Opportunistic Replacement of a Single Part in the Presence of Several Monitored Parts

Roy Radner; Dale W. Jorgenson

Management Science, Vol. 10, No. 1. (Oct., 1963), pp. 70-84.

MANAGEMENT SCIENCE
Vol. 10, No. 1, October, 1963
Printed in U.S.A.

OPPORTUNISTIC REPLACEMENT OF A SINGLE PART IN THE PRESENCE OF SEVERAL MONITORED PARTS*†

ROY RADNER[1] AND DALE W. JORGENSON[2]

In a system with several stochastically failing parts and economies of scale in their maintenance, it may be advantageous to follow an "opportunistic" policy for maintenance. In opportunistic policies the action to be taken on a given part at a given time depends on the state of the other parts of the system.

For the model considered in this paper, it is assumed that all parts of the system but one are inspected continuously (monitored) and that the remaining part cannot be inspected except when it is replaced. If the monitored parts have exponential distributions of time to failure, the optimal replacement policy for the remaining part has the following form:

Let the non-monitored part be labeled 0 and let there be M monitored parts, labeled $1, \cdots, M$; then there are $M + 1$ numbers $n_1, \cdots, n_M, N$, with $0 \leqq n_i \leqq N \leqq \infty$ such that:

(a) if Part i fails at a time when the age of Part 0 is between 0 and n_i, replace Part i alone $(i = 1, \cdots, M)$;

(b) if Part i fails at a time when the age of Part 0 is between n_i and N, then replace Parts 0 and i together;

(c) if Part 0 reaches age N at a time when all monitored parts are good, replace Part 0 alone.

It is demonstrated that the parameter N is finite provided that the reliability of the non-monitored part approaches zero as the age of this part goes to infinity.

These results are proved for several alternative criteria for evaluating replacement policies. Computation of the parameters of the optimal policy is also discussed. Applications of this policy are discussed in [2].

1. Introduction

This paper is one of a series of RAND studies dealing with optimal maintenance policies for manned aircraft and ballistic missile systems and is a companion piece to the following paper, "Operating Characteristics of Opportunistic Replacement and Inspection Policies," by J. J. McCall [5]. Other studies in this series are listed in the references [2 and 4].

The significance of this paper is that it presents a rigorous mathematical proof of the optimality of a class of policies for replacement of a single randomly failing part in the presence of any number of monitored parts. The previously cited studies used these policies in a series of applications, but they did not include the necessary proof of optimality.

* Received November 1962.

† This research was sponsored by the United States Air Force under Project RAND—Contract No. AF 49(638)-700—monitored by the Directorate of Development Planning, Deputy Chief of Staff, Research and Technology, Hq, USAF. Views or conclusions contained in this paper should not be interpreted as representing the official opinion or policy of the United States Air Force.

[1] University of California, Berkeley.

[2] University of Chicago and University of California, Berkeley.

In a system with several stochastically failing parts and economies of scale in their maintenance, it may be advantageous to follow an "opportunistic" maintenance policy, in which the maintenance action to be taken on a given part at a given time depends on the state of the rest of the system.[3] In particular, the following problem has been solved [1]:

Suppose a system has two parts, labeled 0 and 1, both of which fail stochastically and independently according to known probability distributions. Part 0 can be replaced at any time, but cannot be inspected, except possibly at the time of replacement. Part 1 is monitored—continuously inspected—and can also be replaced at any time. Replacement of Part 0 takes K_0 units of time and replacement of Part 1 takes K_1 units of time, but if the two parts are replaced together, the time taken is $K_{01} \leqq K_0 + K_1$. (If $K_{01} < K_0 + K_1$, one says that there are economies of scale in replacement.) The system is in a good state if and only if both parts are good. The criterion by which a given maintenance policy is to be judged is assumed to be the resulting expected total discounted time in which the system is in a good state (or, as a limiting case, the long-run average proportion of the time the system is good). The problem is to determine a replacement policy that is optimal with respect to this criterion.

It has been shown [1] that a solution to this problem is provided by the so-called (n, N) type of policy, provided that the monitored part has an exponential distribution of time to failure. There are two numbers n and N, with $0 \leqq n \leqq N$, such that:

(a) if Part 1 fails at a time when the age of Part 0 is between 0 and n, then replace Part 1 alone;

(b) if Part 1 fails at a time when the age of Part 0 is between n and N, then replace Parts 0 and 1 together;

(c) if Part 0 reaches age N at a time when Part 1 is good, then replace Part 0 alone.

The purpose of this paper is to generalize this result in two directions: (1) to consider a system with any number of monitored parts (but still only one non-monitored part); (2) to consider more general criteria that take account of the costs of maintenance actions as well as the time the system spends in a good state. It will be shown that a solution to the more general problem is a policy of what might be called the (n_i, N) type, which can be described as follows: As before, let the non-monitored part be labeled 0, and let there be M monitored parts, labeled $1, \cdots, M$; then there are $M + 1$ numbers $n_1, \cdots, n_M, N$, with $0 \leqq n_i \leqq N$, such that:

(a′) if Part i fails at a time when the age of Part 0 is between 0 and n_i, then replace Part i alone $(i = 1, \cdots, M)$;

(b′) if Part i fails at a time when the age of Part 0 is between n_i and N, then replace Parts 0 and i together;

(c′) if Part 0 reaches age N at a time when all the monitored parts are good, then replace Part 0 alone.

[3] For a summary of the literature, see [2].

This result is proved under the assumption that all the monitored parts have exponential distributions of time to failure.

The criterion for judging a given maintenance policy is assumed here to be one of the following three:

(1) expected total discounted value of good time minus costs;
(2) expected total discounted time in which the system is in a good state;
(3) the ratio of expected total discounted good time to expected total discounted cost.

For each of these three criteria, there is a corresponding limiting case in which expected total discounted sums are replaced by long-run averages per unit time.

Actually, criterion (2) is a special case of (1), and criterion (3) can be transformed into (1). Section 4 discusses the relations among the three criteria; it is shown that it is sufficient to prove the optimality of the (n_i, N)-type policy under criterion (1).

In Sec. 2 the problem and the assumptions are formulated precisely, and Sec. 3 gives a proof of the optimality of the (n_i, N)-type policy. Section 4 discusses alternative criteria for choosing maintenance policies. Section 5 discusses the computation of optimal policies.

2. Problem Formulation

2.1. *Description of the System*

The system to be considered has $(M + 1)$ parts, numbered $i = 0, \cdots, M$. Part 0 can be replaced[4] at any time, but cannot be inspected except possibly in connection with the replacement action. Until it is replaced its state is uncertain; $R(x)$ denotes its reliability at age x, i.e., $R(x)$ is the probability that it has not failed by age x. It is assumed that:

$$R(0) = 1,$$

$$R(x) \text{ is strictly decreasing as } x \to \infty, \tag{1}$$

$$R \text{ is differentiable.}$$

No further assumptions about R are made here.

Parts $1, \cdots, M$ are each continuously monitored, so that their failures are discovered as soon as they occur. It is assumed that the time-to-failure distribution of Part $i \geqq 1$ is of the negative exponential type, with mean $1/\lambda_i$. Parts $0, \cdots, M$ are assumed to fail independently. The system is assumed to be good if and only if every part is good.

The time to replace Part i alone is K_i, and the cost of replacing Part i alone is C_i $(i = 0, \cdots, M)$. The time and cost of replacing Parts 0 and i together are K_{0i} and C_{0i}, respectively $(i = 1, \cdots, M)$. The (continuous) discount rate is

[4] The term "replacement" will be used here to denote any maintenance action that restores the part to a "new" condition.

denoted by $\alpha > 0$. It is assumed that

$$
\begin{aligned}
&0 \leqq K_0, \qquad K_i \leqq K_{0i} \leqq K_0 + K_i, \\
&0 \leqq C_0, \qquad C_i \leqq C_{0i} \leqq C_i + e^{-\alpha K_i} C_0, \qquad i = 1, \cdots, M.
\end{aligned} \tag{2}
$$

The last inequality on each line of (2) expresses the presence of possible economies of scale in replacement times and costs.

The units in which value and time are measured are chosen so that one unit of time during which the system is in a good state ("good time") is worth one unit of value. The objective is to choose a maintenance policy that maximizes the expected total discounted net value (good time minus cost).[5]

At any particular time let x denote the age of Part 0, and let y denote the state of the set of monitored parts. Since in the optimal policy a monitored part will be replaced as soon as it fails, no more than one monitored part will be failed at any one time. Thus one can have:

$$
y = \begin{cases} i \text{ denotes Part } i \text{ is failed, } i = 1, \cdots, M; \\ 0 \text{ denotes all monitored parts are good.} \end{cases}
$$

It is assumed that no part ages while replacement activity is going on. (Even where this assumption is not strictly justified, it will represent a good approximation if, under the optimal policy, the system spends a sufficiently small fraction of the time in replacement activity.)

2.2. *Alternative Actions and Their Consequences*

At any instant of time at which a replacement action is not in progress the following alternative actions are to be considered:

- (i) Replace Part i alone $(i = 0, \cdots, M)$.
- $(0, i)$ Replace Parts 0 and i together $(i = 1, \cdots, M)$.
- W_Δ ("waiting action"). Do nothing for an interval Δ, and then take action (0), unless some monitored part fails in the interval, in which case take a best one of the actions (i), $(0, i)$, $i = 1, \cdots, M$.

There is no need to consider combined maintenance actions other than those described above. Not more than one monitored part can fail at once (with probability one), and since parts with negative exponential time-to-failure distributions do not "wear out," there is no advantage to be gained by replacing them prior to actual failure.

The consequences of the several actions can be described as follows. Let $g(x, y)$ denote the maximum possible expected total discounted net value, given that one starts with the state of information (x, y), i.e., given that under an optimal policy, the consequences of the several replacement actions can be denoted by

$$
g_0 = -C_0 + e^{-\alpha K_0} g(0, 0), \tag{3}
$$

[5] As will be shown in Sec. 4, the optimality of the (n_i, N)-type policy is valid for all of the criteria described in Sec. 1. This particular criterion has been chosen because of its convenience in the dynamic programming technique.

$$g_i(x) = -C_i + e^{-\alpha K_i} g(x, 0), \tag{4}$$

$$g_{0i} = -C_{0i} + e^{-\alpha K_{0i}} g(0, 0). \tag{5}$$

Assume that g_i and g_{0i} are >0. The consequence of action W_Δ can be described by

$$\begin{aligned} g_w(x \mid \Delta) = \int_0^\Delta [\bar{R}(x, u) + e^{-\alpha u} f(x + u)]\lambda e^{-\lambda u}\, du \\ + [\bar{R}(x, \Delta) + e^{-\alpha \Delta} g_0] e^{-\lambda\Delta}, \end{aligned} \tag{6}$$

where

$$R(x, u) = \int_0^u R(x + t)e^{-\alpha t}\, dt, \tag{7}$$

$$\lambda = \textstyle\sum_{i=1}^M \lambda_i, \tag{8}$$

$$f(x) = \textstyle\sum_{i=1}^M (\lambda_i/\lambda) g(x, i). \tag{9}$$

Expressions (6)–(9) can be understood with the aid of the following interpretations. Let U denote the random variable that is equal either to the time to the first failure of some monitored part or to Δ, whichever is the smaller (starting from a state in which all monitored parts are good); then U has a negative exponential density $\lambda e^{-\lambda u}$ for $0 \leqq u < \Delta$, and a probability mass $e^{-\lambda\Delta}$ at the value Δ. Further, given that some one monitored part has just failed, the probability that it is Part i is (λ_i/λ). The term $\bar{R}(x, u)$ represents the accumulated discounted reliability of Part 0 (and therefore of the system) from x to $(x + u)$. Thus, given that $U = u < \Delta$, the maximum possible expected value is

$$\bar{R}(x, u) + e^{-\alpha u} f(x + u), \tag{10}$$

whereas given that $U = \Delta$, the maximum expected value is

$$R(x, \Delta) + e^{-\alpha\Delta} g_0 . \tag{11}$$

Averaging (10) and (11) with due regard for the probability distribution of U then yields (6). Note that under an optimal policy one would not take action W_Δ (with $\Delta > 0$) unless $y = 0$.

2.3. *The Functional Equation for $g(x, y)$*

Applying the usual technique of dynamic programming one can characterize $g(x, y)$ by the conditions:

$$g(x, 0) = \max\,[g_0 , \sup_\Delta g_w(x \mid \Delta)]. \tag{12}$$

$$g(x, i) = \max\,[g_i(x), g_{0i}], \qquad i = 1, \cdots, M. \tag{13}$$

2.4. *Description of an Optimal Policy*

The "(n_i , N)-Policy"

There are numbers $n_1 , \cdots , n_M , N$, with $0 \leqq n_i \leqq N$, such that

(i) if $y = 0$, then action $W_{(N-x)}$ is best for $0 \leqq x < N$, and action (0) is best for $x \geqq N$;

(ii) if $y = i \geqq 1$, then action (i) is best for $0 \geqq x < n_i$, and action $(0, i)$ is best for $x \geqq n_i$.

3. Proof of Optimality of the (n_i, N)-Policy

We first prove that $g(x, y)$ is non-increasing in x for each y. This is plausible, since the reliability function for Part 0 is non-increasing. Let the initial instant of time be $t = 0$, and let $x(t)$ and $y(t)$ be the age of Part 0 and the state of the monitored parts, respectively, at any time $t \geqq 0$.

Given that an optimal policy is used, let T be the first time $t' \geqq 0$ such that replacement of Part 0 begins at time t'; and let $S(y) = 1$ if all of the monitored parts are good $(y = 0)$, and 0 otherwise $(y > 0)$. Between $t = 0$ and $t = T$ the expected accumulated good time for the system is

$$E \int_0^T e^{-\alpha t} R[x(t)] S[y(t)]\, dt. \tag{14}$$

The accumulated discounted costs from 0 to T are all due to maintenance actions on monitored parts; denote this by C_T. The time used up for the maintenance begun at $t = T$—call it K—will depend upon the state of the monitored parts at $t = T$, and so will the cost—call it C. After the maintenance action is completed, and one continues to use the optimal policy, one can look forward to an expected total discounted net value of $g[0, y(T + K)]$. In sum, if $x(0) = x^*$, then

$$\begin{aligned} g[x^*, y(0)] = E\biggl\{\int_0^T & e^{-\alpha t} R[x(t)] S[y(t)]\, dt - C_T \\ & - e^{-\alpha T} C + e^{-\alpha(T+K)} g[0, y(T + K)]\biggr\}, \end{aligned} \tag{15}$$

where it is understood that the expectation in (15) is conditional on $y(0)$, and $x^*(t)$ is the particular evolution of $x(t)$ starting from $x(0) = x^*$.

Suppose now that the initial age $x(0)$ were less than x^*, say $x^{**} = x^* - h$. Denote by $\phi(x, y)$ the optimal replacement policy, and denote by ϕ^* the policy defined by

$$\phi^*(x, y) = \phi(x + h, y). \tag{16}$$

If the initial age were x^{**} instead of x^*, but the policy ϕ^* were used instead of ϕ up to the first replacement of Part 0, then the timing of replacement actions would be unchanged. In particular T, C_T, C, and $g[0, y(T + K)]$ would be the same. However, the expected accumulated discounted good time between 0 and T would be

$$E \int_0^T e^{-\alpha t} R[x^{**}(t) - h] S[y(t)]\, dt \tag{17}$$

Since R is non-increasing, and since the evolution of $y(t)$ is unchanged, expression (17) is not less than expression (14). Hence, if $g^*[x^{**}, y(0)]$ denotes the expected discounted net value obtained by using the policy ϕ^* from 0 to T, and the policy ϕ thereafter, starting with $x(0) = x^{**}$, and given $y(0)$ it follows that

$$g^*[x^{**}, y(0)] \geqq g[x^*, y(0)]. \tag{18}$$

But $g^*[x^{**}, y(0)]$ is no larger than the maximum that can be achieved starting from x^{**} and $y(0)$, hence

$$g[x^{**}, y(0)] \geqq g[x^*, y(0)] \quad \text{for} \quad x^{**} \leqq x^*, \tag{19}$$

which is the desired result.

We now turn to the characterization of an optimal policy in terms of the numbers n_i and N. Consider any state $y = i \geqq 1$. It follows from the functional equation (13) that action $(0, i)$ is as good as or better than action (i) for all x such that

$$g(x, i) \geqq g_{0i}\,; \tag{20}$$

otherwise action (i) is better than action $(0, i)$. But $g(x, i)$ is non-increasing in x, and g_{0i} is a constant, so that (20) is satisfied either for no x, or for all $x \geqq$ some finite number, say n_i. Hence, part (ii) of the description of the (n_i, N)-policy in Sec. 2.4.

Similarly,

$$g(x, 0) \begin{Bmatrix} > \\ \leqq \end{Bmatrix} g_0 \quad \text{for} \quad \begin{cases} 0 \leqq x < N \\ x \geqq N \end{cases}, \tag{21}$$

for some number N with $0 \leqq N \leqq \infty$. Hence, for $x < N$, action W_Δ is best with $\Delta = \Delta(x)$, say. Indeed, for $x < N$ one must have $\Delta(x) > 0$, since from (6)

$$g_W(x \mid 0) = g_0\,. \tag{22}$$

We would like to show that if $x \leqq N$, then

$$\Delta(x) = N - x$$

(including possibly the case $N = \infty$, with suitable interpretation). By the so-called "optimality principle" of dynamic programming (see Bellman[6]), if $g(0, 0) = g_W(0 \mid \Delta_0)$, with $\Delta_0 > 0$, then for any x such that $0 \leqq x \leqq \Delta_0$,

$$g(x, 0) = g_W(x \mid \Delta_0 - x). \tag{23}$$

Hence, by (6), as $x \to \Delta_0$ from below, $g(x, 0)$ decreases to g_0. Hence $\Delta_0 \geqq N$. On the other hand, suppose $\Delta_0 > N$, and let x' be such that $N < x' < \Delta_0$. It would follow again from the optimality principle that $\Delta(x') = \Delta_0 - x' > 0$, and hence that $g(x', 0) > g_0$, but this would contradict the defining relation (21) for N. Hence part (i) of the (n_i, N)-policy defined in Sec. 2.4. We have shown incidentally that $g(N, 0) = g_0$. (The preceding argument must be suitably interpreted in the case $N = \infty$.)

[6] P. 83 of [3].

It remains to show that $n_i \leqq N$ for $i \geqq 1$, i.e., that if $g(x, 0) = g_0$, then $g(x, i) = g_i$, for $i \geqq 1$. Now if $g(x, 0) = g_0$, then, using (2),

$$\begin{aligned} g_i(x) &= -C_i + e^{-\alpha K_i} g_0 \\ &= -C_i + e^{-\alpha K_i}[-C_0 + e^{-\alpha K_0} g(0, 0)] \\ &= -[C_i + e^{-\alpha K_{0i}} C_0] + e^{-\alpha(K_0 + K_i)} g(0, 0) \\ &\leqq -C_{0i} + e^{-\alpha K_{0i}} g(0, 0) = g_{0i}. \end{aligned}$$

3.1. *Some Further Properties of the Function* $g(x, y)$

We first show that $g(x, 0)$ has a continuous derivative at all x. It suffices to show this for $0 \leqq x \leqq N$. For such values of x it follows from the optimality principle that

$$g(0, 0) = \int_0^x [\bar{R}(0, u) + e^{-\alpha u} f(u)] \lambda e^{-\lambda u} \, du + [\bar{R}(0, x) + e^{-\alpha x} g(x, 0)] e^{-\lambda x}. \tag{24}$$

Solving for $g(x, 0)$, this desired result is obvious, and the derivative is given by

$$g'(x, 0) = \alpha g(x, 0) + \lambda[g(x, 0) - f(x)] - R(x). \tag{25}$$

This differential equation may be useful in solving for the function $g(x, 0)$ in any particular case.

Second, for $i \geqq 1$, the derivative of $g(x, i)$ exists except possibly at $x = n_i$, where in any case the one-sided derivatives exist. This is clear from (4) and the functional equation (13).

Third, if $R(x)$ is itself differentiable, then the second derivative of $g(x, 0)$ exists, except possibly at $x = n_1, \cdots, n_M$.

Fourth, if $R(x)$ approaches zero as x gets large (the typical case) then N is finite. This can be seen as follows. Given x, let T denote the interval of time until the next replacement of Part 0, using an optimal policy; T is of course a random variable. Since the expected accumulated discounted net value during the interval cannot exceed the accumulated discounted reliability of Part 0 during the interval, one has

$$g(x, 0) \leqq E\left[\int_0^T R(x + t) e^{-\alpha t} \, dt + e^{-\alpha T} g_0\right]. \tag{26}$$

Since R is decreasing,

$$\int_0^T R(x + t) e^{-\alpha t} \, dt \leqq R(x) \int_0^T e^{-\alpha t} \, dt = \frac{R(x)}{\alpha} (1 - e^{-\alpha T}),$$

so that, from (14),

$$g(x, 0) \leqq (1 - E e^{-\alpha T}) R(x)/\alpha + (E e^{-\alpha T}) g_0. \tag{27}$$

But $R(x)$ is decreasing to 0, so that for sufficiently large x the right side of (15) does not exceed g_0. Hence, by the functional equation (12) $g(x, 0) = g_0$ for sufficiently large x.

3.2. *Extreme Cases*

Finally, we inquire into when we can expect $n_i = 0$ or $n_i = N$. If, for some $i \geqq 1$, $C_{0i} = C_i$ and $K_{0i} = K_i$ ("perfect economies of scale"), then one sees from (4) and (5) that for all $x \geqq 0$,

$$\begin{aligned} g_i(x) &= -C_{0i} + e^{-\alpha K_{0i}} g(x, 0) \\ &\leqq -C_{0i} + e^{-\alpha K_{0i}} g(0, 0) - g_{0i}, \end{aligned}$$

so that action $(0, i)$ is as good as or better than action (i) for all x. In other words, $n_i = 0$.

On the other hand, if for some $i \geqq 1$

$$C_{0i} = C_i + e^{-\alpha K_i} C_0, \qquad K_{0i} = K_0 + K_i$$

(no economies of scale), then for $0 \leqq x < N$,

$$\begin{aligned} g_i(x) &= -C_i + e^{-\alpha K_i} g(x, 0) \\ &> -C_i + e^{-\alpha K_i} g_0 \\ &= -[C_i + e^{-\alpha K_i} C_0] + e^{-\alpha(K_i + K_0)} g(0, 0) \\ &= -C_{0i} + e^{-\alpha K_{0i}} g(0, 0) \\ &= g_{0i}, \end{aligned}$$

so that $n_i = N$.

4. Alternative Criteria for Judging Maintenance Policies

In this Section we give precise formulations of the three criteria mentioned in Sec. 1, and discuss the relations among them.

For a given maintenance policy p, let $S(t, p)$ denote the state of the system at time t, when policy p is used, i.e.,

$$S(t, p) = \begin{cases} 1 & \text{if the system is good at time } t, \\ 0 & \text{if the system is failed at time } t; \end{cases} \tag{28}$$

$S(t, p)$ is of course a stochastic process. The expected total discounted good time is defined by

$$S_\alpha(p) = E \int_0^\infty e^{-\alpha t} S(t, p)\, dt, \tag{29}$$

where $\alpha > 0$.

On the cost side, let $C^+(t, p)$ denote the *accumulated* acquisition and maintenance cost up to and including time t. As a function of t, $C^+(t, p)$ is nondecreasing, and may be discontinuous; it is taken to be continuous from the right. $C^+(0, p)$ may be interpreted as the initial acquisition cost. The expected total discounted cost is defined by

$$C_\alpha(p) = E\int_0^\infty e^{-\alpha t}\, dC^+(t, p). \tag{30}$$

As α approaches 0, $S_\alpha(p)$ and $C_\alpha(p)$ are typically unbounded. However, one has the following relationship between expected total discounted values and long-run average values per unit time:

$$\lim_{\alpha\to 0} \alpha S_\alpha(p) = \bar{S}(p) \equiv \lim_{T\to\infty} E\left[\frac{1}{T}\int_0^T S(t, p)\, dt\right], \tag{31}$$

$$\lim_{\alpha\to 0} \alpha C_\alpha(p) = \bar{C}(p) \equiv \lim_{T\to\infty} E[(1/T)C^+(t, p)], \tag{32}$$

provided the indicated limits exist.

Criterion 1. Expected total discounted net value, where good time is valued at V dollars (say) per unit time. In terms of the present notation, the criterion is

$$W_\alpha(p) = VS_\alpha(p) - C_\alpha(p). \tag{33}$$

As a limiting case one has the expected long-run average net value per unit time,

$$\bar{W}(p) = V\bar{S}(p) - \bar{C}(p). \tag{34}$$

Criterion 2. Expected total discounted good time. This is clearly a special case of Criterion 1, with $V = 1$ and all costs equal to zero. As a limiting case one has the expected long-run proportion of good time, $\bar{S}(p)$.

Criterion 3. Ratio of expected total discounted good time to expected total discounted cost. In the present notation this is $S_\alpha(p)/C_\alpha(p)$. As a limiting case one has $\bar{S}(p)/\bar{C}(p)$.

For an appropriate choice of the constant V, choosing a policy that is optimal with respect to Criterion 3 is equivalent to choosing a policy that is optimal with respect to Criterion 1, provided there is some solution to the problem under Criterion 3 for which $S_\alpha(p)$ and $C_\alpha(p)$ are positive. Indeed, the appropriate choice of V is the minimum value of $C_\alpha(p)/S_\alpha(p)$.

More precisely, for fixed α, let P be the set of pairs (s, c) such that for some policy p,

$$s = S_\alpha(p),$$
$$c = C_\alpha(p).$$

Suppose that the pair $(\hat{s}, \hat{c})$ is in P, with $\hat{s}, \hat{c} > 0$. Define

$$V = \hat{c}/\hat{s};$$

then

$$1/V = \hat{s}/\hat{c}, \qquad V\hat{s} - \hat{c} = 0.$$

Furthermore, for any positive s and c,

$$s/c \leqq 1/V \quad \text{if and only if} \quad Vs - c \leqq 0.$$

Hence $(\hat{s}, \hat{c})$ maximizes (s/c) on P if and only if $(\hat{s}, \hat{c})$ maximizes $(Vs - c)$ on P.

It might be objected that in order to find the appropriate value of V that transforms a problem of type 3 into a problem of type 1 one must know the value of the maximum in the problem of type 3. However, in this paper the interest has been only in showing that an (n_i, N) type of policy is optimal for problem 3, and for this it is sufficient to show that such a transformation is possible.

Indeed the transformation can be made the basis of an iterative method for solving a problem of type 3, provided the set P is sufficiently regular, but this topic will not be explored here.

5. Computation

Given the structure of the optimal policy, the parameter values of this policy (n_i, N) can be computed directly. The criterion used is to maximize the ratio of expected total discounted good time to expected total discounted cost, in the limiting case that the discount rate α goes to zero. Since the renewal property characterizes the decision process, it suffices to maximize this ratio over a single cycle of operations—from the end of one replacement action to the end of the next. The criterion function is then:

$$G^+ = T/L^+, \tag{35}$$

where T is the expected reliability or good time accumulated over one cycle of operations, and L^+ is the "imputed" length of the cycle, including downtime for repair evaluated at the system's time rate of amortization and repair costs.

We introduce the following notation: K_i^+ and K_{0i}^+ are the (imputed) replacement times for Part i alone ($i = 1 \cdots M$) and for Parts i and 0 together; K_0^+ is the replacement time for Part 0 alone. Then:

$$K_i^+ = K_i + C_i/A,$$

and

$$K_{0i}^+ = K_{0i} + C_{0i}/A,$$

where A is the time rate of amortization of the equipment's money-cost, K_i is the actual time required for replacement or repair of Part i alone, and K_{0i} that for Parts i and 0 together. Similarly, C_i is the money-cost of replacement or repair for Part i alone, and C_{0i} that for Parts i and 0 together.

The first problem is to evaluate the expected length of the interval between replacements of the non-monitored part, L^+. It is assumed that the distribution of times to failure for Part i has an exponential density with rate of failure $\lambda_i (i = 0 \cdots M)$. Let V_i be the time to the first failure of the monitored Part i after n_i, given that the part is good at n_i. The random variable V_i has an exponential density with rate of failure λ_i. Secondly, let X be the age of the non-monitored part when replacement of that part is begun. Then X is defined by: $X = \min [n_1 + V_1, n_2 + V_2 \cdots n_M + V_M, N]$; that is, the age of the non-monitored part when its replacement begins is its age when a monitored part first fails, at which time both parts are to be replaced, or else its age is N, the time at which the non-monitored part is to be replaced even if no monitored part has failed, whichever is smaller. Next, let Z_i be the length of the interval over which Part i alone would be replaced were it to fail. Clearly, $Z_i = \min [X, n_i]$. Then let U_i^+ represent the expected (imputed) time spent replacing Part i alone from 0 to Z_i, where K_i^+ is the (imputed) time required for replacement: $U_i^+ = \lambda_i Z_i K_i^+$. Finally, let $U^+ = \sum_{i=1}^{M} U_i^+$ and let W^+ represent the time spent in replacement at the end of the interval. The expected length of the interval is then:

$$L^+ = E[X + U^+ + W^+]. \tag{37}$$

To evaluate this expression we first consider the expected value of X, which represents the age of the non-monitored part when its replacement is begun. The cumulative density of X for $x < N$ is:

$$\begin{aligned} P\{X \geqq x\} &= P\{n_i + V_i \geqq x, i = 1 \cdots M\}, \\ &= P\{V_i \geqq x - n_i, i = 1 \cdots M\}, \\ &= \textstyle\prod_{i=1}^{M} P\{V_i \geqq x - n_i\}, \\ &= \textstyle\prod_{i=1}^{M} \exp [-\sum_{x-n_i \geqq 0} \lambda_i(x - n_i)], \end{aligned}$$

and for $X = N$:

$$P\{X = N\} = \exp [-\textstyle\sum_{i=1}^{M} \lambda_i(N - n_i)].$$

Now, let

$$\begin{aligned} \mu_i &\equiv \textstyle\sum_{j=1}^{i} \lambda_j , \\ D_i &\equiv \exp \textstyle\sum_{j=1}^{i} \lambda_j n_j , \\ f_i(X) &\equiv D_i \mu_i e^{-\mu_i X}. \end{aligned}$$

The distribution of X is given by the density:

$$X = \begin{cases} 0 & \text{for } X \leqq n_i \\ f_i(X) & \text{for } n_i \leqq X \leqq n_{i+1} , \qquad i + 1 \cdots M - 1 \\ f_M(X) & \text{for } n_M \leqq X \leqq N, \end{cases}$$

with probability mass

$$\exp - \textstyle\sum_{i=1}^{M} \lambda_i(N - n_i) = D_M e^{-\mu_M N},$$

concentrated at $X = N$.

To calculate the expected value of X we must evaluate the expression:

$$E(X) = \sum_{i=1}^{M} \int_{n_i}^{n_{i+1}} X f_i(X)\, dX + N D_M e^{-\mu_M N}, \quad \text{where } n_M + 1 = N.$$

For each of the integrals in the first sum we have:

$$\int_{n_i}^{n_{i+1}} X f_i(X)\, dX = \int_{n_i}^{n_{i+1}} X D_i \mu_i e^{-\mu_i X}\, dX = D_i \mu_i \int_{n_i}^{n_{i+1}} X e^{-\mu_i X}\, dX,$$

where:

$$\int_{n_i}^{n_{i+1}} Xe^{-\mu_i X}\,dX = \left[_{n_i}^{n_{i+1}} -\frac{e^{-\mu_i X}}{\mu_i}\left(\frac{1}{\mu_i}+X\right)\right.$$
$$= \frac{e^{-\mu_i n_i}}{\mu_i}\left[\frac{1}{\mu_i}+n_i\right] - \frac{e^{-\mu_i n_{i+1}}}{\mu_i}\left[\frac{1}{\mu_i}+n_{i+1}\right],$$

so that:

$$E(X) = \sum_{i=1}^{M} D_i\left[e^{-\mu_i n_i}\left(\frac{1}{\mu_i}+n_i\right) - e^{-\mu_i n_{i+1}}\left(\frac{1}{\mu_i}+n_{i+1}\right)\right] + N\,D_M\,e^{-\mu_M N}.$$

It is possible to simplify this formula by observing that since $D_i = e^{\lambda_i n_i} D_{i-1}$ and $\mu_i = \mu_{i-1} + \lambda_i$,

$$D_i\,e^{-\mu_i n_i}\,n_i = D_{i-1}\,e^{-\mu_{i-1} n_i} n_i,$$

so that

$$E(X) = n_1 + \sum_{i=1}^{M}\frac{D_i}{\mu_i}\left[e^{-\mu_i n_i} - e^{-\mu_i n_{i+1}}\right]$$
(37)
$$= n_1 + \sum_{i=1}^{M}\frac{D_i\,e^{-\mu_i n_i}}{\mu_i}\left[1 - e^{-\mu_i(n_{i+1}-n_i)}\right].$$

For future convenience, we introduce the notation

$$m_0 = n_i$$
$$m_i = \frac{D_i\,e^{-\mu_i n_i}}{\mu_i}\left[1 - e^{-\mu_i(n_{i+1}-n_i)}\right], \qquad (i = 1 \cdots M),$$

so that:

$$E(X) = \textstyle\sum_{i=0}^{M} m_i .$$

To evaluate $E(Z_i)$, where Z_i is a random variable with density equal to that of X for $X < n_i$ and probability mass $P\{X = n_i\}$ concentrated at n_i, we proceed as follows:

$$E(Z_i) = \sum_{j=1}^{i-1}\int_{n_j}^{n_{j+1}} Xf_j(X)\,dX + n_i\left[1 - \sum_{j=1}^{i-1}\int_{n_j}^{n_{j+1}} f_j(X)\,dX\right],$$
$$= \sum_{j=1}^{i-1} D_j\left[e^{-\mu_j n_j}\left(\frac{1}{\mu_j}+n_j\right) - e^{-\mu_j n_{j+1}}\left(\frac{1}{\mu_j}+n_j\right)\right]$$
$$+ n_i\left[1 - \sum_{j=1}^{i-1} D_j(e^{-\mu_j n_j} - e^{-\mu_j n_{j+1}})\right],$$

which may be simplified as follows:

$$E(Z_i) = n_i + \textstyle\sum_{j=1}^{i-1} m_i - n_i\,D_{i-1}\,e^{-\mu_{i-1} n_i}$$
(38)
$$+ n_i[1 - \textstyle\sum_{j=1}^{i-1} D_j\,(e^{-\mu_j n_j} - e^{-\mu_j n_{j+1}})]$$
$$= \textstyle\sum_{j=0}^{i-1} m_i .$$

Since

$$E(U^+) = E(\textstyle\sum_{i=1}^M U_i^+) = \sum_{i=1}^M \lambda_i K_i^+ E(Z_i),$$

we have:

$$E(U^+) = \textstyle\sum_{i=1}^M \lambda_i K_i^+ \sum_{j=0}^{i-1} m_i . \tag{39}$$

Finally, to calculate $E(W^+)$, let d_i represent the probability that the cycle ends on the interval $[n_i, n_{i+1}]$:

$$d_i = P\{n_i \leqq X \leqq n_{i+1}\}, \qquad (i = 1 \cdots M),$$

where $n_{M+1} = N$ as before. Then:

$$d_1 = [1 - e^{-\mu_1(n_2-n_1)}],$$

$$\cdots\cdots$$

$$d_i = \textstyle\prod_{j=1}^{i-1} (1 - a_j)a_i ,$$

$$\cdots\cdots$$

$$d_{M+1} \equiv \textstyle\prod_{j=1}^{M} (1 - a_j) = D_M e^{-\mu_M N},$$

where

$$a_j = 1 - e^{-\mu_j(n_{j+1}-n_j)}.$$

Next let p_i represent the probability that the cycle ends with a replacement of Part i and Part 0 together. Then

$$p_i = \textstyle\sum_{j=i}^M d_j(\lambda_i/\mu_j) = \lambda_i \sum_{j=i}^M (d_j/\mu_j),$$

so that:

$$E(W^+) = \textstyle\sum_{i=1}^M p_i K_{0i}^+ + d_{M+1} K_0^+. \tag{40}$$

This completes the computations necessary to evaluate the expected length of the interval, L^+.

To calculate G^+ it is necessary to compute the accumulated reliability of the non-monitored part to replacement, namely:

$$\begin{aligned} T &= E_X\left[\int_0^X R(X)\,dX\right], \\ &= \frac{1}{\lambda_0} - \frac{1}{\lambda_0} E(e^{-\lambda_0 X}). \end{aligned} \tag{41}$$

The expected value of the function $e^{-\lambda_0 X}$ is calculated as follows:

$$\begin{aligned} E(e^{-\lambda_0 X}) &= \sum_{i=1}^M \int_{n_i}^{n_{i+1}} e^{-\lambda_0 X} D_i \mu_i e^{-\mu_i X}\, dX + e^{-\lambda_0 N} D_M e^{-\mu_M N}, \\ &= \sum_{i=1}^M D_i \mu_i \int_{n_i}^{n_{i+1}} e^{-(\lambda_0+\mu_i)X}\, dX + D_M e^{-(\lambda_0+\mu_M)N}, \\ &= \sum_{i=1}^M \frac{D_i \mu_i}{\lambda_0 + \mu_i} [e^{-(\lambda_0+\mu_i)n_i} - e^{-(\lambda_0+\mu_i)n_{i+1}}] + D_M e^{-(\lambda_0+\mu_M)N}, \\ &= \sum_{i=1}^M \frac{D_i \mu_i}{\lambda_0 + \mu_i} e^{-(\lambda_0+\mu_i)n_i}[1 - e^{-(\lambda_0+\mu_i)(n_{i+1}-n_i)}] + D_M e^{-(\lambda_0+\mu_M)N}, \end{aligned} \tag{42}$$

so that:

$$T = 1/\lambda_0\{[1 - \textstyle\sum_{i=1}^{M} (D_i\mu_i)/(\lambda_0 + \mu_i)e^{-(\lambda_0+\mu_i)n_i}(1 - e^{-(\lambda_0+\mu_i)(n_{i+1}-n_i)}) + D_M\, e^{-(\lambda_0+\mu_M)N}]. \tag{43}$$

The maximization of $G^+(n_i, N)$ is straightforward. A set of applications of policies of the (n_i, N) class to a hypothetical missile system is given in [2].

References

1. Radner, R. and Jorgenson, D., "Optimal Replacement and Inspection of Stochastically Failing Equipment," in K. J. Arrow, S. Karlin, and H. Scarf (Eds.), *Studies in Applied Probability and Management Science*, Stanford, California: Stanford University Press, 1962.
2. Jorgenson, D. and McCall, J. J., *Optimal Scheduling of Replacement and Inspection*, The RAND Corporation, Memorandum RM-3098-PR, May, 1962.
3. Bellman, R., *Dynamic Programming*, Princeton, New Jersey: Princeton University Press, 1957.
4. Jorgenson, D. W. and McCall, J. J., "Optimal Replacement Policies for a Ballistic Missile," *Management Science*, Vol. 9, No. 3 (April 1963), pp. 358–379.
5. McCall, J. J., "Operating Characteristics of Opportunistic Replacement and Inspection Policies," *Management Science*, Vol. 10, No. 1 (October 1963).