscosity, and decreasing in r, the discount rate.

Note, too, that although the consumers are acting in a "myopic" manner at those times when they consider whether to buy the service or not, one can show that, given the optimal price strategy of the monopolist, their choices are optimal. (See Section 3.)

In Case 2 (when the conditions of Case 1 do not hold), the target penetration policy described above is dominated by a policy in which the price oscillates rapidly between 0 and m. In fact, *strictly speaking there is no optimal policy*; *we may say that in the "optimal" policy the price oscillates infinitely fast between* 0 *and* m! (One can provide a precise meaning of this statement.) Such a situation is hardly realistic, and provokes a reconsideration of the behavioral assumptions of the model when it has these parameter values. In particular, if prices are oscillating very quickly, one would not expect consumers to react so myopically as they do in the model described above. For example, one might expect (boundedly rational) consumers to forecast prices in some "adaptive" manner, e.g., with a moving average of past prices. Although a complete characterization of the monopolist's optimal policy in the face of consumers with adaptive expectations is unavailable, it is possible to show that it exists and will typically lead (roughly speaking) to cyclical fluctuations of prices and market penetration. Such pricing could be interpreted as a policy of intermittent "sales". This case is discussed briefly at the end of Section 3 and in Section 5.3. However, Case 2 is not the central focus of this paper, and in particular I do not consider the corresponding case in the duopoly model. (For a fuller treatment of Case 2 for the monopoly model, see [9].)

I next consider (Section 4) a model of a duopoly with a law of motion analogous to that of the monopoly model. In a duopoly there are three classes of consumers: (1) customers of firm 1, (2) customers of firm 2, and (3) noncustomers, i.e., consumers who are not customers of either firm. Again, all consumers have the same long-run willingness to pay, but each duopolist controls his own price dynamically. The *state of the system at time t* describes the number (mass) of consumers in each class at that time. Depending on the firms' prices, relative to each other and to the consumers' long-run willingness-to-pay, w, consumers will flow from one class to the other. More precisely, if the lowest price is less than w, then consumers will flow to the firm with that price, whereas if the lowest price exceeds w, then customers will flow from both firms into the class of noncustomers. When both firms charge the same price, and it is less than w, then noncustomers will flow to both firms in proportion to the firms' current stocks of customers. When the firms both charge a price equal to w, the masses of consumers in the two firms will remain constant. The total number of customers of the two firms will be called the *market penetration*, and the ratio of the number of customers of a firm to the market penetration will be called that firms's *market share*. Again, as in the case of a monopoly, if a firm raises its price above that of its competitor and above the WTP, it does not immediately lose all of its customers. For this reason, lowering its price to increase its market share represents a kind of investment.

In the context of such a model I shall describe a dynamic game in which the players are the two duopolists. I shall describe, and demonstrate the existence of, a family of (Nash) equilibria with (roughly) the following properties: (1) the strategies of the two players are *stationary*, i.e., at each time each firm's price depends only on the current state of the system (such an equilibrium is usually called *Markovian*); (2) each equilibrium in the family is characterized by two parameters, which may be interpreted as *the target market penetration* of the two firms and the *target market share of firm* 1 (the target market share of firm 2 is, of course, one minus the target market share of firm 1); (3) if a firm's market share is strictly less than its target, then it charges a price equal to zero, and the other firm charges a price equal to m (the maximum price); (4) if both firms' market shares are equal to their targets, then they both charge a price equal to zero if the (total) market penetration is strictly less than the target market penetration, and a price equal to w if it is greater than or equal to the target. In order for such a strategy-pair to form an equilibrium, the parameters of the model must satisfy certain conditions (similar to those in the monopoly case), and the target penetration and market shares must lie in a certain (nonempty) set. To simplify the analysis, I assume that the cost parameter, c, is zero, so that a firm's profit equals its revenue.

To describe the results for a duopoly more fully, I need some additional notation. Let S denote the target market share of firm 1, and let $(1 - Z)$ denote the target market penetration (in other words, Z is the target mass of noncustomers). I call the pair of strategies described above a (Z, S) *target strategy-pair*. Let $\zeta \equiv 1 - s$ (cf. (1.1) above). Under assumptions that correspond to Case 1 of the monopoly model, I demonstrate that there exists a number $\zeta' < \zeta$ such that, if m is sufficiently close to w, and if

$$\zeta' \leqslant Z \leqslant \zeta,$$

$$1 - s \leqslant S \leqslant s,$$

then the corresponding (Z, S) target strategy-pair is an equilibrium of the game (Theorem 2).

Since the total mass of consumers is unity, we can characterize the system state at any time by the vector (x, z), where x is the mass of customers of firm 1, and z is the mass of noncustomers. A comparison with the monopoly case shows that the equilibrium path is *efficient*, in the sense that the total profit of the two firms is maximized, if and only if (1) the initial state vector is on the line $x = S(1 - z)$, and (2) $Z = \zeta$. Thus, if these conditions are satisfied, then the industry outcome as a whole mimics the monopoly outcome. On the other hand, if $Z < \zeta$, then the asymptotic market penetration will be greater than it would be in the corresponding monopoly, and the system spends more time in the regime in which one or both firms charge a zero price. In this sense, the equilibrium can be more "competitive" than the monopoly outcome. Note that as r/kw approaches zero, the minimum target market penetration, $1-\zeta$, approaches unity.

An implication of Theorem 2 is that a division of the market into shares S and $(1-S)$ is self-sustaining, so that no "explicit collusion" is required once the target S is determined. On the other hand, since thare is a nondegenerate interval of market shares that can be so sustained, some kind of "coordination" on a particular value of S is required. The same is true of the target market penetration, Z.

When the target market penetration and market shares have been reached, if one firm lowers its price below w, the other will do so, too, whereas if a firm raises its price above w, the other firm will not respond. The effect of this is that each firm's demand curve will not be differentiable at the point at which its price equals w. Thus Theorem 2 provides a game-theoretic explanation of the so-called "kinked demand curve" in a duopoly. (I owe this observation to T. Groves.)

Although the models analyzed in Sections 3 and 4 are somewhat special, they illustrate how the consideration of viscous demand introduces interesting new features in the analysis of monopoly and duopoly pricing, and provides additional explanatory power. In Section 5, I describe various extensions of the analysis, as well as some open problems. First, an obvious question is whether there are other equilibria of the duopoly game of Section 4. I cannot characterize the full set of equilibria, nor do I know whether there are other Markovian equilibria. However, I can show that a variant of Anderson's (1985) concept of *quick-response equilibrium* (R.M. Anderson, Quick-Response Equilibrium, Department of Economics, U. of Calif., Berkeley, Feb. 1985, unpublished) yields an equilibrium outcome that is identical to that of the "efficient" (Z,S) target strategy-pair, but with somewhat different strategies. In this equilibrium, (1) if the initial total market penetration of the two firms is less than s, then both firms charge a zero price until the market penetration reaches s, after which they both charge a price equal to w; (2) if the initial total market penetration is at least s, then both firms charge w; (3) once the total market penetration reaches or exceeds s, if either firm charges a price strictly less than w, then the other firm will "immediately retaliate" by charging a price equal to *zero* (in fact, both firms will switch to zero); (4) on the other hand, if in cases (1) and (2) either firm *raises* its price, the other firm will not change its own price. Again, the equilibrium strategies have the effect that the industry as a whole imitates a monopolist's behavior, while the two firms maintain their initial relative market shares.

The relationship of the present model to the classic Bertrand model is discussed in Remark 4 of Section 4.

In the second part of Section 5, I sketch a model of "adaptive expectations", and summarize the results of Radner and Richardson [9], concerning the optimal monopoly pricing strategy corresponding to the model of Section 3.

In the third part of Section 5, I generalize the monopoly model to allow for the possibility that different consumers have different long-run willingness-to-pay. I sketch an argument that if there is a sufficient dispersion of the long-run willingness-to-pay, w, in the population of consumers, then under the optimal pricing policy of the monopolist there is no steady state of the system.

In the last part of Section 5, I discuss some of the issues that arise in the analysis of a general oligopoly and a "competitive" market.

In Section 6, I provide some bibliographic notes on the few previously published papers about somewhat related models of demand, notably by Selten, Phelps and Winter, Rosenthal, and Rosenthal and Chen.

2. A model of demand viscosity

As noted in the Introduction, I envisage the viscosity of demand as resulting from the fact that consumers typically have an "attention budget". For decisions about whether to start or stop a service (such as a subscription to a newspaper), switch suppliers (such as switching from one long-distance carrier to another), or switch brands of a commodity bought repeatedly (such as breakfast cereal), a consumer will devote only limited attention to the decision problem during any day or week. Rather, the consumer rethinks such decisions from time to time, regularly or at some random intervals. We may think of the consumer as a "server" for a queue of decision problems, which are considered according to some system of priorities. The time it takes for a decision problem to be "served" will depend not only on the duration of the service time, but also on the pattern of arrivals of decision problems at the queue. Decisions about a service that belongs to a higher priority class will be served more quickly, and so the demand for that service will display less viscosity. For example, it has been verified empirically that, on the average, consumers who spend more on long-distance telephone service also exhibit less viscosity of demand.

To go into more detail, consider a consumer who is a potential or actual subscriber to home delivery of the New York Times. Let $P(t)$ denote the subscription rate (price) per unit of time (e.g., per week) at date t, and let $x(t)$ be 1 or 0 according as the consumer is or is not a subscriber at date t; call $x(t)$ her *state at date t*. At stochastic decision times, $T_1, T_2, \ldots$, the consumer considers whether to be a subscriber or not, i.e., whether or not to change her state. Suppose that she does so by comparing the current price with her *long-run willingness-to-pay*, say w. Assume further that if w is greater than the current price, then she remains or becomes a subscriber, whereas if w is less than the current price, then she remains or becomes a nonsubscriber. (If w and the price are equal, she leaves her state unchanged.) A decision time may be triggered by various events: receiving a bill in the mail, seeing a commercial on TV, talking to a friend, etc. But even the news of a price change may not be sufficient to engage the consumer's immediate attention; if the change is small enough, she may put off the decision problem until a less busy day. On the other hand, if she learns that the weekly subscription rate is about to go up to \$1000 per week, she will no doubt cancel her subscription immediately.

Notice that there are two aspects of bounded rationality embodied in this model. First, the consumer is not continuously deciding whether or not to subscribe, but only *visits this decision problem from time to time*. Second, when the consumer does reconsider her decision whether or not to subscribe, she makes her decision *myopically*, comparing w with the current price, rather than attempting to forecast what the price will be until her next decision time. The first aspect is inevitable in almost all decision making, although some decisions are programmed to be made automatically by a computer. (Examples are (1) programs used by large businesses to select the "optimal" long-distance carrier for every long-distance call, and (2) "programmed trading" by large traders in securities markets.) Nevertheless, the mean rate at which decisions are reconsidered will be influenced by the "importance" of the decision, so that more important decisions may be reviewed more frequently. The second aspect of bounded rationality, the myopia of the decision criterion, is more plausible the less frequently prices and other relevant variables change in time. As we shall see in Section 3, for some values of the model's parameters, the firms' optimal price strategy may require rapidly oscillating prices, in which case myopic decision behavior by the consumer may no longer be plausible, which will lead me to consider a so-called "adaptive expectations" model of consumer behavior (Section 5.2).

Consider first the case of a monopoly. If there are very many consumers in the market, the behavior that was described (qualitatively) above will imply that consumers will be observed to "flow" stochastically in and out of the customer (subscriber) category, at mean rates that depend on the current price and on the joint distribution of the frequency of decision making and w in the population of consumers. Let $X(t)$ denote the fraction of consumers who are customers at time t. One can show that (under suitable assumptions), in the limit, as the number of consumers increases without bound, this fraction satisfies a deterministic ordinary differential equation of the form

$$X'(t) = \mu[P(t), X(t)]. \tag{2.1}$$

Heuristically, we may envisage a "continuum" of consumers, of total "mass" 1, and $X(t)$ as the mass of customers at time t. This is, however, only a metaphor, and Eq. (2.1) is derived using a law of large numbers for continuous-time Markov processes (see below and Section 9.1).

Even so, the formulation in (2.1) is too general for my purposes. First, except in the model of Section 5.3, I shall assume that *all consumers have the same long-run willingness-to-pay*, $w > 0$. Second, we can expect the *law of motion*, μ, to be nonlinear, with the following properties:

(1) When the price exceeds w, consumers will flow *out* of the customer pool, at a rate that is proportional to the mass of *customers*.

(2) When w exceeds the price, consumers will flow *into* the customer pool, at a rate that is proportional to the mass of *noncustomers*.

(3) The rates of flow will be monotone in the (absolute value of the) difference between w and the price.

(4) If the price exceeds w by too much, consumers will flow out of the customer pool at a very rapid rate.

A simple formula for the law of motion that has these properties is:

$$\mu(p,x) = \begin{cases} 0 \leqslant k(w-p)(1-x) & \text{for } p \leqslant w, \\ -k(p-w)x & \text{for } w \leqslant p \leqslant m, \\ -\infty & \text{for } p > m, \end{cases} \tag{2.2}$$

where $k>0$ and $m>w$ are parameters. Note that I assume that the price must be nonnegative. This law of motion is consistent with a model in which, (1) in any time interval in which the price is p, the decision times of a single consumer form a Poisson process with rate $k|w-p|$; and (2) the demand processes of the different consumers are statistically independent. It is natural to call $1/k$ the *viscosity coefficient*. A derivation of (2.2) is given in Appendix A.

This particular formulation was chosen for its tractability. Other similar formulas might also be reasonable, depending on the application. In particular, the piecewise linearity of (2.2) and the jump down to $-\infty$ should be thought of as an approximation to a smoother function. Smoothness would also be introduced by "noise" in the flows, caused by movements of consumers that are not explained by price and willingness-to-pay alone. Such noise is observed in practice. Furthermore, one would expect μ to be "S-shaped" as a function of price, for each x. For example, a nonlinear law of motion can be derived from the assumption that consumers' choices (when made) are generated by a discrete-choice logit model. It would be desirable to be able to assess the robustness of the results in this paper to such departures from the strict piecewise linearity of (2.2). However, such an assessment will have to await further research.

In the case of a duopoly, there are three groups of consumers: customers of firm 1, customers of firm 2, and noncustomers. If at least one of the firms has a price lower than w, then consumers flow to the lowest-price firm; whereas if both firms' prices exceed w, then their customers flow to the noncustomer group. The precise model will be spelled out in Section 4; in particular, specific assumptions need to be made for the case of ties.

In Section 5, I shall sketch a partial analysis of a monopoly model in which w has a (nondegenerate) distribution in the population of consumers.

3. Monopoly

In this section I shall present the analysis of a model of a monopoly for a service, as described in Section 2. Recall that the total mass of consumers is taken to be unity, the mass of actual customers at time t is denoted by $X(t)$, and the price of the service (per unit time) is denoted by $P(t)$. All of the consumers have the same long-run willingness-to-pay (WTP) for the service, denoted by w. (See Section 5 for a discussion of generalizations of this assumption.) The *law of motion* for $X(t)$ is given by (2.1) and (2.2) in Section 2, which I reproduce here for the convenience of the reader:

$$X'(t) = \mu[P(t), X(t)], \tag{3.1}$$

$$\mu(p,x) \equiv \begin{cases} k(w-p)(1-x) & \text{for } p \leqslant w, \\ -k(p-w)x & \text{for } w \leqslant p \leqslant m, \\ -\infty & \text{for } p > m. \end{cases} \tag{3.2}$$

I assume that the monopolist's cost per unit time is proportional to the mass of customers, i.e., is equal to $cX(t)$, where $c < w$ is a nonnegative constant. (*Note*: One could add a fixed cost, but its magnitude would not affect the optimal pricing policy, although it would influence the net profitability of the service.) The monopolist's total discounted profit is therefore

$$V = \int_0^\infty e^{-rt}[P(t) - c]X(t)\,dt, \tag{3.3}$$

where $r > 0$ is the exogenously given rate of interest. Given the initial mass of customers, $X(0)$, the monopolist wants to choose a price path to maximize the profit V in (3.3). For reasons that will be explained below, I make the following assumptions:

$$0 < r < k(w-c), \tag{3.4a}$$

$$0 < w \leqslant m, \tag{3.4b}$$

$$0 \leqslant P(t) \leqslant m. \tag{3.4c}$$

In view of the third line of (3.2), the second inequality of (3.4c) is not really an assumption, but it is included here for completeness.

By Blackwell's Theorem, one can without loss of generality take the optimal price policy to be *stationary*, in the sense that, for some function Φ,

$$P(t) = \Phi[X(t)]. \tag{3.5}$$

Theorem 1. *If* (3.4) *is satisfied, and m is sufficiently close to w, then the optimal* (*stationary*) *policy is given by*

$$\Phi(x) = \begin{cases} 0 & \textit{if } 0 \leqslant x < s, \\ w & \textit{if } s \leqslant x \leqslant 1, \end{cases} \tag{3.6a}$$

where

$$s \equiv \frac{k(w-c)}{r + k(w-c)} > 1/2. \tag{3.6b}$$

The maximum profit is

$$V = \begin{cases} \dfrac{ws}{r} D(x) - c\left[\dfrac{rx + kw - (1-s)kwD(x)}{r(r+kw)}\right] & \textit{for } x < s, \\ \dfrac{(w-c)x}{r} & \textit{for } x \geqslant s, \end{cases}$$

where

$$D(x) \equiv \left(\frac{1-s}{1-x}\right)^a \quad \textit{and} \quad a \equiv r/kw < 1.$$

(For the proof of the theorem, see Appendix A.)

Remark 1. Call $X(t)$ the *market penetration*, and s the *target penetration*. If the initial market penetration is strictly less than the target, then, under the policy Φ, the penetration will increase monotonically to the target, reaching it in finite time. On the other hand, any penetration greater than or equal to the target is a steady state. (These conclusions hold even if the target, s, does not satisfy (3.6b), i.e., even if it is not optimal.)

Remark 2. Under the optimal policy, the market penetration never reaches, or even approaches, unity (unless it starts there), so that a strictly positive fraction of the consumers never become customers. These results are intuitively plausible in the light of the first line of the law of motion, (3.2); as the market penetration increases, the remaining mass of noncustomers, $[1 - X(t)]$, decreases towards zero, so that eventually the incremental discounted value of adding to the current customer base (market penetration) is unable to make up for the corresponding incremental loss of revenue from the current customer base.

Remark 3. The optimal target penetration, s, is *decreasing* in the marginal cost, c, the viscosity, $(1/k)$, and the discount rate, r, approaching unity as $k \to \infty$ and/or $r \to 0$. However, $s < 1$ even when the marginal cost is zero.

Remark 4. The value function V is increasing and differentiable, strictly convex for $x < s$, and linear for $x > s$.

Remark 5. We have assumed that our boundedly-rational consumers are "myopic", in the sense that whenever they make a decision they do so on the basis of the currently prevailing price, not a projection of future prices. However, one can easily verify that, if the monopolist uses the policy Φ of the theorem, then *the consumers' postulated behavior is an optimal response at those instants of time when they make a decision*. In fact, this last statement will remain true in the face of any "target market penetration" price policy, not necessarily an optimal one. The reason is that the monopolist's price never exceeds the long-run WTP; hence, at any actual decision moment, it does not pay for a noncustomer to wait to switch, nor for a customer to become a noncustomer.

I shall now briefly discuss the nature of the optimal price policy if the assumptions of Theorem 1 are not satisfied. The parameters of the model are: w, k, r, c, and m, all of which are assumed to be nonnegative. In fact, the parameter space can be partitioned into two parts, say R and R', such that (1) the conclusion of the theorem holds in R, whereas (2) in the set R' there is no exactly optimal policy, but the supremum of the profit will be approached as the price oscillates faster and faster between 0 and m. Intuitively, when m is too large, the monopolist can make a large enough profit in a short period of time to compensate for the loss of customers, until the customer base gets too small. At that point it pays to build up the customer base again by lowering the price to zero. A similar phenomenon occurs when the viscosity is large (k is small) relative to the discount rate, r. (The assumptions stated in the hypothesis of Theorem 1 determine a strict subset of R, so that the conclusion is actually valid for a somewhat larger set. For a full treatment of both cases, and a precise characterization of the two sets R and R', see [9].)

In control theory, the kind of policy that is "optimal" in the set R' is sometimes called a *measure-valued* or *generalized* control. It should be clear that the behavior of even boundedly rational consumers facing a very rapidly oscillating price is unlikely to conform to the kind of model of viscous demand described in Section 2 (and in the hypothesis of Theorem 1). When faced with such a price policy, even "myopic" consumers are more likely to react to some (possibly weighted) *average of past prices*, rather than to the current price at the instant of decision. Such a model of consumer behavior, and its implications, will be described briefly in Section 5.

4. Duopoly

In a duopoly there are three classes of consumers: (1) customers of firm 1, (2) customers of firm 2, and (3) noncustomers, i.e., consumers who are not customers of either firm. The *state of the system at time t* describes the number (mass) of consumers in each class at that time. Depending on the firms' prices, relative to each other and the consumers' long-run willingness-to-pay, w, consumers will flow from one class to the other. More precisely, if the lowest price is less than w, then consumers will flow to the firm with that price, whereas if the lowest price exceeds w, then customers will flow from both firms into the class of noncustomers. I assume (somewhat arbitrarily) that when both firms charge the same price, and it is less than w, then noncustomers will flow to both firms in proportion to the firms' current stocks of customers. When the firms both charge a price equal to w, the masses of consumers in the two firms will remain constant. The total number of customers of the two firms will be called the *market penetration*, and the ratio of the number of customers of a firm to the market penetration will be called that firm's *market share*.

In the context of such a model I shall describe a dynamic game in which the players are the two duopolists (see below for a precise mathematical formulation). I shall describe, and demonstrate the existence of, a family of equilibria with (roughly) the following properties: (1) the strategies of the two players are *stationary*, i.e., at each time each firm's price depends only on the current state of the system (such an equilibrium is usually called *Markovian*); (2) each equilibrium in the family is characterized by two parameters, which may be interpreted as *the target market penetration* of the two firms and the *target market share of firm* 1 (the target market share of firm 2 is, of course, one minus the target market share of firm 1); (3) if a firm's market share is strictly less than its target, then it charges a price equal to zero, and the other firm charges a price equal to m (the maximum price); (4) if both firms' market shares are equal to their targets, then they both charge a price equal to zero if the (total) market penetration is strictly less than the target, and a price equal to w if it is greater than or equal to the target. In order for a strategy-pair to form an equilibrium, the parameters of the model must satisfy certain conditions (similar to those in the monopoly case), and the target penetration and market shares must lie in a certain (nonempty) set. To simplify the analysis, I assume that the cost parameter, c, is zero, so that a firm's profit equals its revenue. (See remarks in the next section.)

I now turn to a precise description of the model and results. Let $X(t)$ and $Y(t)$ denote, respectively, the masses of customers of firms 1 and 2. Then, adopting the convention that the total mass of consumers is unity, the mass of noncustomers is $Z(t) = 1 - X(t) - Y(t)$. Hence we can take the state of the system at time t to be $[X(t), Z(t)]$.

Let $P(t)$ and $Q(t)$ denote the prices at time t of firms 1 and 2, respectively. Suppose that at a given time t,

$$P(t) = p, \quad Q(t) = q, \quad M = \min\{w, q\},$$
$$X(t) = x, \quad Y(t) = y, \quad Z(t) = z.$$

To describe the law of motion of the system, let $X'(t)$ denote the time derivative of $X(t)$; then Table 1 shows the values of $X'(t)/k$ for the various cases of the relative magnitudes of p, q, w, and M.

(Recall that k is the inverse of the viscosity coefficient.) The law of motion for $Y(t)$ is determined symmetrically and, since the total mass of consumers is unity, $X'(t) + Y'(t) + Z'(t) = 0$. In particular, if $p = q = w$, then $X'(t) = Y'(t) = Z'(t) = 0$.

Note that $X(0) > 0$ implies that $X(t) > 0$ for all t, and similarly for Y and Z. Unless I explicitly mention otherwise, I shall assume that

$$X(0), Y(0), Z(0) \text{ are all } > 0. \tag{4.1}$$

I shall also assume, as in the monopoly model, that each firm's prices are confined to the closed interval $[0, m]$, where $m \geqslant w$ is an exogenously given parameter.

A *history*, $H(t)$, of the system at time t describes the time-path of the state of the system up to and including time t, and the time path of prices up to but not including

Table 1
$X'(t)/k$ as a function of (p, q)

Case	$X'(t)/k$
$p < M$	$(w - p)z + (q - p)y$,
$p > M$	$-(p - M)x$,
$p = q < w$	$[x/(x + y)](w - p)z$,
$p = w < q$	$[x/(x + z)](q - p)y$,
$p = q = w$	0

time t, i.e.,

$$H(t) = [\{X(s), Z(s); 0 \leqslant s \leqslant t\}, \{P(s), Q(s); 0 \leqslant s < t\}]. \tag{4.2}$$

A *strategy* for a firm is a mapping that determines, for each time t, its price at time t as a function of the history $H(t)$. A pair of strategies is called *feasible* if it determines a time path, $[X(t), Z(t), P(t), Q(t), t \geqslant 0]$, such that the payoffs of the two firms are well defined. The *payoff* (total discounted profit) of firm 1 is given by

$$V = \int_0^\infty \exp(-rt) P(t) X(t)\, dt, \tag{4.3}$$

where $r > 0$ is an exogenously given rate of interest. (Recall that costs are zero, so that profit equals revenue.) Firm 2's payoff is defined analogously.

The set of feasible strategy-pairs is not a product space, so it is not possible to define a game and its associated equilibria in the normal way. (See [13,14].) Instead, I shall use the concept of a *generalized game* [3]. For any strategy ψ of firm 2, let $\Phi(\psi)$ denote the set of strategies ϕ of firm 1 such that the strategy-pair (ϕ, ψ) is feasible; such a strategy ϕ will be called *a feasible response to* ψ. (Note that the set of feasible responses may be empty.) The set $\Psi(\phi)$ of feasible responses by firm 2 to a strategy ϕ of firm 1 is defined analogously. A feasible strategy-pair (ϕ, ψ) is called a (*Nash*) *equilibrium* if neither firm can increase its payoff by unilaterally switching to another feasible response. (Subgame-perfection can be defined analogously.) A firm's strategy is called *stationary* if its current price is a function of the current state of the system only (not the full history at the current date). An equilibrium strategy-pair is called *Markovian* if the strategies are stationary. (Markovian equilibria are automatically subgame-perfect.)

I shall demonstrate the existence (under certain assumptions) of a family of particularly simple Markovian equilibria, indexed by two parameters, a target total market penetration, and a target division of the market between the two firms. Formally, let Z and S be numbers between zero and one, where $(1 - Z)$ is interpreted as the *target market penetration*, and S and $(1 - S)$ are interpreted as the *target market shares* of firms 1 and 2, respectively. The pair (ϕ, ψ) of stationary strategies will be called a (Z, S) *target strategy-pair* if the prices $p = \phi(x, z)$ and $q = \psi(x, z)$ are given by the following table. Table 2 divides the (x, z) state space into four regions, and shows the corresponding prices and laws of motion in each region.

Table 2 indicates the motion of the state vector in the triangle

$$\Delta \equiv \{(x, z) : x \geqslant 0, z \geqslant 0, x + z \leqslant 1\}. \tag{4.4}$$

From any point in Δ the state vector, $[X(t), Z(t)]$, moves to the line $x = S(1 - z)$, with $Z(t)$ decreasing. Once on this line, say at (x, z), if $z > Z$ then the state vector moves down the line until $Z(t) = Z$, and stays there; if $z \leqslant Z$ then the state vector stays at (x, z). Thus any point (x, z) on the line segment

$$x = S(1 - z),$$

$$0 \leqslant z \leqslant Z,$$

Table 2
A (Z, S) target strategy-pair

Case	Region	Prices	Law of motion
Case 1A	$x = S(1 - z)$, $z \leqslant Z$	$p = q = w$	$X'(t) = Z'(t) = 0$
Case 1B	$x = S(1 - z)$, $z > Z$	$p = q = 0$	$X'(t) = [x/(x + y)]kwz,\ Z'(t) = -kwz$
Case 2	$x < S(1 - z)$	$p = 0, q = m$	$X'(t) = kmy + kwz,\ Z'(t) = -kwz$
Case 3	$x > S(1 - z)$	$p = m, q = 0$	$X'(t) = -kmx,\ Z'(t) = -kwz$

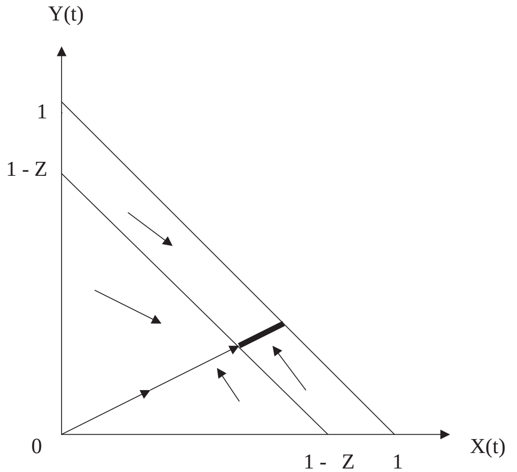

Fig. 1. Phase diagram for Table 2 in the (x, y) plane.

is a steady state of the system. Fig. 1 indicates the corresponding motion of the state in the (x, y) plane.

Define

$$a \equiv \frac{r}{kw},$$
$$s \equiv \frac{1}{a+1},$$
$$\varsigma \equiv \frac{a}{a+1} = 1 - s. \tag{4.5}$$

Make the following assumptions:

$$0 < a < 1,$$
$$0 < w \leqslant m,$$
$$0 \leqslant P(t) \leqslant m,$$
$$0 \leqslant Q(t) \leqslant m. \tag{4.6}$$

Theorem 2. *If assumptions* (4.6) *are satisfied, and if m is sufficiently close to w, then there exists* $\zeta' < \varsigma$ *such that, if*

$$\zeta' \leqslant Z \leqslant \varsigma,$$
$$1 - s \leqslant S \leqslant s,$$

then the (Z, S) *target strategy-pair is an equilibrium.*

(See Appendix A for the proof of the theorem.)

Remark 1. A comparison with Theorem 1 shows that the equilibrium path is *efficient*, in the sense that the total profit of the two firms is maximized, if and only if (1) the initial state vector is on the line $x = S(1 - z)$, and (2) $Z = \zeta$. Thus, if these conditions are satisfied, then the industry outcome as a whole mimics the monopoly outcome. On the other hand, if $Z < \zeta$, then the asymptotic market penetration will be greater than it would be in a corresponding monopoly, and the system spends more time in the regime in which one or both firms charge a zero price. In this sense, the equilibrium can be more "competitive" than the monopoly outcome.

Remark 2. An implication of the theorem is that a division of the market into shares S and $(1 - S)$ is self-sustaining, so that no "explicit collusion" is required once the target S is determined. On the other hand, since there is a nondegenerate interval of market shares that can be so sustained, some kind of "coordination" on a particular value of S is required. The same is true of the target market penetration, Z.

Remark 3. As $a \equiv (r/kw)$ approaches zero, the minimum target market penetration, $1 - \zeta$, approaches unity. In particular, for fixed discount rate and long-run WTP, this happens as the viscosity $(1/k)$ approaches zero.

Remark 4. It is natural to investigate the relationship between the present model and the classic Bertrand model, especially in the limit as the viscosity approaches zero $(k \to \infty)$. Thus, fix the initial state, (x, z), and let $k \to \infty$. From Remark 3 above, the target market penetration, $1 - Z$, must approach unity, so eventually $x < 1 - Z$. One can verify from the value function derived in Appendix A that each firms's profit converges to its target share of the monopoly profit. This contrasts with the static Bertrand duopoly, in which the duopolists made zero profit. I do not know whether there is some other equilibrium of the duopoly with viscous demand such that the firms' profits tend to zero as the viscosity approaches zero. However, suppose that the second firm sets its price equal to zero for all time. One can show that the first firm's best response is not to set its own price equal to zero for all time. On the other hand, for any policy of firm 1 for which its price is bounded, its corresponding profit approaches zero as $k \to \infty$.

5. Extensions and problems

In this section I describe three extensions of the previous analysis, and also discuss some open problems. (See Section 1 for a summary.)

5.1. Quick-response equilibrium

In this subsection I describe a non-Markovian equilibrium of the duopoly model of Section 4 that formalizes behavior in which each firm retaliates against a price cut by the other firm with a price cut of its own. Although the strategies are different, the equilibrium outcome is the same as that for the particular (Z, S) target equilibrium described in Section 4 in which $Z = \zeta$ and S is the initial market share of firm 1. The equilibrium strategies have the following properties: (1) if the initial total market penetration of the two firms is less than $s \equiv (1 - \varsigma)$, then both firms charge a zero price until the market penetration reaches s, after which they both charge a price equal to w; (2) if the initial total market penetration is at least s, then both firms charge w; (3) once the total market penetration reaches or exceeds s, if either firm charges a price strictly less than w, then the other firm will "immediately retaliate" by charging a price equal to *zero* (in fact, both firms will switch to zero); (4) on the other hand, if either firm *raises* its price above the equilibrium price, then the other firm will not change its own price. (Once again, each firm faces a "kinked demand curve".) Note that on the equilibrium path the industry as a whole imitates a monopolist's behavior, while the two firms maintain their initial relative market shares. This property of the equilibrium path contrasts with the seemingly "competitive" behavior of the firms (a point made by Anderson (1985) in a different context).

Technically, the game-theoretic approach differs somewhat from that of Section 4. I use a concept of equilibrium derived from the approaches of T.A. Marschak and R. Selten [5] and R.M. Anderson (1985). Following Anderson, I shall call this a "quick-response equilibrium". This concept formalizes the intuitive notion that if time is continuous then one firm can respond "immediately" to changes in the other firm's price. In order to sidestep some of the difficulties of doing game theory with continuous time (see, e.g., [12,13]) this approach deals with a family of discrete-time approximations to the continuous-time model. The framework is consequently notationally more complicated than that of Section 4. I shall present here only the model and the results. (For proofs, see R. Radner, Viscous Demand, AT&T Bell Laboratories, 1995, unpublished.)

The underlying model used here is that described at the beginning of Section 4, and thus has time varying continuously. In particular, the law of motion is the one given in Table 1. However, in a *quick-response equilibrium* (*QRE*) one does not define a game directly for the situation of continuous time, but rather approximates that situation with a family of discrete-time games. Accordingly, for each number $h > 0$, define a game $G(h)$ as follows: for every nonnegative integer multiple nh of h, the two firms simultaneously choose respective prices that will be operative during the half-open interval, $[nh, (n+1)h)$, and in that interval the masses of customers of the two firms, $X(t)$ and $Y(t)$, evolve according to the law of motion described in Table 1. Thus a *strategy* in the game $G(h)$ is defined in the usual way for a discrete-time game, and each strategy determines a time path, $[P(t), Q(t), t \geqslant 0]$ of the prices of the two firms. The *payoff* (total discounted profit) of firm 1 is given by

$$V = \int_0^\infty \exp(-rt)P(t)X(t)\,dt,$$

where $r > 0$ is the exogenously given rate of interest. (Recall that costs are zero, so that profit equals revenue.) Firm 2's payoff is defined analogously. Note that the last integral is well-defined, since P is a simple function, which we may take to be continuous from the right.

Suppose that $\sum = \{\Sigma(h)\}$ is a family of strategy pairs such that $\Sigma(h)$ is a strategy-pair in the game $G(h)$, and let $[V(h), W(h)]$ denote the corresponding payoffs of the two firms, respectively. The family $\sum$ is a QRE if the following two conditions hold: (1) for every initial state $(x, y) \gg 0$ there exists a number $h(x, y) > 0$ such that, for every strictly positive number $h \leqslant h(x, y)$, the strategy pair $\Sigma(h)$ is a Nash equilibrium of the game $G(h)$; (2) the limit payoffs exist, namely,

$$V = \lim_{h \to 0} V(h), \quad \text{and} \quad W = \lim_{h \to 0} W(h).$$

The numbers (V, W) will be called the *QRE payoffs*.

For each h, let $[X(t; h), Y(t; h)]$ denote the state of the system at time t determined by the (QRE) strategy-pair $\Sigma(h)$ in game $G(h)$, and let the corresponding prices be $[P(t; h), Q(t; h)]$. If, in addition, the limit trajectory exists, namely

$$[X(t), Y(t), P(t), Q(t)] = \lim_{h \to 0} [X(t; h), Y(t; h), P(t; h), Q(t; h)],$$

then I shall call the limit trajectory the *QRE path*. In the QRE of the duopoly model that I shall describe, the QRE path will exist, and furthermore the QRE payoff for each firm will be its discounted profit along the QRE path.

Note that the definition of QRE given thus far does not include any notion of subgame-perfecton, i.e., it is not required that the "threats" of retaliation against price cuts be "credible". It is therefore desirable to define a stronger version of QRE that responds to this need. Accordingly, for every $\theta > 0$, let $\Delta(\theta)$ denote the open triangle,

$$x > \theta, y > \theta, x + y < 1 - \theta.$$

I shall say that the QRE $\sum$ is *quasi-subgame-perfect* (*QSP*) if, for every $\theta > 0$, there is an $H(\theta) > 0$ such that, for every positive $h < H(\theta)$, every time t, and every history of the game $G(h)$ through time t for which

$$[X(s), Y(s)] \text{ is in } \Delta(\theta), \quad 0 \leqslant s \leqslant t,$$

the continuation of the strategy-pair $\Sigma(h)$ from time t on is a QRE of the continuation game.

Here is a heuristic description of the QRE strategies. Fix $h > 0$, and let I be a positive integer. It suffices to describe firm 1's strategy, firm 2's being defined symmetrically. Firm 1 charges a price equal to zero until the first date nh at which

$Z(nh) \leqslant \zeta$ (i.e., total market penetration reaches or exceeds the target). Thereafter, firm 1 charges a price equal to w, with the following exception: if at some date nh firm 2 undercuts firm 1 by charging a price strictly less than w, then firm 1 will retaliate by charging a price equal to zero for the next I periods, and then return to the price w at date $(n + I + 1)h$; by symmetry, firm 2 will do likewise. The sequence of retaliation periods will also be started anew after any failure of either firm to carry out the prescribed retaliation. I can show that by taking I large enough, firm 1 can deter firm 2 from any deviation from the QRE path, since whatever value firm 1 can gain in period nh will be offset by a sufficiently large loss in the subsequent I periods. It will be important to show that the number I can be taken to be independent of h, although it will depend on the state, $[X(nh), Y(nh)]$, in which the deviation occurs. Note that firm 1 does *not* respond if firm 2 raises its price above firm 1's price (no matter what the value of $Z(t)$ at the time). (For a precise description of the QRE strategies, see Radner, 1995.)

Theorem 3. *There exists a choice of the function I such that the family $\sum$ of strategy-pairs is a quasi-subgame-perfect quick-response equilibrium of the family of games $\{G(h)\}$. Furthermore, the QRE path exists, and the QRE payoff for each firm is its total discounted profit along the QRE path. Firm* 1*'s QRE payoff is given by*

$$V = \begin{cases} \left(\frac{wx}{r}\right)\frac{(1-\zeta)\zeta^a}{(1-z)z^a} & \text{for } z > \zeta, \\ \left(\frac{wx}{r}\right) & \text{for } z \leqslant \zeta. \end{cases}$$

A corresponding equation holds for firm 2.

We see that, for fixed z, firm 1's QRE payoff is linear in its initial market penetration, x. On the other hand, by Lemma 6 of Section 9.4, the function f defined by

$$f(z) \equiv (1-z)z^a$$

is decreasing if $z > \zeta$ (and, incidentally, increasing if $z < \zeta$); use the fact that

$$\zeta = a/(1+a).$$

Hence, for fixed x, firm 1's QRE payoff is independent of z for $z < \zeta$, and is increasing in z for $z > \zeta$.

5.2. Adaptive expectations

As noted in Section 3, for some parameter values the monopolist's "optimal" price oscillates "infinitely fast" between zero and the maximum value. As noted in Sections 1 and 2, purely myopic choices by consumers would be implausible under such circumstances. Suppose, therefore, that—when considering whether or not to purchase the service—a consumer forecasts the future price to be some moving average of past prices, and makes the purchase decision on the basis of that forecast.

Accordingly, let $\tilde{P}(t)$ denote the price "forecast" at time t, and suppose that $\tilde{P}(t)$ is determined by

$$\tilde{P}(t) = \theta \int_0^{\infty} e^{-\theta s} P(t-s)\, ds, \tag{5.2.1}$$

where $\theta > 0$ is a given parameter of the model. Assume that the law of motion (3.1) is modified to read

$$X'(t) = \mu[\tilde{P}(t), X(t)]. \tag{5.2.2}$$

Following a precedent by Arrow and Nerlove in the literature on expectations [1], I shall call this the *adaptive expectations model*, or more precisely, the θ-AR model (since the model is parametrized by θ). Note that

$$\lim_{\theta \to \infty} \tilde{P}(t) = P(t),$$

so that the model of Section 3 may be considered a limiting case of the adaptive expectations model, which one might denote the ∞-AR model. In fact, although an explicit solution for the monopolist's optimal pricing strategy is not known for the θ-AR model (with $\theta < \infty$), one can show that (1) an optimal policy exists, and (2) for large finite θ the corresponding optimal policy is approximately optimal for $\theta = \infty$. In particular, when the optimal price for the ∞-AR model oscillates infinitely fast, the optimal price for large finite θ also oscillates, but at a finite rate (for these results, see [9]). Such behavior by the monopolist might be interpreted as a policy of "intermittent sales".

5.3. A monopolist facing consumers with a distribution of willingness-to-pay

Up to this point, the entire analysis for both monopoly and duopoly has been carried out under the assumption that all the consumers have the same long-run willingness-to-pay (WTP) for the service. Extending the analysis to the case in which the consumers are heterogeneous with respect to WTP would be desirable, but thus far I have been unable to do this in any generality in the context of the present model. In this subsection I give a heuristic argument that suggests that oscillatory pricing may be common when the WTP is sufficiently dispersed in the population of consumers.

I shall say that *a consumer is of type w* if his WTP for the service is w. Although the consumers are heterogeneous with respect to WTP, I assume that the monopolist can charge only a single price at any given time. In other words, the monopolist is unable to discriminate among the consumers according to their type. Of course, if the monopolist could do so, then the problem would reduce to that of Section 3, for each type of consumer.

Suppose that the type w lies in the interval $[0, W]$ and has an absolutely continuous distribution in the population of consumers, and let g denote the density function of this distribution. The absolute continuity of the distribution expresses the assumption that the WTP is "dispersed" in the population of consumers. Let $X(w,t)$ denote the "fraction of consumers of type w" who are customers of the monopolist at time t, or to be more precise, the total mass of customers of type not exceeding w at time t is given by the integral,

$$\int_0^w X(u,t)g(u)\,du.$$

Let $X(t)$ denote the function $X(\cdot,t)$; then $X(t)$ is the *state of the system at time t*. Thus a state of the system is a function, say x, from $[0,W]$ to $[0,1]$ such that the integral

$$\int_0^w x(u)g(u)\,du$$

exists for each w. Assume that, for each w, the state variable $X(w,t)$ obeys the law of motion (3.1)–(3.2).

A *steady state* for a particular pricing policy would be a state-price pair, say (ξ,φ), such that

$$[X(0),P(0)] = (\xi,\varphi) \Rightarrow [X(t),P(t)] = (\xi,\varphi) \quad \text{for all } t>0.$$

(For example, in the case of a single WTP (Section 3), if $s \leqslant x \leqslant 1$, then (x,w) is a steady state.) The law of motion implies that a steady state (ξ,φ) satisfies

$$\xi(w) = 0 \text{ or } 1 \text{ according as } w< \text{ or } >\varphi. \tag{5.3.1}$$

(Note that the law of motion implies nothing about $\xi(p)$.) I shall give a heuristic argument that suggests that *there is no steady state for an optimal policy*.)

The heuristic argument uses the "Bellman Optimality Conditions" (see Section A.2, (A.2.14)–(A.2.16)). Suppose that an optimal pricing policy exists. For any state x , let $V(x)$ denote the monopolist's maximum profit, starting from the state x; V is the monopolist's *value functional*. Note that V is a mapping from the infinite-dimensional space of states to the real numbers. Although the state-space is infinite dimensional, one can still (under suitable conditions) formulate the notion of *the partial derivative of V with respect to* $x(w)$, which I shall denote by $V'(w,x)$. Correspondingly, the "Bellmanian Functional" (or "Bellmanian", for short) for the monopolist's optimization problem is (again, see Section A.2)

$$\begin{aligned} B_V(p,x) \equiv p \int_0^W x(w)g(w)\,dw - rV(x) - \int_{w<p} k(p-w)x(w)V'(w,x)\,dw \\ + \int_{w>p} k(w-p)[1-x(w)]V'(w,x)\,dw. \end{aligned}$$

If (ξ,φ) is a steady state for the optimal policy, then by (5.3.1), if $p>\varphi$,

$$B_V(p,\xi) \equiv p \int_\varphi^W g(w)dw - rV(\xi) - \int_\varphi^p k(p-w)V'(w,x)\,dw,$$

and hence

$$\frac{\partial B_V(p,\xi)}{\partial p} = \int_{\varphi}^{W} g(w)\,dw - \int_{\varphi}^{p} kV'(w,x)\,dw$$
$$\to \int_{\varphi}^{W} g(w)\,dw \text{ as } p \searrow \varphi.$$

Therefore, if $\varphi < W$, then the Bellmanian is strictly increasing in the price p for $p > \varphi$ and sufficiently close to φ. Hence (under suitable regularity conditions) $\varphi < W$ cannot be optimal at the state ξ.

On the other hand, if (ξ, W) were a steady state for the optimal policy, then by (5.3.1), $\xi(w)$ would be zero for all $w < W$, and hence $V(\xi)$ would be zero. However, starting from such a state ξ it is possible for the monopolist to make a strictly positive profit, e.g., by setting the price equal to zero for a positive amount of time, and then setting the price equal to any positive value thereafter. Hence $\varphi = W$ cannot be optimal, either.

To make this heuristic argument rigorous, one could formulate a model with a large but finite number of consumer types, whose masses are uniformly small. This can be done, but I omit the details. An alternative approach would be: (1) postulate conditions on the model such that a theory of the Bellmanian Functional for an infinite-dimensional state space is valid; (2) provide conditions such that the corresponding Bellman Conditions for optimality are necessary rather than sufficient. Such an analysis is beyond the scope of this paper.

5.4. *Oligopoly and competition*

In this subsection I sketch some of the problems to be faced in generalizing the analysis to the cases of oligopoly (with more than 2 firms) and "competition".

The formulation of a model with more than 2 identical firms is straightforward, if one assumes that the consumers always flow to the firm(s) with the lowest price, or flow into the noncustomer category if all firms charge a price greater than the WTP (assuming for the moment that all consumers have the same WTP). One might even conjecture that there are Markovian equilibria analogous to that of the duopoly of Section 4, in which each firm has a target market share, and sets its price low or high according as its current market share is less than or greater than its target.

The assumption that the firms are identical is of course problematic, even in the case of a duopoly. Differentiation of the firms could have two consequences: (1) each consumer would have different WTPs for the services of different firms; (2) consumers could differ in their WTP profiles, with some consumers preferring the services of one firm, and others preferring the services of another firm (different firms have different "clienteles"). Given the difficulty suggested in Section 5.3 above, I shall not even venture a conjecture about the nature of equilibria in such cases.

If by "competition" (perfect or imperfect) one means a "large" number of "small" firms, then one might want to reconsider the law of motion. First, with a law of motion similar to that described above, if a small firm's price were the lowest among a large number of firms, then that firm would face a relatively enormous rate of increase in demand, which it might not be able to meet in the short run. Second, a consumer facing a large number of firms might not immediately be able to identify the most preferred one (e.g., the one with the lowest price), and hence might migrate in stages from less preferred firms to more preferred ones. (Models studied by Selten and by Phelps and Winter have the flavor of the latter phenomenon; see Section 6.)

The analysis of models that incorporate all these considerations would appear to present daunting difficulties. In fact, my limited exposure to practical problems of this kind suggests to me that managers are not (knowingly) following optimal policies in the pricing situations that they face. If this impression is correct, then a satisfactory theory would have to model the bounded rationality of the managers, as well as that of the consumers.

6. Bibliographic notes

The earliest theoretical paper that I am aware of that is related to the present one in spirit is by Selten [12]. In Selten's model, finitely many firms repeatedly face a market with an exogenously given total "demand potential" (see below). Each firm has a linear cost function (they may be different). Time is discrete. If we specialize the Selten model to the case of a duopoly, with total "demand potential" constant in time, then in the notation of Section 4 of the present paper, the demand (quantity sold) for firm 1 obeys the law of motion,

$$X(t+1) - X(t) = KP(t) + (1-K)Q(t) - P(t+1),$$

where K is a constant, the same for both firms. (Recall that firm 1's demand in period t is $X(t)$, and the prices of firms 1 and 2, resp., are $P(t)$ and $Q(t)$.) The law of motion for firm 2's demand is determined symmetrically. The initial conditions are

$$X(0) = m - P(0),$$

$$Y(0) = n - Q(0),$$

where m and n are given constants, or equivalently,

$$X(0) + P(0) = m,$$

$$Y(0) + Q(0) = n.$$

One can verify that, for all periods t,

$$X(t) + P(t) + Y(t) + Q(t) = m + n.$$

Selten calls

$$M(t) \equiv X(t) + P(t) \text{ firm 1's "demand potential"},$$

and

$$N(t) \equiv Y(t) + Q(t) \text{ firm 2's "demand potential"}.$$

We may interpret a single firm's demand potential in a given period as what its demand would be if its price were zero in that period. The model implies that the *total* demand potential remains constant, i.e., equal to the initial value, $(m+n)$.

For a game with a fixed (and known) number of periods, Selten demonstrates the existence of a subgame-perfect equilibrium. (An infinite-horizon game is studied in Part II of the paper.) The model is related to the present one in that customers do not instantaneously react fully to changes in prices. However, it is not clear to me how to reconcile this law of motion with a model of consumer behavior like that sketched in Section 2 of the present paper. (Some readers will recognize Selten's paper as the one in which he introduced the concept of "subgame perfection".)

In a series of three papers [2,10,11], Rosenthal and Chen have studied related models of duopoly with "customer loyalties". Each of the models is a discrete-time, infinite-horizon, symmetric, non-zero-sum stochastic game, in which the players are the two firms, and there are finitely many identical customers who act in accordance with a fixed "rule of thumb" (different in each paper). In [10], in each period "each buyer purchases from the same seller from whom he purchased in the last period unless that seller has raised his price, in which case the buyer purchases from the current-period low-pricesetter". In this game, Rosenthal demonstrates the existence of a Markov equilibrium in mixed strategies, and studies its properties. In [10], customer loyalties are "weaker". In each period, "after the sellers have set their current-period prices, a random device... determines whether (with probability α) each buyer will remain loyal whenever his previous-period seller has not raised price, or whether (with probability $(1-\alpha)$) all buyers abandon their loyalties and purchase from the current-period low-price seller". In this game, Rosenthal demonstrates the existence of an epsilon-equilibrium in stationary mixed strategies. (There is an exact equilibrium from an initial state in which one seller has all the customers.) In [2], in each period, if one seller's price is strictly less than the other's, then one customer shifts from high-price seller to the other one (unless, of course, the high-price seller has no customers). Again, there is a Markov equilibrium in mixed strategies, and the authors study the effect on this equilibrium of changing the parameters of the model. (*Note*: I have omitted a description of the detailed assumptions in each paper.)

Phelps and Winter [7] studied a model at the other end of the spectrum from duopoly, namely, one in which there is a very large number of small sellers. Each seller i "subjectively assumes" that the (continuous-time) law of motion for his market share, $X_i(t)$, is

$$X_i'(t) = \delta[p_i(t), P_i(t)]X_i(t),$$

where $p_i(t)$ is his current price, $P_i(t)$ is the customer-weighted mean of the other firms' prices, and δ is a skew-symmetric function with plausible properties. In fact, the "true" law of motion in the model is different from, and more complicated than, the one "subjectively perceived" *by the firms*; I omit the details. At time $t=0$, each firm chooses a price path that maximizes the present value of its discounted profits, under the assumption that the average price $P_i(t)$ will remain equal to its initial value, $P_i(0)$, for all time, in other words, assuming that the law of motion for its market share is

$$X_i'(t) = \delta[p_i(t), P_i(0)]X_i(t).$$

This represents a further simplification of the firm's subjective perception of the law of motion of its demand. (Alternatively, it may represent an approximation that may be reasonable for a short enough time interval.) In particular, each firm assumes that, if its price remains constant during some interval of time, then its demand will grow or decrease exponentially in that interval. This model is sufficiently far from the one in the present paper that I shall not attempt to summarize the results of the authors' analysis, except to report that any steady state of the system will depart from the standard picture of the equilibrium of a ''neoclassical'' model.

I have already noted the difference between the model of viscous demand presented here, and models intended to capture consumers' '' switching costs''. For a recent treatment, and references to earlier literature, see [6].

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Appendix A

A.1. Derivation of the model of Eqs. (2.1) and (2.2)

In this appendix I sketch a derivation of the model of Eqs. (2.1) and (2.2), using the law of large numbers. Suppose that there are potentially countably many consumers, indexed by the nonnegative integers n, and let $X_n(t)$ be 1 or 0 according as consumer n is or is not a customer at time $t \geqslant 0$. All the consumers have the same long-run willingness-to-pay, w. Let $P(t)$ denote the monopolist's price at time t, and recall that

$$0 \leqslant P(t) \leqslant m.$$

Assume that $P(\cdot)$ is measurable, and define

$$k(t) = k[P(t) - w], \tag{A.1.1}$$

then $k(\cdot)$ is bounded and measurable on $[0, \infty)$. Define the positive and negative parts of $k(\cdot)$ by

$$k^+(t) = \max\{k(t), 0\},$$
$$k^-(t) = \max\{-k(t), 0\}.$$

I now specify the properties of the "demand processes", $X_n(t)$. For each nonnegative integer n, let $\{X_n(t) : t \geqslant 0\}$ be a continuous-time Markov process, with $X_n(t)$ taking values 0 or 1. The processes $\{X_n(\cdot)\}$ are mutually independent and identically distributed. Each $\{X_n(\cdot)\}$ is a "generalized Poisson process", defined as follows. Conditional on a given time t, let T be a random time. First, condition on $X_n(t) = 0$:

$$\Pr\{T \leqslant s \mid X_n(t) = 0\} = 1 - \exp\left(-\int_0^s k^-(t+u)\,du\right) \equiv F_0(s), \tag{A.1.2}$$

$$X_n(t+u) = 0, \quad 0 \leqslant u < T,$$
$$X_n(t+T) = 1. \tag{A.1.3}$$

Similarly, condition on $X_n(t) = 1$:

$$\Pr\{T \leqslant s \mid X_n(t) = 1\} = 1 - \exp\left(-\int_0^s k^+(t+u)\,du\right) \equiv F_1(s), \tag{A.1.4}$$

$$X_n(t+u) = 1, \quad 0 \leqslant u < T,$$
$$X_n(t+T) = 0. \tag{A.1.5}$$

Finally, for a given p, with $0 < p < 1$,

$$\Pr\{X_n(0) = 1\} = p. \tag{A.1.6}$$

It is straightforward to verify that Eqs. (A.1.2)–(A.1.6) determine a Markov process. It will be useful to denote by Ω the underlying probability space on which these processes are defined, suppressing the notation for the corresponding probability measure and sigma-field of measurable sets. Thus $X_n(t) = X_n(t, \omega)$.

Note that, for each consumer n, Eqs. (A.1.2)–(A.1.6) determine a (finite or infinite) sequence $\{T_{nm}\}$ of random times. At time T_{nm} consumer n switches from being a customer to being a noncustomer, or vice versa.

For any positive integer N define

$$(1)\quad X(t,N,\omega)=\frac{1}{N}\sum_{n=1}^{N}X_n(t,\omega).$$

Let $x(t)$ denote the expected value of $X_n(t,\omega)$, i.e.,

$$x(t)\equiv EX_n(t,\omega). \tag{A.1.7}$$

By the strong law of large numbers, for every $t\geqslant 0$ there is a subset $S(t)$ of Ω with probability one such that, for every $\omega\in S(t)$,

$$\lim_{N\to\infty} X(t,N,\omega)=x(t). \tag{A.1.8}$$

In fact, the following stronger statement is true:

Proposition A.1. *There exists a subset S of Ω with probability one such that, for every $\omega\in S$, and every $t\geqslant 0$, Eq.* (A.1.8) *holds.*

To prove the proposition, let I be any finite time interval. By Theorem 8.1 of Parthasarathy [8, p. 255], there is a measurable subset $S(I)$ of Ω with probability one such that, for every $\omega\in S(I)$ and every $t\in I$, Eq. (A.1.8) holds. Since the half-line $[0,\infty)$ is a union of countably many finite intervals, the proposition follows.

The question is whether, when the number of consumers is "large", it is legitimate to replace the actual evolution of demand with its expected value. I shall call the resulting model the "deterministic model". There are two aspects of this question that need to be resolved.

First, we want to show that, in a sense to be made precise, when N is "large" we can calculate the monopolist's profit approximately by replacing $X(t,N,\omega)$ by $x(t)$. For any given N, the monopolist's total discounted profit per consumer is

$$V(N,\omega)=\int_0^\infty e^{-rt}U(t,N,\omega)\,dt,$$
$$U(t,N,\omega)\equiv[P(t)-c]X(t,N,\omega),$$

where $r>0$ is the exogenously given rate of interest, and c is his constant per-unit cost. Think of the nonnegative reals as a measure space with the finite measure generated by

$$\rho([s,u])=\int_s^u e^{-rt}\,dt$$

for every interval $[s,u]$. Fix $\omega\in S$; then for every $t\geqslant 0$,

$$\lim_{N\to\infty} U(t,N,\omega)=[P(t)-c]x(t).$$

Since $U(t,N,\omega)$ is bounded, it follows from Lebesgue's dominated convergence that

$$\lim_{N\to\infty} V(N,\omega)=\int_0^\infty e^{-rt}[P(t)-c]x(t)\,dt.$$

Note that the right-hand side is independent of $\omega \in S$, and recall that S has probability measure one.

Second, we want to show that the function $x(\cdot)$, which describes the evolution through time of the expected demand per consumer, satisfies the differential equation,

$$x'(t) = \begin{cases} -k(t)x(t) & k(t) \geqslant 0, \\ k(t)[1 - x(t)] & k(t) \leqslant 0. \end{cases}$$

This property of the expected demand makes the model amenable to standard control theory.

Proposition A.2. *If $k(\cdot)$ is bounded and measurable; then $x(\cdot)$ satisfies the differential equation*

$$x'(t) = -k^+(t)x(t) + k^-(t)[1 - x(t)].$$

Proof. For $h > 0$ "small",

$$\begin{aligned} &\Pr\{X_n(t+h,\omega) = 1 | X_n(t,\omega) = 1\} \\ &\quad = 1 - F_1(h) + o(h^2) \\ &\quad = 1 - hF_1'(0) + o(h^2) \\ &\quad = 1 - hk^+(t) + o(h^2), \end{aligned}$$

and

$$\begin{aligned} &\Pr\{X_n(t+h,\omega) = 1 | X_n(t,\omega) = 0\} \\ &\quad = F_0(h) + o(h^2) \\ &\quad = hF_0'(0) + o(h^2) \\ &\quad = hk^-(t) + o(h^2). \end{aligned}$$

Now observe that

$$\begin{aligned} x(t+h) = &\, x(t)\Pr\{X_n(t+h,\omega) = 1 \mid X_n(t,\omega) = 1\} \\ &+ [1 - x(t)] \Pr\{X_n(t+h,\omega) = 1 \mid X_n(t,\omega) = 0\}. \end{aligned}$$

From the above equations we get

$$x(t+h) - x(t) = h\{-k^+(t)x(t) + k^-(t)[1 - x(t)]\} + o(h^2),$$

which leads immediately to the desired result.

I note that it would be tempting to proceed directly to a model with a "continuum" of independent and identically distributed (IID) consumer-demand processes, invoking some sort of law of large numbers, so that the consumers flow smoothly in and out of the customer pool according to the differential equation in the last proposition. However, the difficulties dealing with an uncountable family of IID random variables (not to mention stochastic processes) are well known.

A.2. Proof of Theorem 1

The proof of Theorem 1 needs the following lemma.

Lemma. *Let* $x \equiv X(0)$, *and let* $V(x)$ *be the profit for any policy of the form* (3.6a), *for any value of* s *in* $[0, 1]$ (*not necessarily the value given in* (3.6b)), *starting from state* x. *Define*

$$D(x) \equiv \left(\frac{1-s}{1-x}\right)^{r/kw}, \tag{A.2.1}$$

$$V_0(x) \equiv D(x)\left(\frac{ws}{r}\right), \tag{A.2.2}$$

$$Q(x) \equiv \frac{rx + kw - (1-s)kwD(x)}{r(r+kw)}, \tag{A.2.3}$$

then the profit is

$$V(x) = \begin{cases} V_0(x) - cQ(x), & x \leqslant s, \\ (w-c)x/r, & x \geqslant s, \end{cases} \tag{A.2.4}$$

furthermore, if $x < s$, *let* T *be the first* t *such that* $X(t) = s$, *then*

$$e^{-rT} = D(x). \tag{A.2.5}$$

Remark A.1. For, $x \leqslant s$,

$$Q(x) = \int_0^\infty e^{-rt} X(t)\, dt.$$

Remark A.2. The value of s that maximizes $V(x)$ in (A.2.4), *for all* x, is given by (3.6b), as is easily verified; that value is

$$s \equiv \frac{k(w-c)}{r + k(w-c)} \quad (3.6b).$$

Remark A.3. Note that V is linear in x for $x \geqslant s$. We shall see that, for $x < s$, V is increasing and strictly convex. For arbitrary s the derivative of V may be discontinuous at s, but for the optimal value of s, (3.6b), the derivative of V is continuous at all x.

Proof of Lemma. *Case* 1: $x \geqslant s$. In this case, x is a steady state, and $p(t) = w$, for all t. Hence, by the definition of profit,

$$V(x) = \int_0^\infty e^{-rt}(w-c)x\,dt = \frac{(w-c)x}{r}, \tag{A.2.6}$$

which proves the second part of (A.2.4).

Case 2: $0 \leqslant x < s$. In this case, $p(t) = 0$ for $0 \leqslant t < T$, so by (3.1) and (3.2),

$$X(t) = 1 - (1-x)e^{-kwt}, \quad 0 \leqslant t \leqslant T, \tag{A.2.7}$$

and T is determined by

$$s = 1 - (1-x)e^{-kwT},$$

recall that $x = X(0)$. Hence

$$e^{-kwT} = \frac{1-s}{1-x}, \tag{A.2.8}$$

$$T = \left(\frac{1}{kw}\right) ln\left(\frac{1-x}{1-s}\right), \tag{A.2.9}$$

which also implies (A.2.5). Keeping in mind that $p(t) = 0$ for all $t < T$,

$$V(x) = -c\int_0^T e^{-rt}X(t)\,dt + e^{-rT}V(s). \tag{A.2.10}$$

By (A.2.7), the integral in this last expression equals

$$\int_0^T e^{-rt}[1-(1-x)e^{-kwt}]\,dt = \frac{1-e^{-rT}}{r} - \frac{(1-x)(1-e^{-(r+kw)T})}{r+kw} = \frac{1-D(x)}{r} - \frac{(1-x)-(1-s)D(x)}{r+kw}, \tag{A.2.11}$$

the last step uses (A.2.1) and (A.2.5). On the other hand, from Case 1 and (1.5), the second term in the right-hand side of (A.2.10) is

$$\frac{D(x)(w-c)s}{r}. \tag{A.2.12}$$

Combining (A.2.10)–(A.2.12) yields (A.2.2)–(A.2.4), which completes the proof of the Lemma. □

Proof of Theorem 1. For any stationary policy ψ, let $V_\psi(x)$ denote the corresponding total discounted profit, given $X(0) = x$. Let ϕ be a particular stationary policy, and let V denote its corresponding value function. Corresponding to V, define the *Bellmanian functional*, $B(p,x)$ by

$$B(p,x) \equiv -rV(x) + (p-c)x + \mu(p,x)V'(x), \tag{A.2.13}$$

where V' denotes the first derivative of V, and μ is the law of motion. The following two facts about the Bellmanian functional are well known (see, e.g., [4, Chapter 4]): (1) if V is C^1, then for all x,

$$B[\phi(x), x] = 0, \tag{A.2.14}$$

and (2) if V is C^1, and for all x and p,

$$B(p, x) \leqslant 0, \tag{A.2.15}$$

then

$$V(x) = \sup_{\psi} V_{\psi}(x). \tag{A.2.16}$$

In other words, the policy ϕ is optimal. Following standard usage, these will be called the *Bellman optimality conditions*.

In the light these facts, to prove Theorem 1 it is sufficient to show that (A.2.15) is verified for the function V of the Lemma, with s given by (3.6b).

I first note that B is piecewise-linear in p, for each x. Let $B' \equiv \partial B/\partial p$; then

$$B'(p, x) = \begin{cases} x - k(1-x)V'(x), & p < w, \\ x[1 - kV'(x)], & p > w. \end{cases} \tag{A.2.17}$$

We shall see below that there exists $x_1 < s$ such that we must consider 3 cases:

(1) $0 < x < x_1$, (2) $x_1 < x < s$, and (3) $s < x$.

From (1.1)–(1.4) it is straightforward to calculate V'. For $x < s$ we have

$$V'(x) = \left(\frac{s}{k}\right)\frac{(1-s)^a}{(1-x)^{1+a}} - \frac{c[1 - (\frac{1-s}{1-x})^{1+a}]}{r + kw}, \tag{A.2.18}$$

where

$$a \equiv r/kw, \tag{A.2.19}$$

whereas for $x > s$,

$$V'(x) = \frac{w - c}{r}. \tag{A.2.20}$$

Note that $V'(x)$ is increasing in x for $x < s$, and that

$$\begin{aligned} V'(0) &= \left(\frac{s}{k}\right)(1-s)^a - \frac{c[1-(1-s)^{1+a}]}{r+kw} \\ &< \left(\frac{s}{k}\right)(1-s)^a \\ &< 1/k, \end{aligned} \tag{A.2.21}$$

also,

$$V'(s-) = \frac{s}{k(1-s)}. \tag{A.2.22}$$

By (3.6b)

$$\frac{s}{1-s} = \frac{k(w-c)}{r}, \tag{A.2.23}$$

$$V'(s-) = \frac{w-c}{r} = V'(s+), \tag{A.2.24}$$

and so V' is continuous at s.

From (A.2.17) we see that for $p > w, B'(p, x)$ is positive or negative according as $kV'(x)$ is less than or greater than 1. From (A.2.21), $kV'(0) < 1$, whereas from (A.2.24) and (3.4a),

$$kV'(s) = \frac{k(w-c)}{r} > 1.$$

Hence there exists x_1 such that $0 < x_1 < s$ and

$$kV'(x) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 1 \text{ as } x \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} x_1. \tag{A.2.25}$$

I now consider the above 3 cases in turn.

Case 1: $0 \leqslant x < x_1$.

Case 1.1: $0 < p < w$.

From the first line of (A.2.17),

$$\begin{aligned} B'(p, x) &= x - k(1-x)V'(x) \\ &\equiv f(x). \end{aligned} \tag{A.2.26}$$

From (A.2.15), we wish to show that $f(x) < 0$, because this would imply that

$$B(p, x) < B(0, x) = B[\phi(x), x] = 0 \tag{A.2.27}$$

(note that we have used (A.2.14) in the last equality). Since

$$f(s) = s - \frac{k(1-s)(w-c)}{r} = 0, \tag{A.2.28}$$

it is sufficient to show that f is increasing on $[0, s]$. One can verify from (A.2.26) and (A.2.18) that

$$f'(x) = 1 + kV'(x) - k(1-x)V''(x),$$

$$V''(x) = \frac{(r+kw)}{kw(1-x)}\left[V'(x) + \frac{c}{r+kw}\right],$$

and hence

$$f'(x) = \frac{(w-c) - rV'(x)}{w} > 0, \tag{A.2.29}$$

the last inequality following from (A.2.24) and the strict convexity of V on $(0, s)$.

Note that we have now shown that (A.2.27) holds for x in the entire open interval $(0, s)$, not just the interval $(0, x_1)$.

Case 1.2: $w < p < m$. In this case, from (A.2.25) and the second line of (A.2.17),

$$B'(p,x) = x[1 - kV'(x)] \equiv g(x) > 0, \quad 0 < x < x_1. \tag{A.2.30}$$

Hence, to verify the Bellman optimality condition (A.2.15) it is sufficient to verify that $B(m,x) \leqslant 0$. Observe that

$$B(m,x) = B(w,x) + (m-w)g(x).$$

On the one hand, from (A.2.27) and the continuity of B,

$$B(w,x) < 0.$$

On the other hand, g is strictly positive on the open interval $(0, x_1)$, and is continuous on the closed interval $[0, x_1]$, so that it attains a maximum there, say $\gamma > 0$; then $B(m,x) \leqslant 0$ for all x in $[0, x_1]$ if

$$m - w \leqslant -\frac{B(w,x)}{\gamma}. \tag{A.2.31}$$

Note that the right-hand-side of the last inequality is positive. We can rewrite this inequality as

$$m \leqslant w - \frac{B(w,x)}{\gamma}, \tag{A.2.32}$$

which, as in the statement of the theorem, can be paraphrased as "m is sufficiently close to w".

Case 2: $x_1 < x < s$. First note that the argument in Case 1.1 shows that $B(0,x) = 0$, and $B'(p,x) < 0$ for $p > w$ as well, since $x > x_1$. Hence the optimality condition (A.2.15) is satisfied in this case.

Case 3: $x > s$. Recall that in this case, by (A.2.20),

$$V'(x) = \frac{w-c}{r}.$$

Hence, by the first line of (A.2.17), if $p < w$ then

$$\begin{aligned} B'(p,x) &= x\left[1 - \frac{k(w-c)}{r}\right] \\ &= \left(\frac{x}{r}\right)[r - k(w-c)], \end{aligned}$$

which is < 0 by assumption (3.4a). Finally, by (A.2.14), $B(w,x) = 0$, so that the optimality condition (A.2.15) is also satisfied in this case. This completes the proof of Theorem 1.

A.3. Theorem 2: Calculation of the value function

I start by assuming that $m = w$. (Recall that m is the maximum price, and w is the willingness-to-pay.) The calculation of the value function is organized in three cases, according to the location of the state of the system, (x, z), in the

Table 3

Case 1A	$x = S(1-z), z \leqslant Z,$	$p = q = w$	$X'(t) = Z'(t) = 0$
Case 1B	$x = S(1-z), z > Z,$	$p = q = 0$	$X'(t) = [x/(x+y)]kwz,\ Z'(t) = -kwz$
Case 2	$x < S(1-z)$	$p = 0, q = w$	$X'(t) = kw(1-x),\ Z'(t) = -kwz$
Case 3	$x > S(1-z)$	$p = w, q = 0$	$X'(t) = -kwx,\ Z'(t) = -kwz$

triangle defined by

$$x + z \leqslant 1, \quad x \geqslant 0, \ \ z \geqslant 0,$$

where x is the mass of customers of firm 1, and z is the mass of noncustomers. Recall also that $y = 1 - x - z$ is the mass of customers of firm 2. The three cases are displayed in Table 3, along with the corresponding prices and the law of motion of the state of the system.

Each case is further subdivided into two subcases, labelled A and B, whose characterizations will be given below.

Case 1A: $x = S(1-z), z \leqslant Z$. Recall that S is the target market share of firm 1, and $1 - Z$ is the target (total) market penetration. In this case, $p = q = w$, so the state of the system does not change, i.e.,

$$X(t) = x, \ \ Z(t) = z, \quad \text{for all } t.$$

Hence

$$V(x,z) = \frac{wx}{r}. \tag{A.4.1}$$

Case 1B: $x = S(1-z), z > Z$. In this case, $p = q = 0$, so customers flow into the two firms in proportion to their current respective masses, until the mass of noncustomers falls to Z and Case 1A is reached. Let T be the first time t such that $Z(t) = Z$. Firm 1 earns no revenue until time T, at which time its value from that time on is given by Case 1A. Hence

$$V(x,z) = e^{-rT}\frac{wX(T)}{r}.$$

First note that $X(T) = S(1-Z)$. Second, by the law of motion,

$$Z(T) = -ze^{-bT}, \quad \text{where } b \equiv kw,$$

and hence

$$e^{-bT} = Z/z,$$

$$e^{-rT} = \left(\frac{Z}{z}\right)^a, \quad \text{where } a \equiv r/b = r/kw.$$

It follows that

$$V(x,z) = \left(\frac{w}{r}\right)\left(\frac{Z}{z}\right)^a S(1-Z). \tag{A.4.2}$$

Case 2: $x < S(1-z)$. In this case, $p = 0, q = w$. Let T be the first t such that $X(t) = S[1 - Z(t)]$. Firm 1 earns no revenue until time T. From the law of motion for this case,

$$X(T) = 1 - (1-x)e^{-bT},$$
$$Z(T) = ze^{-bT},$$

and hence, since $X(T) = S[1 - Z(t)]$,

$$1 - (1-x)e^{-bT} = S[1 - ze^{-bT}],$$
$$e^{-bT} = \frac{1-S}{1-x-Sz}.$$

From the above one obtains:

$$Z(T) = \frac{z(1-S)}{1-x-Sz},$$
$$X(T) = \frac{S(1-x-z)}{1-x-Sz},$$
$$e^{-rT} = \left(\frac{1-S}{1-x-Sz}\right)^{a}. \tag{A.4.3}$$

The value function is determined by

$$V(x,z) = e^{-rT} V[X(T), Z(T)]. \tag{A.4.4}$$

At time T the state of the system will be in either of the two cases, 1A or 1B, and this will determine the value on the right-hand side of the preceding equation. From (A.4.3) one has

$$Z(T) \leqslant Z \quad \text{if and only if} \quad \frac{z(1-S)}{1-x-sz} \leqslant Z. \tag{A.4.5}$$

Case 2A: $Z(T) \leqslant Z$. In this case, the state of the system at time T is in Case 1A. Hence, by (A.4.4), (A.4.3), and (A.4.1), the value function is given by

$$V(x,z) = \frac{wS(1-S)^a(1-x-z)}{r(1-x-Sz)^{a+1}}. \tag{A.4.6}$$

In the equilibrium analysis I shall need formulas for the two first-order partial derivatives of the value function:

$$V_1(x,z) = \frac{wS(1-S)^a[a(1-x)-(1+a-S)z]}{r(1-x-Sz)^{a+2}},$$
$$V_2(x,z) = \frac{wS(1-S)^a\{[S(a+1)-1](1-x)-aSz\}}{r(1-x-Sz)^{a+2}}. \tag{A.4.7}$$

Case 2B: $Z(T) > Z$. At time T the state of the system is in Case 1B. Let T' be the first time t such that $Z(t) = Z$. Firm 1 earns no revenue until time T'. Furthermore, in Cases 2B and 1B the law of motion of $Z(t)$ is the same. Hence the formula for the value function for Case 2B is the same as for Case 1B, namely

$$V(x,z) = \left(\frac{w}{r}\right)\left(\frac{Z}{z}\right)^a S(1-Z). \tag{A.4.8}$$

(This can also be verified using Eq. (A.4.4).) The first-order partial derivatives are

$$V_1(x,z) = 0,$$
$$V_2(x,z) = -\frac{awS(1-Z)Z^a}{rz^{a+1}}. \tag{A.4.9}$$

Case 3: $x > S(1-z)$. In this case, $p = w, q = 0$. Again, let T be the first t such that $X(t) = S[1-Z(t)]$. By the law of motion for this case, for $0 \leqslant t \leqslant T$,

$$X(t) = xe^{-bt},$$
$$Z(t) = ze^{-bt}. \tag{A.4.10}$$

Hence, since $X(T) = S[1-Z(T)]$,

$$xe^{-bT} = S(1-ze^{-bT}),$$
$$e^{-bT} = \frac{S}{x+Sz}. \tag{A.4.11}$$

At time t for $(0 \leqslant t \leqslant T)$ Firm 1 earns at the rate $wX(t)$. Hence the value function for this case is determined by

$$V(x,z) = \int_0^T e^{-rt} wX(t)\, dt + e^{-rT} V[X(T), Z(T)]. \tag{A.4.12}$$

From (A.4.10) to (A.4.11) we have

$$\begin{aligned}\int_0^T e^{-rt} wX(t)\, dt &= wx \int_0^T e^{-(r+b)t}\, dt \\ &= \frac{wx}{r+b}(1-e^{-(r+b)T}) \\ &= \frac{wx}{r+b}\left[1-\left(\frac{S}{x+Sz}\right)^{a+1}\right].\end{aligned} \tag{A.4.13}$$

One also has

$$e^{-rT} = \left(\frac{S}{x+Sz}\right)^a. \tag{A.4.14}$$

The value of $V[X(T), Z(T)]$, in the second term of (A.4.12), depends upon whether the state of the system at time T is in Case 1A or Case 1B, i.e., whether $Z(T) \leqslant Z$ or $> Z$. Note that

$$Z(T) \leqslant Z \;\Leftrightarrow\; \frac{Sz}{x+Sz} \leqslant Z \;\Leftrightarrow\; Sz \leqslant \frac{xZ}{1-Z}. \tag{A.4.15}$$

Case 3A: $Z(T) \leqslant Z$. In this case $V[X(T), Z(T)]$ is given by Case 1A, and hence by (A.4.10)–(A.4.11) and (A.4.1),

$$V(x,z) = \frac{wx}{r+b}\left[1 - \left(\frac{S}{x+Sz}\right)^{a+1}\right] + \frac{wx}{r}\left(\frac{S}{x+Sz}\right)^{a+1}$$
$$= \frac{wx}{r(1+a)}\left[a + \left(\frac{S}{x+Sz}\right)^{a+1}\right]. \tag{A.4.16}$$

One can verify that the first-order partial derivatives of the value function are:

$$V_1(x,z) = \frac{w}{r(1+a)}\left[a + \frac{S^{a+1}}{(x+Sz)^{a+2}}(Sz - ax)\right],$$
$$V_2(x,z) = -\frac{wx}{r}\left(\frac{S}{x+Sz}\right)^{a+2}. \tag{A.4.17}$$

Case 3B: $Z(T) > Z$. In this case $V[X(T), Z(T)]$ is given by Case 1A, and hence by (A.4.10)–(A.4.11) and (A.4.2),

$$V(x,z) = \frac{wx}{r+b}\left[1 - \left(\frac{S}{x+Sz}\right)^{a+1}\right] + \left(\frac{w}{r}\right)S(1-Z)\left(\frac{Z}{z}\right)^{a}. \tag{A.4.18}$$

For the equilibrium analysis we shall need only the partial derivative with respect to the first argument, x;

$$V_1(x,z) = \left(\frac{w}{r+b}\right)\left[1 - \frac{S^{a+1}}{(x+Sz)^{a+2}}(Sz - ax)\right]. \tag{A.4.19}$$

A.4. Theorem 2: Equilibrium analysis. Method of proof

I must show that each firm's strategy is a best response to the other firm's strategy. Since each firm's strategy is stationary, and the discount rate is constant, each firm faces a standard dynamic program in continuous time. I shall use the Bellman optimality condition to verify the optimality of firm 1's strategy (Section A.3). The optimality of firm 2's strategy will follow by symmetry.

At the risk of confusing the reader with an abuse of notation, for the purposes of this section let $X'(x,z;p,q)$ and $Z'(x,z;p,q)$ denote the time derivative of X and Z, respectively, when the state of the system is (x,z) and the firms' prices are p and q, respectively. Let ϕ and ψ be the price strategies of firms 1 and 2, respectively. Finally, let V be firm 1's value function implied by these strategies. The Bellmanian functional is

$$B_V(p;x,z) \equiv px - rV(x,z) + X'[x,z;p,\psi(x,z)]V_1(x,z)$$
$$+ Z'[x,z;p,\psi(x,z)]V_2(x,z) \tag{A.4.20}$$

(provided the indicated derivatives exist). Note that the Bellmanian functional depends on the value function, V, which in turn depends on firm 1's price strategy, ϕ. Recall that (cf. (A.2.14))

$$B_V[\phi(x,z);x,z]=0, \tag{A.4.21}$$

provided that V is C^1. In what follows, I shall suppress the subscript, V, on the symbol for the Bellmanian Functional, and denote it simply by B.

Recall that the following two conditions together are sufficient for ϕ to be an optimal response to ψ: for all relevant values of (x,z,p),

$$V \text{ is } C^1, \tag{A.4.22}$$

$$B[p;x,z]\leqslant 0 \quad \text{for all } p \text{ in } (0,w), \tag{A.4.23}$$

cf. the Bellman optimality conditions, (A.2.15)–(A.2.16). In other words, for each state of the system (x,z), the Bellmanian Functional (or *Bellmanian*, for short) is maximized at $p=\phi(x,z)$.

I have already shown that the value function V is C^1. The remainder of this section is devoted to showing that () is satisfied. In fact, it will be sufficient to show that it is satisfied in Cases 2 and 3 above. This follows from the observation that in Case 1 any departure of firm 1's price from that prescribed by the strategy ϕ instantaneously moves the state of the system into Case 2 or Case 3.

Case 2: $x<S(1-z)$. Recall that in this case,

$$q=\psi(x,z)=w,$$

$$X'[x,z;p,\psi(x,z)]=k(w-p)(1-x),$$

$$Z'[x,z;p,\psi(x,z)]=-k(w-p)z.$$

Hence the Bellmanian takes the form

$$B=px-rV(x,z)+k(w-p)(1-x)V_1(x,z)-k(w-p)zV_2(x,z),$$

and

$$\frac{\partial B}{\partial p}=x-k(1-x)V_1+kzV_2.$$

Hence, to show that $\phi(x,z)=0$ is optimal it is sufficient to show that

$$x-k(1-x)V_1+kzV_2\leqslant 0$$

or

$$k(1-x)V_1-kzV_2\geqslant x. \tag{A.4.24}$$

Case 2A: A straightforward but tedious computation shows that

$$k(1-x)V_1(x,z)-kzV_2(x,z)=\frac{S(1-S)^a(1-x-z)}{(1-x-Sz)^{a+1}}.$$

Hence (2.5) is equivalent to

$$\frac{S(1-S)^a(1-x-z)}{(1-x-Sz)^{a+1}} \geqslant x. \tag{A.4.25}$$

Note that

$$x < S(1-z) \Rightarrow 1 - x - SZ > 1 - S > 0, \tag{A.4.26}$$

so that (1) is equivalent to

$$A \equiv S(1-S)^a(1-x-z) - x(1-x-Sz)^{a+1} \geqslant 0. \tag{A.4.27}$$

The following lemma is stated without proof.

Lemma A.1. *For* $0 \leqslant u \leqslant 1, 0 \leqslant a \leqslant 1$, *define* $f(u) \equiv u(1-u)^a$; *then* $f(u)$ *is increasing for* $u < s$, *and decreasing thereafter*, *where*

$$s \equiv \frac{1}{a+1}. \tag{A.4.28}$$

In particular,

$$\max_u f(u) = f(s) = \frac{a^a}{(a+1)^{a+1}}. \tag{A.4.29}$$

Fix $x < S$, and note that

$$0 < x < S(1-z) \;\Leftrightarrow\; 0 < z < 1 - \frac{x}{S}.$$

Consider A as a function of z. First observe that

$$A'(z) = -S(1-S)^a + (a+1)Sx(1-x-Sz)^a,$$

which is decreasing in z. Hence, for fixed x, *A is concave in z.*

Second, observe that

$$\begin{aligned} A(0) = A &\equiv S(1-S)^a(1-x) - x(1-x)^{a+1} \\ &= (1-x)[S(1-S)^a - x(1-x)^a]. \end{aligned}$$

Recall that $S \leqslant s$. Since $x < S$, the Lemma implies that

$$A(0) > 0.$$

Third,

$$\begin{aligned} &z = 1 - \frac{x}{S} \Rightarrow 1 - x - z = \frac{(1-S)x}{S} \text{ and } 1 - x - Sz = 1 - S \\ &\Rightarrow A\left(1 - \frac{x}{S}\right) = 0. \end{aligned}$$

Together with the concavity of A, the last two observations imply that $A > 0$ for $0 < z < 1 - (x/S)$. Hence $\phi(x, z) = 0$ is optimal in Case 2A.

Case 2B: In this case, by (A.4.9), $V_1 = 0$ and

$$k(1-z)V_1 - kzV_2 = kz\left[\frac{awS(1-Z)Z^a}{rz^{a+1}}\right] = S\left(\frac{Z}{z}\right)^a.$$

By (A.4.24), we want to show that

$$x < S(1-z),$$
$$\frac{z(1-S)}{1-x-Sz} > Z,$$

implies that

$$S\left(\frac{Z}{z}\right)^a \geqslant x. \quad \text{(A.4.30)}$$

For this it is necessary and sufficient to show that (A.4.30) holds if

$$x = S(1-Z),$$
$$z > Z. \quad \text{(A.4.31)}$$

In other words, we want to show that (A.4.31) implies that

$$S\left(\frac{Z}{z}\right)^a \geqslant S(1-z) \quad \text{(A.4.32)}$$

or

$$(1-z)z^a \leqslant Z^a.$$

By the Lemma, taking $u = 1 - z$,

$$(1-z)z^a \leqslant \frac{a^a}{(a+1)^{a+1}},$$

and the maximum of the left-hand-side is attained when

$$z = \frac{a}{a+1} = 1 - s.$$

Hence,

$$\left(\frac{1}{a+1}\right)^{1/a}(1-s) \leqslant Z \leqslant 1 - s \quad \text{(A.4.33)}$$

implies that (A.4.32) is satisfied for all $z > Z$, which completes the proof for Case 2B.

Define

$$g(a) \equiv \left(\frac{1}{a+1}\right)^{1/a}, \quad \text{(A.4.34)}$$

and note that, for $0 \leqslant a \leqslant 1$,

$$\min_a g(a) = g(0) = 1/e \approx .368,$$
$$\max_a g(a) = g(1) = 1/2. \tag{A.4.35}$$

Case 3: $x > S(1-z)$. Recall that in this case,

$$q = \psi(x,z) = 0,$$
$$X'[x,z;p,\psi(x,z)] = -kpx,$$
$$Z'[x,z;p,\psi(x,z)] = -kwz.$$

Hence the Bellmanian takes the form

$$B = px - rV - kpxV_1 - kwzV_2$$
$$px(1-kV_1) - rV - kwzV_2,$$

$$\frac{\partial B}{\partial p} = x(1-kV_1).$$

Therefore, to show that $p = w$ is optimal, it is sufficient to show that

$$1 - kV_1 \geqslant 0$$

or

$$kV_1 \leqslant 1. \tag{A.4.36}$$

Case 3A: From (A.4.17),

$$kV_1(x,z) = \frac{kw}{r(1+a)}\left[a + \frac{S^{a+1}}{(x+Sz)^{a+2}}(Sz-ax)\right]. \tag{A.4.37}$$

By (A.4.15), in Case 3A

$$Sz \leqslant \frac{xZ}{1-Z}.$$

Recall that $Z \leqslant 1 - s = a/(1+a)$; hence $Z/(1-Z) \leqslant a$, and so

$$Sz - ax \leqslant \frac{xZ}{1-Z} - ax \leqslant ax - ax = 0.$$

Hence, from (A.4.37)

$$kV_1 = \frac{kwa}{r(1+a)} = \frac{1}{1+a} < 1. \tag{A.4.38}$$

Note that the inequality is strict.

Case 3B: From (A.4.19),

$$kV_1(x,z) = \left(\frac{kw}{r+b}\right)\left[1 - \frac{S^{a+1}}{(x+Sz)^{a+2}}(Sz-ax)\right]$$
$$= \left(\frac{1}{a+1}\right)\left[1 - \frac{S^{a+1}}{(x+Sz)^{a+2}}(Sz-ax)\right]. \quad \text{(A.4.39)}$$

By (A.4.15), in Case 3B

$$Sz \geqslant \frac{xZ}{1-Z},$$

$$Sz - ax \geqslant x\left(\frac{Z}{1-Z} - a\right).$$

Define

$$\zeta \equiv 1 - s = \frac{a}{a+1}. \quad \text{(A.4.40)}$$

Observe that

$$Z = \zeta \Rightarrow Sz - ax \geqslant 0 \Rightarrow kV_1(x,z) \leqslant \frac{1}{a+1} < 1. \quad \text{(A.4.41)}$$

Since the inequality is strict, there exists $\zeta'$with

$$0 < \zeta' < \zeta \quad \text{(A.4.42)}$$

such that

$$\zeta' \leqslant Z \leqslant \zeta \Rightarrow kV_1 \leqslant 1. \quad \text{(A.4.43)}$$

In the light of (A.4.33) in Case 2B, choose ζ' to satisfy

$$\left(\frac{1}{a+1}\right)^{1/a} < \zeta' < \zeta. \quad \text{(A.4.44)}$$

This completes Case 3B, and hence the proof that ϕ is an optimal response to ψ. Hence (ϕ, ψ) is an equilibrium.

The case of $m > w$: If $m > w$, one encounters a problem in Case 2 similar to that in the case of monopoly, when firm 1's market penetration, x, is sufficiently small. However, as in the monopoly case, for m sufficiently close to w the equilibrium analysis goes through. Roughly speaking, when $m = w$, the required inequalities are actually strict, so that (by the continuity of the relevant functions), these inequalities remain valid for m sufficiently close to w. I omit the details.

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